Richness and Reflection[*]

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Abstract

A pervasive thought in contemporary Philosophy of Mathematics is that in order to justify reflection principles, one must hold Universism: the view that there is a single universe of pure sets. I challenge this kind of reasoning by contrasting Universism with a Zermelian form of Multiversism. I argue that if extant justifications of reflection principles using notions of richness are acceptable for the Universist, then the Zermelian can use similar justifications. However, I note that for some forms of richness argument, the status of reflection principles as *axioms* is left open for the Zermelian.

Introduction.

The development of Set Theory since Gödel's seminal result on the consistency with ZF of the Axiom of Choice and Continuum Hypothesis has been marked by a significant phenomenon: independence results. Set-theoretic practice is now replete with model-theoretic techniques that facilitate the study of diverse models of ZF and its variants, thereby providing witnesses to the consistency of various statements with ZF. Broadly speaking, there have been two philosophical

reactions to the independence phenomenon in set theory: those who think that the independence of statements (such as CH) from ZFC is indicative of a failure of bivalence[1] and those that do not.

Before proceeding any further, we should note that bivalence in Set Theory is a tricky subject. Indeed, one may hold that some independent statements (such as CH) are bivalent while still asserting that there are (in a set-theoretic context) non-bivalent statements. For example, it is a reasonable position to hold that CH is bivalent, while at the same time asserting that it is not bivalent whether or not there exists an inaccessible cardinal. Indeed this is a position that has been advocated by several authors, including recently by [Isaacson, 2011]. I am concerned here with two views that do hold that statements concerning the 'width' (*id est* the subsets formed at V_α for successor ordinals α) at stages indexed by (relatively) small α are bivalent, and hence CH is bivalent.

The two views are the following:

> [Universism] There is a single, unique, universe of sets which is maximal with respect to height and width. Each V_α is to be understood as an initial segment of this universe.

This contrasts sharply with the following:

> [Zermelian[2] Multiversism] There is no one unique universe of pure sets that is maximal with respect to height, but rather a series of universes, all of which satisfy second-order ZFC_2. Each universe, when viewed from a taller universe (or from the Universist's perspective), is of the form V_κ for strongly inaccessible κ, and no one V_κ is especially privileged.

It will be useful to introduce some terminology from the outset in order to make discussion more precise. By the term 'PZ-universe[3]', I mean any model of ZFC_2 (with the full semantics), and denote these universes using $\mathcal{V}$, $\mathcal{V}'$, $\mathcal{V}''$ *et cetera*. Occasionally, it will be more convenient to refer to a particular $\mathcal{V}$ by viewing it as a particular V_κ in

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[1] A sentence ϕ is *bivalent* iff it is either true or false (and not both). There is a rich literature on bivalence in the Philosophy of Mathematics and Philosophy more widely: an excellent recent discussion of some of these issues is available in [Rumfitt, 2015].

[2] One must specify that the view to be discussed is Zermelian in order to distinguish it from Multiversisms of significantly different character, such as the views presented in [Hamkins, 2012].

[3] For reasons to be made clear later, 'PZ-universe' stands for 'Primordial Zermelian Universe'.

some taller $\mathcal{V}'$ (where meaning is clear from context). By the term 'CZ-universe[4]' I mean any $\mathcal{V}$ that satisfies the Zermelian's currently best justified theory of sets.[5] We assume that the Zermelian has an adequate justification for the use of ZFC_2 (with the full semantics) from the outset. The philosophical and metamathematical issues here are thorny[6], but (as we shall see below) the acceptance of ZFC_2 as our initial best theory of sets is part of the Zermelian's motivation for her[7] position. In order then to tackle the Zermelian from her own perspective, we shall put aside issues concerning the justification of ZFC_2 and use of the full semantics.

We should start by noting that each view has much to recommend it. The former, held by Gödel and Cantor[8], has a certain aesthetic and semantic simplicity to it: our theories of sets describe a single universe about which we may reason and prove facts absolutely[9]. The latter, held by Zermelo, neatly avoids the problem of proper classes by having the 'proper classes' of one $\mathcal{V}$ be sets in some taller $\mathcal{V}'$. Further, if one thinks that categoricity is important for an account of reference to mathematical objects[10], then Zermelian Multiversism is motivated by the fact that ZFC_2 is only *quasi-categorical* in the sense that for any two models of ZFC_2, either they are isomorphic or one is isomorphic to a proper initial segment of the other. This has been seen by several authors (including Isaacson and Zermelo himself) as indicative of the failure of our thought and language to uniquely determine one universe of sets rather than a plurality thereof.[11]

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[4]Here, 'CZ-universe' stands for 'Canonical Zermelian Universe'.

[5]As we shall see, the distinction between PZ-universes and CZ-universes will be important for understanding the sense in which reflection principles can be viewed as *axioms* for the Zermelian Multiversist.

[6]See, for example, [Hamkins, 2012] and [Meadows, 2013].

[7]To avoid ambiguity, I will use the female pronoun for the Zermelian (when Zermelo himself is not being denoted) and the male pronoun to denote the Universist.

[8]There is *some* disagreement in the literature whether or not Cantor was actually a Universist in the sense outlined here. For a view that argues that in fact he was not, see [Linnebo, 2013].

[9]Throughout this paper, I speak as if sets are mind-independent, abstract entities, and put aside problems standardly associated with Platonism. It is an interesting question how much of the discussion could be formulated in the absence of this assumption, but one that I lack the space to address here.

[10]For just such a view see [Isaacson, 2011].

[11]It should be noted that there are extant attempts to provide *full* categoricity results for set theory. A good example here is [McGee, 1997] who uses a theory augmented with *urelemente* to prove the categoricity of the pure sets. As the significance of the result is controversial (the proof requires both unrestricted first-order quantification and that the *urelemente* form a set) and the philosophical issues subtle, we set this result aside. For the purposes of seeing the Zermelian's motivations, it suffices

We now have two incompatible views before us, each of which appears to be reasonably well motivated. In order to try and inform the debate, a natural methodology is to see how each view behaves with respect to mathematical practice. In particular, we shall examine here how the views interpret certain kinds of mathematical statements, and how each addresses issues of *justification*.

Often reflection principles are cited as a problematic case for the Zermelian.[12] This paper provides criticism of this line of argument. In particular, I shall argue that if it is assumed that extant Universist motivations for reflection principles are satisfactory, then the Zermelian can make use of similar motivating ideas in justifying her own forms of reflection. Despite this, I address some limitations: though the Zermelian is able to capture many of the pleasing theoretical features of reflection, the status of principles justified through ineffability as *axioms* is left open. My strategy is as follows: In §1 I outline reflection principles, in particular the distinction between *strong* and *weak* varieties. §2 then explains the problem that some have seen for the Zermelian Multiversist. In §3 I argue that justifications of reflection principles based on the closure of the ordinals apply equally well on the Zermelian's view, putting her in the same position as the Universist. Finally, §4 argues that motivating reflection through notions of *direct ineffability* allow the Zermelian to also justify the non-vacuity of both weak and strong forms of reflection. However, I note that it is unclear that the Zermelian can argue that these principles are *axioms*. I conclude that if the Universist can motivate reflection using these notions of richness, then the Zermelian Multiversist can both (i) motivate the non-vacuity of this discourse, and (ii) provide justification for many of the pleasing theoretical consequences of reflection arguments.

1   Varieties of reflection.

Reflection principles present an excellent test case for studying particular philosophies of set theory. There are two main reasons for this. First, such principles are *mathematically fruitful* (in ways to be made precise later). Second, many set theorists and philosophers regard these new 'axioms' as *natural* principles.[13]

We should first be precise about the kinds of principle with which we are interested here. Reflection principles are of the following gen-

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to note that ZFC_2 is only quasi-categorical.

[12]See, for example, the excellent [Koellner, 2009].

[13]See, for example, [Bernays, 1961], [Reinhardt, 1974], and [Fraenkel et al., 1973].

eral form:

$$[RP1]\ (\exists\alpha)(\phi \to \phi^{V_\alpha})$$

This states that if some formula ϕ is satisfied *simpliciter* then it is satisfied by some V_α. More informally, one can view a reflection principle as saying that any property held by the universe is held by some initial segment of the universe.

