# Indeterminateness and 'The' Universe of Sets: Multiversism, Potentialism, and Pluralism

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**Abstract**

In this article, I survey some philosophical attitudes to talk concerning 'the' universe of sets. I separate out four different strands of the debate, namely: (i) Universism, (ii) Multiversism, (iii) Potentialism, and (iv) Pluralism. I discuss standard arguments and counterarguments concerning the positions and some of the natural mathematical programmes that are suggested by the various views.

## Contents



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# Introduction

This chapter deals with the question of what we mean when we talk about "the" universe of sets: Do we talk about one determinate structure/universe when we do so, or many? Perhaps the most natural view, before we have deeply engaged with the mathematical details, is to assume that our set-theoretic axioms are about a unique maximal set-theoretic universe. What are axioms for set theory about? Why the sets of course!

Developments in mathematics from the latter half of the 19$^{\text{th}}$ century to the present day have challenged this natural idea, however. In particular, the discovery of the set-theoretic paradoxes and use of extensions (e.g. forcing) in proving independence results have been argued to put pressure on the believer in a unique maximal universe of sets. A popular response has been to admit that our talk of "the" universe of sets is really indeterminate in various ways. In this chapter, I'll explain the different ways authors have answered the challenges of paradoxes and extensions and the mathematics that has resulted. Since this article is meant to be expository rather than argumentative, my aim is to give a map of the terrain rather than argue for one particular route through it (though of course I will give some evaluation as we go). My discussion will be guided by the following questions:

(1.) What kinds of views are there about our talk concerning "the" universe of sets?

(2.) What is the status of independent questions (e.g. CH) on each view?

(3.) What are the salient challenges for the proponent of each view, and how do they respond to them?

(4.) What mathematical programmes are suggested by the views?

Here's the plan: In §1 I outline the core problems in detail, in particular the set-theoretic paradoxes and response via the iterative conception of sets (§1.1), the use of extensions in proving independence (§1.2), and the role extensions play in proving theorems (§1.3) and formulating axioms (§1.4). I'll then (§2) explain the Universist position that there is a unique maximal universe of set theory determining set-theoretic truth, some arguments for and against it, and how it interprets some of the constructions and challenges from §1 as well as some natural mathematical programmes motivated by the position. Next (§3) I'll examine positions that deny that there is a single maximal universe of set theory that determines the truth value of every sentence of set theory (let's call this class of views Anti-Universism). Here, we'll consider the multiversist position that the subject matter of set theory is actually constituted by a plurality of universes (§3.1), the potentialist viewpoint that takes set theory to be inherently modal (§3.2), the idea that there is one universe of sets that is indeterminate (§3.3), and we'll briefly explore some mathematical programmes associated with the views (§3.4). I'll then (§4) consider the links between the views essayed and the pluralist idea that we should investigate many different set theories, and should not treat a particular one as foundationally privileged. Finally (§5) I'll conclude with some brief remarks and suggestions for further research.

Two short remarks are needed before we get into the details:

The first concerns the use of the term "universe" in this debate and more widely. On the one hand "universe" is often used to mean a structure in the model-theoretic sense as a set coding a tuple of domain, constants, functions, and relations (and in the case of "universe" specifically one with a membership relation and satisfying some specified set-theoretic axioms). On the other hand, by "universe" one could mean something more general and philosophical; namely the place(s) where our set-theoretic talk is interpreted. In this paper, I mean the broader philosophical sense of the term (though, as we'll see, for some views there is a collapse between the set-theoretic and philosophical notion).

Second, a quick remark on the scope of the paper and how to read it. We'll cover a lot of ground and we'll touch on enough material to easily fill a textbook. Obviously, this means that I've had to sacrifice depth for the sake of breadth, and that some areas will be either too easy or too difficult, contingent on the reader's level of expertise. My aim is threefold: First, the last twenty years (since the early 2000s) has seen something of an explosion in the literature on the philosophy of set theory and how it interacts with more mathematical considerations, and I hope that the piece can help students and researchers by consolidating various ideas and concepts into one place. Second, given the current depth of literature out there, I hope that the chapter can serve as a general roadmap for the neophyte interested in entering into some of the mentioned debates by helping them to navigate this difficult terrain more easily. Where depth has been sacrificed, I hope to have provided sufficient references to guide the reader to the details. Third, by bringing together a wide variety of material, we can draw some connections between them that would not be possible in a more specific research-focussed piece. Important here is the relationship between the various positions (e.g. between Potentialism, Multiversism, and Pluralism) that will form the backbone of the chapter. For this reason, I hope the piece will be of interest to experts as well as relative newcomers.

# 1 The Core Problems

This section will lay out the mathematical data on which the rest of the rest of the paper will be based. In particular we'll explain the set-theoretic paradoxes and the iterative conception as a response (§1.1), the adding of subsets to prove independence (§1.2), and uses of extensions proving theorems (§1.3) and formulating axioms (§1.4).

## 1.1 Paradoxes

The set-theoretic paradoxes have been known since the late 1800s, and are elementary by today's standards. The core point for the perspective of mainstream contemporary set theory is that it not the case (contra early Frege) that for a universe of set theory and well-formed condition $\phi$, there is a set containing just the sets satisfying $\phi$ in the universe.

Given Frege's second-order system, one could derive the following principle about sets (rendered in today's set-theoretic notation):

> **Unrestricted Comprehension.** Let $\phi$ be a formula in the language of set theory, then:
>
> $$\exists x \forall y (y \in x \leftrightarrow \phi(y))$$

As is well known nowadays, this leads to contradictions by considering the conditions "$x \notin x$", "$x = x$", and "$x$ is an ordinal". Thus, in **ZFC**-based set theory, there is no set of non-self-membered sets, universal set or set of all ordinal numbers (conceived of as transitive sets well-ordered by $\in$).[1]

**Remark 1.** There are several other upshots one might take from the paradoxes. One might, for example, take them as evidence for dialethism (e.g. [Priest, 2002]). Since this chapter is (mostly) focussed on classical **ZFC**-based set theory, I'll set this issue aside, despite its interest.

Of course, there is then the challenge of saying *why* these well-formed conditions do not define sets. That brings us on to:

**Responding to the paradoxes: The iterative conception of set.** **ZFC** set theory blocks the paradoxical reasoning by placing restrictions on the level of Comprehension allowed. Specifically, instead of full Unrestricted Comprehension, we have the:

> **Axiom Scheme of Separation.** If $\phi$ is a formula in $\mathscr{L}_\in$ with $y$ not free then:
>
> $$\forall x \exists y \forall z [z \in y \leftrightarrow (z \in x \land \phi(z))]$$

[1] I won't in general include proofs, since doing so will take up far too much space. The proofs of the paradoxes can be found in several texts, but [Giaquinto, 2002], Part II, Ch. 1 is especially thorough. The case of ordinals in the **ZFC** is slightly vexed, since one depends on a particular coding of the ordinal numbers in order to cash out the exact content of the theorem (as I've presented it in the text, it pertains to von Neumann ordinals). Some authors have suggested that we might view ordinals as Urelemente, and hence the Burali-Forti paradox as a paradox of property theory rather than set theory. See, for example, [Menzel, 1986] and [Menzel, 2014].

Essentially Separation restricts Comprehension so that the domain we extract the set of $\phi$ from is also a set. We therefore can't collect *all* satisfiers of $\phi$ into a set, just the ones in some given set or other.

Explaining *why* we cannot collect all satisfiers of a condition is a problem some have seen for the philosophy of set theory (i.e. what is it about the nature of sets that prevents having a set of all $\phi$?). There is some debate as to whether or not the mere fact of the contradiction is enough.[2] However, one important idea that has been mobilised in answering this questions is that sets are given to us by the *iterative conception of set*. Shoenfield expresses it as follows:

> Sets are formed in *stages*. For each stage $S$ there are certain stages which are before $S$. At each stage $S$, each collection consisting of sets formed at stages before $S$ is formed into a set. There are no sets other than the sets which are formed at the stages. ([Shoenfield, 1977], p. 323)

There's lots to say about the iterative conception of set, in particular how it might relate to the justification of axioms like **ZFC**.[3] For now, let us note that the conception *seems* to block the paradoxes. If we are thinking of sets as formed in a well-founded sequence of stages, by starting with the empty set and then taking all available sets of sets at successor stages, then we always get new sets at successor stages. All sets are non-self-membered, and there is no set of all non-self-membered sets or universal set since the satisfiers of the two conditions appear unboundedly in the stages. Similarly for the Burali-Forti paradox; standard ways of picking representatives for the ordinals (such as the von Neumann representation) have representatives unboundedly in the stages, and so there is no set of all of them.[4]

[2]Dummett, for example, refers to merely pointing to the fact of contradiction as to merely "wield the big stick" ([Dummett, 1994], p. 26) an idea in turn taken up by [Linnebo, 2010]. Soysal, on the other hand, provides a close relative of the idea that the contradiction is enough, but with additional content given by the underlying iterative conception (see [Soysal, 2020]).

[3]The literature here is huge, but (for example) [Boolos, 1971] motivates the axioms on the basis of the iterative conception, [Boolos, 1989] doubts how far it can take us, and [Paseau, 2007] examines Boolos' arguments. [Potter, 2004] doubts Replacement on the basis of the iterative conception, and [Maddy, 2011] argues that we should be doubtful of its justificatory force. An introduction to some of these ideas is available in [Linnebo, 2017], esp. Chs. 10 and 12.

[4]As mentioned in a previous footnote, [Menzel, 1986] argues that ordinals might be allowed to be Urelemente. As we're restricting to pure set theory, this isn't so important, however if we allow arbitrary collections of Urelemente at the first stage, then we would get a set of all ordinals at the second stage. This idea is explored in [Menzel, 2014], with a restriction on the axiom of Replacement used to keep things consistent.

We then (so the thinking goes) can see why Separation is true instead of Unrestricted Comprehension: Given some set $x$ first formed at stage $S_\alpha$, all members $y$ of $x$ such that $\phi(y)$ are available earlier than $S_\alpha$, and so (assuming that we take all possible sets at successor stages) there should be a set of all of the $\phi$s in $x$ at latest at stage $S_\alpha$. In this sense, it seems that the iterative conception licences in favour of Separation and tells us why Unrestricted Comprehension should be false.[5] We will see some further discussion of this 'solution' in §2.2 and §3.1.1.

The 'stage theory' description of the iterative conception (which can be formally expressed[6]) has a corresponding theorem in **ZFC**. There we can define:

**Definition 2.** The *Cumulative Hierarchy of Pure Sets composed of the* $V_\alpha$ is defined in **ZFC** as follows:

(i) $V_0 = \emptyset$

(ii) $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, for successor ordinal $\alpha + 1$.

(iii) $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$, for limit ordinal $\lambda$.

(iv) $V = \bigcup_{\beta\in On} V_\beta$.

We can then prove:

**Theorem 3.** (**ZF**) For any set $x$, there is an $\alpha$ such that $x \in V_\alpha$.

There is a sense then in which the iterative conception is more than just a philosophical motivation for the **ZFC** axioms, but also (given **ZF**) it is just a mathematical fact of life; every set quite simply *has* to belong to some $V_\alpha$.

There is a philosophical question as to what extent the use of quantification over the ordinals and the $V_\alpha$ allows us to divorce the iterative conception from the temporal terms in which it is initially couched. This will be treated differently by different theorists, and we shall see some discussion of it throughout this chapter (especially §3.2). Still, articulating an interpretation of the iterative conception and what role it plays is an important task for each view concerning 'the' universe of sets.

[5] Interestingly though, the history here is not quite as neat as one might like. It is not really until Zermelo (in [Zermelo, 1930]) that we start to see the idea of cumulative hierarchy appear. This was then further integrated in Gödel's work on $L$ (in [Gödel, 1940]), but it was not until the late 1960s and 1970s that the idea of the iterative conception and its relation to **ZFC** were fully isolated (e.g. [Boolos, 1971]). Separation, however, was in currency long before the iterative conception was widely accepted (it appears, for example, in [Zermelo, 1908]). See [Kanamori, 1996] for a summary of the history.

[6] e.g. in [Boolos, 1971].

## 1.2 Independence

At the turn of the 20th century, at the International Congress of Mathematicians in Paris, Hilbert presented ten of what he considered to be the 23 most important problems facing mathematics. Number one on his list was the resolution of the Continuum Hypothesis ($\mathsf{CH}$); the claim that there are no cardinalities intermediate between that of the natural numbers and that of the reals (i.e. $2^{\aleph_0} = \aleph_1$). Since reals are coded by subsets of $\omega$, and functions from $\mathcal{P}(\omega)$ to its subsets are coded by sets of ordered pairs, questions like $\mathsf{CH}$ (as well as many other questions in set theory) depend upon what subsets exist. This observation, combined with the understanding that we can have different models of **ZFC** (a fact already known by the work of Leopold Löwenheim and Thoralf Skolem prior to 1920), lead to the unusual resolution of the Continuum Hypothesis (as far as the axioms of **ZFC** were concerned). It was shown that $\mathsf{CH}$ is *independent* from the axioms of **ZFC** (i.e. $\mathbf{ZFC} \nvdash \mathsf{CH}$ and $\mathbf{ZFC} \nvdash \neg\mathsf{CH}$). Number two on Hilbert's list of problems was to show that arithmetic was consistent. Instrumental in answering this question was Gödel's Incompleteness Theorems; Gödel showed that for any recursive theory $\mathbf{T}$ capable of representing Primitive Recursive Arithmetic (assuming its $\omega$-consistency), $\mathbf{T}$ could not prove its own consistency sentence (i.e. the claim that there is no natural number coding a proof of $0 = 1$ from $\mathbf{T}$).

These results have lead to two different kinds of independence. (i) $\mathsf{CH}$ is independent from **ZFC**, but adding either $\mathsf{CH}$ or $\neg\mathsf{CH}$ to **ZFC** does not increase its strength in the sense that both $\mathbf{ZFC} + \mathsf{CH}$ and $\mathbf{ZFC} + \neg\mathsf{CH}$ are consistent just in case **ZFC** is (i.e. $\mathsf{CH}$ is an *Orey* sentence for **ZFC**). (ii) Adding $Con(\mathbf{ZFC})$, on the other hand, *does* increase theory strength; there are theories (e.g. **ZFC**) that we can prove consistent in $\mathbf{ZFC} + Con(\mathbf{ZFC})$ that we couldn't prove consistent in **ZFC**.

There are several mathematical principles and techniques that have been developed as means to exploring these kinds of independence. We survey some of them here.

**Large cardinals.** One challenge when considering large cardinal axioms is that there is no fully precise definition of what they are. However, the rough idea is that there are certain axioms of set theory that imply the existence of cardinal numbers with closure properties in certain models. For example, we can consider the following:

**Definition 4.** A cardinal $\kappa$ is *strongly inaccessible* iff it is an uncountable regular strong limit cardinal.

Any such $\kappa$ satisfying this definition cannot be reached from below by the operations of Powerset and Replacement. The existence of at least one such object yields $Con(\mathbf{ZFC})$ (and in fact much more), since if $\kappa$ is strongly inaccessible then $V_\kappa \models \mathbf{ZFC}$.

Far stronger large cardinal axioms have been defined, and there is a whole hierarchy of increasing consistency strengths, that appear, as far as we can see, to be linearly ordered. Often strength and size considerations go hand in hand, as often a cardinal of a particular kind will contain many cardinals of another kind below it. For example, a *Mahlo* cardinal is a cardinal $\kappa$ that is strongly inaccessible and the set of regular cardinals below $\kappa$ is stationary. This implies the existence of many inaccessible cardinals (in fact $\kappa$-many) below $\kappa$. However, size and strength do not always go hand in hand, if both superstrong and strong cardinals exist, then the least strong cardinal is smaller than the least superstrong, however one can build a model of a strong cardinal from a superstrong cardinal.[7]

The details are dealt with in detail in several places, and an excellent survey is available in [Koellner, 2011]. Important for later (e.g. §3.1.1) is the observation that there is no largest consistent axiom at the top of the large cardinal hierarchy. For example, given a large cardinal axiom $\phi$, one can come up with a stronger (in terms of consistency strength) large cardinal axiom:

**Axiom 5.** There is an $\alpha$ such that $V_\alpha \models \mathbf{ZFC} + \phi$.[8]

Of course this template will immediately prove the consistency (with **ZFC**) of any large cardinal axiom $\phi$ one desires. Whilst there is nothing mathematically deep here, it shows that there is no limit to the consistency strength of the usual (consistent) large cardinal axioms.

**Inner models.** One of the first and most well-known inner model constructions was developed in [Gödel, 1940]. What Gödel showed was that by controlling very precisely the subsets allowed, we could (assuming that **ZFC** is consistent) generate a model $L$ of **ZFC** satisfying CH, and hence $\neg$CH is not a consequence of **ZFC**. More precisely, he defined the following structure via transfinite recursion:

**Definition 6.** The *constructible hierarchy* or $L$ is defined as follows:

(i) $L_0 = \emptyset$

[7] See [Kanamori, 2009], p. 360.

[8] In fact, simply asserting the existence of a set-sized model $\mathfrak{M} \models \phi$ would do. All we require to get $Con(\mathbf{ZFC} + \phi)$ is for *some* model to satisfy the axiom.

(ii) $L_{\alpha+1} = \{x| \text{"}x \text{ is definable over } L_\alpha \text{ with parameters in } L_\alpha\text{"}\}$.

(iii) $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ (for limit $\lambda$).

(iv) $L = \bigcup_{\beta \in On} L_\beta$

and then showed that $L \models$ **ZFC** + CH and hence **ZFC** $\nvdash \neg$CH (assuming **ZFC** is consistent). As one can see from its construction, we define $L$ by keeping a very tight control on what subsets are formed at successor stages in the recursion; we only allow those subsets that are definable in the language of **ZFC** with parameters available at prior stages.

Since Gödel's pioneering work, an enormous diversity of similar models, so called 'inner models' have been studied and discovered. This can be done by relaxing the parameters allowed in the construction (e.g. by allowing arbitrary ordinal parameters in defining $HOD$), allowing additional predicates into our notion of definability, or building $L$ over some initially specified set rather than the empty set. In particular we can build $L$-like models that contain certain large cardinals ($L$ itself cannot tolerate very strong large cardinals), a discipline known as *inner model theory*. This represents some exceptionally sophisticated constructions with some difficult open questions.[9] The point that will be relevant for later is just that we study a wide variety of different structures on which we try and carefully control the subsets present.

**Set forcing.** Whilst the use of inner models seems to place *constraints* on the subsets we take at successor stages, forcing looks to *expand* the subsets we have, given some antecedently given model. More specifically, for *set forcing* we begin with a partial order with domain $P$, ordering $\leq_{\mathbb{P}}$, and maximal element $1_{\mathbb{P}}$, denoted by '$\mathbb{P} = \langle P, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$', and have $\mathbb{P} \in \mathfrak{M}$ for some **ZFC** model[10] $\mathfrak{M}$. The relevant $p \in P$ are known as *conditions* and provide partial information about how objects are defined in an extension. Using a generic filter $G$ on $\mathbb{P}$ we then, via a

[9]Current open problems are to build an $L$-like model for a Woodin limit of Woodin cardinals and (assuming the Unique Branch Hypothesis) a supercompact cardinal. See [Sargsyan, 2013] for an overview.

[10]One does not need full **ZFC**, and forcing over models of weaker theories is well studied. Indeed, in several mathematics texts (such as [Kunen, 2013]), one uses the reflection theorem to obtain a model of 'enough' **ZFC** to conduct the independence proof. We will talk a little about this strategy in §2.

careful choice of names (known as '$\mathbb{P}$-*names*')[11], and evaluation procedure[12] add a filter $G$ on $\mathbb{P}$ that intersects all dense sets of $\mathbb{P}$ in $\mathfrak{M}$ to $\mathfrak{M}$. The end result is a model $\mathfrak{M}[G]$ that (i) satisfies **ZFC**, (ii) has exactly the same ordinal height as $\mathfrak{M}$, and (iii) is strictly larger than $\mathfrak{M}$ (in the sense that $\mathfrak{M} \subset \mathfrak{M}[G]$).[13]

Set forcing is historically significant in that it has been used to settle many open questions (the most famous examples being the independence of CH and AC).

In particular we can note the following:

**Theorem 7.** [Cohen, 1963] If **ZFC** is consistent, then so is **ZFC** $+ \neg$CH (and hence, **ZFC** $\nvdash$ CH).

The proof proceeds by taking a model $\mathfrak{M}$ of **ZFC** $+$ CH ($L$ will do)[14] and adding $\kappa$-many reals for some $\mathfrak{M}$-cardinal greater than $\aleph_1$ (this poset is often denoted $Add(\mathcal{P}(\omega), \kappa)$). One can then show, by looking at properties of the partial order (namely that it has the countable chain condition and hence it does not destroy cardinals) that the resulting model (that we call $\mathfrak{M}[G]$) satisfies $\neg$CH.

The situation is in fact even more extreme. Once we have destroyed CH, we can resurrect it again. To do this, we take a partial order $Col(\omega_1, |\mathcal{P}(\omega)|)$ that will collapse the cardinality of $\mathcal{P}(\omega)$ back to $\aleph_1$, restoring CH. And one can repeat the process, turning CH off and on like a light switch. In fact, more generally:

**Theorem 8.** Let $\mathfrak{M}$ be any model of **ZFC**. Then there are forcing extensions:[15]

(i) $\mathfrak{M}[G]$, adding no new reals, such that $\mathfrak{M}[G] \models$ CH.

(ii) $\mathfrak{M}[H]$, collapsing no cardinals, such that $\mathfrak{M}[H] \models \neg$CH.

[11] A $\mathbb{P}$-name is a relation $\tau$ such that $\forall \langle \sigma, p\rangle \in \tau$["$\sigma$ is a $\mathbb{P}$-name" $\wedge p \in \mathbb{P}$]. In other words, $\tau$ is a collection of ordered pairs, where the first element of each pair is a $\mathbb{P}$-name and the second is some condition in $\mathbb{P}$.