Of course, what one allows as the relevant ϕ in one's reflection principle is going to affect the results one gets. If we consider only *first-order* formulae ϕ, the following principle of reflection is as uncontroversial as the axioms of ZFC, following as it does from them:

$$[RP2]\ \forall\alpha\exists\beta > \alpha\forall\bar{x} \in V_\beta[\phi(\bar{x}) \leftrightarrow \phi^{V_\beta}(\bar{x})]$$

Allowing higher-order formulae produces stronger and stronger reflection principles. For instance, allowing second-order sentences to be reflected yields inaccessible cardinals. If one then asserts that formulae are reflected to a V_κ with κ strongly inaccessible, one produces Mahlo cardinals. Reflecting then to Mahlo cardinals results in α-Mahlo cardinals.[14] One can move to a higher-order language, thus allowing stronger and stronger reflection principles and thereby producing a hierarchy of cardinals known as the indescribable cardinals.[15] However, one will always stay below the first ω-Erdős cardinal.[16]

One might try to achieve extra strength *via* the introduction of parameters. Care is required, however, as it has been shown by Tait that admitting unrestricted third-order parameters results in a contradiction [Tait, 2005][17]! Tait instead tries to gain extra strength whilst

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[14]A *Mahlo* cardinal is a cardinal κ such that every normal (*id est* continuous and increasing) function on κ has a strongly inaccessible fixed point. Similarly we iterate this definition for α-Mahlo cardinals, a cardinal κ being α-Mahlo iff every normal function on κ has an $(\alpha - 1)$-Mahlo fixed point.

[15]A cardinal κ is Q-indescribable (where Q is of the form Σ_n^m or Π_n^m) iff for any $X \subset V_\kappa$ and sentence ϕ of Q order and complexity, if $\langle V_\kappa, \in, X \rangle \models \phi$ then there is an $\alpha < \kappa$ such that $\langle V_\alpha, \in X \cap V_\alpha \rangle \models \phi$.

[16]The first ω-Erdős cardinal is the least cardinal satisfying certain combinatorial properties on a partition into its finite subsets. As the definition of the cardinal is somewhat involved and is not central to my argument I omit it here, however details are available in a wide variety of texts including [Drake, 1974], [Kanamori, 2009], and [Jech, 2002].

[17]The result that reflection with unrestricted third-order parameters is inconsistent is often attributed to Reinhardt in [Reinhardt, 1974]. However, the proof is not explicitly given and arguing that Reinhardt published the result depends on a generous interpretation of the following comment: "It does not appear to be possible to generalize (S3) [a reflection principle] in this direction [third-order parameters] very easily. (The reader who does not believe this is welcome to try.)." ([Reinhardt, 1974], p196).

avoiding contradiction using *restricted* forms of higher-order parameters in reflection principles. However, Koellner has since showed that all of Tait's principles are either inconsistent or remain below the first ω-Erdős cardinal [Koellner, 2009]. Let us call such reflection principles (whose strength is below the first ω-Erdős cardinal) *weak reflection principles*.

Despite these setbacks, recently Welch has proposed a (much stronger) *global* reflection principle. Welch's principle has its conceptual roots in the reflection arguments used by Reinhardt in [Reinhardt, 1974], which in turn are somewhat similar to the ideas at play in [Magidor, 1971]. Reinhardt's principles, though technically fascinating, face deep philosophical challenges (see [Koellner, 2009]), and so we focus on Welch's presentation here.

Welch proceeds by considering a structure denoted by '$(V, \in, \mathcal{C})$'. Here we augment the standard structure $(V, \in)$ (where V is the universe of all sets), with the collection of all parts of V, *id est* all classes (to be denoted by '$\mathcal{C}$'). Welch then (for reasons to be discussed later), argues that there should be a reflection of $(V, \in, C)$ to some $(V_\kappa, \in, V_{\kappa+1})$. This may be stated as follows:

> [$RP3$] There is a non-trivial elementary embedding[18] j and ordinal κ with $crit(j) = \kappa$ such that:
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> $j : (V_\kappa, \in, V_{\kappa+1}) \longrightarrow (V, \in, \mathcal{C})$

This states that there is a non-trivial elementary embedding j from some $(V_\kappa, \in, V_{\kappa+1})$ to $(V, \in, \mathcal{C})$. In this way the structure which we may talk about in our language of $(V, \in, \mathcal{C})$ is *reflected* in some $(V_\kappa, \in V_{\kappa+1})$.

It should be noted here that this principle differs substantially from standard reflection principles in that is posits the existence of an elementary embedding, rather than simply asserting that formulae are reflected. It is important to bear in mind that the very kind of ontological commitment being made is much stronger than in standard reflection principles. In particular, when using strong reflection principles, we commit ourselves not just to the satisfaction of some formula(s) by a particular V_α, but also to the existence of the relevant j. This might lead one to hold that the moniker of 'reflection principle' is somewhat misleading, and a different term would be appropriate to make clear the distinction between the two. However, one can see the sense in which such principles imply reflection of a sort: they assert that there

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[18]The level of elementary insisted upon results in different technical consequences: see [Welch, 2014] for details. These technical details have no bearing on my argument here, for any level of elementarity desired the arguments carry over immediately.

is a great deal of resemblance between V and some initial segment thereof.[19]

2 Motivating reflection principles: The problem for the Zermelian Multiversist.

The central problem that the Universist raises for the Zermelian Multiversist is direct and simple: reflection principles involve mention of the truth of a formula ϕ *simpliciter*. A natural way to understand this is through holding that ϕ is true in *the* Universe of sets, an entity that is not countenanced by the Zermelian's philosophy. Tait expresses this point as follows:

> "From this viewpoint [Zermelian Multiversism] the 'paradoxes of set theory' vanish. But also from this viewpoint...reflecting *down* from the universe of all sets, becomes problematic. For it seems to require that we know what it means to say that a sentence $\phi(t)$ is true in the universe of all sets." ([Tait, 2003], p473)

We may characterise the issue in the following way. The Zermelian Multiversist asserts that there is no maximal universe of sets but rather an unbounded sequence thereof, in which we are always implicitly or explicitly restricted. Worries of paradox are thereby removed[20]: apparently 'paradoxical proper classes' are merely sets in some expanded domain. However, by denying the existence of the 'real V' we lose the coherency of statements involving V that we wish to reflect to particular V_α: we cannot yet say what it is for ϕ to be true in *the* Universe of all sets.

Let us draw the problem out a little further, in order to be precise as to where the difficulty lies. There are two challenges the Universist might try to raise for the Zermelian:

1. *Naturalism.* The view is unable to underpin (even in principle) formalisations of the relevant principles.

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[19]Many thanks to Peter Koellner and Sy Friedman for emphasising to me the importance of this distinction. As we shall see in §3 and §4, another substantial difference between weak and strong reflection is the kinds of justification that have been offered for the principles.

[20]There is a substantial debate as to whether or not the problem is truly avoided: in stating that there is an unbounded sequence of models we seemingly have to be able to understand what 'absolutely unbounded' means, and hence understand the totality of ordinals. The issue is well-worn in the literature on absolute generality, and I set it aside here.

2. *Motivation.* The view is unable to adequately motivate and justify the principles which we would like to be true for doing set-theoretic mathematics.

Now, for the Zermelian Multiversist (1.) is not a problem. For, though V does not exist on the Zermelian standpoint, this does not mean that they do not ascribe any meaning to the use of the term 'V'. Indeed it would be strange if the Zermelian rejected all set theory that mentioned 'V': such use is ubiquitous across set-theoretic discourse.[21]

We should pause briefly to see how a Zermelian interprets the use of the term 'V' within their philosophical framework. Central here is the emphasis placed by the Zermelian on the *quasi-categoricity theorem.* As noted earlier, the Zermelian regards the failure of our use of set-theoretic language to pin down a single intended model up to isomorphism as significant. This is one of the main motivations for the Zermelian view, and is indicative of the tight link she draws between *ontology* and the *use of language.* Since, the Zermelian argues, it is impossible[22] to give a mathematical characterisation of *the* universe of sets (for any characterisation of some $\mathcal{V}$, one can assume that it is some V_κ in a larger universe), there simply is no one single universe, rather the universes are many.