[12] We evaluate $\mathbb{P}$-names by letting the value of $\tau$ under $G$ (written '$val(\tau, G)$' or '$\tau_G$') be $\{val(\sigma, G) | \exists p \in G(\langle \sigma, p\rangle \in \tau)\}$. The valuation operates stepwise by analysing the valuation of all the names in $\tau$ and then either adding them to $\tau_G$ (if there is a $p \in G$ and $\langle \sigma, p\rangle \in \tau$) or discarding them (if there is no such $p \in G$).

[13] It should be noted that in order for the forcing to be non-trivial, $\mathbb{P}$ has to be *non-atomic* (i.e. every $p \in P$ has incompatible extensions in $\mathbb{P}$).

[14] Of course, if the model already satisfies **ZFC** $+ \neg$CH we are done immediately.

[15] Exactly what it means to be a generic extension might need to be coded, depending on your philosophical position. For example, if we take $V$ to be starting model and you think that there is just one universe of sets, we can't add any $G$ to $V$. We'll see some more discussion of this in §2.

However, especially *philosophically* interesting is that as long as a generic is available, forcing preserves *standardness*. A model $\mathfrak{M}$ is normally called *standard* iff it has the real $\in$-relation. When there is a generic available, the resulting forcing extendion $\mathfrak{M}[G]$ will be standard if $\mathfrak{M}$ is.[16] Thus, assuming that the ground model $\mathfrak{M}$ is transitive, well-founded, and satisfies **ZFC**, and that there is a generic $G$ available, the forcing extension $\mathfrak{M}[G]$ (i) has the same ordinals as $\mathfrak{M}$, (ii) satisfies **ZFC**, and (iii) is transitive and well-founded. In this way, generic extensions of a standard model of **ZFC** are also **ZFC**-satisfying cumulative hierarchies and look like more legitimate models compared to models of $\neg Con(\mathbf{ZFC})$, which can only be true on an $\omega$-nonstandard model of **ZFC**.

**Class forcing.** *Class forcing* is very similar to set forcing, except here we do not insist that $\mathbb{P}$ is a *member* of $\mathfrak{M}$ and also allow *proper-class-sized* partial orders that are *subclasses* of $\mathfrak{M}$. Apart from that, things are somewhat similar; the technique also uses partial orders with maximal elements $\langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$, and adds a generic $G$ to our ground model $\mathfrak{M}$. Of course with class forcing, $\mathbb{P}$ (and hence any associated dense classes and $G$) can now be proper-class-sized rather than just set-sized. There are some additional intricacies and features concerning the details of how it functions, but we'll suppress these for ease of reading.[17]

An important early application of class forcing was *Easton forcing*. Here we use a proper class of conditions and the notion of *Easton product* to define a notion of forcing coding any desired pattern (consistent with the constraints provided in **ZFC** provided by König's Theorem) into the continuum function (i.e. the function $f : \kappa \mapsto 2^{\kappa}$) on regular cardinals. More precisely, we can show:

**Theorem 9.** (Easton)[18] Let $\mathfrak{M}$ be a transitive model of **ZFC** such that the Generalised Continuum Hypothesis holds. Let $F$ be a function that is defined on all regular cardinals and outputs cardinal numbers such that:

1. $F(\kappa) > \kappa$.

[16] See [Kunen, 2013], §IV.2 for verification of the basic properties of forcing.

[17] One option here is to force over models of the form $L(A) = \bigcup\{L(A \cap V_{\alpha}) | \alpha \in On\}$. Any model $(M, A)$ of **ZF** (where we include Replacement for formulas mentioning $A$) can be changed to a model of this form by expanding it to a model $(M, A^*)$ where $A^* = \{\langle 0, x\rangle | x \in A\} \cup \{\langle 1, V_{\alpha}^M\rangle | \alpha \in On^M\}$. Details of this presentation are available in [Friedman, 2000], Chapter 2. A second (more recent) option is to proceed directly in a second-order set theory. See [Antos, 2018] for explanations of approaches of this method.

[18] See here [Jech, 2002] pp. 232–237.

2. If $\kappa \leq \lambda$ then $F(\kappa) \leq F(\lambda)$.

3. The cofinality of $F(\kappa)$ is larger than $\kappa$.

Then there is an extension $\mathfrak{M}[G]$ such that $\mathfrak{M}$ and $\mathfrak{M}[G]$ have the same cardinals and cofinalities, but for every regular $\kappa$:

$$\mathfrak{M}[G] \models 2^{\kappa} = F(\kappa)$$

Class forcing has some interesting properties when contrasted with standard set forcing. For example, there are reals we can construct using class forcing that cannot be added by set forcing.[19] Further, class forcing can violate **ZFC**. Consider the partially ordered class $Col(\omega, On)$ (i.e. functions $p$ from finite subsets of $\omega$ into $On$ ordered by reverse inclusion). This is perfectly legitimate as class forcing partial order. But forcing using it constructs a model $\mathfrak{M}[G]$ that: (i) satisfies **ZFC** as long as $G$ is *not* allowed as a class predicate, as the first-order domains of $\mathfrak{M}$ and $\mathfrak{M}[G]$ are identical[20], and (ii) if we admit $G$ as a predicate into the language Replacement fails since $G$ codes a cofinal sequence from $\omega$ to $On^{\mathfrak{M}[G]}$.[21] If we wish to restrict to class partial orders that preserve **ZFC** Replacement and Powerset, we have to consider partial orders that are *pretame* and *tame* respectively.[22]

**Hyperclass forcing.** As it turns out, we can go even further. Recently [Antos and Friedman, 2017] showed that one can define forcing that takes *classes* as the conditions, and so the forcing partial order is a *hyperclass* (i.e. a collection of classes). This is done by starting with a $\beta$-model of a strengthening of Morse-Kelley class theory,[23] before coding

[19] This is a very important result of Jensen, see [Friedman, 2010], p. 559 for details.

[20] To see this, note that for any $\mathbb{P}$-name $\sigma$ for this poset and for each condition $p$ in the intersection of the transitive closure of $\sigma$ with $\mathbb{P}$, $ran(p) \subseteq rank(\sigma)$. We then define the dense set $D = \{p \in \mathbb{P} | rank(\sigma) \in ran(p)\}$. $D$ is then both dense and definable over $\mathfrak{M}$. Letting $\sigma^p = \{\tau^p | \exists q \in \mathbb{P}[\tau, q \in \sigma \wedge p \leq_{\mathbb{P}} q]\}$. We then have $\sigma^p = \sigma^G \in M$ whenever $G$ is $\mathbb{P}$-generic over $\mathfrak{M}$ and $p \in D \cap G$, because $p$ either extends or is incompatible with any condition in the transitive closure of $\sigma$. Hence, whenever $G$ is $\mathbb{P}$-generic over $\mathfrak{M}$, they contain exactly the same first-order objects.

[21] For details, see [Holy et al., 2016].

[22] See [Friedman, 2000] for details.

[23] The relevant additional axiom is the Class Bounding principle:

$$\forall x \exists A \phi(x, A) \rightarrow \exists B \forall x \exists y \phi(x, (B)_y)$$

where $(B)_y$ is defined as follows:

$$(B)_y = \{z | \langle y, z \rangle \in B\}.$$

a height extension of $\mathfrak{M}$ (with a largest cardinal) denoted by $\mathfrak{M}^*$ and performing a definable class forcing there. Whilst the implications for independence results are not yet clear, there are reals that can be added via hyperclass forcing that are not class generic.[24]

## 1.3 Proving theorems with extensions

In the last subsection we talked about three different but related techniques for adding subclasses and/or subsets to models, *set* forcing, *class* forcing, and *hyperclass* forcing. The kinds of results we discussed there largely pertained to *independence* results; we want to show that for some extension $\mathbf{T}$ of $\mathbf{ZFC}$ and some sentence $\phi$ of set theory $\phi$, $\mathbf{T} \nvdash \phi$ and $\mathbf{T} \nvdash \neg\phi$. The way forcing allows you to do this is by building models that showed the consistency of $\mathbf{T}$ with $\phi$ (or $\neg\phi$) given some antecedently accepted consistency statement.

In the next two subsections we'll see that extensions are not just used to build models for witnessing number-theoretic consistency sentences. Rather, situating a universe within a framework of extensions can also be useful for proving theorems (§1.3) and formulating axioms (§1.4) about infinite sets.

Within $\mathbf{ZFC}$, there are a wide number of questions concerning the universe that can be settled on the basis of considering forcing extensions. The rough strategy of such theorems is to show that if $V$ has a forcing extension such that $\phi$ (for some particular $\phi$) then some other sentence $\psi$ holds of $V$ (say by using absoluteness facts). For example:[25]

**Theorem 10.** [Baumgartner and Hajnal, 1973] $\omega_1 \longrightarrow (\alpha)^2_n$ for all finite $n$ and countable $\alpha$ (i.e. For all finite $n$ and countable $\alpha$, every partition of the two-element subsets of $\omega_1$ into a finite number of pieces has a homogeneous[26] set of order-type $\alpha$).

The proof proceeds by finding a homogeneous set in a forcing extension $V[G]$ where MA holds. This then establishes that a certain tree from $V$ is non-well-founded in $V[G]$. We then know, by the absoluteness of well-foundedness, that the tree is also non-well-founded in $V$, establishing the theorem.

[24]For the details, see [Friedman, 2000], §5.1. The rough idea is to produce a real $r$ by hyperclass forcing such that for any class $A$ of the ground model $V$, the satisfaction predicate $Sat(V, A)$ is definable over $(V[r], A)$. By the Truth Lemma for class-forcing, if $r$ were class-generic then $Sat(V[r], A)$ would be definable over $(V[r], Sat(V, A))$ for some class $A$ of $V$; but then $Sat(V[r], A)$ would be definable over $(V[r], A)$, contradicting Tarski's Theorem.

[25]I am grateful to Andrés Caicedo for pointing out this example.

[26]Here, a homogeneous set is a subset $X$ of $\omega_1$ such that every 2-element subset of $X$ is in the same member of the partition.

The theorem is reasonably representative of how one can use the perspective of extensions to prove facts about the ground model. One moves to an extension where one has ensured the existence of objects of a certain desirable kind. One then uses absoluteness facts (e.g. Lévy-Shoenfield absoluteness) along with the objects in the extension to show that the desired theorem holds in the ground model. Importantly here, theorems like the above are not straightforwardly about *independence* (which is naturally interpreted number-theoretically). Rather they rather concern large infinitary objects in the ground model. A compendium of similar theorems is available in [Todorčević and Farah, 1995]. A recent further proof of this kind is the [Malliaris and Shelah, 2016] result that the two cardinal characteristics $\mathfrak{p}$ and $\mathfrak{t}$ are in fact equal, settling a major open question about the possible relative sizes of uncountable sets.[27] Their proof depends crucially on supposing for contradiction that $\mathfrak{p} < \mathfrak{t}$ in $V$, and then tracing out some consequences of this assumption (and finding a contradiction) in a forcing extension $V[G]$.

## 1.4 Formulating axioms using extensions

Extensions are also useful for formulating axioms *extending* **ZFC**. In this way the flexibility afforded by extensions often provides us with additional resources for expressing axioms with interesting properties.

**Generic embeddings.**[28] Earlier (§1.2) we discussed *large* cardinals. For many large cardinals, one way of asserting their existence is through the use of elementary embeddings. The cardinals *measurable*, *strong*, *supercompact* (among others) are all naturally defined by positing the existence of elementary embeddings from $V$ into transitive inner models. These represent *strong* axioms, in that they imply $V \neq L$. When defining a large cardinal through an embedding $j : \mathfrak{N} \longrightarrow \mathfrak{M}$, the strength of the embedding depends mainly on two parameters:[29]

(i) The size of $\mathfrak{N}$ and $\mathfrak{M}$.

(ii) Where $j$ sends the ordinals.

[27] I thank Jonathan Schilhan, Daniel Soukup, and Vera Fischer for discussion here.

[28] See here [Foreman, 2010] and [Foreman, 1986] for several key results, and [Foreman, 1998] for a more informal overview. [Eskew, 2020] provides an argument against their use as axioms for settling CH.

[29] See here [Foreman, 2010], p. 887.

To see some examples of how this works, consider the minimal case of such an embedding for proper class models; namely the existence of a non-trivial $j : L \longrightarrow L$. This suffices suffices to define the principle that "$0^\sharp$ exists".[30] If we instead assume that $dom(j) = V$ and the target model is some transitive proper class model we obtain something stronger (namely a *measurable* cardinal). A theorem of [Kunen, 1971] shows that the existence of a non-trivial $j : V \longrightarrow V$ is inconsistent with **ZFC**, and so there are some limits (within **ZFC**) for what can be defined this way. Intermediate cardinals are obtained by modifying the properties of $j$ and $\mathfrak{M}$. For example, we can increase the similarity between $V$ and $\mathfrak{M}$ as in the following definition:

**Definition 11.** A cardinal $\kappa$ is *$\lambda$-strong* iff $\kappa$ is the critical point of a non-trivial elementary embedding $j : V \to \mathfrak{M}$ such that $V_\lambda \subseteq \mathfrak{M}$.

This strengthens the definition of measurables (for $\lambda > \kappa + 1$)[31] by insisting that $V_\lambda$ be contained in $\mathfrak{M}$, increasing the similarity between $V$ and $\mathfrak{M}$. Along the other dimension we can consider:

**Definition 12.** [Hamkins, 2009] $\kappa$ is $\lambda$-tall iff $\kappa$ is the critical point of a non-trivial elementary embedding $j : V \to \mathfrak{M}$ such that $j(\kappa) > \lambda$ with ${}^\kappa\mathfrak{M} \subseteq \mathfrak{M}$.

In the definition of a $\lambda$-tall cardinal, we require that $j$ sends $\kappa$ above $\lambda$ increasing strength (for suitably large $\lambda$) beyond that of measurables. As it turns out, tall cardinals ($\lambda$-tall for every $\lambda$) and strong cardinals ($\lambda$-strong for every $\lambda$) are equiconsistent (see here [Hamkins, 2009]). Of course, the two dimensions are not completely independent. For one, we may need to insist on strength in both of the dimensions in tandem to get the desired properties (for example in the definition of $\lambda$-tallness, we need to insist that the embedding has a target model that is at least closed under $\kappa$-sequences). For another, increasing along one dimension may *imply* increasing along the other (e.g. an embedding for a $\lambda$-strong cardinal will also send $\kappa$ above $\lambda$). Similar remarks apply to the following often given definition of a *supercompact* cardinal:

**Definition 13.** A cardinal $\kappa$ is *$\lambda$-supercompact* iff it is the critical point of a non-trivial elementary embeddings $j : V \longrightarrow \mathfrak{M}$, such that $j(\kappa) > \lambda$ and ${}^\lambda\mathfrak{M} \subseteq \mathfrak{M}$ (i.e. $\mathfrak{M}$ is closed under $\lambda$-sequences).

[30] For details of $0^\sharp$, see [Jech, 2002], Ch. 18.

[31] We only get something stronger when $\lambda$ is suitably large because when $\kappa$ is measurable, there is an embedding $j : V \to \mathfrak{M}$ such that $V_{\kappa+1} = (V_{\kappa+1})^{\mathfrak{M}}$. See [Schindler, 2014], p. 51, Lemma 4.52.

The definition of $\lambda$-supercompactness postulates (i) a higher degree of similarity between $V$ and $\mathfrak{M}$ (in terms of closure under $\lambda$-sequences for the relevant $\lambda$), and (ii) stipulates that $j$ sends $\kappa$ above $\lambda$. But in fact one does not have to insist that $j(\kappa) > \lambda$—we get an equivalent definition if we remove this requirement.[32]

Given a forcing construction adding a generic $G$ over a model $\mathfrak{N}$, there is the possibility of considering *generic embeddings* $j : \mathfrak{N} \longrightarrow \mathfrak{M} \subseteq \mathfrak{N}[G]$. In other words, we begin to study embeddings from structures to *inner models of their forcing extensions*, and the embedding lives in the *forcing extension*.

These kinds of embeddings represent new possibilities for studying large cardinal like properties. Moreover, they provide a third parameter in which we can vary the strength of the relevant cardinal to be defined:

(iii) The nature of the forcing required to define $j$.[33]

Especially interesting here is also the fact that the critical points of these embeddings can be quite small (even $\omega_1$ is possible), despite their strength.[34] Thus, these embeddings provide significant combinatorial power whilst facilitating proof concerning *small* uncountable sets. In this way, they provide a kind of information that the normal variety of large cardinal defined through an embedding cannot; the criticial points of usual embeddings are always at least inaccessible.[35]

Whilst this has received slightly less attention in the literature, generic embeddings can also be defined using class forcing. In particular, by using class-sized stationary tower forcing (on the assumption that $V$ satisfies large cardinal properties), we can define generic

[32] I am grateful to an anonymous reviewer for this observation and pointing out that the reason one needn't insist that $j(\kappa) > \lambda$ is that given an embedding $j : V \to \mathfrak{M}$ with ${}^{\lambda}\mathfrak{M} \subseteq \mathfrak{M}$, there must be a finite iterate of $j$ with $j(\kappa) > \lambda$ (if not, one can obtain a violation of the Kunen inconsistency).

[33] For further exposition of this line of thinking, see [Foreman, 1998] and [Foreman, 2010].

[34] For example, the existence of both a saturated ideal on $\omega_1$ (and associated generic embedding) and a measurable cardinal implies the existence of an inner model with a Woodin cardinal, whereas the consistency strength of a measurable cardinal is far below that of a single Woodin. See [Steel, 1996] for details.

[35] For example, concerning accessible cardinals and generic embeddings, Foreman writes:

> The advantage of allowing the embeddings to be generic is that the critical points of the embeddings can be quite small, even as small as $\omega_1$. For this reason they have many consequences for accessible cardinals, settling many classical questions of set theory. ([Foreman, 2010], p887)

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embeddings. An example: Suppose that $V$ contains a proper class of completely Jónsson cardinals. Letting $\mathbb{P}_\infty$ be the class tower forcing, and $G \subset \mathbb{P}_\infty$ be $V$-generic, and $V[G]$ be:

$$V[G] = \bigcup_{\alpha \in Ord} L(V_\alpha, G \cap V_\alpha)$$

there exists a generic embedding $j : V \longrightarrow V[G]$ such that for every $a \in \mathbb{P}_\infty$, $a \in G$ iff $j[\cup a] \in j(a)$.[36]

**Virtual large cardinals.**[37] Another recent development has been the study of *virtual* large cardinals. These cardinals are defined by postulating that a particular ordinal or initial segment of $V$ has a certain large cardinal property *in an extension* of $V$ as opposed to *within* $V$. In this vein, work has been done studying the notions of *virtually supercompact*, *virtually strongly compact*, *virtually strong*, *virtually Woodin*, and *virtually extendible*. As it turns out, the inconsistency of having a $j : V \to V$ does not hold for virtual embeddings, and so one can even have a notion of *virtually rank-into-rank*.

Let us look at an example:

**Definition 14.** [Schindler, 2000] A cardinal $\kappa$ is *remarkable* iff in the $Col(\omega, <\kappa)$ forcing extension $V[G]$, for every regular $\lambda > \kappa$ there is a cardinal $\lambda_0 < \kappa$, $\lambda_0$ regular in $V$, and $j : H^V_{\lambda_0} \longrightarrow H^V_\lambda$ such that $crit(j) = \gamma$ and $j(\gamma) = \kappa$.

Remarkability of $\kappa$ is thus a property that concerns the embeddings that exist in the extension if we collapse all cardinals less than $\kappa$ to $\omega$. In this way, by studying how sets behave *in the extension*, we are able to ascribe large cardinal properties to ordinals *in* $V$. The definition turns out to be a characterisation of the notion of *virtual supercompactness*[38]. The consistency strength of a remarkable cardinal lies between a 1-iterable and 2-iterable cardinal. While not strong enough to push us outside $V = L$, it is substantially stronger than a weakly compact

[36] I thank Monroe Eskew for bringing this example to my attention. The details can be found in [Larson, 2004], §2.3, p. 59.

[37] An excellent survey of the recent developments in virtual large cardinals mentioned here is available in [Gitman and Schindler, 2018]. Two particular uses of these cardinals are to study *Silver indiscernibles* in $L$ and index the consistency strength of other kinds of virtual axioms. In fact, we could have spoken for longer about different kinds of virtual principle, such as *virtual forcing axioms*, but considerations of space prevent a presentation of the full picture.

[38] See [Gitman and Schindler, 2018], p. 2.

cardinal[39]. Thus, whilst all known virtual large cardinals are consistent with $V = L$ they can still have substantial strength in comparison to the usual cardinals consistent with $V = L$ (e.g. inaccessible, Mahlo, etc.). This feature makes them useful for studying the hierarchy of large cardinals between ineffability and $0^\sharp$.

**Absoluteness principles.** The next kind of way we can formulate axioms using extensions is as *absoluteness principles*. These state that certain principles that hold in extensions are already true in an appropriate context in $V$.

One way absoluteness principles are useful is in providing equivalent characterisations of forcing axioms. For example, Bagaria characterises Martin's Axiom (MA) and the Bounded Proper Forcing Axiom (BPFA) as follows:

**Definition 15.** [Bagaria, 1997] *Absolute*-MA. We say that $V$ satisfies *Absolute*-MA iff whenever $V[G]$ is a generic extension of $V$ by a partial order $\mathbb{P}$ with the countable chain condition in $V$, and $\phi(x)$ is a $\Sigma_1(\mathcal{P}(\omega_1))$ formula (i.e. a first-order formula containing only parameters from $\mathcal{P}(\omega_1)$), if $V[G] \models \exists x \phi(x)$ then there is a $y$ in $V$ such that $\phi(y)$.

**Definition 16.** [Bagaria, 2000] *Absolute*-BPFA. We say that $V$ satisfies *Absolute*-BPFA iff whenever $\phi$ is a $\Sigma_1$ sentence with parameters from $H(\omega_2)$, if $\phi$ holds a forcing extension $V[G]$ obtained by proper forcing, then $\phi$ holds in $V$.

These formulations make it perspicuous how some forcing axioms respond to the intuition of maximising the universe under 'possibly forceable' sets; if we could force there to be a set of kind $\phi$ (for a particular kind of $\phi$ and $\mathbb{P}$), one already exists in $V$. Some authors (e.g. [Bagaria, 2005]) see this fact as evidence for the claim that such axioms are natural in virtue of their making precise a notion of maximality.

Absoluteness characterisations of forcing axioms depend upon a careful calibration between the kinds of parameters, dense sets, and partial orders considered. For example, if one allows $\omega_1$ as a parameter and arbitrary set forcings, one immediately obtains a contradiction with **ZFC** by collapsing $\omega_1$ in $V$. A move to considering *arbitrary* extensions has been considered recently by Friedman. He considers the following:

[39] Weakly compact cardinals are so named in virtue of being characterisable through compactness properties on infinitary languages. They admit of a diverse number of equivalent characterisations. For details, see [Kanamori, 2009], p. 37.

**Definition 17.** ([Friedman, 2006]) *The Inner Model Hypothesis.* Let $\phi$ be a parameter-free first-order sentence. By an *outer model* of a model $\mathfrak{M}$, we mean a model $\mathfrak{O}$ satisfying **ZFC** with the same ordinals as $\mathfrak{M}$, and such that $\mathfrak{M} \subseteq \mathfrak{O}$. Then the *Inner Model Hypothesis* for $\mathfrak{M}$ states that if $\phi$ is true in an inner model of an outer model of $\mathfrak{M}$, then $\phi$ is already true in an inner model of $\mathfrak{M}$.

An interesting feature of this axiom is the consequences it has for large cardinals. On the one hand it implies that there are no inaccessible cardinals in $V$. On the other hand, it implies that there are measurable cardinals in inner models of arbitrarily large Mitchell order, and is consistent relative to the theory $\mathbf{ZFC} + \mathsf{PD}$. In this way it presents a somewhat different perspective on the nature of set theory, on which large cardinals are consistent but false.

The Inner Model Hypothesis, as proposed by [Friedman, 2006], is meant to apply to *arbitrary* width extensions (i.e. models with the same ordinals but more subsets) of a model, as well as *arbitrary* inner models. It is therefore often stated using higher-order resources as concerned with countable transitive models $\mathfrak{M} = (M, \in, \mathcal{C}^{\mathfrak{M}})$ in some ambient universe (possibly $V$), where quantification over $\mathfrak{M}$ and its outer models is uncontroversial. Recently, however, Antos, Barton, and Friedman showed that by using infinitary logics to code satisfaction in outer models, and coding the infinitary logic using using proper-class-sized trees, one can formulate versions of the full IMH in a variant of **MK**. We'll talk more about these kinds of coding later (see §2.3).[40]

**$\sharp$-generation.** A further use of extensions is in stating reflection properties of universes. Useful here is the notion of $\sharp$-generation. Before we begin, we need the following definition on the kinds of iterations we can perform along class well-orders:

**Definition 18.** (NBG) Let ETR (for '**E**lementary **T**ransfinite **R**ecursion') be the statement that every first-order recursive definition along any well-founded binary class relation has a solution.[41]

Given ETR we are guaranteed the ability to construct recursions along class well-orders present in a model of **NBG**. We can then define the notion of a structure being generated by a sharp with a particular kind of iteration:

[40] There are also restricted forms of the IMH that one can consider (for example as restricted to set-forcing extensions or tame class forcing extensions). For some philosophical discussion of these variants, see [Barton, 2020].

[41] For discussions of ETR, see [Fujimoto, 2012] and [Gitman and Hamkins, 2017].

**Definition 19.** (**NBG** + ETR)[42] A transitive structure $\mathfrak{N} = (N, U)$ is called a *class-iterable sharp with critical point $\kappa$* or just a *class-iterable sharp* iff:

(i) $\mathfrak{N}$ is a transitive model of $\mathbf{ZFC}^-$ (i.e. **ZFC** with the power set axiom removed) in which $\kappa$ is the largest cardinal and is strongly inaccessible.

(ii) $(N, U)$ is amenable (i.e. $x \cap U \in N$ for any $x \in N$).

(iii) $U$ is a normal measure on $\kappa$ in $(N, U)$.

(iv) $\mathfrak{N}$ is iterable in the sense that all successive ultrapower iterations along class well-orders (over the ambient model containing the sharp) starting with $(N, U)$ are well-founded, providing a sequence of structures $(N_i, U_i)$ (for $i$ a set or class well-order) and corresponding $\Sigma_1$-elementary iteration maps $\pi_{i,j} : N_i \longrightarrow N_j$ where $(N, U) = (N_0, U_0)$.

Using the existence of the maps $\pi_{i,j} : N_i \longrightarrow N_j$, we can then provide the following definition:

**Definition 20.** (**NBG** + ETR) A transitive model $\mathfrak{M} = (M, \in)$ is *class iterably sharp generated* iff there is a class-iterable sharp $(N, U)$ and an iteration $N_0 \longrightarrow N_1 \longrightarrow N_2 ...$ such that $M = \bigcup_{\beta \in On^{\mathfrak{M}}} V_{\kappa_\beta}^{N_\beta}$.

In other words, a model is class iterably sharp generated iff it arises through collecting together the $V_{\kappa_i}^{N_i}$ (i.e. each level indexed by the largest cardinal of the model with index $i$) resulting from the iteration of a class-iterable sharp through the ordinal height of $\mathfrak{M}$.

If a model is class iterably sharp generated, it satisfies several reflection properties. In particular, it implies that any satisfaction (possibly with parameters drawn from $\mathfrak{M}$) obtainable in height extensions of $\mathfrak{M}$ adding ordinals (through the well-orders in the class theory of the ambient universe) is already reflected to an initial segment of $\mathfrak{M}$.[43] For example if a model $\mathfrak{M}$ is class iterably sharp generated then it satisfies reflection from $\mathfrak{M}$ to initial segments of $n^{\text{th}}$-order logic for any

[42] This way of defining sharps is modified from the discussion in [Friedman, 2016] and [Friedman and Honzik, 2016]. This work defined sharps in terms of 'all' successive ultrapowers being well-founded in a framework where any universe can be extended in height. Since philosophically relevant here is looking at how these sorts of axioms interact with the view that there is just one universe of sets, we will only allow the iteration of the ultrapower along any class well-order, and hence make the definitions in **NBG** + ETR.

[43] See [Friedman, 2016] and [Friedman and Honzik, 2016] for discussion.

$n$.[44] In this way, we are able to coalesce many reflection principles into a single property of a model. However, claiming that a model is $\sharp$-generated seems to refer to sets outside that model. In particular, the sharp generating $V$ cannot live within $V$.[45] A further interesting feature of sharps is that they cannot be reached by known forcing constructions.[46] They thus present an interesting way of using extensions that cannot be captured by forcing using current technology.

# 2 Universism: The most natural interpretation?

We now have several features of set theory before us:

(1.) There are issues of paradoxes besetting naive set theory.

(2.) The Iterative Conception of Set provides us with a putative solution to these paradoxes by viewing sets as formed in *stages*.

(3.) Independence is both widespread and splits into (at least) two different kinds: *Orey* sentences (e.g. CH) that neither the addition of the statement nor the addition of its negation increase consistency strength, and axioms that do increase consistency strength (e.g. $Con(\mathbf{ZFC})$ or large cardinal axioms).

(4.) We can prove theorems about uncountable sets *within* **ZFC** using extensions of universes.

(5.) We can use extensions to formulate new axioms *extending* **ZFC**.

The first philosophical position we shall consider is the following:

> **Universism.** There is (up to isomorphism) just one maximal unique universe of set theory, and it contains all the sets. Every set-theoretic sentence has a definite truth value in this universe.

In this section, we'll mainly consider how Universism responds to these factors. But first, let's quickly review two arguments in its favour.

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[44]See here [Friedman and Honzik, 2016] and [Friedman, 2016].

[45]If the sharp were in $V$, one could obtain a class club resulting from iterating the sharp (namely the class of $\kappa_i$), which in turn forms a club of *regular* $V$-cardinals. The $\omega^{\text{th}}$ element of any club of ordinals with proper initial segments in $V$ must be *singular* with cofinality $\omega$, and so we would obtain a contradiction at $\kappa_\omega$; it would have to be both regular and singular.

[46]See [Friedman, 2000], §5.2 for details.

## 2.1 Arguments for Universism

**Naturality.** The first 'argument' is just the flat observation that Universism seems natural without arguments to the contrary. After all, doesn't the iterative conception just tell us to take all sets at successor stages and then iterate this construction through all the ordinals? Doesn't such a process just define an absolute Universe for us? Gödel, in the second (1964) version of his paper on the Continuum Hypothesis, writes:

> "It is to be noted, however, that on the basis of the point of view here adopted, a proof of the undecidability of Cantor's conjecture from the accepted axioms of set theory (in contradistinction, e.g., to the proof of the transcendency of $\pi$) would by no means solve the problem. For if the meanings of the primitive terms of set theory ... are accepted as sound, it follows that the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must either be true or false. Hence its undecidability from the axioms being assumed today can only mean that these axioms do not contain a complete description of that reality. ([Gödel, 1964], p. 260)

Whilst Gödel's intuition that the axioms of set theory describe some well-determined reality is shared by many philosophers and mathematicians, it will (of course) be wholly unconvincing to anyone who doubts Universism. This highlights a common theme debates concerning 'the' universe of sets: Many of the arguments and intuitions provided are only convincing given that you already hold the view in question.

**Categoricity arguments.** A second line of argument is to use some version of a categoricity argument to show that there is a restricted class of structures that conform to our concept of set.

The first kind of categoricity argument is a *semantic* categoricity argument. (Semantic, because it will involve the comparison of models, rather than a proof-theoretic approach using second-order logic that we examine below.) This was originally put forward by Zermelo (in [Zermelo, 1930]), and developed by [Shepherdson, 1951], [Shepherdson, 1952], and [Shepherdson, 1953], but a version was proposed (and put to philosophical application) more recently in [Martin, 2001]. The key theorem is:

**Theorem 21.** (*The Quasi-Categoricity Theorem*) Let $\mathfrak{M}$ and $\mathfrak{N}$ be models of $\mathbf{ZFC}_2$ with the full semantics for the second-order variables. Then either:

1. $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic.

2. $\mathfrak{M}$ is isomorphic to proper initial segment of $\mathfrak{N}$, of the form $V_\kappa$ for inaccessible $\kappa$.

3. $\mathfrak{N}$ is isomorphic to proper initial segment of $\mathfrak{M}$, of the form $V_\kappa$ for inaccessible $\kappa$.

There are a couple of different ways of presenting the proof which we shall not go into here.[47]

A different approach to categoricity is *internal*, and has been developed recently by [Väänänen and Wang, 2015] and [Button and Walsh, 2018]. These proceed *proof-theoretically* in impredicative second-order logic. The core point is that *within* the set theory $\mathbf{T}$ we wish to show is categorical (say $\mathbf{ZFC}_2$) we can write out a second-order formula Structure$_{\mathbf{T}}(D, E)$ asserting that domain $D$ and binary relation $E$ on $D$ satisfy the axioms of $\mathbf{T}$. Given appropriate $\mathbf{T}$ (the impredicative second-order versions of $\mathbf{ZFC}$ or Scott-Potter set theory $\mathbf{SP}$ suffice), one can prove that given $D_1, E_1$ and $D_2, E_2$ for which Structure$_{\mathbf{T}}(D_i, E_i)$ holds, either $D_1$ and $E_1$ are isomorphic to a proper initial segment of $D_2$ and $E_2$ (where isomorphism is cashed out as a particular second-order sentence holding) or vice versa. Moreover, if $D_1$ and $E_1$ and $D_2$ and $E_2$ agree on what ordinals there are, then they are isomorphic. This way of proving categoricity does *not* appeal to the kinds of semantics involved (that the semantics is full), but rather pertain to what can be *proved* in our second-order theories.[48]

The thrust of the argument is then as follows: By the categoricity arguments, our canonical theory of sets (namely $\mathbf{ZFC}_2$) determines the structure of the sets up to any ordinal $\alpha$. Thus, one might think, the categoricity arguments lend support to the claim that there is just one universe of sets; given the ability to quantify over all ordinals, we

[47]See [Button and Walsh, 2018], §8A for a modern presentation of a proof. A slightly different version (given in [Martin, 2001]) proceeds stepwise by comparing different levels of the hierarchy. In fact, full $\mathbf{ZFC}_2$ is not necessary for quasi-categoricity, Scott-Potter set theory is enough to guarantee quasi-categoricity between *stages*, just not necessarily ones of inaccessible rank (see here [Button and Walsh, 2018], §8.5 and §8.C).

[48]See [Button and Walsh, 2018] (Ch. 11) for an overview of the internal categoricity results, and connections between these and the results of [McGee, 1997] and [Martin, 2001]. I am grateful to Chris Scambler for some helpful discussion here.

can be confident (so the Universist argues) that we have picked out a unique structure with our reasoning.

Many authors argue that this appeal to categoricity is dialectically ineffective and begs the question against the theorist who rejects that we have a determinate conception of the powerset operation. For semantic categoricity to go through, we require that the semantics be *full*—the range of the second-order quantifiers to be *all* subclasses of the domain. The rejoinder is then that in order for the semantic categoricity argument to work, one must put in at least as much expressive resources as one gets out in the proof of categoricity. Adopting Henkin semantics, for example, will result in a theory equivalent to two-sorted first-order logic, and be (as far as meta-logical results like compactness are concerned) the same as first-order logic. Thus, insofar as one regards the powerset operation as indeterminate, one is likely to hold that our conception of the range of the second-order quantifiers is also indeterminate. For this reason, some authors regard appeals to semantic categoricity as dialectically unconvincing (e.g. [Hamkins, 2012], [Koellner, 2013], [Meadows, 2013], [Hamkins and Solberg, 2020]) even if they might tell us that our axiomatisation has been successful ([Meadows, 2013] presses this claim especially strongly). Similar points apply to the internal categoricity results, any model of set theory will satisfy its own version of the internal (and semantic) categoricity argument, even though they can radically disagree on what subsets there are and possibly other set-theoretic facts.[49] Further analyses of this question are likely to depend on how we conceive of second-order logic, and so we will set that thorny issue aside here.

## 2.2 Universist responses to the paradoxes and the iterative conception

With some motivations in hand, let's now examine how the different aspects of our set-theoretic reasoning outlined in §1 can be interpreted by a Universist.

We'll start with how the Universist responds to the paradoxes and interprets the iterative conception of set.

For the Universist, description of sets being formed in stages is metaphorical, and simply refers to how the universe can be structured. There is no non-metaphorical sense of sets being 'formed' in stages,

[49] A recent detailed examination of this line of thought is provided in [Hamkins and Solberg, 2020] who also contrast the internal and external perspectives on categoricity in the context of set theory.

rather we have the theorem that every set belongs to some $V_\alpha$, and for any particular $V_\alpha$ there are sets not in $V_\alpha$. The iterative conception is seen as describing an abstract structure of sets that stands free of any literal *process* of formation.

In this framework we do not have sets for the problematic conditions "$x \notin x$", "$x = x$", and "$x$ is an ordinal", since the structure of the $V_\alpha$ prohibits formation of all sets of this kind at any point in $V$. However, it seems like I can make claims about these classes that are not sets (so-called 'proper classes'), e.g.

"The Universal class is the same class as the Russell class."

Since we know that *every* set is non-self-membered under the iterative conception, and assuming an extensionality principle for classes, this claim looks *true*.

More interestingly, it seems like set-theoretic practice is laden with the use of proper classes.[50] For example, using the notion of *embedding*, we can define the notion of *measurable cardinal* as follows:

**Definition 22.** An uncountable cardinal $\kappa$ is *measurable* iff it is the critical point of a non-trivial elementary embedding $j : V \to \mathfrak{M}$ for some transitive inner model $\mathfrak{M} \models \mathbf{ZFC}$.

In this definition, both $V$ and $\mathfrak{M}$ are proper classes, and if we code $j$ by ordered pairs, then it is too. A whole hierarchy of cardinals are definable in this way, and proper-class-sized elementary embeddings provide one of the natural contexts in which to talk about them.

There is then a puzzle here: We are able to make seemingly true claims about proper classes that are *useful* in our set-theoretic practice.[51] One might thus think that the Universist is under some pressure to interpret talk of proper classes. There are several options here, and we lack the space to go into full details. Nonetheless, since the interpretation of proper classes is philosophically open and leads to some interesting mathematical problems, we will survey a few options and challenges here.

The standard approach is to regard talk of proper classes as shorthand for some formula or other holding of particular sets. Instead of

[50] I am enormously grateful to Sam Roberts for many helpful discussions concerning the role of classes and embeddings and his permission to include some of these remarks here.

[51] The *usefulness* of proper classes is perhaps interesting for scholars working in the tradition of Penelope Maddy, who regard the *fruitfulness* of the introduction of entities as important for their acceptance (see [Maddy, 2011]). A full analysis is outside the scope of this paper, but it merits further consideration.

countenancing class talk as legitimate in its own right, we might try to paraphrase the class talk through the use of the relevant $\phi$ that define the classes. Hamkins, for example, says the following:

> One traditional approach to classes in set theory, working purely in **ZFC**, is to understand all talk of classes as a substitute for the first-order definitions that might define them... ([Hamkins, 2012], p. 1873)

To take some simple examples, we can paraphrase "$x \in R$", "$x \in V$", and "$x \in On$", as "$x$ is non-self-membered", "$x$ is a set", and "$x$ is an ordinal" respectively. Similarly, if we wish to state that $V = R$, we can do by stating that "$\forall x(x = x \leftrightarrow x \notin x)$".

Interestingly, we can provide first-order definitions for more complicated kinds of class. For example, let $j : V \longrightarrow \mathfrak{M}$ be an embedding witnessing the measurability of an uncountable cardinal $\kappa$. We can (using a parameter $U$ for a $\kappa$-complete non-principal ultrafilter on $\kappa$) define a first-order formula $\phi(x, y, z)$ such that $j(x) = y$ iff $\phi(x, y, U)$ holds in $V$. Then, one can show:

(1) $\phi(x, y, U)$ relates every $x$ to at most one $y$ (i.e. $\phi(x, y, U)$ is function-like).

(2) $\phi(x, y, U)$ relates no two $x$ to the same $y$ (i.e. $\phi(x, y, U)$ is one-to-one).

(3) $\phi(x, y, U)$ relates every set in $V$ to a set in $\mathfrak{M}$ (i.e. $\phi(x, y, U)$ is total on $V$).

(4) There is at least one $x$ and $y$ such that $\phi(x, y, U)$ and $x \neq y$ (i.e. $\phi(x, y, U)$ is non-trivial).

(5) For any $x_0, ..., x_n$ and $y_0, ..., y_n$ if $\phi(x_n, y_n, U)$ holds for both sequences then for any parameter-free first-order formula $\psi(z_0, ...z_n)$:

$$\psi(x_0, ..., x_n) \leftrightarrow \psi(y_0, ..., y_n)$$

(i.e. $\phi(x, y, U)$ preserves first-order truth).[52]

(6) There is an ordinal $x$ (namely $\kappa$) such that $\phi(x, y, U)$ and $y$ is an ordinal greater than $x$ (i.e. $\phi(x, y, U)$ identifies the critical point of $j$).

[52]As this holds for any first-order formula $\psi$, this will be a schema of theorems.

All this can be shown in a first-order fashion[53]. We can thus use the relation $\phi(x, y, U)$ to do the work of the prima facie second-order entity $j$, whilst only talking about sets. The above formula effectively moves through the hierarchy of sets relating the sets in $V$ and $\mathfrak{M}$, identifying a critical point along the way, without ever talking about actual proper classes.

However, there are questions surrounding the interpretation of all classes as definable, and plausibly reasons to think that a Universist might want to countenance talk of non-definable classes. Several areas of set-theoretic research seem, without further re-interpretation, to commit us to such classes. Reflection principles are just one such area. A reflection principle is of the following general form:

$$\exists\alpha(\phi \rightarrow \phi^{V_\alpha})$$

In other words, if $\phi$ is true then $\phi$ is satisfied by some initial segment $V_\alpha$ (with quantifiers and parameters restricted to $V_\alpha$). A salient fact is that often Universists consider reflection properties that are given by second-order parameters over $V$, and use the principles to study small large cardinals. For example, the second-order reflection principle states that, for any second-order parameter $A$ over $V$:

$$(V, \in, A) \models \phi \rightarrow (V_\alpha, \in, A \cap V_\alpha) \models \phi^{V_\alpha}$$

Such a principle is most naturally understood when quantification and the parameter $A$ are able to refer to non-first-order definable parameters over $V$, and produces many orders of large cardinals consistent with $V = L$. Without the use of such non-definable classes, we lose interpretation of the relevant $A$, and hence lose the consequences we would like within $V$ (such as, in the case of second-order reflection, inaccessibles and Mahlo cardinals). Rather the kinds of cardinal we get are merely definable shadows of their full second-order relatives (e.g. definably inaccessible, definably Mahlo etc.). Moreover, second-order reflection as it is normally understood actually *reverses* to the truth of full impredicative comprehension in the class theory. To see this, note that if any instance of impredicative compehension fails in the class theory of $V$, then (by the second-order reflection principle) there must be a $V_\alpha$ for which impredicative comprehension fails. However, this is impossible: since the restricted second-order variables are interpreted as restricted to subsets of $V_\alpha$ (i.e. as ranging over $V_{\alpha+1}$),

[53]See [Suzuki, 1999] and a very clear exposition in [Hamkins et al., 2012] for the full technical details.

the truth of impredicative comprehension in the second-order theory of $V_\alpha$ is guaranteed by the strength of the Power Set Axiom.[54]

Further, the study of large cardinal embeddings also raises questions for the paraphrase of class talk in terms of definable classes. As noted earlier, we can characterise a measurable cardinal as the critical point of a non-trivial elementary embedding $j$ from $V$ to some transitive inner model $\mathfrak{M}$. We could also characterise this embedding using a parameter for an ultrafilter $U$ and a first-order formula $\phi(x, y, U)$. A natural question is whether or not this method makes good sense of *all* theorems concerning embeddings.