Thus, on a given occasion of utterance of set-theoretic sentences, the models that can serve as referents of the term 'V' are determined by the sentences uttered and the given context. In this way the Zermelian's account of reference is *schematic,* in the sense that the universes that can be referred to by 'V' must be of the correct form to satisfy the uttered sentences. To see this more clearly, consider a different kind of schematic phenomenon: axiom schema. To take a relatively simple example, we may state the Comprehension Scheme for systems of second-order arithmetic as follows:

$\exists X \forall n[n \in X \leftrightarrow \phi(n)]$

Where $\phi(n)$ is a formula of second-order arithmetic in which X is not free.

This axiom scheme allows us to substitute formulae just in case they are of the right 'shape'. Now, what counts as the right shape is

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[21]A good example being the characterisation of large cardinal properties in terms of non trivial elementary embeddings $j : V \longrightarrow \mathfrak{M}$, for transitive inner models $\mathfrak{M}$. The exact nature of these embeddings and the relationships they bear to their set-like surrogates (such as ultrafilters and extenders) is a knotty philosophical issue in itself, and one I shall not consider in this paper.

[22]This only holds of course, if we disallow the addition of *ad hoc* anti-large cardinal axioms or the controversial results in [McGee, 1997].

going to vary from context to context. We may only allow, for example, arithmetical formulae into the Comprehension Scheme when working in the context of ACA_0. However a shift in context may liberalise what we allow into the Comprehension Scheme (say to Π_1^1 formulae in the context of working in $\Pi_1^1 - CA_0$). From an expanded context of utterance , it is legitimate to substitute different formulae for $\phi(n)$ compared to the context[23] of ACA_0.

A somewhat similar (though inverse) phenomenon is occurring on the Zermelian picture. On a particular use of the term 'V', we are able to interpret 'V' as referring to a PZ-universe of the correct form. So, given an utterance of ZFC_2 not involving any large cardinal notions, any PZ-universe that satisfies ZFC_2 is able to serve as a referent for the term 'V'. However, as we start to involve stronger notions, fewer universes can be referred to by the term 'V'. For example, if a Zermelian is working in the context of ZFC_2+ "There exists an inaccessible cardinal", then (assuming that some inaccessibles do in fact exist), the smallest PZ-universe satisfying ZFC_2 cannot serve as a referent of the term 'V' as it does not contain any inaccessibles.

We thus see why, in a given context, (1.) is no problem. The utterance of some sentences of second-order set theory involving the use of reflection (either weak or strong) may simply be interpreted *locally* where 'V' is taken to denote *schematically* some PZ-universe satisfying the principle in question. In the context of weak reflection, the relevant universe $\mathcal{V}$ will reflect higher-order sentences down to appropriate initial segments V_α. Viewed from a universe $\mathcal{V}'$ in which $\mathcal{V}$ appears as some initial segment V_β, there will be reflection from V_β to various V_α (for $\alpha < \beta$). For example, if κ is totally indescribable[24], then V_κ will model ZFC_2 plus reflection for any second-order sentence. Such a PZ-universe would therefore be able to serve as a referent of the term 'V' in the context of talking about the (weak) reflection of any and all second-order sentences. A similar move can be made in the case of strong reflection. Letting '$\mathcal{C}'$' denote the classes of an appropriate PZ-universe $\mathcal{V}'$, and V_β an initial segment of $\mathcal{V}'$, we can interpret $[RP3]$ as an embedding $j : (V_\beta, \in, V_{\beta+1}) \longrightarrow (\mathcal{V}', \in, \mathcal{C}')$. Again, when viewed from a larger universe $\mathcal{V}''$ in which $\mathcal{V}' = V_\alpha$ for some strongly inaccessible α, the embedding will be from[25] $(V_\beta, \in, V_{\beta+1})$ to $(V_\alpha, \in, V_{\alpha+1})$.

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[23]Indeed often notation is used to keep track of these contexts and associated subsystems.

[24]A cardinal κ is *totally indescribable* iff it is Q_n^m-indescribable for both $Q = \Pi$, $Q = \Sigma$, and any $m, n \in \omega$.

[25]The existence of such an embedding implies that β is 1-extendible. For details, see [Kanamori, 2009].

Thus, we see that as long as the Zermelian can secure the existence of *some* PZ-universe satisfying the relevant reflection, she can secure the non-vacuity of mathematical discourse involving the use of such principles (thereby avoiding problems of *Naturalism*). However, some authors have seen a problem for the Zermelian Multiversist in the form of *Motivation*, and it is to this issue that I now turn.

2.1 Reflection and justification.

Let us take stock. We have seen that as long as the Zermelian can justify the existence of the required PZ-universes, she can secure an interpretation of set-theoretic discourse involving reflection principles over structures satisfying ZFC_2. However, though the talk of reflection principles can be formulated in terms of the relevant PZ-universes, she has not yet developed an explanation of *why* the relevant reflection principles should be *true*. Koellner presses the worry as follows:

> "On the actualist view one can refer to the totality of sets and thus one can articulate the idea that this totality cannot be described from below and hence satisfies the reflection principles...On the potentialist[26] view the closest one can come to speaking of the totality of sets is through speaking of some V_α. One can certainly make sense of higher-order quantification over V_α but now the difficulty lies in motivating and justifying reflection principles." ([Koellner, 2009], p209)

Here, Koellner claims that the difficulty with reflection is precisely the issue of *Motivation*. Given that V does not exist for the Zermelian, why should we say that there are *any* PZ-universes that facilitate the reflection? Since, on her view, any particular PZ-universe does not represent 'the absolute', why should we think that any $\mathcal{V}$ is ineffable in the sense that properties reflect from $\mathcal{V}$ to its initial segments? The situation for the Universist is different: they may very well hold that V is in some sense uncharacterisable and hence there are reflections from V down to particular V_α. If the Zermelian cannot motivate the existence of $\mathcal{V}$ of the desired kind, then the mathematics that they wish to interpret becomes vacuous.

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[26]Koellner's Potentialist and my Zermelian differ subtly: the Potentialist need not hold that the subject matter of Set Theory is constituted by only the models of ZFC_2, they merely hold that the subject matter of set theory is "open-ended [in height]". While one may be a Potentialist but not a Zermelian, it is true that Zermelian Multiversism represents a form of Potentialism, and so Koellner's arguments carry over to the present case.

There are difficult philosophical issues here surrounding what is meant by Koellner's use of the terms '*justification*' and '*motivation*'. I shall take these terms to refer to roughly the same phenomenon: we are concerned with arguments that purport to establish that particular principles hold on certain kinds of structure. There are at least two dimensions in which the meaning of such terms can vary for a Zermelian.

The first concerns a well-rehearsed distinction[27] between two kinds of justification: *intrinsic* and *extrinsic*. A full examination would require a significant literature on its own, however in order to clarify the notions (especially with respect to Koellner's arguments) we shall nonetheless examine the issue a little here. We begin with some infamous quotations from Gödel. Characterising intrinsic justification, Gödel writes:

> "First of all the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the very concept of set on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation "set of"...These axioms show clearly, not only that the axiomatic system of set theory as used today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which only unfold the content of the concept of set explained above." ([Gödel, 1964], pp260-261)[28]

and

> "also there may exist, besides the ordinary axioms,...other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by those concepts" ([Gödel, 1964], p261)[29]

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[27]See, for example, [Maddy, 1988a], [Maddy, 1988b], [Maddy, 1990], and [Koellner, 2009]. The roots of this distinction go back at least to [Gödel, 1947] and plausibly even [Russell, 1907].

[28]Here, I use [Gödel, 1964] rather than [Gödel, 1947] (p181) for a couple of reasons. First [Gödel, 1964] represents Gödel's more mature philosophical views (and indeed he was more satisfied with his command of English during this period: see [Moore, 1990]). Second, his wording leaves the kinds of justification he has in mind more open: in [Gödel, 1947] he is clearly more concerned with weak reflection principles, referring to "axioms which are only the natural continuation of the series of those set up so far", rather than those that "unfold the concept of set".

[29]See p182 of [Gödel, 1947] for the relevant passage, essentially similar in content.