There are reasons to think that the definable formula interpretation does not. We mention two such theorems, one negative and one positive. We deal with the negative first:

**Theorem 23.** [Kunen, 1971] There is no non-trivial elementary embedding $j : V \longrightarrow V$.

Kunen's Theorem is relatively involved. It was conjectured by Reinhardt that there could be such an embedding, and took roughly a year to solve.[55] Moreover, the theorem built on other results in infinitary combinatorics (such as [Erdős and Hajnal, 1966]). Recent presentations use a result of Solovay that any stationary set $S$ on a regular cardinal $\kappa$ can be partitioned into $\kappa$-many stationary sets, and although they substantially simplify the proof[56] the result remains non-trivial. Contrast this with the result for *first-order definable* elementary embeddings:

**Theorem 24.** [Suzuki, 1999] There is no non-trivial elementary embedding $j : V \longrightarrow V$ definable from parameters.

*Proof.* (Sketch) This result is far simpler than any proof of Kunen's Theorem. Consider a $j$ with $\kappa = crit(j)$. Let $\phi(x, y)$ define $j$ (we suppress any parameters). We know that since $\phi$ is first-order, then we can define a first-order formula $\psi(x)$ that holds iff $x$ is the least ordinal moved by $j$. Since $\psi(\kappa)$, by the elementarity of $j$ we have that $\psi(j(\kappa))$ in the target model. But since $dom(j) = V$ and $ran(j) = V$, we have

[54] I am grateful to Sam Roberts for this observation.

[55] The timings are somewhat hard to determine in virtue of the fact that [Solovay et al., 1978] was 'about' to be published from 1970 at the latest (Kunen himself mentions Reinhardt and cites the paper in [Kunen, 1971]). The philosophically relevant point still stands; the possibility of a $j : V \longrightarrow V$ was conjectured, relatively well-known, and took some time to refute.

[56] See [Schindler, 2014], Theorem 4.53 for an updated proof (attributed to Woodin) that uses the Solovay Splitting Lemma and [Kanamori, 2009] §23 for several different other proofs of Kunen's Theorem.

that $V \models \psi(\kappa)$, $V \models j(\kappa) > \kappa$, and $V \models \psi(j(\kappa))$. Hence $\kappa$ both is and is not the least ordinal moved by $j$, $\perp$.[57] $\square$

The proof does not require any deep analysis of the nature of sets to prove. All we do is follow through the consequences of $j$ being first-order definable and make some elementary observations about the nature of $j$ in terms of its domain and range. Thus there seems to be some discord between the claim that all embeddings are first-order definable and the complexity involved in Kunen's Theorem. On the subject of Kunen's Theorem and the definability of $j$, Hamkins, Kirmayer, and Perlmutter say the following:

> "Our view is that this way of understanding the Kunen inconsistency does not convey the full power of the theorem. Part of our reason for this view is that if one is concerned only with such definable embeddings **j** in the Kunen inconsistency, then in fact there is a far easier proof of the result, simpler than any of the traditional proofs of it and making no appeal to any infinite combinatorics or indeed even to the axiom of choice." ([Hamkins et al., 2012], p. 1873)

There are several points to note here. First, it is simply a fact that many set theorists *are* interested in the possibility of non-definable elementary embeddings and this perhaps lends weight to the idea that we should find an interpretation that matches their discourse as closely as possible. This is especially so if one holds some variety of naturalism (e.g. [Maddy, 1997]) or second philosophical perspective (e.g. [Maddy, 2007]). Second, the view that all elementary embeddings are first-order definable substantially *trivialises* Kunen's Theorem, in that it makes his result relatively easy when it appears to concern deep facts about the combinatorial nature of the sets. Third, definability is unaffected by whether or not the Axiom of Choice holds, and Kunen's Theorem and many of its generalisations depend essentially on use of the Axiom of Choice. Currently, it is regarded as an open question whether or not there could be a non-trivial elementary embedding $j : V \longrightarrow V$ if AC turns out to be false in $V$ (or indeed in any proper-class-sized model of **ZF** where AC fails). Regarding all embeddings as first-order definable would immediately answer this question: since there can be no definable embedding with or without AC, there is no embedding in the particular case where AC is false.[58]

[57] For full thoroughness (including checking that the notion of elementary embedding can be formalised in a first-order theory), see [Suzuki, 1999].

[58] I am grateful to Sam Roberts for emphasising the importance of triviality and

It is not just with respect to *negative* theorems concerning the non-existence of elementary embeddings that we see this problem, however. *Prima facie*, set theorists talk about the existence of embeddings that cannot be first-order definable. The following is a good example:

**Theorem 25.** [Vickers and Welch, 2001] Suppose $I \subseteq On$ witnesses that the ordinals are Ramsey[59]. Then, definably over $(V, \in, I)$, there is a transitive model $\mathfrak{M} = (M, \in)$, and an elementary embedding $j : (M, \in) \longrightarrow (V, \in)$ with a critical point.

Here, $I$ is a proper class of good indiscernibles for $On$. If we introduce a predicate '$I(x)$' into the language to talk about those indiscernibles (so $I(x)$ holds iff $x \in I$), we can define (using $I(x)$) a non-trivial elementary embedding from $\mathfrak{M}$ to $V$. However, we should also be mindful of the following result:

**Theorem 26.** [Suzuki, 1999] Let $j : \mathfrak{M} \longrightarrow V$ be a definable elementary embedding such that $\mathfrak{M}$ is transitive and $On \subset \mathfrak{M}$. Then $j$ has no critical point.

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the settling of open questions to me, and also for pointing out Kunen's Theorem as a place where these issues arise. I am also grateful to an anonymous reviewer for the following observation regarding this point: This tension raises its head in some set theory textbooks where classes are assumed to always be definable. For example, the wonderful [Kanamori, 2009] only allows definable classes (see Ch. 0) and embeddings (see Ch. 5). Presumably aware of the way in which regarding embeddings as definable will result in substantial trivialisation, Kanamori then opts for an interpretation of the theorem as about set-sized structures, namely the non-existence of a $j : V_{\delta+2} \rightarrow V_{\delta+2}$ for any $\delta$. One might think that this deforms the apparent content of the Kunen's Theorem on metamathematical grounds, especially when we try to generalise to the **ZF**-context (left as an open question about the possibility of a *proper-class-sized* $j : V \rightarrow V$ in §23). For Kanamori's remarks, see [Kanamori, 2009], especially pp. 318–324.

[59]The details of Ramsey properties are somewhat technical and inessential for seeing the philosophical issues, and so we relegate them to a footnote:

To define Ramseyness, we first need the notion of a *good set of indiscernibles*. Let $I \subseteq \mathfrak{A} = L_\kappa[A, \in, \vec{B}, ...]$ be a first-order structure. Then $I$ is a *good set of indiscernibles* for $\mathfrak{A}$ if for any $\gamma \in I$:

(i) $\mathfrak{A} \restriction =_{df} L_\gamma[A \restriction \gamma, \in, \vec{B} \restriction \gamma, ...] \prec \mathfrak{A}$,

(ii) $I \backslash \gamma$ is a set of indiscernibles for $\langle \mathfrak{A}, \langle \zeta \rangle_{\zeta < \gamma} \rangle$

We then say that $\kappa$ is *Ramsey* iff any first-order structure with $\kappa \subseteq |\mathfrak{A}|$ has a good set of indiscernibles of length $\kappa$. To define Ramseyness for the particular case of the proper class $On$ (the previous definitions only apply to set-sized structures). We say that $On$ is *Ramsey* iff there is a class $I \subseteq On$, unbounded, of good indiscernibles for $(V, \in)$. More details and uses of these definitions are available in [Vickers and Welch, 2001].

By this theorem, the Vickers-Welch embedding cannot be first-order definable over $V$. However, it seems that we are able to talk about such an embedding in a perfectly rigorous manner. It is not just $j$ that cannot be definable in the above theorem. $I$ cannot be definable as one can define a satisfaction relation for $(V, \in)$ over $(V, \in, I)$.[60] One might thus think that insisting on all classes being first-order definable prohibits an area of study that may produce fruitful mathematics with consequences for $V$.

Given an acceptance of the use of non-definable classes, one substantial question is how we should interpret talk concerning them. One option is simply to regard all talk about non-definable classes as simply implicitly restricted to some $V_\kappa$ where non-definable classes for $V_\kappa$ are uncontroversially available in $\mathcal{P}(V_\kappa)$. If, however, one wishes to have "$V$" denoting $V$ in the relevant theorems, we need a more subtle interpretation. This seems like no easy task, as Penelope Maddy identifies:

> The problem is that when proper classes are combinatorially determined just as sets are, it becomes very difficult to say why this layer of proper classes on top of $V$ is not just another stage of sets we forgot to include. It looks like just another rank; saying it is not seems arbitrary. The only difference we can point to is that the proper classes are banned from set membership, but so is the $\kappa$th rank banned from membership in sets of rank less than $\kappa$. ([Maddy, 1983], p. 122)

Many accounts that seek to provide an interpretation of non-definable class talk thus attempt to provide a way of justifying but reconciling the following two desiderata (that seem to be in tension):

(1.) We should be able to interpret talk about non-definable classes for the Universist.

[60] See [Vickers and Welch, 2001] for details. In the Introduction to the paper containing the above result, Vickers and Welch say the following:

> It is quite natural to study the properties of elementary embeddings $j : V \longrightarrow M$ for $M$ some inner model, since many such embeddings, if they exist, have first order formulations within **ZFC**. The question of reversing the arrow and looking at a non-trivial $j : M \longrightarrow V$ in general does not readily admit of such formulations. So we study in this paper what might be considered the **ZFC** consequences of the second order statement that there are proper classes $j$, $M$ such that... ([Vickers and Welch, 2001], p. 1090)

(2.) We should make clear how our interpretation of non-definable classes makes them different from sets, and why classes cannot be members of sets (or if they can be, why this isn't problematic).

Several accounts have been advanced in the literature. One could interpret classes using properties (one could extract such an account from [Linnebo, 2006]) or Fregean concepts ([Linnebo, 2010]). Another option is to interpret the classes as possible predicates (this direction is suggested by [Parsons, 1974], though he is somewhat circumspect about the possibility of interpreting lots of non-definable class talk this way). A further methodology is to regard class talk as interpreted through plural reference and quantification, and the impredicative class comprehension schema as underwritten by the plural comprehension schema (see here [Uzquiano, 2003]).

A separate challenge concerning classes, one that does not depend on the non-definability of the classes in question, is the possibility of considering well-orders longer than all the ordinals. The following is an example (let $\Omega$ denote the length of the ordering of all ordinals[61]):

$\alpha \prec_{\Omega+1} \beta$ iff either:

(i) $\alpha \geq 1 \wedge \beta \geq 1 \wedge \alpha < \beta$, or

(ii) $\alpha \geq 1 \wedge \beta = 0$.

Such an ordering (expressible in $\mathscr{L}_{\in}$) effectively puts $\emptyset$ past the end of all the ordinals, defining a well-order of length $\Omega + 1$. The example can be pushed further:

$\alpha \prec_{\Omega.2} \beta$ iff either:

(i) $\alpha$ is a successor and $\beta$ is a limit, or

(ii) $\alpha$ and $\beta$ are both limits and $\alpha < \beta$, or

(iii) $\alpha$ and $\beta$ are both successors and $\alpha < \beta$.

Such a definition (expressible in $\mathscr{L}_{\in}$) *prima facie* defines an ordering of length $\Omega.2$. We can provide definitions of still longer well-orderings. The following defines an ordering on ordered pairs of ordinals that is (*prima facie*) $\Omega$ times as long as $\Omega$:

$\langle \alpha, \beta \rangle \prec_{\Omega.\Omega} \langle \gamma, \delta \rangle$ iff

[61] If this makes the reader feel metamathematically queasy, one can interpret the talk over some $V_\alpha$ to see the problem in the case of $V$.

(i) $\alpha < \gamma$, or
(ii) $\alpha = \gamma$ and $\beta < \delta$.[62]

Intuitively speaking, such an ordering defines an $\Omega$-length sequence of ordered pairs for every ordinal. Clearly it is possible (by moving to ordered triples, quadruples, etc.) to iterate the definition to ordinally multiplying $\Omega$ by itself over and over. Moreover, 'long' definable well-orders appear in more mathematically deep contexts. An example is the standard ordering on mice—certain kinds of small object useful for defining embeddings[63]—which is defined as follows:

**Definition 27.** Let $J_\alpha^U = Hull_n^{J_\alpha^U}(\gamma \cup p)$ and $J_{\alpha'}^{U'} = Hull_m^{J_{\alpha'}^{U'}}(\gamma' \cup p')$ be mice and $\lambda$ be any sufficiently large regular cardinal. Let $i_{0,\lambda} : J_\alpha^U \longrightarrow J_\beta^{\mathcal{C}_\lambda}$ and $i'_{0,\lambda} : J_{\alpha'}^{U'} \longrightarrow J_{\beta'}^{\mathcal{C}_\lambda}$ be the respective iterated ultrapowers witnessing their mice-hood. Then $J_\alpha^U = Hull_n^{J_\alpha^U}(\gamma \cup p) <_M J_{\alpha'}^{U'} = Hull_m^{J_{\alpha'}^{U'}}(\gamma' \cup p')$ iff:

(i) $\beta < \beta'$, or

(ii) $\beta = \beta'$ and $\gamma < \gamma'$, or

(iii) $\beta = \beta'$, $\gamma = \gamma'$, and $q < q'$ in the descending lexicographic ordering.

The definition is technical, but the philosophically important point is that it provides a mathematically useful definition of an order-type of length $\Omega.3$.

A puzzle then emerges; we *appear* to be defining orders longer than $\Omega$, but there is no set-theoretic representative corresponding to these order-types. The problem is a philosophical rather than mathematical one. It is not mathematically incoherent to say that there are formulae defining well-orders longer than $\Omega$. It is simply that these well-orders cannot have von Neumann ordinal representatives in $V$ (on pain of the Burali-Forti contradiction). This then raises a *conceptual* problem: If we are able (prima facie) to coherently compare these long well-orders and they have mathematical use, then what is the underlying ontology behind the order-types?

A substantial open philosophical question for the Universist is thus the following: It seems that there are ways we talk mathematically

[62]The orderings $\prec_{\Omega+1}$, $\prec_{\Omega.2}$, and $\prec_{\Omega.\Omega}$ are taken from [Shapiro and Wright, 2006], who also consider the philosophical ramifications and options for interpretation in detail.

[63]See [Schimmerling, 2001] for a survey of mice.

about classes, and there are ways these classes can be ordered that do not have representatives. This raises several further questions:

1. What *theory* should we use to underwrite talk of classes?
2. What philosophical *interpretation* of classes should we provide?
3. What *constraints* are there on interpretations of classes?
4. How should we think of the *ontology* of well-orders 'longer than' $\Omega$?

Later (§3) we shall see how views that deny Universism attempt to answer these difficult questions. For now, we move on to consideration of the options for the Universist in interpreting the *extensions* we talked about in §1.

### 2.3 Universist interpretations of extensions

Let us move on to the question of how the Universist might interpret the various kinds of extensions we talked about in §1. There we saw four main kinds of extensions (i) set forcing extensions, (ii) class forcing extensions, (iii) hyperclass forcing extensions, and (iv) extensions with sharps, which in turn were useful for showing (a) independence, (b) proving theorems, and (c) formulating axioms.

Whence the problem? Well each of set, class, and hyperclass forcing, as well as sharps (1.) appear to require sets external to the structures they concern, and (2.) often "$V$" is used to denote the relevant ground model. But if "$V$" really denotes $V$ here, then we appear to be considering sets outside the 'one true universe', contrary to the Universist's position.

Obviously for the Universist, they require some re-interpretation of the talk (or to claim that the discourse never claimed to discuss extensions of $V$ in the first place). Here we survey some extant possibilities.

**An abuse of notation.** One option is just to argue that use of the term "$V$" is merely an abuse of notation, and refers to any model of **ZFC** that can support the relevant construction. For example, Koellner writes:

> Set theorists often use '$V$' instead of '$M$' and so write '$V[G]$'. But if $V$ is the entire universe of sets then $V[G]$ is an "illusion". What are we to make of this? Most set theorists would say that it is just an abuse of notation. When one is

> proving an independence result and one invokes a transitive model $M$ of $\mathbf{ZFC}$ to form $M[G]$ one wants to underscore the fact that $M$ could have been any transitive model of $\mathbf{ZFC}$ and to signal that it is convenient to express the universality using a special symbol. The special symbol chosen is '$V$'. This symbol thus has a dual use in set theory—it is used to denote the universe of sets and (in a given context) it is used as a free-variable to denote any countable transitive model (of the relevant background theory). ([Koellner, 2013], p. 19)

Koellner's point is that we could interpret talk of "$V$" as some countable transitive model or other of $\mathbf{ZFC}$ (let's denote it by $\mathcal{V}$). There is then no problem to interpret any of set-forcing, class forcing, hyperclass forcing, or sharps over $\mathcal{V}$; it can be extended by any of these constructions in width. Moreover, $Ord(\mathcal{V})$ is just some tiny countable ordinal. We can thus perfectly well consider models $\mathfrak{M}$ such that $\mathcal{V} \in V_\alpha^{\mathfrak{M}}$, and there are many height extensions of $\mathcal{V}$ in $\mathfrak{M}$. So there is no trouble to interpret extensions if this is the interpretation of the use of the symbol "$V$" prescribed.

One issue of this response is the extent to which it preserves the 'naturality', 'intendedness', or 'aboutness' of the interpretation. *If* one is moved by such considerations, we can generate problems for the response as follows: Such an interpretation is fine for interpreting independence results, when all we want to show is that for some extension $\mathbf{T}$ of $\mathbf{ZFC}$, some model of $\mathbf{T} + \neg\phi$ exists and thus there cannot be a proof (code) of $\phi$ within models of $\mathbf{T}$. In this case, we don't really care what the model looks like; we're just providing a countermodel for the claim that $\mathbf{T} \vdash \phi$. However, when we are formulating axioms or proving theorems *about* objects within $V$, perhaps we want a tighter connection between those objects we are predicating a property of and the use of extensions. As an example, suppose that we predicate the property of remarkability for some ordinal $\alpha$ in $V$. This uses extensions, and so using the countable transitive model strategy, we interpret this as about some small countable ordinal $\alpha_{\mathcal{V}} \in \mathcal{V}$. But then it does not seem that our talk is *about* $\alpha$ at all.[64]

**Use of the forcing relation.** Different options are available for the specific case of forcing. Here we can let "$V$" denote $V$, but reinterpret what is meant by talk of 'forcing extension'. One option is to move

[64]See [Barton, 2020] for further examination of this point.

*syntactically* by defining the *forcing relation*, that captures the consequences of extensions without actually committing to the existence of any models. Roughly put, letting $\mathbb{P} = \langle P, \leq_{\mathbb{P}} \rangle$ be a forcing poset, $p \in P$, and $\phi$ be in the forcing language for $\mathbb{P}$,[65] we can define a relation $p \Vdash^*_{\mathbb{P}}$ recursively such that:[66]

(1) If $\phi_1, ..., \phi_n \vdash \psi$ and $p \Vdash^*_{\mathbb{P}} \phi_i$ for each $i$, then $p \Vdash^*_{\mathbb{P}} \psi$.

(2) $p \Vdash^*_{\mathbb{P}} \phi$ for every axiom of **ZFC**.

(3) If $\phi(x_1, ..., x_n)$ is a formula known to be absolute for transitive models, then for every $p$ and all sets $a_1, ..a_n$; $p \Vdash^*_{\mathbb{P}} \phi(\check{a}_1, ..., \check{a}_n)$ iff $1_{\mathbb{P}} \Vdash^*_{\mathbb{P}} \phi(\check{a}_1, ..., \check{a}_n)$ iff $\phi(a_1, ..., a_n)$ is true in $V$.

This relation lets us talk about what would be satisfied in the extension $V[G]$ by analysing what sentences conditions $p \in P$ force. By (3), any theorem proved 'in $V[G]$' will be verified by the check names and hence by specific sets in $V$. Similarly, if we wish to formulate an axiom about $V$ using a forcing extension, we can do so by finding a $p$ that forces the required sentence about objects in the ideal extension.

There are two main challenges to this interpretation, as far as forcing in general is concerned. The first is an issue of scope. Whilst the interpretation does not aim to interpret talk of sharps, there are still difficulties as far as *class* forcing is concerned, for the simple reason that the forcing relation is not always definable when the forcing poset is *proper-class-sized*.[67]

A second question is the extent to which such an interpretation preserves as much 'intendedness' of the original forcing interpretation as

[65]The *forcing language of* $\mathbb{P}$ is the collection of all formulas that can be formed by the usual logical operators from the language $\mathscr{L}_\in$ combined with a constant symbol for every name in $V^{\mathbb{P}}$ (the $\mathbb{P}$-names).

[66]See [Kunen, 2013] for details of the forcing relation and verification of the relevant proofs.

[67]For example, consider the following forcing:

**Definition 28.** Let $\mathfrak{M}$ be a model for **ZFC**. Then the *Friedman* poset (denoted by '$\mathbb{F}^{\mathfrak{M}}$') is a partial order of conditions $p = \langle d_p, e_p, f_p \rangle$ such that:

(i) $d_p$ is a finite subset of $\omega$.

(ii) $e_p$ is a binary acyclic relation on $d_p$.

(iii) $f_p$ is an injective function with $dom(f_p) \in \{\emptyset, d_p\}$ and $ran(f_p) \subseteq \mathfrak{M}$.

(iv) If $dom(f_p) = d_p$ and $i, j \in d_p$, then $ie_pj$ iff $f_p(i) \in f_p(j)$.