These quotations require some unpacking before we have a precise characterisation. Key to the above passages is the thought that intrinsic justifications are concerned with unfolding and explaining particular concepts. We determine, *via* conceptual analysis, what principles are implied by the mathematical conception with which we are working. Such a characterisation, as it stands, is somewhat unclear: what constitutes a satisfactory 'unfolding' of a concept is itself in need of explaining. Koellner makes the issues a little more precise:

> "One can also gain a sharper understanding of the notion of intrinsic justification by pointing to some of its properties. First, an intrinsically justified statement need not be self-evident, in part because the justification may be quite involved (for example, in the case of arithmetic, this would be the case with reflection principles at the level of some large ordinal approaching Γ_0), in part because it is possible that the underlying conception is problematic (as, for example, was the case with the Fregean conception of extension). On the other hand, the notion of intrinsic justification is intended to be more secure than mere "intrinsic plausibility"... in that whereas the latter merely adds credence, the former is intended to be definitive (modulo the tenability of the conception)." ([Koellner, 2009], p207)

So we have a picture on which intrinsic justifications proceed *via* (possibly quite involved) conceptual analysis, to provide definitive justification of certain principles.

It is unclear whether or not Gödel himself took intrinsic justifications to be "definitive" (certainly many of the terms he used, such as "intrinsic necessity", indicate that this may well have been the case). There is also the separate question of whether or not *we* should take intrinsic justification to be definitive or simply a matter of degree or adding credence. I do not wish to become entangled in these tricky issues here. Since our interest is in comparative issues of justification between Universist and Zermelian Multiversist, we may remain neutral on the issue of whether intrinsic justification is an 'all or nothing' matter. What we wish to show is that whatever level of justification the Universist has, the Zermelian has also.[30]

The notion of intrinsic justification contrasts with that of *extrinsic* justification. Rather than providing reasons to think that a principle

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[30]As we shall see later, this holds only *modulo* some uncertainties concerning the meaning of the term 'axiom' and the notion of 'truth' for a Zermelian.

results from conceptual analysis of a particular mathematical conception, extrinsic justification concerns its theoretical consequences. Explicating the notion, Gödel writes:

> "Secondly, however, even disregarding the intrinsic necessity of some new axiom, and even in case it had no intrinsic necessity at all, a probable decision about its truth is possible also in another way, namely inductively by studying its "success". Success here means, fruitfulness in consequences, in particular in "verifiable" consequences, i.e., consequences demonstrable without the new axiom, whose proofs with the help of the new axiom however, are considerably simpler and easier to discover, and make it possible to contract into one proof many different proofs." ([Gödel, 1964] p261)[31]

and

> "A much higher degree of verification than that, however, is conceivable. There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving given problems (and even solving them constructively, as far as that is possible) that no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory." ([Gödel, 1964], p261)[32]

Again, the philosophical and exegetical issues here are both difficult and subtle. For our purposes, it suffices to note that extrinsic justification is concerned with the consequences of a particular principle rather than whether or not the principle follows from a particular conception.

Now it is clear that the kind of justification with which we will be interested is primarily *intrinsic*. The facts of mathematical theorems are not disputed between the Universist and Zermelian Multiversist and so any extrinsic evidence for particular assumptions will carry over equally well between the two views. Instead, we are concerned here whether or not the Zermelian can provide *intrinsic* motivation for the relevant reflection principles.

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[31]Here p182 of [Gödel, 1947] does not differ substantially from [Gödel, 1964].

[32]Again here, [Gödel, 1947] (pp182-183) is not significantly different from the [Gödel, 1964] revision.

It would be dialectically significant if it were the case that the Zermelian Multiversist could not intrinsically justify reflection principles where the Universist could. For reasons I outline below, we might think that if the Universist can motivate reflection principles she can in turn motivate a *better* theory of sets on his ontology. One might think this is important: if I can motivate a better theory of sets on my philosophy where you cannot, then we might think that (*ceteris paribus*), I have a better philosophy.

Several well-trodden facts from the set-theoretic literature are relevant here. We survey just two, one for strong reflection principles, and one for weak. It will serve first to briefly remark on the genesis of reflection principles originating in the work of [Levy, 1960] and [Bernays, 1961]. Recall the first-order reflection scheme:

$$[RP2] \; \forall \alpha \exists \beta > \alpha \forall \bar{x} \in V_\beta [\phi(\bar{x}) \leftrightarrow \phi^{V_\beta}(\bar{x})]$$

As noted earlier, such a principle is provable from first-order ZFC. However, there is slightly more to the issue than this. For, one can prove from $ZC + [RP2]$ with the Axiom of Infinity removed, both the Axiom Scheme of Replacement and the Axiom of Infinity. Thus, insofar as one regards ZFC as explicating our conception of set, some form of reflection is also essential: principles that are often held to follow from this conception turn out to be jointly equivalent to first-order reflection.

Indeed, the progression is natural as we move through the orders of reflection principles. For, many of cardinals produced by weak reflection principles are equivalent to the existence of fixed-points on normal functions. For example, second-order reflection is equivalent to Mahlo's principle that any unbounded sequence of ordinals has at least one initial segment which tends to a fixed-point.

Why is this significant? Many (such as [Bernays, 1961] and [Drake, 1974]) have seen the existence of fixed-points of normal functions as indicative of the 'absolutely' unbounded nature of the ordinals, a thought which one might think is desirable given the Iterative Conception of Set.[33] The key idea here is that by indexing the stages of the Cumulative Hierarchy by the ordinal number sequence, we are committing ourselves to the principle that there is no way of exhausting the ordinals with an enumerating function without hitting a fixed-point. Such cardinals are precisely what is delivered by the higher-order weak reflection principles, and might lead one to think that if

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[33]This is, it must be admitted, a controversial assumption. However, it is one that is not without its adherents.

the Universist can motivate these principles where the Zermelian cannot, then the Universist has a better theory of the Iterative Conception of Set.

The theoretical benefits of strong reflection principles concern the *significant reduction in incompleteness* provided by the substantive large cardinals implied. There is a large and detailed literature on the matter, so we mention just one case here.[34]

The statements which constitute much of the discussion are so called axioms of *definable determinacy*. For ease of presentation, we restrict our attention to determinacy concerning *projective* sets.

Projective Determinacy is the following statement:

> [PD] Every ω-length game of perfect information whose winning condition is a projective set is determined.

The details surrounding PD are rather technical[35] , and so I suppress them here. For our philosophical purposes, it suffices to note three facts concerning PD:

1. PD is independent from ZFC.

2. The truth value of PD is settled in the first few levels above V_ω (and hence has the same truth value for both the Universist and Zermelian).

3. PD has pleasing theoretical consequences.

It is a well known fact that PD is independent from the axioms of ZFC.[36] (2.) is true in virtue of the fact that PD is a statement concerning sets of infinite sequences of natural numbers, the games (conceived of as trees) that may be played over them, and strategy functions on these trees. Hence, all objects involved in the statement of PD occur a finite number of levels above V_ω. Since, these levels are the same on both the Universist and Zermelian Multiversist's ontology, they should both assert that PD is bivalent (and indeed has the same truth value).

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[34]For further philosophical and technical exposition, the reader is directed to [Woodin, 2001], [Koellner, 2009], [Koellner, 2010], and [Welch, 2014].

[35]For details see [Kanamori, 2009]. For a comprehensive (but challenging) presentation of the relationship between determinacy axioms and large cardinals, see [Koellner and Woodin, 2010].

[36]To see this note that PD is false in L (Determinacy in fact fails in L at level of Π_1^1 sets of reals). The fact that $ZFC \nvdash \neg PD$ follows from the Martin-Steel Theorem outlined below.

The issues surrounding (3.) are again technical, but we mention just one consequence of PD. As [Woodin, 2001] notes, PD provides us with a complete theory of the countable sets in the same way that PA provides us with a complete theory of the natural numbers: there are no known sentences involving the hereditarily countable sets (other than Gödelian diagonal sentences) that are independent of the theory $ZFC^- + PD + V = HC$.[37] In this way, both PA and $ZFC^- + PD + V = HC$ pin down the truths about the *objects* of their intended model, even if there are still independent statements concerning the theory. It seems then that there are reasons to want PD to be true: the near-completeness of PD over the countable sets is theoretically desirable.

It is thus natural for both the Universist and the Zermelian Multiversist to look for additional axioms to settle statements such as PD. The following is a theorem:

Theorem 1. [Martin and Steel, 1989] If there are infinitely many Woodin cardinals[38] then PD holds.

So, there is a proof available of PD. However, one should note the resources required to prove the theorem. The existence of infinitely many Woodin cardinals is, philosophically speaking, a significant ontological assumption deserving of scrutiny. Therefore, aside from the proof provided, we would like *justification* of why PD should hold.

PD may well be argued to have some extrinsic justification in virtue of its pleasing theoretical consequences, insofar as the results above can be considered kinds of 'fruitfulness'.[39] However, one might not be satisfied with justifications of this kind. Indeed, both the Universist and Zermelian Multiversist would like to argue that it is *their* ontology that motivates the best possible theory of sets. Thus, each might seek *intrinsic* justification for PD by showing that either it can

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[37]Here ZFC^- refers to ZFC with the Power Set Axiom removed (usually a subset collection scheme is substituted instead), and $V = HC$ refers to the statement that the universe is all and only the hereditarily countable sets. For additional details, see [Welch, 2014].

[38]A Woodin cardinal is an ordinal κ such that for all functions $f : \kappa \longrightarrow \kappa$ there is some $\alpha < \kappa$ such that $\{f(\beta)|\beta < \alpha\} \subseteq \alpha$ and there is a non-trivial elementary embedding j from V to some transitive inner model $\mathfrak{M}$ with critical point α such that $V_{j(f)(\alpha)} \subseteq \mathfrak{M}$. For discussion and alternative characterisations, the reader is directed to [Kanamori, 2009].

[39]The status of extrinsic justifications for PD turns out to be very subtle and technically involved, depending upon interrelations between axioms of definable determinacy and (inner models of) large cardinals. The reader is directed to [Maddy, 1988b] and [Koellner, 2010] for a discussion.

be given direct intrinsic justification, or it follows from other intrinsically justified principles.[40]

It is unlikely that the former plan is will be successful. Before understanding its pleasing theoretical consequences, there is little to choose between PD and its negation. For example, Donald Martin (who jointly with John Steel provided the proof of PD from the existence of infinitely many Woodin cardinals) wrote the following:

> "Is PD true? It is certainly not self-evident."
> ([Martin, 1977], p813)[41]

Given then that PD does not seem to admit of direct intrinsic justification, the only other alternative for providing intrinsic justification is to show that it follows from principles that *are* so justified. As remarked earlier, the existence of infinitely many Woodin cardinals will secure the result, but seems a weighty ontological assumption in need of further justification of its own. Here we find a rôle for reflection principles.

Indeed, Welch's global reflection principle delivers the result almost immediately:

> **Theorem 2.** [Welch, 2014] $ZFC + [RP3] \vdash$ "There exist proper-class-many measurable Woodin cardinals."

Hence, $[RP3]$ delivers (more than) the required Woodin cardinals to prove PD.[42]

The distinction between intrinsic and extrinsic justification represents one respect in which justifications can differ for a Zermelian. The challenge there is to provide *intrinsic* justification for both the reflection principles and their pleasing theoretical consequences (PD in the context of strong reflection, and ordinal unboundedness for the weak variety). There is, however, another dimension in which justification varies on a Zermelian picture. Once she has settled on a method (or

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[40]Of course, one might think that we should not even be looking for intrinsic justification of statements such as PD. This is certainly a tenable viewpoint, and one with which the author has a some sympathy. However, one should accept that if an ontological view can intrinsically motivate a better theory of sets than another, then it is *in this respect* a better view of ontology.

[41]There are several other places similar comments by Martin (and others) can be found, for example: [Martin, 1976], p90 [Moschovakis, 1980], p. 610. For an excellent discussion of PD and justification, the reader is once again directed to [Maddy, 1988b].

[42]The existence of a proper class of measurable Woodin cardinals has some other nice theoretical results, see [Woodin, 2001], [Woodin, 2010], and [Welch, 2014] for examples and discussion.

methods) of justification, she has to decide precisely what it is that she is trying to establish. First, she could be justifying the claim that mathematical discourse involving certain principles is *non-vacuous*. Second, she could be answering the question of whether such principles are justified as *axioms*.

Recall earlier the schematic account of reference provided, on which a given utterance of some set-theoretic sentences can refer to some $\mathcal{V}$ satisfying these sentences. There, we saw that the range of PZ-universes that can figure as referents of the term 'V' are determined by the sentences uttered. So, if the Zermelian is to motivate reflection principles as non-vacuous, she has to motivate the existence of *at least one* universe $\mathcal{V}$ of the required kind. Then, for some reflection principle $[RP]$, when a set theorist utters sentences of the form $ZFC_2 + [RP]$ there will be at least one $\mathcal{V}$ which can be taken as the schematic referent of 'V'. Since, for the Zermelian, every $\mathcal{V}$ is an initial segment of the form V_α in some taller $\mathcal{V}'$, the challenge amounts to justifying the existence of at least one V_α satisfying the relevant properties (*id est*, for $\beta < \alpha$, reflection to V_β in the case of weak reflection, and for strong reflection an embedding $j : (V_\beta, \in V_{\beta+1}) \longrightarrow (V_\alpha, \in, V_{\alpha+1})$).

It should be noted here that justification of the existence of *one PZ-universe* satisfying reflection secures much of the desirable theoretical consequences of that principle. In the case of PD, a single PZ-universe satisfying $[RP3]$ will ensure the existence of the necessary Woodin cardinals to prove that PD is true in every PZ-universe (by the Quasi-Categoricity Theorem and the fact that the set-theoretic objects determining the truth value of PD appear in the first few levels above V_ω). In the case of weak reflection, a single PZ-universe satisfying the required principle will ensure that there are at least some contexts in which the ordinals are unbounded in the relevant sense.[43]

However, there is an additional question for the Zermelian: can she motivate the *truth* of reflection principles as *axioms*? Here, we find a rôle for the distinction between PZ-universes and CZ-universes.

Consider again the claim that there exists an inaccessible cardinal (let the corresponding sentence be denoted by 'ϕ'). Now ϕ is *false* in the smallest $\mathcal{V}$: if κ is the least inaccessible then $V_\kappa \models \neg\phi$. However, we would be reluctant to say (assuming that the existence of an inaccessible is independently motivated) that ϕ's status as a set-theoretic *axiom* is thereby vitiated. Rather, it seems correct to say that our canonical best theory has changed. It is no longer simply ZFC_2, but rather $ZFC_2 + \phi$. This is reflected, for the Zermelian Multiversist, in how we

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[43]We shall see later that the issues here are subtle, the justificatory status of PD (under $[RP3]$) and ordinal unboundedness (under weak reflection) may come apart.

best interpret set-theorists when they intend to speak in the broadest possible context. Thus, through justified theory expansion, the range of CZ-universes can change, shrinking as we independently motivate more axioms and augment ZFC_2. What was, primordially (for Zermelo), the range of canonically justified universes, is changeable based on justified theory expansion.[44] Thus ϕ retains its status as an *axiom* in virtue of being true in every CZ-universe satisfying the independently motivated principles. To sum up, our original challenge of Motivation diverges into two salient problems:

1. Can the Zermelian Multiversist motivate the existence of at least one PZ-universe satisfying weak/strong reflection, thereby providing referents for contexts involving the principles and securing their pleasing theoretical consequences (*id est PD* in the case of $[RP3]$ and for weak reflection the existence of certain ordinals and iterative levels)?

2. Can the Zermelian motivate the *truth* of these principles as *axioms* in the sense that they are true in all CZ-universes?

As we shall see, an analysis of the *Universist's* motivations for accepting reflection principles yields responses for the Zermelian Multiversist (though questions concerning point (2.) remain). For, there is an extant challenge *for the Universist*: I have said nothing yet about *their* motivation for reflection principles. What is required by the Universist before the discussion can proceed is an adequate bridging principle that tells us *why* reflection principles should hold.

Consider one of the original motivations for reflection principles given by Gödel:

> "The Universe of sets cannot be uniquely characterized (i.e. distinguished from all its initial segments) by any internal structural property of the membership relation in it, which is expressible in any logic of finite or transfinite type, including infinitary logics of any cardinal number." (Gödel in [Wang, 1977])

Here, Gödel argues that the universe is 'ineffable': it is impossible for us to develop a sentence in any language that applies to the universe and not one of its initial segments. Merely asserting this is one thing, but why should it hold?

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[44]Hence the use of the terms 'primordial' and 'canonical' to describe the different kinds of Zermelian universes.