(v) The ordering on $\mathbb{F}^{\mathfrak{M}}$ is given by:

$$p \leq_{\mathbb{F}^{\mathfrak{M}}} q \leftrightarrow d_q \subseteq d_p \wedge e_p \cap (d_q \times d_q) = e_q \wedge f_q \subseteq f_p.$$

possible. [Barton, 2020], for example, argues that an alternative paraphrase is desirable since the interpretation fails to explicitly mention models, and thus is insufficiently 'natural' from a forcing perspective.

**Boolean-valued models and Boolean ultrapowers.** An alternative but closely related paraphrase of forcing constructions is via the use of Boolean-valued models.[68] Given a forcing poset $\mathbb{P}$, we can find a separative[69] partial order $\mathbb{Q}$, equivalent to $\mathbb{P}$ for forcing, and a (unique up to isomorphism) *Boolean completion* of $\mathbb{Q}$ (denoted by '$\mathbb{B}(\mathbb{Q})$').[70] We then consider the class of $\mathbb{B}(\mathbb{P})$-names (denoted by '$V^{\mathbb{B}(\mathbb{P})}$'), assign values from $\mathbb{B}(\mathbb{P})$ to atomic relations between them, and provide an inductive definition for the quantifiers.[71]

It is then routine to show that $V^{\mathbb{B}(\mathbb{P})}$ is a Boolean-valued model of $\mathbf{ZFC}$. In particular every axiom (and hence every theorem) of $\mathbf{ZFC}$ has Boolean-value $1_{\mathbb{B}(\mathbb{P})}$ in $V^{\mathbb{B}(\mathbb{P})}$. Moreover, for the purposes of consistency proofs, we know that if we can assign $\phi$ a Boolean-value greater than $0_{\mathbb{B}(\mathbb{P})}$, then $\neg\phi$ is not a consequence of $\mathbf{ZFC}$ (as if $\neg\phi$ is a consequence of $\mathbf{ZFC}$, then $\phi$ receives Boolean value $0_{\mathbb{B}(\mathbb{P})}$). In fact, an assignment of a Boolean value greater than $0_{\mathbb{B}(\mathbb{P})}$ to $\phi$ exactly mimics the satisfaction of $\phi$ in some $V[G]$, for $V$-generic $G$. Thus, even if we are Universists, we might just interpret talk of forcing and generics as about the relevant Boolean-valued models.

This defines a proper-class-sized partial order as the individual $f_p$ include every function from some finite subset of $\omega$ to a (sub)set of $\mathfrak{M}$, and hence there are proper-class-many such ordered triples (relative to $\mathfrak{M}$). The partial order adds a bijection $F_{\mathbb{F}}$ between $\omega$ and $M$, and a relation $E_{\mathbb{F}} \in \mathfrak{M}[G]$ such that $\langle \omega, E_{\mathbb{F}} \rangle$ and $\langle M, \in \rangle$ are isomorphic. If the forcing relation for $\mathbb{F}$ were definable, $\mathfrak{M}$ would then have access to its own truth definition (contradicting Tarski's Theorem). For the details of the proof, and further discussion of the Truth and Definability lemmas in context of class forcing, see [Holy et al., 2016]. One might instead *postulate* that a definition can be given for $\mathbb{F}^V$ and other class-sized partial orders with non-first-order definable forcing relations. Such definitions could not be first-order, but interestingly such a hypothesis fits naturally in the space of second-order set theories between $\mathbf{NBG}$ and $\mathbf{MK}$. [Gitman et al., 2017] showed that the hypothesis that every class partial order has a forcing relation is equivalent to the principle that transfinite recursions of class relations for ordinal length are legitimate (so called '$ETR_{Ord}$').

[68]The Boolean-valued approach was developed by Scott and Solovay, with additional contributions by Vopěnka (among others). See [Smullyan and Fitting, 1996], p. 273 for historical details and references.

[69]A partial order $\mathbb{P} = (P, <_{\mathbb{P}})$ is *separative* iff for all $p, q \in P$, if $p \not\leq_{\mathbb{P}} q$ then there exists an $r \leq_{\mathbb{P}} p$ that is incompatible with $q$.

[70]For details of Boolean algebras (from which our presentation is derived) see [Jech, 2002], Chapter 7. A discussion of Boolean completions is available in ibid. Chapter 14.

[71]See here, [Jech, 2002], Ch. 14.

Again, there is a question of scope for the Boolean-valued model approach (independently of considerations concerning sharps). The main issue is that the existence of Boolean-completions for class-sized partial orders is not a trivial matter. The usual way we obtain a Boolean completion is to find a separative partial order equivalent to $\mathbb{P}$ for forcing (known as the separative quotient), and embed it into a Boolean algebra. Effectively, we add a bottom element and the required suprema to form $\mathbb{B}(\mathbb{P})$.[72]

If one is a Universist, however, things are not so simple where class-sized partial orders are concerned. Since the partial order is unbounded in $V$, one can not always assume that there will be space to add a bottom element and suprema (without committing oneself to the existence of hyperclasses). As it turns out, a class partial order has a class Boolean-completion in a model of **MK** precisely when all antichains are at most set-sized (known as the $Ord$-chain condition).[73] Thus, the kinds of class forcing we can interpret using Boolean-valued models is somewhat restricted.

Further, there is again the issue of whether or not the interpretation preserves as much as possible of the forcing idea. Though the approach is model-theoretic (rather than syntactic as with the forcing relation), the model is not two-valued, and no model is actually being extended when we consider what holds in a Boolean-valued context. In this respect, we might question whether the Boolean-valued models approach is a philosophically satisfactory paraphrase for many forcing constructions.

There are, however, ways of modifying Boolean-valued models to proper-class-sized two-valued structures via Boolean-ultrapowers and quotient structures. The technique is studied in detail in [Hamkins and Seabold, 2012] (and plays a role in motivating Hamkins' multiverse view that we will discuss later). Importantly, the method provides a way of finding models internal to $V$ that bear forcing relationships to one another. More formally, one can prove:

**Theorem 29.** [Hamkins and Seabold, 2012] *The Naturalist Account of Forcing*. If $V$ is the universe of set theory and $\mathbb{B}$ is a notion of forcing, then there is in $V$ a definable class model of the theory expressing

[72] More formally, for any set-sized partial order $\mathbb{P}$, there is a Boolean algebra $\mathbb{B}(\mathbb{P})$ and an embedding $e : \mathbb{P} \longrightarrow \mathbb{B}(\mathbb{P})^+$ (where $\mathbb{B}(\mathbb{P})^+$ is the set of non-zero elements of $\mathbb{B}(\mathbb{P})$) such that for $p, q \in \mathbb{P}$: (i) if $p \leq_{\mathbb{P}} q$, then $e(p) \leq_{\mathbb{B}(\mathbb{P})} e(q)$, (ii) $p$ and $q$ are compatible iff $e(p) \wedge e(q)$, and (iii) $\{e(p) | p \in \mathbb{P}\}$ is dense in $\mathbb{B}(\mathbb{P})$. For the full details, see [Jech, 2002], Chapter 14.

[73] See [Holy et al., 2016] and [Holy et al., 2018] for the result (attributed to Hamkins).

what it means to be a forcing extension of $V$. Specifically, in the forcing language with $\in$, constant symbols $\check{x}$ for every $x \in V$, a predicate symbol $\check{V}$ to represent $V$ as a ground model, and a constant symbol $\mathring{G}$, the theory asserts:

(1) The full elementary diagram of $V$, relativised to the predicate $\check{V}$, using the constant symbols for elements of $V$.

(2) The assertion that $\check{V}$ is a transitive proper class in the (new) universe.

(3) The assertion that $\mathring{G}$ is a $\check{V}$-generic ultrafilter on $\check{\mathbb{B}}$.

(4) The assertion that the new universe is $\check{V}[\mathring{G}]$, and **ZFC** holds there.

This can be done by first taking an ultrafilter $U$ on the relevant Boolean-algebra $\mathbb{B}$ (for convenience sake, we now drop the notation $\mathbb{B}(\mathbb{P})$) and using it to build a particular ultrapower embedding $j_U$ (the so-called Boolean ultrapower map) between $V$ and an inner model $\check{V}_U$. When we then form the *quotient structure* $V^{\mathbb{B}}/U$ of $V^{\mathbb{B}}$ (formed by taking the standard quotient structure), we find an interesting relationship between $\check{V}_U$ and $V^{\mathbb{B}}/U$— $V^{\mathbb{B}}/U$ is precisely the forcing extension of $\check{V}_U$ by $U$. One can verify that $V^{\mathbb{B}}/U \models \mathbf{ZFC}$ and also that if $\phi$ has Boolean-value greater than $0_{\mathbb{B}}$ in $V^{\mathbb{B}}$, then $V^{\mathbb{B}}/U \models \phi$.

Importantly, this can be done with a *non-$V$-generic* ultrafilter. Hence $U$ can perfectly well be in $V$. In fact, when one constructs the Boolean ultrapower over some model of set theory $\mathfrak{M} = (M, E)$, the claim that $U$ is $\mathfrak{M}$-generic is equivalent to the Boolean ultrapower $j_U$ being trivial (i.e. letting $E_U$ be the 'membership' relation defined by the Boolean ultrapower, $j_U$ is an isomorphism between $\mathfrak{M}$ and $(\check{M}_U, E_U)$).

If $j_U$ is non-trivial on $V$, we map $V$ to a *subclass* of itself (much as we do with a measurable cardinal embedding). Since $\check{V}_U$ is not the whole of $V$ when $U$ is in $V$ (and hence not $V$-generic), it is possible for a set external to $\check{V}_U$ to be our generic for $\check{V}_U$. One might then use $\check{V}_U$ as our interpretation of "$V$" and $V^{\mathbb{B}}/U$ as our interpretation of $V[G]$. This seems attractive, whilst $\check{V}_U$ is not isomorphic to $V$, it does nonetheless looks a *lot* like $V$; it is a proper-class-sized two-valued elementary extension of $V$, and $V^{\mathbb{B}}/U$ *really is* the forcing extension of $\check{V}_U$ by $U$.

Aside from the remaining issue of finding Boolean-completions in the class theory, there are some further challenges. For instance, often $\check{V}_U$ is non-well-founded (especially when the forcing in question alters the subsets of small $V_\alpha$) and part of what was interesting about forcing was that it kept the relevant models standard. This raises questions as to whether it is a good candidate for interpreting the 'naturality' of forcing constructions, whatever its independent mathematical interest.[74]

The options discussed so far only touch *forcing* constructions for the Universist, and there seem to be challenges for interpreting both class forcing and hyperclass forcing (where we may not have a definable forcing relation or a Boolean-algebra in the class theory), or sharps (which we do not yet know how to obtain through forcing).

**$V$-logic.** Recently, we have discovered that infinitary logics have application for interpreting many outer models, including sharps. It has been known since [Barwise, 1975] that infinitary logics can be used to interpret satisfaction in outer models over a given model of set theory. This was then used by [Antos et al., 2015] in providing an interpretation of extension talk in a framework where *height extensions* are available.[75] [Antos et al., F] then showed how the relevant height extensions required could be coded using impredicative class theory.

There are three main components to this strategy:

(1.) Define the relevant infinitary logic ($V$-logic) for coding satisfaction in outer models.

(2.) Show that this logic can be represented in $Hyp(V)$, the least admissible 'set' containing $V$ as an element.

(3.) Code $Hyp(V)$ using class-theoretic machinery.

In defining the logic, we first need to set up the language:

**Definition 30.** $\mathscr{L}_{\in}^{V}$ is the language consisting of **ZFC** with the following symbols added:

(i) A predicate $\bar{V}$ to denote $V$.

(ii) A constant $\bar{x}$ for every $x \in V$.

[74]For the details and these arguments, see [Barton, 2020]. Koellner also raises the issue:

> "There are three important things to note about $[V^{\mathbb{B}}/U]$—it need not be transitive, it need not be well-founded, it is a definable class in $V$. For all three reasons it is as non-standard a model of set theory...one sees by construction that the model produced is not of the appropriate type to count as the universe of sets." ([Koellner, 2013], pp. 19–20)

[75]This idea is also discussed in [Antos et al., 2015], [Friedman, 2016], and [Barton and Friedman, 2017].

We can then define $V$-logic:

**Definition 31.** *$V$-logic* is a system in $\mathscr{L}_{\in}^{V}$, with provability relation $\vdash_V$ (defined below) that consists of the following axioms:

(i) $\bar{x} \in \bar{V}$ for every $x \in V$.

(ii) Every atomic or negated atomic sentence of $\mathscr{L}_{\in} \cup \{\bar{x}|x \in V\}$ true in $V$ is an axiom of $V$-logic.

(iii) The usual axioms of first-order logic in $\mathscr{L}_{\in}^{V}$.

For sentences in $\mathscr{L}_{\in}^{V}$, $V$-logic contains the following rules of inference:

(a) *Modus ponens*: From $\phi$ and $\phi \rightarrow \psi$ infer $\psi$.

(b) *The Set-rule*: For $a, b \in V$, from $\phi(\bar{b})$ for all $b \in a$ infer $\forall x \in \bar{a}\phi(x)$.

(c) *The $V$-rule*: From $\phi(\bar{b})$ for all $b \in V$, infer $\forall x \in \bar{V}\phi(x)$.

Proof codes in $V$-logic are thus (possibly infinite) well-founded trees with root the conclusion of the proof. Whenever there is an application of the $V$-rule, we get proper-class-many branches extending from a single node. Using this definition, one can then set up a notion of proof as follows:

**Definition 32.** For a theory $\mathbf{T}$ and sentence $\phi$ in the language of $V$-logic, we say that $\mathbf{T} \vdash_V \phi$ iff there is a proof code of $\phi$ in $V$-logic from $\mathbf{T}$. We furthermore say that a set of sentences $\mathbf{T}$ is *consistent in $V$-logic* iff $\mathbf{T} \vdash_V \phi \wedge \neg\phi$ is false for all formulas of $\mathscr{L}_{\in}^{V}$.

One can then show how the apparatus of $V$-logic can interpret extensions. For example, letting $\Phi$ be a condition in any particular formal language on universes we wish to simulate in an extension, we then introduce the following 'axioms' into our theory of $V$-logic:

(i) *$\bar{W}$-Width Axiom.* $\bar{W}$ is a universe satisfying **ZFC** with the same ordinals as $\bar{V}$ and containing $\bar{V}$ as a proper subclass.

(ii) *$\bar{W}$-$\Phi$-Width Axiom.* $\bar{W}$ is such that $\Phi$.

We can then have the following axiom to give meaning to the notion of an extension such that $\Phi$:

$\Phi^{\vdash_V}$-*Axiom.* The theory in $V$-logic with the $\bar{W}$-Width Axiom and $\bar{W}$-$\Phi$-Width Axiom is consistent under $\vdash_V$.[76]

One can then interpret the existence of set forcing generics by claiming that the theory with the following axiom added is consistent:

**Definition 33.** *$\bar{W}$-$G$-Width Axiom.* $\bar{W}$ is such that it contains some $\bar{V}$-$\bar{\mathbb{P}}$-generic $G$.

Similarly with class forcing, with the addition of further predicates $\bar{\mathbb{P}^C}$ and $\bar{G^C}$ for $\mathbb{P}^C$ and $G^C$ into the usual syntax of $V$-logic:

**Definition 34.** *$\bar{W}$-$G^C$-Width Axiom.* $\bar{W}$ is such that $\bar{G^C} \subseteq \bar{W}$ and $\bar{G^C}$ is $\bar{\mathbb{P}^C}$-generic over $V$.

Importantly, this method also allows us to formulate axioms that capture *non*-forcing extensions. For example:

**Definition 35.** *$\bar{W}$-Class-$\sharp$-Width Axiom.* $\bar{W}$ has the same ordinals as $\bar{V}$, satisfies $\mathbf{NBG} + \mathsf{ETR}$, and contains a class sharp that generates $V$.

This then allows us to express the claim that $V$ is sharp generated:

**Definition 36.** *The Class Iterable Sharp Axiom$^{\vdash_V}$.* The theory in $V$-logic with the $\bar{W}$-Width Axiom and $\bar{W}$-Class-$\sharp$-Width Axiom is consistent under $\vdash_V$.

as well as the Inner Model Hypothesis:

**Definition 37.** $\mathsf{IMH}^{\vdash_V}$. Suppose that $\phi$ is a parameter-free first-order sentence. Let $\mathbf{T}$ be a $V$-logic theory containing the $\bar{W}$-Width Axiom and also the $\bar{W}$-$\phi$-Width Axiom (i.e. $\bar{W}$ satisfies $\phi$). Then if $\mathbf{T}$ is consistent under $\vdash_V$, there is an inner model of $V$ satisfying $\phi$.

In this way, if we allow the use of $V$-logic, we are able to syntactically code satisfaction in arbitrary extensions of $V$ in which $V$ appears standard, and hence the effects of extensions of $V$ on $V$.

One still has to show that this can be done within the Universist framework. This can be done by showing that (a) $Hyp(V)$—the least admissible set (i.e. model of Kripke-Platek set theory) containing $V$—can be coded in class theory, and (b) if there if a proof code

[76]Strictly speaking, this will involve a new consequence relation $\vdash'_V$, that includes mention of any axioms involving $\bar{W}$. In fact, *any* collection of additional axioms will result in a new consequence relation involving those axioms. The consequence relation is simply $\vdash_V$ but with any additional axioms added to our original definition of $V$-logic. For clarity we suppress this detail, continue to use $\vdash_V$ (thereby mildly abusing notation).

in $V$-logic, then there is one in $Hyp(V)$. The former is accomplished by coding sets 'above' $V$ via proper-class-sized trees (since trees can code the membership relation of transitive closures), and requires $\Sigma^1_1$-Comprehension over $\mathbf{NBG}$. The latter is lengthy but routine, so we do not include it here.[77]

This coding goes substantially beyond the previous examples in that it can incorporate all the extensions we talked about in §1. However, it bears mentioning that the move to use of higher-order resources is significant, and $Hyp(V)$ cannot be coded without the use of impredicative classes. Moreover, the coding is very *syntactic*; it concerns how sentences and a consequence relation in infinitary logic interact. In this regard it is not unlike the forcing relation and the same worries about the 'intendedness' or 'naturality' of the interpretation transfer immediately across.

**Countable transitive models: redux.** A response to these kinds of problems (for the forcing relation, Boolean-valued, and $V$-logic strategies) is to provide some way of representing similar relations over countable transitive models (where extensions are easily available), but provide enough of a link to $V$ to justify the claim that the interpretation is still 'about' the right (large) objects. For example, given a sufficiently rich class theory, we can show:[78]

**Fact 38.** Let $\phi$ be a sentence of $V$-logic with no constant symbols apart from $\bar{V}$. Then there is a countable transitive model $\mathfrak{V}^*$ such that the following are equivalent (when every instance of $\bar{V}$ is replaced by $\bar{\mathfrak{V}}^*$):

(1.) $\phi$ is consistent in $V$-logic.

(2.) $\phi$ is consistent in $\mathfrak{V}^*$-logic.

(3.) $\mathfrak{V}^*$ has an outer model with $\phi$ true.

This is shown by using a truth predicate and the reflection theorem to reduce the parameter-free theory of $Hyp(V)$ to a countable transitive model. Claims about large infinite sets can then be rendered in a syntactic way in $V$-logic as about the real sets, but with the acknowledgement that when model-theoretic reasoning about the sets occurs we have to move to a smaller countable transitive interpretation in order to represent the reasoning. It is an open philosophical question to what degree this is a satisfactory move.

[77]For the details of both, see [Antos et al., F].

[78][Antos et al., F] use a variant of $\mathbf{NBG} + \Sigma^1_1$-Comprehension. A similar move is considered by [Barton, 2020].

## 2.4 Universist foundational programmes

Thus far, we've surveyed how a Universist can respond to various kinds of challenges coming from different aspects of set theory. Responding to challenges is just one dimension of a foundational set-theoretic view, however. It is also important to consider the mathematical programmes that are naturally motivated by Universism.

Of course, if you hold Universism, then you hold that every sentence of set theory has a definite truth-value. Given the independence results, a significant part of many universist programmes is motivating the acceptance of some axiom(s) resolving independence. Whilst it would take us too far afield to go through every possible programme in detail, there are some that should be highlighted. This list should not therefore not be treated as exhaustive.

**The Inner Model Programme.** One strategy for developing new axioms and studying the internal structure of $V$ has come from the inner model programme. Here, we consider the kinds of inner model that can be built assuming the existence of cardinal numbers of particular kinds. For example, assuming the existence of a measurable cardinal, we can build $L[U]$, an $L$-like (in that it satisfies GCH and the Condensation Lemma[79]) model containing a measurable cardinal. Moving higher up the large cardinal hierarchy, we find other kinds of $L$-like inner model. The details become complex quickly, but set theorists are currently working on constructing inner models for many Woodin cardinals and (with additional assumptions on the kind of iteration available[80]) at supercompact cardinals.[81]

**Ultimate-$L$.** Standing at the end of this road is the idea that $V$ *just is* one such model. This has been recently proposed by Woodin, who shows that if there is an $L$-like inner model for a supercompact cardinal (call it Ultimate-$L$) then all cardinals that exist in $V$ are inherited by Ultimate-$L$. This lies in stark contrast to known $L$-like inner models that are transcended by stronger large cardinal assumptions. For example, $V = L[U]$ is refuted by the existence of two measurable cardinals. He then proposes that we take $V =$ Ultimate-$L$ as a new axiom

[79] *The Condensation Lemma* for $L[U]$ (or, mutatis mutandis, any other model) states that if $\mathfrak{M} \prec (L_\delta[U], \in, U \cap L_\delta[U])$ for limit ordinal $\delta$, then the transitive collapse of $\mathfrak{M}$ is $L_\gamma[U]$ for some $\gamma \leq \delta$.

[80] Namely the *Unique Branch Hypothesis*.