One way authors have tried to justify ineffability is through assumptions of *richness*. We can distinguish two main kinds of richness assumption to which Universe Theorists have appealed: arguments from *ordinal closure* and notions of *direct ineffability*.

3   Richness as ordinal closure.

The first way in which we might justify (weak) reflection principles is simply to reverse the direction of argument concerning reflection principles and the Iterative Conception of Set. It was noted earlier that a motivation of weak reflection principles was that they produce a 'better' theory of the Iterative Conception in that they give mathematical content to the claim that the stages of the Cumulative Hierarchy are 'absolutely unbounded'. Instead, we might argue that it is part of the *Iterative Conception* that V is closed under certain kinds of operation, and thus (by the work of Montague, Levy, and Bernays) satisfy the various forms of reflection.

Let us be a little more precise. There are three distinct but closely linked ways of postulating the closure of sequences of ordinals. The main contenders (and those that have received the most attention) are the use of *fixed-points*, *indescribability properties*, and *Tait's approach*. I consider each here, and show how the Zermelian can avail themselves of very similar justifications in order to motivate the existence of particular $\mathcal{V}$ that can act as referents for the term 'V' in discussions of reflection principles.

3.1   Functions and fixed points.

The simplest kind of ordinal closure I shall consider is in terms of derivatives of enumerating functions.[45] We begin by postulating the existence of inaccessibles by arguing that the intersection of the regular cardinals and Beth numbers is non-empty. Effectively this states the closure of the ordinals under the two main size-generating principles of ZFC_2 after the Axiom of Infinity: the Power Set Axiom and Replacement Axiom. The justification behind this is that the operations of Power Set and Replacement are not enough to characterise the whole ordinal number sequence: the process will always 'run out' in the sense that we arrive at a fix-point.

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[45] I am indebted to Peter Koellner for explaining to me both justifications involving derivatives and Tait's approach.

Once we have one cardinal of this variety, it is natural to postulate that there is a proper class of elements of the intersection, corresponding to repeated applications of this process, thereby leading to a proper class of inaccessible cardinals.

We now wish to iterate this definition to obtain higher levels of inaccessibles. Let I_0 be the proper class of regular Beth numbers. For any class X, let F_X be a (class-sized) function enumerating the elements of X. Further, let F'_X be the fixed points of F_X. We may proceed by asserting that F'_{I_0} has a proper class of elements (these are the 1-inaccessibles). This can clearly be iterated through the ordinals. More formally we have the following recursive definition:

Definition 3. $I_0 =$ The class of regular beth cardinals.

$I_{\alpha+1} = F'_{I_\alpha}$ for successor α.

$I_\lambda = \bigcap_{\beta < \lambda} I_\beta$

As we move through the ordinals we describe higher and higher levels of inaccessibility, gradually thinning out the class of original inaccessibles.

A similar kind of process is visible for stronger forms of ordinal closure. For example, we might argue for Mahlo cardinals in the following way. We first note that:

Definition 4. κ is *Mahlo* iff $\{\beta < \kappa \,|\, \text{``}\beta \text{ is inaccessible''}\}$ is stationary (in that it intersects every closed unbounded subset of κ) in κ.

or equivalently:

κ is *Mahlo* iff every normal (*id est* continuous and increasing) function on κ has a strongly inaccessible fixed-point.

We can then postulate that there is a proper class of Mahlo cardinals. We are asserting here that each κ in this class extends so far that the set of inaccessibles less than κ must meet any club set of κ. Equivalently, we assert that every normal function has a strongly inaccessible fixed-point: κ continues 'so far' that no matter what normal function we pick on κ there is a fixed-point of inaccessible cardinality.

Of course the definition can be iterated through the ordinals:

Definition 5. κ is 0-Mahlo iff κ is inaccessible.

κ is $(\alpha + 1)$-Mahlo iff $\{\beta < \kappa \,|\, \beta \text{ ``is } \alpha\text{-Mahlo''} \}$ is stationary in κ (equivalently: every normal function has an α-Mahlo fixed point).

κ is λ Mahlo iff κ is β-Mahlo for all $\beta < \lambda$.

So we have a characterisation of large cardinals in terms of fixed-points of normal functions. This suggests the following Axiom:

Axiom 6. Every normal function on the ordinals has a regular fixed point.

By the work of Levy and Bernays, this turns out to be equivalent to:

Axiom 7. $[RP4] \ \forall \alpha \exists \beta > \alpha [Inac(\beta) \wedge \forall \bar{x} \in V_\beta (\phi \leftrightarrow \phi^{V_\beta})]$

which asserts that reflection occurs between V and V_β for inaccessible β. Of course, we can generalise the closure under fixed points to include cardinals of any kind whose existence we have already accepted, essentially asserting that the type of cardinal does not exhaust the ordinals without hitting a fixed-point. Letting $\Phi(x)$ be one of, "x is inaccessible", "x is α-inaccessible", "x is Mahlo", or "x is α-Mahlo" we can then generalise to the following axiom schema:

Axiom 8. Every normal function on the ordinals has a fixed-point of kind Φ.

Again, by the work of Levy and Bernays, this is equivalent to having a version of $[RP4]$, where $Inac(\beta)$ is replaced by $\Phi(\beta)$. This justifies reflection principles that increase in strength as we push the reflection up through the inaccessible, α-inaccessible, Mahlo, and α-Mahlo cardinals.

Whatever one thinks of the plausibility of these closure arguments, let us assume that they are satisfactory on the Universist's picture. How does this justification for weak reflection principles affect the arguments against the Zermelian?

There are two obvious problems with taking this route if the Universist is to establish superiority over the Zermelian with respect to motivating weak reflection. The first is that it makes one of the main arguments for wanting to intrinsically justify weak reflection hopelessly circular. Earlier it was noted that the Universist would be in a better dialectic position if she could motivate weak reflection where the Zermelian could not, as she would have motivated from her own view the 'absolute unboundedness' (in the sense of closure under fixed-points) of the stages of the Cumulative Hierarchy, and hence have a better theory of the Iterative Conception of Set. But now this form of 'absolute unboundedness' is being assumed and taken as the *starting point* in intrinsically motivating the reflection principles. The

desired pleasing theoretical output of the reflection principles (the unboundedness of the ordinals under fixed-points of normal functions), is being used in order to motivate the principles themselves.

The second problem is that the Zermelian Multiversist can perfectly well appeal to this sort of motivation also. For, the Zermelian may subscribe to the Iterative Conception as much as the Universist, thereby transferring any intrinsic motivation conferred from unfolding said conception. So if closure under various kinds of fixed-point is entailed by the Iterative Conception *simpliciter*, then the Zermelian is allowed to appeal to said closure.

Let us explore this in a little more detail. The Universist is asserting that there should be no conceivable end to the ordinal number sequence. For example, Drake puts the point as follosets of ws:

> "We look for justification for these axioms [*id est* reflection principles] from the point of view of the cumulative type structure, where we want to say that the collection of levels, which is indexed by the ordinals, is a very rich structure with no conceivable end." ([Drake, 1974], p123)

We see the justification then: since there is no conceivable end to processes of generating ordinals from others, we should assert that the ordinal number sequence is closed under formation of these fixed-points, and hence V satisfies the relevant reflection principles.

If the Universist may say this about the ordinal numbers and closure of the Cumulative Hierarchy, then so may the Zermelian. Thus she may perfectly well say that the same fix-points on normal functions exist, and hence the same cardinals exist. Thus on this justification, at least from the point of view of motivating a theory of the ordinals for the Iterative Conception, the Zermelian is able to produce exactly the same objects and degree of unboundedness.

There are two natural responses on behalf of the Universist that admit of a uniform solution. The first is to note that the Zermelian holds that she is always implicitly restricted to particular V_κ. Thus (the Universist argues) she cannot say that "Every normal function has a fixed point of kind Φ", due to the restriction she fails to talk about 'all' functions.

The second objection is to point out that even if the Zermelian manages to motivate the existence of the relevant cardinals, she still has not motivated the existence of the relevant V to witness reflection on her ontology. It is one thing to motivate the existence of the cardinals *produced* by reflection, and another to motivate the (suitably interpreted) principles themselves.