[81] For discussion here, see [Sargsyan, 2013] or [Woodin, 2017].

of set theory, seeing as it would solve known problems of independence and could not be refuted by large cardinal axioms.[82]

**Forcing axioms.** A very different approach is provided by proponents of so-called *forcing axioms*. We mentioned *absoluteness* characterisations of some of these axioms in §1. In general, however, forcing axioms assert that there are generics in $V$ for certain families of dense sets and kinds of partial order. A popular one such is:

**Axiom 39.** (*The Proper Forcing Axiom* or PFA) If $\mathbb{P}$ is a proper[83] forcing poset, and $\mathcal{D}$ is an $\aleph_1$-sized family of dense sets for $\mathbb{P}$, then there is a $\mathcal{D}$-generic filter $G$ intersecting every member of $\mathcal{D}$.

The idea of such an axiom is to hold that $V$ is closed under the formation of generics for families of dense sets for some appropriate class of forcings, and in this way saturate the universe under as much forcing as possible. Strengthenings of PFA are possible, for example to Martin's Maximum (where we allow arbitrary stationary set preserving posets instead of proper posets) and its variants[84]. Of course, it is not possible to saturate the universe with generics for arbitrary families of dense sets, such an axiom would imply (per impossibile) a $V$-generic filter. However, the idea can be given intuitive content for Universists by using the idea that every possible set exists. Magidor writes the following:

> "Forcing axioms like Martin's Axiom (MA), the Proper Forcing Axiom (PFA), Martin's Maximum (MM) and other variations were very successful in settling many independent problems. The intuitive motivation for all of them is that the universe of sets is as rich as possible, or at the slogan level: A set [whose] existence is possible and there is no clear obstruction to its existence [exists]...
>
> ...What do we mean by "possible"? I think that a good approximation is "can be forced to [exist]"...
>
> I consider forcing axioms as an attempt to try and get a consistent approximation to the above intuitive principle by

[82] The full case for Ultimate-$L$ is somewhat more sophisticated than what I have presented here. For the mathematical details, see [Woodin, 2017]. A concise overview of the key philosophical details is available in [Koellner, 2019], with further background in [Koellner, 2014], [Koellner, 2011].

[83] A forcing poset $\mathbb{P}$ is *proper* iff for every uncountable cardinal $\kappa$, every stationary subset of $[\kappa]^\omega$ is stationary in the generic extension.

[84] For variants of Martin's Maximum, see [Viale, 2016] and [Viale, 2016].

> restricting the properties we talk about and the the forcing extensions we use. ([Magidor, U], pp. 15–16)

The issue of exactly how far this intuition can take us is an open question. However, forcing axioms represent a markedly different approach to the Ultimate-$L$ programme, the former implies CH, whereas most forcing axioms that settle a value for the continuum imply that $2^{\aleph_0} = \aleph_2$.

**Reflection principles.** A final variety of axioms we'll look at consists of *reflection principles*. Many set theorists and philosophers regard these axioms as *natural* principles[85], and it has been argued that they represent axioms that the Universist can justify better than her counterparts.

What exactly counts as a reflection principle is a somewhat difficult question. Roughly, they assert that the universe cannot be distinguished (in some precise sense) from one of its initial segments. For example we might consider the first-order reflection schema:

**Axiom 40.** (First-Order Reflection Schema) Let $\phi(\vec{x})$ be a formula with variables $\vec{x}$ and let $\phi^{V_\beta}(\vec{x})$ denote the restriction of quantifiers and variables in $\phi(\vec{x})$ to $V_\beta$. Then

$$\forall\alpha\exists\beta > \alpha\forall\vec{x} \in V_\beta[\phi(\vec{x}) \leftrightarrow \phi^{V_\beta}(\vec{x})]$$

This schema is equivalent (modulo the other axioms of **ZFC**) to the Axiom of Infinity and Replacement Scheme. Allowing higher-order formulae and parameters produces stronger and stronger reflection principles. For instance, allowing second-order sentences to be reflected yields inaccessible cardinals. If one then asserts that formulae are reflected to a $V_\kappa$ with $\kappa$ strongly inaccessible, one produces Mahlo cardinals. Reflecting then to Mahlo cardinals results in $\alpha$-Mahlo cardinals. One can move to a higher-order language, thus allowing stronger and stronger reflection principles and thereby producing a hierarchy of cardinals known as the indescribable cardinals.[86] However, as [Koellner, 2009] shows, all known cardinals of this form are below the least $\omega$-Erdős cardinal.[87] Generalising to third-order parameters

[85]See, for example, [Bernays, 1961], [Reinhardt, 1974], and [Fraenkel et al., 1973].

[86]A cardinal $\kappa$ is $Q$-indescribable (where $Q$ is of the form $\Sigma^m_n$ or $\Pi^m_n$) iff for any $X \subset V_\kappa$ and sentence $\phi$ of $Q$ order and complexity, if $\langle V_\kappa, \in, X\rangle \models \phi$ then there is an $\alpha < \kappa$ such that $\langle V_\alpha, \in X \cap V_\alpha\rangle \models \phi$.

[87]The first $\omega$-Erdős cardinal is the least cardinal satisfying certain combinatorial properties on a partition into its finite subsets. As the definition of the cardinal is somewhat involved I omit it here; details are available in a wide variety of texts including [Drake, 1974], [Kanamori, 2009], and [Jech, 2002].

yields a contradiction.[88]

Recently, Welch has proposed a (much stronger) *global* reflection principle in an effort to overcome this boundary. The principle has its conceptual roots in the reflection arguments used by Reinhardt in [Reinhardt, 1974], which in turn are somewhat similar to the ideas at play in [Magidor, 1971]. He uses *elementary embeddings* to characterise:

**Axiom 41.** (GRP) Let $(V, \in, \mathcal{C})$ denote the structure of $V$ with all its classes. The *Global Reflection Principle* states that there is a non-trivial elementary embedding[89] $j$ and ordinal $\kappa$ with $crit(j) = \kappa$ such that:
$j : (V_\kappa, \in, V_{\kappa+1}) \longrightarrow (V, \in, \mathcal{C})$

The GRP is very strong, implying the existence of a proper class of Woodin cardinals. There is a question, however, as to the extent that this is a 'reflection' principle, depending as it does on the $j$ used to define the embedding. However recently Roberts has shown that by implementing reflection with a satisfaction predicate, one can obtain a strong reflection principle (implying the existence of 1-extendible cardinals) that looks more like the traditional reflection principles.[90]

Interesting here is that some authors see this as a possible difference between the Universist and her counterparts; she can motivate reflection more easily. The reason for this is that a reflection principle asserts that the universe (or the 'absolute') in some sense evades being captured. If we deny Universism and hold that there is no absolute, the motivation seems less clear.[91]

**Overlapping consensus.** The Universist thus has several competing foundational programmes to choose from. One question then is the extent to which they overlap. For example, Koellner points to the fact that $\mathsf{AD}^{L(\mathbf{R})}$ is implied by all theories that provide a fine structure theory for Woodin cardinals.[92] This includes, for example, theories that

[88]See here [Tait, 2005] or [Koellner, 2009].

[89]The level of elementary insisted upon results in different technical consequences: see [Welch, 2014] for details.

[90]See [Roberts, 2017] and [Welch, 2019] for discussion of this issue.

[91]For example, Tait writes that if we deny Universism:

> "...reflecting *down* from the universe of all sets, becomes problematic. For it seems to require that we know what it means to say that a sentence $\phi(t)$ is true in the universe of all sets." ([Tait, 1998], p. 473)

This idea is echoed by [Koellner, 2009]. However, [Tait, 1998] goes on to provide a way of obtaining reflecting universes within an Anti-Universist background, and [Barton, 2016b] argues that Welch's motivations can be used without holding Universism.

[92]See here [Koellner, 2014] (esp. §4.5) and the references contained therein.

imply the outright existence of Woodin cardinals (such as the strong reflection principles considered by Roberts and Welch) but also axioms like PFA.[93] These results seem to suggest that there might be certain levels of the hierarchy that can be filled out under the Universist's programme, even if it is unclear exactly what axioms we should pick globally.[94]

# 3 Anti-Universism

Thus far, we have examined Universism; the idea that there is one universe of sets that settles all truths about set theory. The time has come to consider some alternatives. There are several ways to cash out a denial of Universism. We'll consider variations of the following three:

> **Multiversism.** There are multiple equally legitimate universes of set theory, and no-one universe is especially privileged.
>
> **Potentialism.** The subject matter of set theory is *modally indefinite* in different ways (we say 'subject matter' here, rather than 'universe of sets', because as we'll see later, it's not really clear whether we should identify Potentialism as a view that thinks that there is one universe of sets that is indefinite, or that thinks that there are multiple universes of sets that we can talk about modally).
>
> **Universe Indeterminism.** There is one universe of sets, but it is indefinite (i.e. not every claim about this one universe is determinately either true or false, and possibly classical logic is not appropriate for reasoning in set theory across the board).

We'll consider variations of these views, arguments for and against them, and the mathematical programmes they suggest in the rest of this section.

[93] See [Steel, 2005] for the result.

[94] We lack the space to discuss this here, but looking at what can be 'freezed' (and hence lies in the overlapping consensus) is part of the motivation behind Woodin's use of $\mathbb{P}_{max}$ and $\Omega$-logic. For a concise introduction and further reading, see [Koellner, 2019], §3.

## 3.1 Multiversism

As stated above, Multiversism is the idea that our set-theoretic talk does not determine a unique universe up to isomorphism, but rather a plurality thereof. There are various ways that we can cash this out, we'll consider adding more ordinals (*height* multiversism: §3.1.1), adding more subsets (*width* multiversism: §3.1.2), or even admitting non-standard universes as interpretations (§3.1.3).

### 3.1.1 Height multiversism

The first kind of multiversism we shall consider is multiversism concerning the *height* of the hierarchy. This can be stated as follows:

> **Height Multiversism.** For any universe of sets $V$, there is another universe of sets $V'$ such that $V \in V'$ and $Ord(V) \in V'$.

One natural species of Height Multiversism is *Level Multiversism*:

> **Level Multiversism.** The universes of sets are well-ordered and any universe $V$ is of the form $V_\kappa \in V'$ for some height-extension $V'$ of $V$.

Under these kinds of multiversism; our reference to "the" universe of sets or "$V$", should be understood as something like a free variable letter, standing for any universe of the appropriate form. But what are the reasons for adopting Height Multiversism?

**The Paradoxes.** The first is simply the paradoxes. On the Height Multiversist picture, it looks like there is no question of the interpretation of proper classes. This is because, for any universe $V$, the 'proper classes' of $V$ are garden-variety sets in some height extension $V'$ of $V$.

Similar consideration apply to long-well-orders over 'the' universe. If we are Height Multiversists, we seem to have an easy answer to the problem of providing an underlying ontology for a given definable long well-order. In particular, given a particular universe $V$, a long-well-order defines an ordering longer than $Ord(V)$ that is represented by an ordinary set in some height extension $V'$.

Moreover, irrespective of whether or not the Height Multiversist accepts an ontology of width extensions as well, she then has a way of interpreting width extensions for any particular universe $V$. Because height extensions are always available, they can always take $Hyp(V)$

and use the resources of $V$-logic without any need for coding in the class theory.[95]

However, one might question exactly how far the Height Multiversist response to the paradoxes takes us. For the Height Multiversist, I want to say things like:

> "Any universe $V$ has an extension $V'$ such that $V \in V'$."

However, in making this claim it seems like I quantify over *all* the universes. What then is *this* domain over which I quantify? Isn't 'the collection of all universes' just a proper class in different clothes? Exactly how to understand these issues and whether this still constitutes a version of the problem of proper classes is a difficult question. Since similar questions are well worn in the literature on absolute generality,[96] we'll set this aside, however later (§3.2) we discuss the Potentialist response to the problem via the use of *modal* resources.

**The Quasi-Categoricity Theorem.** A different motivation for a particular version of Height Multiversism in fact comes from the Quasi-Categoricity Theorem. If one thinks that categoricity is important for an account of how we refer to mathematical objects[97], then the following version of Level Multiversism is pertinent:

> **Inaccessible Level Multiversism.** By *Inaccessible Level Multiversism* we mean the view that the universes are well-ordered, and that any universe $V$ is of the form $V_\kappa$ in some larger universe $V'$, in which $\kappa$ is inaccessible.[98]

One way of motivating Inaccessible Level Multiversism is to argue that the *quasi*-categoricity (as opposed to *full* categoricity) of $\mathbf{ZFC}_2$ is indicative of the failure of our thought and language to uniquely determine one universe of sets rather than a plurality thereof. If one thinks that in order to determinately refer to or understand a domain we have to provide a theory that pins it down, the quasi-categoricity theorem

[95] See here [Antos et al., 2015] and [Barton and Friedman, 2017].

[96] See [Rayo and Uzquiano, 2006b] for a useful survey of the subject, and the essays contained in [Rayo and Uzquiano, 2006a] for further detail.

[97] For just such a view see [Isaacson, 2011].

[98] This kind of Multiversism is often attributed to [Zermelo, 1930], but is clearly recently advanced by [Hellman, 1989] (see Ch. 2), [Rumfitt, 2015] (see Ch. 9) and [Isaacson, 2011].

shows that no maximal domain is possible without bounding the number of inaccessible cardinals.[99]

This analysis of the significance of the quasi-categoricity theorem can be linked to a diagnosis of the set-theoretic paradoxes. The paradoxes, one might think, are another manifestation of the claim that our thought and language cannot pin down a unique structure in terms of height, to do so (according to the Height Multiversist) is incoherent and would produce a contradiction with other principles she holds. The paradoxes thus show that our thought *cannot* pin down an intended structure and the quasi-categoricity theorem shows *why* it does not; any natural theory of sets (without anti-large cardinal axioms added that bound the number of inaccessibles) will only ever be quasi-categorical. So Zermelo writes:

> Scientific reactionaries and anti-mathematicians have so eagerly and lovingly appealed to the 'ultrafinite antinomies' in their struggle against set theory. But these are only apparent 'contradictions', and depend solely on confusing set theory itself, which is not categorically determined by its axioms, with individual models representing it. What appears as an 'ultrafinite non- or super-set' in one model is, in the succeeding model, a perfectly good, valid set with both a cardinal number and an ordinal type, and is itself a foundation stone for the construction of a new domain. To the unbounded series of Cantor ordinals there corresponds a similarly unbounded double-series of essentially different set-theoretic models, in each of which the whole classical theory is expressed. The two polar opposite tendencies of the thinking spirit, the idea of creative advance and that of collection and completion [Abschluß], ideas which also lie behind the Kantian 'antinomies', find their symbolic representation and their symbolic reconciliation in the transfinite number series based on the concept of well-ordering. This series reaches no true completion in its unrestricted advance, but possesses only relative stopping-points, just those 'boundary numbers' which separate the higher model types from the lower. Thus the set-theoretic 'antinomies', when correctly understood, do not lead to a cramping and mutilation of mathematical science,

[99]See [Hamkins and Solberg, 2020] for a fine-grained analysis of the kinds of theories and cardinals that can be categorical given $\mathbf{ZFC}_2$ as a base theory. A further question raised by Hamkins and Solberg is whether this desire for categoricity pulls in the opposite direction to reflection.

> but rather to an, as yet, unsurveyable unfolding and enriching of that science. ([Zermelo, 1930], p. 1233)

This response to the paradoxes, appealing to both the quasi-categoricity theorem and the paradoxes as revealing an unbounded sequence of equally legitimate universes, works in tandem with the observation, identified in §1.2, that there is no known greatest consistent large cardinal axiom. Because one can always assert that some axiom or other holds within an inaccessible rank, we obtain a picture of set theory on which we come to be able to define larger and larger domains on the basis of stronger and stronger large cardinal axioms, but without ever isolating a unique maximal universe.

### 3.1.2 Width Multiversism

Where Height Multiversism concerned the addition of *ordinals* (and hence ranks) to models, *Width* Multiversism rather concerns how *subsets* can be added to universes. We can state it as follows:

> **Width Multiversism.** For any universe of sets $V$, there is a (are) universe(s) of sets $V'$ such that $V \subset V'$ and $Ord(V) = Ord(V')$.

A quick remark first: Width Multiversism and Height Multiversism are not necessarily orthogonal, and can be combined with one another. In fact certain mathematical facts may require one to commit to both. For example, suppose that one thinks that the Shepherdson-Cohen minimal model is a legitimate universe.[100] Even adding two extra $L$-levels to this model will necessarily add reals.[101] So this universe cannot be extended in height to a new well-founded model without adding subsets too. In contrast, any version of Level Multiversism will be inconsistent with Width Multiversism.

Many natural versions of Width Multiversism are those that talk about the relevant kinds of construction. For example, we might consider:

[100] The Shepherdson-Cohen minimal model is a countable transitive model of the form $L_\alpha \models \mathbf{ZFC}$ where $\alpha$ is the least such ordinal. This is the *minimal* transitive model of **ZFC**; it contains no other transitive models of **ZFC**.

[101] This is because in a model $L_\beta \models \mathbf{ZFC}$, a first order sentence $\phi$ is true iff for some $n$, $\phi$ is $\Sigma_n$ and there exists a satisfaction predicate for $\Sigma_n$ formulas witnessing this. These partial satisfaction predicates are definable over $L_\beta$, and hence a full satisfaction predicate exists in $L_{\beta+2}$. Since every set in the Shepherdson-Cohen minimal model $L_\alpha$ is definable, the satisfaction predicate appears as a new real at $L_{\alpha+2}$.

**Set Forcing Multiversism.** For any universe of sets $V$, and any forcing partial order $\mathbb{P} \in V$, there is a universe $V'$ such that there is a generic filter $G \in V'$ for $\mathbb{P}$ and $V' = V[G]$.

For class forcing, the situation becomes a little more complex, since we have to say how the classes are expanded:

**Class Forcing Multiversism.** Let $(V, \in, \mathcal{C}^V)$ be a universe with some collection of classes $\mathcal{C}^V$ over this universe. Then if $\mathbb{P}$ is a class forcing partial order in $\mathcal{C}^V$, then there is a universe $(V[G], \in, \mathcal{C}^{V[G]})$, such that $G$ is generic for $\mathbb{P}$ and is one of the classes of $\mathcal{C}^{V[G]}$, and where $V[G]$ consists of the interpretations of set-names in $V$ using $G$, and $\mathcal{C}^{V[G]}$ consists of the interpretations of class-names in $\mathcal{C}^V$ using $G$.

Hyperclass Forcing Multiversism can be defined similarly, since current accounts of hyperclass forcing (e.g. [Antos and Friedman, 2017]) depend on performing class forcings:

**Hyperclass Forcing Multiversism.** Let $(V, \in, \mathcal{C}^V)$ be a universe satisfying **MK** with the Class Bounding Axiom added, that is correct about well-founded relations (i.e. it is a $\beta$-model). Then if $\mathbb{P}$ is a hyperclass forcing partial order codable in $\mathcal{C}^V$, then there is a universe $(V[G], \in, \mathcal{C}^{V[G]})$, such that $G$ codes a $V$-hyperclass-generic for $(V, \in, \mathcal{V})$.

The situation is slightly more difficult when it comes to sharps since we need to talk about iterations, but it can be done:

**$\sharp$-Multiversism.** Suppose that $V$ contains enough large cardinals to support being generated by a sharp via an iteration of some length $\alpha$. Then there is a universe $V'$ containing a sharp for $V$ that generates $V$ with an iteration of length $\alpha$.

We can see then that Width Multiversism refers to a large diversity of positions (some of which imply each other; for instance Class Forcing Multiversism trivially implies Set Forcing Multiversism). What then are the kinds of argument advanced for Width Multiversism?

**Naturality and forcing.** One source of considerations mobilised in favour of Width Multiversism is the idea that it provides an especially natural interpretation of width extensions such as forcing. For example, Hamkins writes:

> "This abundance of set-theoretic possibilities poses a serious difficulty for the universe view, for if one holds that there is a single absolute background concept of set, then one must explain or explain away as imaginary all of the alternative universes that set theorists seem to have constructed. This seems a difficult task, for we have a robust experience in those worlds, and they appear fully set theoretic to us. The multiverse view, in contrast, explains this experience by embracing them as real, filling out the vision hinted at in our mathematical experience, that there is an abundance of set-theoretic worlds into which our mathematical tools have allowed us to glimpse." ([Hamkins, 2012], p. 418)

and

> "...a set theorist with the universe view can insist on an absolute background universe $V$, regarding all forcing extensions and other models as curious complex simulations within it. (I have personally witnessed the necessary contortions for class forcing.) Such a perspective may be entirely self-consistent, and I am not arguing that the universe view is incoherent, but rather, my point is that if one regards all outer models of the universe as merely simulated inside it via complex formalisms, one may miss out on insights that could arise from the simpler philosophical attitude taking them as fully real." ([Hamkins, 2012], p. 426)

The thought here is that a large part (if not the majority) of set-theoretic practice involves the study of different universes satisfying axioms, and what can obtain from these universes. So, for example, the original result of Cohen can be viewed as what one can obtain given a universe satisfying $\mathsf{CH}$. Whilst this can be coded (as outlined in §2.3) often this is *unnatural* requiring "contortions" and not respecting the "mathematical experience". On a Forcing Multiversist approach, however, both $V$ and $G$ are uncontroversially available, and so reference can be completely transparent and one can simply view the original universe being extended.

Further in this direction, we can point to the fact that many theorems and axioms can be formulated in some variety of Width Multiversit Framework (as noted earlier in §1.3 and §1.4). Whilst these theorems can be coded within the Universist framework, it seems reasonable to say that the thinking that underlies them is Multiversist in flavour. Perhaps then the most *natural* interpretation of *these* aspects of the underlying subject matter is multiversist.

**Analogy with other areas of mathematics.** A different motivation for Width Multiversism is through analogies with historical episodes in mathematics. Two that have been mobilised (especially in [Hamkins, 2012]) are the use of the complex plane with respect to the real numbers, and independence of the Parallel Postulate (PP) with respect to geometry.