Both these objections ignore the account of reference and quantification on the Zermelian Multiversist's understanding. Earlier it was noted that use of the term 'V', and any associated universal quantification, is schematic: 'V' applies to any appropriate $\mathcal{V}$. So the Zermelian can perfectly well say that "there should be no conceivable end to 'the ordinals' and hence every normal function on 'the ordinals' has a fixed point of kind Φ". It is simply that in saying so, this holds relative to some $\mathcal{V} = V_\beta$, under which every normal function on On^{V_β} has a fixed point of kind Φ. V_β will then satisfy all of the requisite reflection properties given by the Bernays-Levy equivalences, thereby guaranteeing the Zermelian the PZ-universe she needs to ensure the non-vacuity of discourse involving the weak reflection principle.

What then of the status of the principle as an *axiom*? Well, if it is intrinsically justified (in the sense of being implied by the concept of) the Iterative Conception of Set that the ordinals should be unbounded in the sense that every normal function on the ordinals has a fixed-point of kind Φ, then we should incorporate this fact into what we hold as our canonically justified best theory of sets. Hence, our canonical best theory will be at least ZFC_2+ "Every normal function on the ordinals has a fixed-point of kind Φ", and hence every CZ-universe will satisfy the corresponding reflection principle, thereby justifying its status as an axiom.

3.2 Ordinal closure as indescribability.

In fact, by extending the Universist's appeal to ordinal closure to *indescribability properties*, the Zermelian can provide *direct* motivation for the *non-vacuity* of reflection principles. We begin by noting the following definition scheme:

> **Definition 9.** For $Q = \Sigma$ or $Q = \Pi$ and $m, n \in \omega$, a cardinal κ is Q_n^m-indescribable iff for any formula ϕ of Q_n^m complexity and parameter $A \subseteq V_\kappa$ with m-order over any V_α quantifiers interpreted as ranging over $V_{\alpha+m}$ we have:
>
> $$(V_{\kappa+0}, ..., V_{\kappa+m}, A, \in) \models \phi \rightarrow \exists \beta < \kappa (V_{\beta+0}, ..., V_{\beta+m}, V_\beta \cap A, \in) \models \phi$$

Often authors will make similar claims about the ordinals under indescribability as was made concerning fixed-points of normal functions. Turning again to Drake, he writes:

> "A natural way of looking for larger cardinals is to take a process which generates larger ordinals (or cardinals) and to look for ordinals α where the process closes: or we could

look at ways of generating sets and look for ordinals α such that the process is closed on V_α.

An extension of this method, which perhaps comes almost as naturally when one remembers that the aim is to say that there is *no* end to the ordinals is to ask for ordinals such that any process we have has already closed before this stage." ([Drake, 1974], p267)

A little exposition is required to see the link between these 'processes for generation' and indescribability. To begin with, consider inaccessibles. Their definition ensures that they are closed under repeated application of the Axiom of Replacement and Power Set. Hyperinaccessibles are also closed under the 'processes' of Replacement and Power Set, but moreover are closed under the formation of inaccessibles. We can link these closure properties to (in)describability. We begin by noting that these ordinals are often *described* by processes as follows:

> **Definition 10.** An ordinal κ is *described by formula* ϕ *and parameters* $A_0, ..., A_n$ iff $(V_{\kappa+0}, ..., V_{\kappa+m}, A, \in) \models \phi \wedge \neg \exists \beta < \kappa (V_{\beta+0}, ..., V_{\beta+m}, V_\beta \cap A, \in) \models \phi.$

Describability provides one way of expressing the idea that an ordinal can be generated from another. To take a simple case, consider the case when κ is singular (and so can be formed by application of Replacement and Union from a smaller ordinal[46]). Let $\lambda = cf(\kappa) < \kappa$ and a function $f : \lambda \longrightarrow \kappa$ witness κ's singularity. Further, let $A_0 = f$ and $A_1 = \{\lambda\}$ be parameters. We then describe κ using the following (first-order) sentence:

$$\phi =_{df} A_1 \neq \emptyset \wedge \forall x \in A_1 x = dom(A_0)$$

We then have that $(V_\kappa, \in, A_0, A_1) \models \phi$, but for all $\alpha < \kappa$, $(V_\alpha, \in, A_0 \cap V_\alpha, A_1, \cap V_\alpha) \not\models \phi.$[47] Here, the fact that κ is generated by an application of Replacement and Union using f and λ provides us with the tools to give a description (with parameters) of κ. To say then

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[46]We use Replacement to obtain $ran(f)$ as a set, and then use Union to get $\cup ran(f) = \kappa$.

[47]To see this, note that we may divide into two cases: (i) $\kappa > \alpha > \lambda$, (ii) $\alpha \leq \lambda$. For (i) We may assume that f is monotone, as any non-monotone cofinal sequence will contain a monotone part. Then, $V_\alpha \cap A_0$ will always remove some ordered pairs from A_0 (f is proper-class-sized in V_κ) and hence $dom(A_0) \neq \lambda$. For (ii) note that $\alpha \leq \lambda$ implies that $A_1 \cap V_\alpha = \emptyset$.

that κ is of a certain level of *indescribability*, is to say that there are no operations and parameters for generating ordinals (of the relevant complexity) that allow us to describe κ in this way. The parameters have already 'closed out' within a V_α for $\alpha < \kappa$, and it is to this phenomenon that Drake is alluding.

Here we see that the ordinal closure appealed to is very similar to the derivatives approach: we are claiming that certain ways of generating ordinals must already have 'run out' at some earlier stage in the sense that there are smaller ordinals indexing stages closed under these methods. There are close affinities between the cardinals described previously and those generated by indescribability. Indeed, indescribability is simply the more general property: Π_0^1-indescribability is equivalent to being inaccessible, and every Π_1^1-indescribable cardinal is Mahlo. Arguing for indescribable cardinals through ordinal closure, while subtly different from justification involving fixed-points, is nonetheless similar.

If the generalisation of ordinal closure to indescribability is accepted, we now have the resources to provide direct justification for the non-vacuity of certain reflection principles on the Zermelian's view. Suppose that we can argue for the existence of indescribable cardinals on the basis of the unboundedness of the ordinals given by the Iterative Conception. Just as before, the Zermelian Multiversist is able to say that she agrees that the ordinal number sequence should be closed in this sense. Hence there are PZ-universes containing these indescribable cardinals. Indeed, if it is *entailed* by our conception of set that there are cardinals of a certain level of indescribability, then our canonical set theory should reflect this, and hence we should hold that the CZ-universes contain such cardinals.

Now, let $\mathcal{V}$ be one such CZ-universe. $\mathcal{V}$ can see a cardinal κ such that for any property ϕ of the relevant level of describability and parameter A,
$(V_{\kappa+0}, ..., V_{\kappa+m}, A, \in) \models \phi \rightarrow \exists \beta < \kappa (V_{\beta+0}, ..., V_{\beta+m}, V_\beta \cap A, \in) \models \phi$. But this is just to say that V_κ is a PZ-universe satisfying that fragment of reflection. In particular, if it is part of our conception of set that there are *totally* indescribable cardinals, then every CZ-universe can see a PZ-universe satisfying reflection of all second-order formulae.

Of course, this does nothing to establish the truth of reflection axioms in the sense of being true in every CZ-universe. Even if a CZ-universe $\mathcal{V}$ can see PZ-universes satisfying a certain amount of reflection, this is no guarantee that $\mathcal{V}$ itself satisfies the same principles. However, we should be mindful that the position is the same for the Universist: the generation of indescribable cardinals merely provides

strongly inaccessible V_α that satisfy reflection, without guaranteeing that V itself reflects to initial segments.[48] Assuming that the justification of these cardinals can stand on its own, the non-vacuity of discourse involving the relevant reflection is assured for both Zermelian and Universist.

3.3 Tait's 'Bottom Up' approach.

Before discussing strong reflection principles, it will serve to mention one more kind of justification for weak reflection principles, namely Tait's 'bottom up' approach. Tait's method is very closely linked to the previous discussion, asserting that initial segments of ordinals are closed under particular conditions. We shall see again that if the Universist is able to use this motivation, then so can the Zermelian, for very similar reasons as in the case of indescribability. The approach is particularly useful as it highlights a feature that the Universist will have to appeal to if she is to motivate reflection where the Zermelian cannot, namely *global* considerations. We start from a condition C (which Tait calls an 'existence condition'). We then use the condition to construct an ordinal Ω_C closed under C using the following principle:

> "If the initial segment Σ of Ω_C satisfies the condition C, then it has a least strict upper bound $S(\Sigma) \in \Omega_C$" ([Tait, 2005], p2)

Selecting the appropriate existence condition C then allows us to construct cardinals Ω_C such that V_{Ω_C} satisfies ϕ reflection. For example, if we take the condition V on $\Sigma \subseteq \Omega_C$ to be:

$$\exists X \subseteq V_\Sigma [V_\Sigma \models \phi(X) \wedge \forall \alpha \in \Sigma \neg \phi^{V_\alpha}(X^{V_\alpha})].$$

we obtain a V_{Ω_C} that satisfies full second-order reflection.[49] Choosing different existence conditions then facilitates the generation of cardinals satisfying various forms of reflection.