The case of the complex numbers is roughly as follows. Shortly after the imaginary numbers were introduced, they were viewed as ontologically not on the same footing as the ordinary real numbers. What after all is $\sqrt{-1}$? Descartes, for example, writes:

> Neither the true nor the false roots are always real; sometimes they are imaginary; that is, while we can always conceive of as many roots for each equation as I have already assigned, yet there is not always a definite quantity corresponding to each root so conceived of. Thus, while we may conceive of the equation $x^3 - 6x^2 + 13x - 10 = 0$ as having three roots, yet there is only one real root, $2$, while the other two, however we may increase, diminish, or multiply them in accordance with the rules just laid down, remain always imaginary. ([Descartes, 1637], p. 175)

Thus Descartes holds that whatever the usefulness of reasoning using the relevant algebraic rules and $\sqrt{-1}$, there were not necessarily actual objects that underwrote the mathematics in question. Imaginary numbers were, according to Descartes, literally *not real*.

It was soon realised, however, that they could be modelled using pairs of real numbers (with a slightly modified definition of multiplication) and often they were useful for solving problems about the reals. [Painlevé, 1900], for example, remarks:

> The natural development of this work soon led the geometers in their studies to embrace imaginary as well as real values of the variable. The theory of Taylor series, that of elliptic functions, the vast field of Cauchy analysis, caused

a burst of productivity derived from this generalization. It came to appear that, between two truths of the real domain, the easiest and shortest path quite often passes through the complex domain.

The analogy then is the following: We should think of the reals in Panlevé's example as analogous to a universe of sets $V$, $\sqrt{-1}$ as analogous to $G$, and $V[G]$ as analogous to the field extension of $\mathbb{R}$. Similar to how $\mathbb{C}$ can be simulated within $\mathbb{R}$, $V[G]$ can be simulated within $V$ (by one of the relevant coding methods). However, moving to a perspective on which the complex plane is its own entity as well is theoretically more elegant, and so we should do the same with $V$ and $V[G]$.

A salient response here comes from the *foundational* and *universal* role that set theory is meant to play (especially from the Universist's perspective). For the Universist, $V$ is meant to encapsulate all the sets there are, which is quite unlike the situation for those interested in $\mathbb{R}$. There is no pressure, for example, to accept that $\mathbb{R}^2$ is part of $\mathbb{R}$, though the former can be coded in the latter. Similar considerations apply to $\mathbb{C}$, which was eventually understood in planar terms. This contrasts sharply with $V$, which is meant to encapsulate *all* sets.

This response on behalf of the Universist will be unconvincing to the Width Multiversist, however. For the Width Multiversist, the situation is very analogous; there are $V$-sets and $V[G]$-sets, just like there are real numbers and complex numbers. Here we see a common theme amongst debates between the different foundational viewpoints; namely a dialectic standoff. The analogy between complex numbers is very convincing (and pedagogically helpful) to the theorist sympathetic to the Width Multiversist's project. However for the Universist, the response begs the question; the analogy breaks down under her position. Similar considerations are at play if an Anti-Width Multiversist appeals to the quasi-categoricity theorem for $\mathbf{ZFC}_2$. This is convincing if one thinks that there is a domain of sets with a definite conception of the available subsets of a set which is maximal under inclusion. However, if one rejects this (as the Width Multiversist does) then all a quasi-categoricity theorem tells you is that *within a universe* there is, up to isomorphism, just one conception of the powerset operation up to some ordinal.

A different analogy is made between the independence of the Parallel Postulate (PP) from the axioms of geometry and the independence of CH from $\mathbf{ZFC}$. Prior to the 19th century, Euclidean geometry was largely regarded as *true* (of nature) and especially epistemologically certain.[102] In the 19th and 20th the independence of PP from the other geometrical axioms was shown by the existence of non-Euclidean geometries. The observation that we can represent these geometries by modelling them within Euclidean space along with the discovery that space-time was non-Euclidean contributed to the acceptance of non-Euclidean geometries as equally legitimate alternative geometrical structures. So, the thinking goes, with $V$ and $V[G]$; we have simulations of $V[G]$ within $V$ (e.g. via a Boolean-ultrapower and quotient structures) and so should accept them.[103]

Again, the advocate of the Universist position is likely to be unmoved. A key factor here (as identified by [Kreisel, 1967]) is in the different behaviour of categoricity arguments with respect to PP and CH. For, the independence proofs for geometry show that PP is independent of the *second*-order axiomatisation. This is not so for bounded set-theoretic statements like CH, where they have the same truth value in all structures of $\mathbf{ZFC}_2$ with the full semantics. This is a salient difference between the independence of CH and PP whether or not one thinks that CH has a definite truth-value; the latter (within any model $\mathfrak{M}$ of set theory) always has one truth value for whatever $\mathfrak{M}$ thinks is the full semantics for $\mathbf{ZFC}_2$, whereas the former always takes different truth values in different models of the second-order axioms for geometry within $\mathfrak{M}$.

### 3.1.3 Radical Multiversism

A further kind of Multiversism is the following:

> **Radical Multiversism.** Any first-order structure satisfying the axioms of **ZFC** is an equally legitimate universe of set theory (ontologically speaking).

Here, universes may disagree even on what holds concerning the natural numbers (for example, they might disagree on whether $Con(\mathbf{ZFC})$ is true or false). One universe can thus be non-well-founded relative to another. For the Radical Multiversist it is not just that the meaning of the powerset operation can vary between different equally legitimate universes, but also that our notions of well-foundedness and even natural number are indeterminate and admit of various interpretations contingent upon the universe under consideration. Hamkins, for example, expresses himself as follows:

[102]e.g. In Descartes, Hobbes, Spinoza, Locke, Hume, and Kant. See [Torretti, 2019] for a short introduction.

[103]This idea, formally expressed, it part of what [Hamkins, 2012] sees as a philosophical ramification of his Naturalist Account of Forcing.

> ...although it may seem that saying "1, 2, 3, ... and so on," has to do only with a highly absolute concept of finite number, the fact that the structure of the finite numbers is uniquely determined depends on our much murkier understanding of which subsets of the natural numbers exist. So why are mathematicians so confident that there is an absolute concept of finite natural number, independent of any set-theoretic concerns, when all of our categoricity arguments are explicitly set-theoretic and require one to commit to a background concept of set? My long-term expectation is that technical developments will eventually arise that provide a forcing analogue for arithmetic, allowing us to modify diverse models of arithmetic in a fundamental and flexible way, just as we now modify models of set theory by forcing, and this development will challenge our confidence in the uniqueness of the natural number structure, just as set-theoretic forcing has challenged our confidence in a unique absolute set-theoretic universe. ([Hamkins, 2012], p. 428)

There is a sense then in which a motivation for Radical Multiversism is a kind of Skolemite position; only those notions that can be characterised absolutely in a first-order manner have determinate meaning.[104] Hence there is no determinate notion of powerset, natural number, and finiteness, and the conception of each can vary between universes.

This denial of the definiteness of some of the most basic notions of set theory has some interesting ramifications. The fact that so much is relative has been used by some authors as an objection to Radical Multiversism as an ontological position. [Barton, 2016a] argues, for example, that there is a problem for Hamkins in that the claim:

> "Any first-order structure satisfying **ZFC** is a universe."

is indeterminate. This is because, by the Radical Multiversist's own lights, the notion of finiteness is indeterminate and hence **ZFC** is indeterminate (since we depend on the notion of finiteness for characterising the notion of well-formed formula and proof). So it is unclear what the Radical Multiversist takes herself to be claiming.[105]

[104]This is suggested as a way of interpreting Hamkins' responses to the categoricity arguments by [Koellner, 2013], [Koellner, 2019], and [Barton, 2016a].

[105][Koellner, 2019] (esp. §4.1) makes a similar point articulating the underlying theory for the Radical Multiversist.

One proposed solution is to view Radical Multiversism not as a view about describing an ontology of a multiverse, but rather as proposing an algebraic concept of set. We might think of set theory on the Radical Multiversist's perspective as analogous to group theory in standard mathematics—it is studying algebraic properties attaching to structures satisfying whatever their own version of **ZFC** is. In this way we avoid the problem of having to articulate what is meant by **ZFC** independently of a schematic commitment to the way the world of sets appears from the perspectives of certain structures.

There are still significant problems for this view. Even if it resolves what is meant by **ZFC**, it is still unclear what sense can be made of a universe 'taking itself to satisfy' whatever is denoted by its own version of **ZFC**. This is because the analysis of satisfaction also becomes highly non-absolute once we admit universes that look non-standard from each other. Consider the following theorem:

**Theorem 42.** [Hamkins and Yang, 2013] Every consistent extension of **ZFC** has two models $\mathfrak{M}_1$ and $\mathfrak{M}_2$, which agree on the natural numbers and on the structure $\langle \mathbb{N}, +, \times, 0, 1, < \rangle^{\mathfrak{M}_1} = \langle \mathbb{N}, +, \times, 0, 1, < \rangle^{\mathfrak{M}_2}$, but which disagree their theories of arithmetic truth, in the sense that there is in $M_1$ and $M_2$ an arithmetic sentence $\sigma$, such that $\mathfrak{M}_1$ thinks $\sigma$ is true, but $\mathfrak{M}_2$ thinks it is false.[106]

This shows that *even if we settle on what* **ZFC** *is*, what a structure satisfies is still contingent upon a particular notion of satisfaction built over it. It seems hard to resolve this issue without using resources substantially beyond those that are countenanced as determinate by the Radical Multiversist. Whether there is a philosophically satisfactory response on behalf of the Radical Multiversist is an open question.

## 3.2 Potentialism

Multiversists (of various stripes) take it that our set-theoretic thought and language is about not one universe of sets, but many. Given a particular set-theoretic utterance, it is natural for her to say that we are always restricted to some universe or other.

However, given this position, old worries about absolute generality in philosophy emerge. For instance, the Multiversist might claim that we are always restricted to a particular universe, and hence that we cannot quantify over all sets. But in making this latter claim, they precisely do what they think is forbidden and seem to quantify over

[106]This can be quite extreme, even to the point where $\mathfrak{M}_1$ thinks a particular number is even whereas $\mathfrak{M}_2$ thinks that it is odd.

some 'super-universe' of universes. We noted a similar problem when discussing whether the Height Multiversist really resolves the problem of proper classes (see §3.1.1). Another example, the Multiversist (of whatever kind) wants to say things like:

> "Every universe satisfies the Axiom of Extensionality."

and this might be viewed as incoherent: She wants to quantify over the 'collection' of all universes whilst denying that there is any such domain. As mentioned before, this issue has been considered extensively in the literature on absolute generality.[107] One popular choice in that literature is the use of modal resources as devices of generality, and Potentialism can be seen as an expression of this approach in the set-theoretic case. Instead of holding that our talk about the universe of sets is always implicitly restricted to some universe or other (and the multiverse visible from it) one might hold that set theory is inherently *modal*.[108]

This idea behind Potentialism—that our talk about sets incorporates some notion of *possibility*—suggests an approach on which we speak about possibilities using modal operators without explicitly first-order quantifying over all sets. On this view, our talk about sets is *indefinitely extensible*; a notion with a long history and contributions by multiple authors[109]. The rough idea is that any time we have some definite collection of sets then we *can* form a set of all of them. This idea of it being *possible* to take a set of any given things whatsoever appears, at first blush, to be modal.

The usual methodology is then to use modal operators into our language and interpret set-theoretic quantification modally. So, for example, given a level version of Potentialism (e.g. [Linnebo, 2010]), we can introduce the operators $\Box$ and $\Diamond$ and lay down an axiomatisation

[107] See [Rayo and Uzquiano, 2006b].

[108] This idea concerning infinity and plausibly goes back as far as Aristotle. See, for example, his remarks in the *Physics* about infinity. e.g.

> Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversable. In point of fact they do not need the infinite and do not use it. They postulate only that a finite straight line may be produced as far as they wish. It is possible to have divided into the same ratio as the largest quantity another magnitude of any size you like. Hence, for the purposes of proof, it will make no difference to them whether the infinite is found among existent magnitudes. (*Physics*, Book III, 207b28–207b34)

[109] See [Shapiro and Wright, 2006] for an overview.

for their use (see §3.4.3 for more detail here). The intended meaning of "$\Diamond\phi$" is "It is possible to go on to form sets so as to make it the case that $\phi$" and that of "$\Box\phi$" is "no matter what sets we go on to form it will remain the case that $\phi$". We might then take the set theorist's quantification in the language of set theory to be modalised, so $\exists x\phi(x)$ should be interpreted as $\Diamond\exists x\phi(x)$ and $\forall x\phi(x)$ interpreted as $\Box\forall\phi(x)$.

The exact manner in which we spell out the modal commitments of the Potentialist can vary along two main arcs. For example, we could use operators looking in different directions (e.g. [Studd, 2013] uses forward and backward modal operators based on tense logic). A second direction in which these views can be modified is by allowing different conceptions for the possible worlds and hence different modal axioms. One might instead of viewing the possible worlds as constituted by levels (in the manner of [Linnebo, 2010]) view any forcing extension of a given world as possible. We discuss some technical ramifications of these different options in §3.4.3.

A salient question and challenge for the Potentialist is how to interpret the relevant mathematical modality. If one is already a Multiversist of some stripe or other, then it is relatively easy; one already has a collection of worlds (the relevant universes) and one can consider different kinds of accessibility relation between them (we consider these in detail in §3.4.3). Also, given an account of modality for the Potentialist, one can extract some interpretation of Multiversism simply by taking the possible worlds to constitute elements of a multiverse.

Whilst the there are clear links between Multiversism and Potentialism, it is not clear that they are the same philosophically. Firstly, Potentialism seems to have some notion of *operation iteration* at its core and in its motivation; if we *can form* sets such that $\phi$ then we should go ahead and do so. This seems to be partly what lies behind the level version of Potentialism (where the generating operation is powerset) and the forcing version of Potentialism (where the generating operation is the addition of some generic set). Some versions of Multiversism, however, do not clearly have a potentialist operation of generation underlying them (e.g. Radical Multiversism).

A different alternative to seeing the potentialist modal operators as underwritten by some kind of Multiversism is to countenance a specifically *mathematical* kind of modality. Often this is taken to be motivated by the iterative conception (at least in its level form) which seems to have a modal flavour. However, this raises a challenge: We require a *philosophical* explanation of how this mathematical modality should be cashed out. This is not least because on many ordinary understandings of mathematics, mathematical truth (and objects) are metaphysically necessary, and this rules out the possibility of using standard metaphysical necessity to do the job. Nonetheless, some authors do take this approach, or opt for logical possibility (e.g. [Hellman, 2002]). A different option is to understand the modality as a way of individuating mathematical objects (e.g. [Linnebo, 2010]), arguing that the existence of a condition for determining the extensionality of some plurality licences the introduction of a new object. Alternatively one might view the modality as *postulational* (e.g. [Fine, 2005]), arguing that we can expand ontologies by postulating new objects and expanding our ontology, and the modality consists in moving between legitimate postulations. [Studd, 2013] opts instead for cashing out the modality in linguistic terms, arguing that it can be explained as liberalising the interpretation of our lexicon. It is an open question to what extent these interpretations are philosophically satisfactory, if there are others, and whether or not there are relationships between them.

The introduction of modal resources leads to another possible philosophical difference between Multiversism and Potentialism. The Multiversist clearly claims that there are *multiple* equally legitimate universes of set theory. However, the Potentialist might claim that there is *just one* universe of set theory, but that it is *modally* indefinite, and the way we achieve generality about the universe is via modal claims rather than outright quantification.[110]

## 3.3 Universe Indeterminism

A different way of examining indefiniteness is to by holding that there is one fixed universe, but it is not bivalent (rather than that there are multiple universes or that the universe is inherently modal). We will call this view Universe Indeterminism:

> **Universe Indeterminism.** There is just one universe of sets, but it is indefinite (that is, not every sentence of set theory is either true or false in it; some are neither true nor false).

This has recently suggested by [Scambler, 2020], who argues that a variant of Feferman's Semi-Constructive Set Theory[111] (that Scambler denotes $\mathbf{SCS}^+$) provides an axiomatisation of this idea and can be motivated via holding that the legitimate interpretations of set theory are the standard transitive models. One can then view $\mathbf{SCS}^+$ as telling us what the determinate 'core' of set-theoretic truths are, whereas the

[110]Whether these modal operators commit one to quantification is also open, and presents a possible line of objection to the Potentialist.

[111]See here [Feferman, 2010].

study of **ZFC** and its extensions tells us about the *relative determinacy* of sentences—what becomes determinate when we take other sentences to be determinate (e.g. the Powerset Axiom).

In this respect, there are similarities to multiversim—we view the study of **ZFC** as studying different set-theoretic possibilities and thus philosophically valuable even in the absence of a determinate background universe of sets. However, the attitude to *truth* is markedly different. Often the multiversist account is *supervaluational*: $\phi$ is true (false) iff it is true (false) in every member of the relevant multiverse, and indeterminate otherwise. For this reason, statements like "$\omega_1$ exists" are true for the multiversist (being theorems of **ZFC**). This is not so for Scambler's version of Universe Indeterminism. Critical here is that since the Universe Indeterminist has a single universe of sets, she is committed to further principles about truthmaking. Scambler writes:

> As there is only one universe of sets for the Universe-Indeterminist ... unicity of truthmakers is important. On the multiverse picture, a theorem of **ZFC**—the existence of $\omega_1$, say—was a determinate truth because it held in all universes of the multiverse. The fact that the entities satisfying the definite description "least uncountable ordinal" differ in extension from universe to universe is, on a multiverse metaphysics, not particularly troubling; after all, the entities in question are envisaged as belonging to wholly distinct universes. But for the Universe-Indeterminist, if $\omega_1$ exists it must be unique; for there can only be one least uncountable ordinal in the universe $V$. In contrast to the Multiverse-Pluralist, then, the plenitude of differing witnesses for this existential claim in legitimate interpretations inhibits her taking it as a determinate truth. ([Scambler, 2020], p. 564)

Universe Indeterminism has received relatively little attention; many constructive mathematicians see their view as a repudiation of the notion of powerset concerning infinite sets, rather than as an elucidations thereof. However, Universe Indeterminism offers an alternative perspective on the categoricity arguments in comparison to its multiversist counterparts. The Multiversist's normal response to categoricity is to deny that there is a unique interpretation of the second-order variables for a semantic categoricity proof, and for an internal categoricity proof to deny that a second-order theory can apply outside of a particular universe. The Universe Indeterminist can perfectly well accept quantification over all sets, however she can reject the use of classical logic required to conduct the categoricity proof (including an internal one).

## 3.4 Mathematical programmes associated with Anti-Universism

Given the various versions of Anti-Universism that we have seen, there are various mathematical programmes associated with them. The list we shall provide is not exhaustive, but it is useful to survey some in order to see the kinds of mathematics that have arisen in the contexts of particular philosophies of mathematics.

### 3.4.1 Multiverse axiomatisations

The first kind of programme we shall consider are multiverse axiomatisations. These seek to provide theories of multiverses, with variables ranging over universes as well as sets. We mention those of Hamkins and Steel.

Hamkins' list of Multiverse axioms is designed to provide a characterisation of some of the principles held by the Radical Multiversist. He suggests the following axiomatisation:

**Definition 43.** The *Radical Multiverse Axioms*[112] consist of:

(i) **The Realizability Principle.** Whenever $M$ is a universe and $N$ is a definable class of $M$, with a set-like membership relation, satisfying **ZFC** from the perspective of $M$, then $N$ is also a universe.

(ii) **The Forcing Extension Axiom.** Whenever $M$ is a universe and $\mathbb{P}$ is a forcing notion in $M$, then $M$ has a forcing extension of $M$ by $\mathbb{P}$, a model of the form $M[G]$, where $G$ is an $M$-generic filter for $\mathbb{P}$.

(iii) **The Class Forcing Extension Axiom.** Whenever $M$ is a universe and $\mathbb{P}$ is a **ZFC**-preserving class forcing notion $\mathbb{P} \subseteq M$, then $M$ has a forcing extension of $M$ by $\mathbb{P}$, a model of the form $M[G]$, where $G$ is an $M$-generic filter for $\mathbb{P}$.

(iv) **The Countability Axiom.** For every universe $M$ there is another universe $N$ such that $M$ is a countable set in $N$.

[112]These are called just the 'Multiverse Axioms' by [Gitman and Hamkins, 2010] and [Hamkins, 2012], but we change their name (and their formulation, ever so slightly) to fit the present context.

(v) **The Wellfoundedness Mirage Axiom.** For every universe $M$, there is another universe $N$ which thinks $M$ is a set with an non-well-founded $\omega$.

Given the existence of a measurable cardinal, there will also be universes corresponding to various ultrapowers, and so one can also consider:

(vi) **The Reverse Ultrapower Axiom.** For every universe $M$ there is a universe $N$ such that $M$ is the internal ultrapower of $N$ by an ultrafilter on $\omega$ in $N$.

(vii) **The Strong Reverse Ultrapower Axiom.** Every universe $M_1$ and every ultrafilter $U_1$ in $M_1$ on a set $X_1 \in M_1$, there is a universe $M_0$ with an ultrafilter $U_0$ on a set $X_0$ such that $M_1$ is the internal ultrapower of $M_0$ by $U_0$, sending $U_0$ to $U_1$.

(viii) **The Reverse Embedding Axiom.** For every universe $M_1$ in $M$ and every embedding $j_1 : M_1 \to M_2$ definable in $M_1$ from parameters and thought by $M_1$ to be elementary, there is a universe $M_0$ and similarly definable $j_0 : M_0 \to M_1$ in $M_0$ such that $j_1$ is the iterate of $j_0$, meaning $j_1 = j_0(j_0)$.

[Gitman and Hamkins, 2010] show that the Radical Multiverse Axioms are realised (within some model of **ZFC**) in the collection of all countable computably saturated models of **ZFC**. One can see how the axioms (with variables for universes) realise some of the key features of Radical Multiversism; the natural numbers are not determinate since every universe is $\omega$-nonstandard from the perspective of another, and universes can disagree wildly on claims about satisfaction and whether a universe satisfies **ZFC**. For example, since a universe $N$ may disagree with another $M$ on what the axioms of **ZFC** are, $N$ might think that $M$ (which satisfies its own versions of **ZFC**) violates **ZFC**.[113]

Steel's approach is slightly different. He gives the following axioms:

**Definition 44.** The *Steel Multiverse Axioms* have variables for *sets* $(x, y, z, x_0, ..., x_n, ...)$ and *universes* $(V, W, V_0, ..., V_n, ...)$, and are as follows:

(i) $\phi^W$, for every world $W$ (for each axiom $\phi$ of **ZFC**).

[113]This indicates part of the problem for the Radical Multiversist; the correctness of the axioms for a certain class of universes can only be evaluated from a prior fixed model of set theory.

(ii) (a) Every world is a transitive proper class. An object is a set only in the case that it belongs to some world.

(b) If $W$ is a world and $\mathbb{P} \in W$ is a forcing poset, then there is a world of the form $W[G]$, where $G$ is $\mathbb{P}$-generic over $W$.

(c) If $U$ is a world, and $U = W[G]$, where $G$ is $\mathbb{P}$-generic over $W$ for a forcing poset $\mathbb{P} \in W$, then $W$ is a world.

(d) (Amalgamation) If $U$ and $W$ are worlds, then there are sets $G$ and $H$ that are generic filters in them (for some $\mathbb{P}_U \in U$ and $\mathbb{P}_W \in W$) such that $W[G] = U[H]$.

Steel's axiomatisation thus corresponds to a species of Width Multiversism (namely Set Forcing Multiversism), but a denial of Height Multiversism and Radical Multiversism. The relevant universes are given by transitive proper class models of **ZFC**, but universes cannot be extended in height, and the natural numbers are determinate (in contrast to the Radical Multiversist).

Part of the motivation for Steel's system is that he does not regard CH as definite mathematical problem. He grounds this claim on the nature of the independence phenomenon concerning CH, and in particular the fact that it cannot be resolved on the basis of known large cardinal axioms.[114] Nonetheless he feels it is important to provide a unified foundation on which there is no natural analogue of CH that can be formulated as about the universes of his multiverse (even if there is a fact of the matter whether or not $\mathsf{CH}^W$ holds relative to some world $W$). Similar remarks apply to the Radical Multiversist—there is no obvious analogue of CH for the Radical Multiverse Axioms and the answer to CH (for Hamkins for instance) consists in our detailed knowledge of how it behaves within the multiverse (even if there are some open questions as to whether or not it holds in certain universes).

However, Steel wants a *greater* degree of absoluteness in his axiomatisation. He additionally accepts that the natural numbers are determinate (so there is a fact of the matter about $Con(\mathbf{ZFC})$ ). He also holds that a central objective of a foundation is to maximise *interpretive power*—the ability of our foundation to interpret mathematics in a *meaning preserving way*—in a unified framework. He therefore accepts that we should add a proper class of Woodin cardinals to our base theory (and hence every universe contains such a class, since one can only destroy boundedly-many Woodin cardinals using set forcing). Thus every universe will additionally satisfy $\mathsf{AD}^{L(\mathbb{R})}$.

[114] This is because given a measurable cardinal $\kappa$ there is a forcing that modifies the truth value of CH whilst leaving the measurability of $\kappa$ untouched. See [Lévy and Solovay, 1967] for the result, and [Steel, 2014] (p. 163) for Steel's remarks.

A further salient difference between Steel's axiomatisation and that of the Radical Multiversist is the prohibition of class forcing extensions and non-amalgamable generics. For the case of class forcing Steel argues that since the ground model need not be definable in the class forcing extension[115], allowing class forcing extensions implies that what is first-order visible can depend upon where one is located in the multiverse (thus the prohibition on class forcing is important in formulating (ii)(c)). Steel argues that this means that information can be 'lost' when moving to a class forcing extension.

Regarding amalgamation, within the Radical Multiversist's framework one can have universes $V[G]$ and $V[H]$ such that there is no third universe satisfying **ZFC** containing both $G$ and $H$, and with the same ordinal height as $V$. This is because $G$ and $H$ can be $V$-generic Cohen reals that individually are unproblematic, but when copresent can be used to recover a cofinal sequence of $\omega$ in $Ord(V)$.[116] This is prohibited from Steel's multiverse via the amalgamation axiom, which he motivates on the grounds that we want an *axiomatisable* framework, something that non-amalgamation prohibits.[117]

One can obtain a model for Steel's axioms by taking a countable transitive model $\mathfrak{M}$ of **ZFC**, and considering the extension $\mathfrak{M}[G]$ where $G$ is generic for $Col(\omega, < Ord(\mathfrak{M}))$. We can then, in $\mathfrak{M}[G]$, consider a multiverse composed of the worlds $W[H] = \mathfrak{M}[G \upharpoonright \alpha]$, for some $W$-generic $H$ and $\alpha \in Ord(\mathfrak{M})$.[118] A recent in depth treatment of Steel's project and further discussion of how it compares to other multiversist ideas has been proposed recently by [Maddy and Meadows, 2020] who provide a way of eliminating the use of *meaning preservation* from the account.

### 3.4.2 Multiverse accounts of truth

Woodin's approach is slightly different again. Rather than providing an explicit axiomatisation, he proposes to examine the perspective on which a sentence is true iff it is true in all models of the generic multiverse obtained by set forcing. He shows how this is accessible from any given universe, in particular showing that:

**Theorem 45.** [Woodin, 2011] For each sentence $\phi$ there is a sentence $\phi^*$, recursively depending on $\phi$, such that for each countable transitive set $M$ such that $M \models$ **ZFC**, the following are equivalent:

[115] See here [Antos, 2018].

[116] See [Mostowski, 1976] for the result.

[117] See here Theorem 34 of Appendix B in [Maddy and Meadows, 2020].

[118] For the details, see [Steel, 2014], p. 166.

(1.) $M \models \phi^*$

(2.) For every $N$ in the generic multiverse generated by $M$, $N \models \phi$.

Whilst this analysis of truth is multiversist in spirit, it is nonetheless examined for the sake of an attempted reductio:[Woodin, 2011] then uses his characterisation of the generic multiverse to argue that (modulo the $\Omega$-conjecture) the position is not satisfactory.

A different option to a set-generic multiversist analysis of truth is to consider the *hyperuniverse*; the collection of all countable transitive models of **ZFC**. Here [Arrigoni and Friedman, 2013] argue that we can distinguish between 'de facto' and 'de jure' set-theoretic truths. The former are those axioms that we take to be non-revisable (e.g. the axioms of **ZFC**) and the latter comprise those truths that we may come to accept on the basis that they hold in certain elements of the hyperuniverse with preferable properties (e.g. the kind of powerset maximality one might think follows from the IMH).[119] The use of the hyperuniverse is one on which all universes are well-founded, but nonetheless is both height and width multiversist. Again, whilst no explicit axiomatisation is provided (what the hyperuniverse looks like will depend a good deal on the initial universe in which it is analysed), the contention is that examination of this structure is useful for elucidating set-theoretic truth. There is a question of whether or not the use of the hyperuniverse in analysing set-theoretic truth is committed to Height or Width Multiversism (see [Antos et al., 2015] and [Antos et al., F] for arguments that it may be used in the absence of one or both these ontologies), but nonetheless it is *methodologically* speaking multiversist; seeking to analyse set-theoretic truth by studying a multiplicity of universes.

Both these approaches are not explicit axiomatisations[120], but provide the resources to analyse different set theories, assess what higher-order conditions (e.g. absoluteness) we might want universes to satisfy, and provide accounts of set-theoretic truth (e.g. via supervaluation). In this sense, by elucidating a particular system of structures on Multiversist grounds, they seek to inform philosophical considerations with mathematical ones.

[119] See [Friedman, 2016] for some of these conditions.

[120] Indeed Woodin's Multiverse *cannot* be axiomatised, again by Theorem 34 of Appendix B of [Maddy and Meadows, 2020].

### 3.4.3 Using modality

Given any sort of Potentialism (either obtained from some variety of Multiversism or by taking modal resources as primitive) we can examine various projects involving properties of the relevant modalities involved.

**Modal results in potentialism.** One direction is to study the kinds of modal logics that arise from different potentialist systems. Hamkins and Löwe for example found that the set of modal validities of set-theoretic forcing (i.e. where "$\Box\phi$" is the claim that in every set-forcing extension $\phi$ is true, and "$\Diamond\phi$" means that $\phi$ is forceable) was $\mathsf{S}4.2$. More recently [Hamkins and Linnebo, 2018] showed that Level Potentialism and Inaccessible Level Potentialism (with inclusion the accessibility relation) validate exactly $\mathsf{S}4.3$, and the natural potentialism associated with the hyperuniverse is $\mathsf{S}4.2$.[121]

A second kind of project is to provide a modal theory of sets, and see what kinds of normal set theory can be interpreted in this framework (with the set theorist's existential quantifier $\exists$ interpreted as $\Diamond\exists$, and the universal quantifier interpreted as $\Box\forall$ in the modal set theory). Results here include the fact that a level-based version of Potentialist set theory formulated using plural resources proves potentialist translations of **ZF** ([Linnebo, 2013] building on work in [Parsons, 1983]) and is relatively consistent with it. A similar result was shown independently by [Studd, 2013] who uses a tenselike modal logic (with forward and backward looking operators and, unlike Parsons and Linnebo, no higher-order resources) to formalise a modalised stage theory which is able to derive the (modalised) axioms of **ZF**. These formalisations of Level Potentialism have recently been extended by [Scambler, ] who considers a version of Potentialism with a vertical and horizontal modality and shows that a natural modal set theory of this form interprets **ZFC**-Powerset+"Every set is countable", but can interpret **ZFC** when the modality is restricted.

**Potentialist Maximality Principles.** We can also use Multiverse/Potentialist resources in formulating axioms. Some natural candidates have already been discussed in the context of axioms that make apparent use of extensions in their formulations (§1.4). However, the explicit use of modal resources allows us to formulate some additional axioms using these resources.

[121] See [Hamkins and Linnebo, 2018] for details and some further modal validities.

One example comes from potentialist *maximality principles*. Earlier, we discussed how CH was like a 'switch' that could be turned off or on by forcing. As a counterpoint to switches we have *buttons*; statements that can be made true by moving to some possible world, but once turned on cannot be turned off again. Examples of buttons (given the modal structure of Set-Forcing Potentialism) include $V \neq L$, or $|\alpha| = \omega$, for some ordinal parameter $\alpha$.

These modal resources allow us to state axioms in the modalised language. For instance:

**Axiom 46.** [Hamkins, 2003] (The Potentialist Maximality Principle.) $\Diamond\Box\phi \rightarrow \phi$.

This axiom scheme states that every button has been pushed, and over S4 (the lower bound for modal validities of most potentialist systems) is equivalent to the additional axioms for S5. In the context of Set Forcing Potentialism (where the modal operators are interpreted as $\Box\phi =$ "$\phi$ is true in every set-generic forcing extension" and $\Diamond\phi =$"$\phi$ is true in some set-generic forcing extension) if no parameters are allowed in $\phi$, then the statement is equiconsistent with **ZFC**. However, introduction of parameters results in increased strength: unrestricted parameters results in contradiction (just collapse $\omega_1$), and allowing real parameters yields some large cardinal strength (it is equiconsistent with statement "$Ord$ is Mahlo"). The assertion that the Maximality Principle with real parameters is necessary (i.e. true in every forcing extension) has consistency strength well above $0^\sharp$ (in fact above infinitely many Woodin cardinals). Interestingly, even the lightface principle implies that for known large cardinal principles, either there is a proper class of them, or none (since any bounded number of some kind of large cardinal can be collapsed by forcing).[122] Moreover, a shift in the kind of Potentialism being considered and interpretation of the modality results in non-equivalent versions of the Maximality Principle. For example, if one is a Level Potentialist (where $\Diamond\phi =$ "true in some larger $V_\beta$" and $\Box\phi =$ "true in all larger $V_\beta$") the assertion that a world $W = V_\alpha$ satisfies the maximality principle is equivalent to the claim that $\alpha$ is $\Sigma_3$-correct (in the non-modalised set theory). If parameters from $V_\alpha$ are allowed, we obtain the result that $\alpha$ is a correct cardinal. These results and further different ways of interpreting the potentialist maximality principles are discussed in detail in a recent paper of Hamkins and Linnebo ([Hamkins and Linnebo, 2018]).

[122] See [Hamkins, 2003] for these results.

**Modal Structural Reflection.** A second kind of axiom that can be defined using modal resources are principles arising in the context of a modal *structuralism*. Briefly put, Modal Structuralism is the idea that mathematics is concerned with logically possible *structures* composed of non-abstract objects. If the Modal Structuralist then wishes to interpret normal mathematics, she must provide a translation of mathematics—including set theory—that eliminates the apparent reference to abstract objects in favour of talk of logical possibility and non-abstract objects.

Some Modal Structuralist views of this kind (e.g. [Hellman, 1989] and [Hellman, 1996]) represent versions of Height Potentialism, since they assert that whenever I have some objects, it is logically possible for those objects to form a set (under the relevant modal paraphrase). The Modal Structuralist, in this context[123], aims to interpret (second-order) $\mathbf{ZFC}_2$, using logically possible structures, where a structure is thought of as a (coded) pair of pluralities; some things as a domain together with some ordered pairs (coded via mereological fusions) as a membership relation. The language thus contains a modal operator $\Diamond$ expressing logical possibility, and then the required logical resources of first-order logic, plural logic, and a suitable mereology. Using these resources, the Modal Structuralist lays down axioms concerning structures, consisting of the usual axioms of the logics above (with a positive free version of S5 and the Plural Comprehension Schema expanded to the larger vocabulary), the usual axioms concerning the behaviour of the codings of ordered pairs (e.g. that two pairs are identical just in case they have the same elements at the same coordinates), modal axioms concerning the behaviour of pluralities and fusions *across* worlds (namely that (i) pluralities cannot exist without their elements and without continuing to comprise these elements, (ii) an extensionality principle for pluralities holds, and (iii) the mereological fusions playing the role of ordered pairs cannot change their parts) and the following axioms concerning structures (letting upper case Latin variables $X$, $Y$, etc. range over coded structures):[124]

First the Modal Structuralist wants an axiom asserting the existence of at least one structure:

**Existence.** $\Diamond\exists M(M = M)$

Second, they provide an axiom that diagnoses the paradox along

[123]From hereon out, by 'Modal Structuralism' I mean the versions presented in [Hellman, 1989] and [Hellman, 1996], as well as the cluster of views considered by [Roberts, 2019].

[124]This characterisation follows [Roberts, 2019].

potentialist lines: Any structure could form a set in some end-extension. Let "$X' \sqsupseteq X$" denote the paraphrase of the claim that $X'$ is an end-extension of $X$, "$x \equiv X$" denote the paraphrase that the set $x$ and plurality $X$ contain exactly the same things, and "$M \models \phi$" denote the claim that the structure $M$ satisfies $\phi$. Then we can define:

**The Extendability Principle.** $\Box \forall M \forall X \subseteq M \Diamond \exists M' \sqsupseteq M \exists x \in M'(M' \models x \equiv X)$

This theory (plus a claim concerning the stability of modal paraphrases between structures) is known as *Modal Structural Set Theory* or **MSST**. Recently, after presenting **MSST**, [Roberts, 2019] proved:

**Theorem 47.** [Roberts, 2019] **MSST** interprets exactly **Z**+"Every set belongs to some $V_\alpha$"+"There are unboundedly many inaccessible cardinals". However, **MSST** does not prove all $\Pi_1$-instances of the Collection scheme.

So this version of Potentialism, whilst it has some strength, fails to interpret some important set-theoretic axioms. An attempt to strengthen **MSST** is proposed by [Hellman, 2015], who considers the following *Modal* Reflection Principle (let $\phi^{pt}$ stand for the modal paraphrase of the set-theoretic sentence $\phi$)[125]:

> **Modal Structural Reflection.** If $\phi$ is syntactically consistent with $\mathbf{ZFC}_2$ (i.e. $(\mathbf{ZFC}_2 \not\vdash \neg\phi)^{pt}$) then:
> $\phi^{pt} \rightarrow \Diamond \exists M(M \models \phi)$.

This provides a modal structural version of the usual idea of a reflection principle reflecting statements from the universe to initial segments thereof. However recently Roberts showed the following:

**Theorem 48.** [Roberts, 2019] **MSST** with Modal Structural Reflection added is inconsistent.

It may be that there are strong reflection principles that can be formulated under the kind of Potentialism provided by Modal Structuralism (though [Roberts, 2019] presents some difficult challenges for an advocate of this idea). The question remains open.

[125] *pt* here stands for "Putnam-translation" since the relevant translation first occurs in [Putnam, 1967]. See [Hellman, 1989] (p. 76) for the details.

**Set-theoretic geology.** Another kind of mathematics suggested by the Multiversist perspective (in particular various kinds of Forcing Multiversism) is *set-theoretic geology*. The metaphor of geology suggests delving deep underground, and this programme concerns the study of how a particular universe $V$ can arise through forcing, and how it sits within a particular forcing multiverse.

In the original paper of [Fuchs et al., 2015], the authors are almost exclusively concerned with set-generic forcing extensions and the set-generic multiverse. Here we can define:

**Definition 49.** (**ZFC**) [Fuchs et al., 2015] A class $W$ is a *ground* of $V$ iff $V$ is obtained by set forcing over $W$, that is if there is some $\mathbb{P} \in W$ and $W$-generic filter for $\mathbb{P}$ such that $V = W[G]$.

We can then consider various geological structures such as *bedrocks* (a ground that is minimal with respect to the forcing-extension relation) and the *mantle* (the intersection of all grounds). [Fuchs et al., 2015] prove several facts about the geological properties models may possess. In particular, whilst these properties seem to be second-order (since they concern relationships between classes), many can be given first-order formulations (often using the Laver definability of the ground model in a forcing extension).

Whilst one can study set-generic multiverses from within a given model in first-order terms, we might consider generalisations of the idea of geology to other extensions. [Fuchs et al., 2015] goes some way towards this, considering the structure present when we allow *pseudo-grounds* into the picture: models that have certain covering and approximation properties that facilitate the definability of the ground model in the (possibly class) forcing extension, despite the fact that in general Laver definability can fail in class forcing extensions.[126]

Recently, Usuba showed that the *Downward Directed Grounds Hypothesis* (that any two grounds have a common ground) and the *Set-Downward Directed Grounds Hypothesis* (that a set-sized parameterised family of grounds have a common ground) follow from **ZFC**. This result resolves many of the open questions concerning set-theoretic geology.[127] It is thus an interesting open question exactly how things can be developed by relaxing the consideration of the set-generic multiverse and its grounds to something more broad such as a consideration of pseudo-grounds.

[126] See [Antos, 2018].

[127] See [Usuba, 2017] for details.

# 4 Pluralism

Thus far we have considered interrelated *ontological* positions; Universism, Multiversism, Potentialism, and Universe Indeterminism. We can, however, distinguish a further related position:

> **Pluralism.** We should tolerate multiple competing *theories* of sets, and not necessarily identify one as privileged.

One would think, right off the bat, that Pluralism best meshes with a Multiversism (or Potentialism) that countenances the satisfaction of multiple different theories in different equally legitimate universes (or worlds). In this case, we have different equally legitimate places to interpret set theoretic discourse, and different theories are true there.

Things are more complicated than this for both the Multiversist, Potentialist, and Universist, however. Firstly, for the Potentialist, as we saw in the last section, she can find interpretations of her potentialist framework with other systems of set theory, and might therefore maintain that there is a privileged theory of sets given the modal paraphrase. There is thus not necessarily pressure for certain kinds of Potentialist to be pluralists. For example, the Potentialism provided by [Hellman, 1989] accepts a determinate truth value for $\mathsf{CH}$, in virtue of the acceptance of full second-order quantification.

For the Multiversist, again things are not quite so simple as having multiple different universes entailing multiple different theories. For, in certain contexts, she might assert that all preferential universes in the multiverse satisfy the same theory. For example, if a multiversist programme nonetheless allows for the justification of new axioms (e.g. [Arrigoni and Friedman, 2013]), then the view can be more or less pluralist, as more axioms come to be accepted. In this way, it might be that one is a Multiversist without being a Pluralist (say if one thinks that there is an optimal set theory, even if one thinks that there are multiple equally legitimate universes in which it is instantiated).

Universism, we might think, cuts in exactly the opposite direction. If one thinks that every statement of first-order set theory has a determinate answer, then one can easily argue that there is a unique privileged set theory; the true one. However, whilst there is an argument for an Anti-Pluralism of this truth-theoretic kind, one might nonetheless endorse a *methodological* Pluralism and tolerate the use of different theories. For example [Barton, 2017] argues that a Universist should nonetheless be a methodological Pluralist in virtue of the kinds of ways in which she might be ignorant, and should tolerate many different theories of sets (at least for now) in making her justificatory case.

# 5 Concluding remarks

In this chapter, we've seen that there are several considerations concerning how we should interpret talk about 'the' universe of sets, and several ways of interpreting these considerations via either Universism, Multiversism, Potentialism, or Universe Indeterminism, with ramifications for Pluralism. Fundamentally, these questions concern what our talk of set theory is *like*. Is it algebraic (like group theory) and concerned with multiple different non-isomorphic structures, or rather is it concerned with a single intended structure? If it is concerned with multiple non-isomorphic structures, how *much* is determinate? Answers to these questions motivate many different philosophical and mathematical questions.

I want to leave the reader with a final point, and possible direction for future research. Each kind of view advocates a different way of interpreting set-theoretic language. But should we really suggest that set-theoretic language has a univocal best interpretation? Perhaps we can view each philosophical viewpoint concerning the nature of subject matter as advocating a new and different concept of set, each of which can serve as a legitimate interpretation in different contexts. For example, perhaps the Radical Multiversist can be viewed as advocating an algebraic concept of set using the language of set theory, whereas the Universist uses set-theoretic language in a very different way. It is not clear, without the further commitment that set-theoretic language should always be interpreted similarly in foundations, that the different philosophical views concerning set theory are truly in tension with one another. Nonetheless, it is also not obvious how one might combine these different perspectives into a unified foundational framework, and there are many open questions to be resolved concerning how the perspectives relate, and indeed if there is in fact an optimal such view.

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