Certainly, there are deep problems with this approach, not least because it is unclear which existence conditions are satisfactory. For instance, if we take C to be $V_\Sigma \not\models$ "There exists a Reinhardt cardinal" we get a V_{Ω_C} that satisfies that there is a Reinhardt cardinal. This

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[48] As an example of this, suppose that there is a measurable cardinal κ. We claim that for α totally indescribable, V_α need not even be Σ_2^0-correct. For, there are many such V_α below V_κ, all of which satisfy that there are no measurable cardinals (a Σ_2^0 statement).

[49] For a more detailed exposition see [Tait, 2005] and [Koellner, 2009] (the example in question is Koellner's).

is clearly problematic, not least because it contradicts the Axiom of Choice ([Koellner, 2009], p209)!

Let us assume that, however, we can differentiate the 'good' conditions from the bad in a principled fashion. For exactly the same reasons as in the indescribability case, the non-vacuity of reflection is assured: Tait's principles generate PZ-universes satisfying different kinds of reflection (namely the relevant V_{Ω_C}). Such an approach will be dialectically available to the Zermelian Multiversist: the satisfaction of reflection is achieved *through* generating particular V_{Ω_C} that satisfy the reflection principle locally (indeed Tait's approach is explicitly *designed* to work within the Zermelian Multiversist's framework). Thus, even if the Universist can make sense of Tait's method, it can still be used by the Zermelian to justify weak reflection principles on exactly the same grounds. Again, the construction of the relevant V_{Ω_C} will occur relative to some ambient $\mathcal{V}$, and for the same reasons as in the case of indescribability, we do not necessitate the truth of reflection in every CZ-universe. However, once more the position of Zermelian and Universist is the same: they both have secured the non-vacuity of discourse involving the relevant reflection, rather than the truth of the reflection principles as axioms.

4 Richness as ineffability.

Let us take stock. We have seen that if the Universist can justify weak reflection principles *via* arguments involving the closure of the ordinals under particular kinds of operation, the Zermelian Multiversist is able to justify very similar local reflection principles that hold relative to PZ-universes of the appropriate kind. There is a different method of justifying reflection principles *via* richness assumptions, one that applies to both weak *and* strong reflection principles. In this section, I will analyse this kind of Universist argument, and argue that the Zermelian Multiversist can also motivate similar principles. In particular, if the Universist can use a particular motivation to motivate $[RP3]$, the Zermelian can motivate a form of local reflection that allows her to say that PD is true.

The method simply takes ineffability to *be* our criterion of richness. This would then allow us to motivate both strong and weak reflection principles as characterisations of this ineffability.

Welch develops the point as follows:

"We have simply declared that the whole universe $(V, \in, \mathcal{C})$ is so rich that there is some κ so that the collection of parts

over V_κ, namely $V_{\kappa+1}$, is in turn sufficiently rich so that any sentential truth we can formulate about the realm V with all of its parts (in the given language) reflects to a truth about V_κ with all of its parts." ([Welch, 2014], p16)

Here we see Welch claim that the set-theoretic universe should be sufficiently *rich* (in the sense that there are sufficiently many sets of varied kinds) that we are unable to distinguish the universe from one of its initial segments. In the case of weak reflection principles, we assert that no one sentence of any language can pin down the universe, and hence each sentence held by the universe must be reflected down to some V_α. With $[RP3]$ we assert that the universe is so rich that not even a fragment of an entire language (not just a sentence thereof) can pin it down: thus the true sentences of the whole fragment are reflected down to some $(V_\beta, \in, V_{\beta+1})$. Such a justification (as Welch is clear) goes substantially beyond the *Iterative* Conception of Set: it depends on not just principles of iteration but also advocates a global richness concerning V. In this sense then, the conceptual analysis involved in the intrinsic justification occurs with respect to an enriched conception of set, combining both iterativity and this notion of richness.

One might think that there is a real problem for the Zermelian Multiversist here. For Welch's justification, unlike the specific cases of indescribability and Tait's approach, is explicitly *global*: it depends on an overall richness in V facilitating the embedding. The Zermelian Multiversist, on the other hand, does not have V to which she can appeal as sufficiently rich to facilitate the satisfaction of $[RP3]$ by any particular $\mathcal{V}$.

4.1   Using richness to respond to the Universist.

We should not be satisfied with this simplistic characterisation, however, and should probe a little deeper into how reflection principles, in particular those of sufficient strength to prove PD, are motivated through richness.

As it stands the notion of 'richness' facilitating the level of ineffability required is exceptionally unclear. If the Universist can appeal to a general principle of 'richness' that follows from an unfolding of our concept of set, then so can the Zermelian Multiversist. This provides the material for the Zermelian to generate a response to the Universist. The Zermelian may assert that she agrees, on the basis of our (enriched) concept of set the subject matter of Set Theory should be as structurally rich as possible and contain many and varied sets. Given

this, she can say, it is entirely reasonable that we should expect (as part of a conceptual analysis of this 'rich' conception of set) there to be a $\mathcal{V} = V_\beta$ that reflects properties to its initial segments.

For any particular notion of weak reflection supported by direct ineffability arguments arising from the enriched conception of set on the Universist picture, there is then a $\mathcal{V}$ satisfying the same reflection principle for the Zermelian. The case of strong reflection is exactly analogous. The Zermelian can argue, on the basis of the enriched concept of set, that there should be a $\mathcal{V} = V_\beta$ and an ordinal $\kappa < \beta$ such that there is a non-trivial $j : (V_\kappa, \in, V_{\kappa+1}) \longrightarrow (V_\beta, \in, V_{\beta+1})$. Any such V_β must, by $[RP3]$, contain β-many measurable Woodin cardinals and satisfy PD. Moreover, since the first few levels above V_ω are the same in every PZ-universe of the Zermelian Multiversist's ontology, and she has justified the claim that there is a $\mathcal{V}$ in which PD is true, she may assert that PD is true in every CZ-universe and hence true *simpliciter*.

There is a possible response that the Universist may give. All the Universist need appeal to, he could argue, is a structural richness about V: his universe of sets. He makes *no* assumptions about the height of V. From this, he argues for a particular reflection principle that reflects properties down to some V_α. The argument, the Universist contends, is exactly *not* analogous for the Zermelian Multiversist. For she *does* have to make assumptions about the height of the particular $\mathcal{V}$ with which she is working. In order to motivate the reflection principles, she must *assume* that their PZ-universes extend far enough to make it true that there is a $\mathcal{V}$ that reflects the relevant properties to its initial segments.

The Universist might bolster this by asking us to consider the argument from $[RP3]$. There he merely stated that the universe was structurally rich enough for there to be a V_κ that reflected V and all its parts. We do not assume that the universe need have *any* particular height. However, the Universist might contend that the Zermelian Multiversist does have to make assumptions about the height of the universes in question. For in any PZ-universe in which there do not exist the required cardinals (of which there are quite a few), $[RP3]$ will be *false*. Therefore, in justifying $[RP3]$ as satisfied by some $\mathcal{V} = V_\beta$ we must in fact *assume* that there is a β above β-many measurable Woodin cardinals. To put it another way, if there were not proper-class-many measurable Woodin cardinals relative to some $\mathcal{V} = V_\beta$, we would not get inconsistency for the Zermelian Multiversist. We would rather just get no models of the required height. The Zermelian Multiversist's commitment to richness thus does not *in itself* (the Universist contends) imply the existence of proper-class-many measurable Woodin cardi-

nals on her ontology. Even given a commitment to the richness of the stages by the Zermelian Multiversist, the Universist will argue that she must assume the very thing she is trying to prove.

This is an example of a particular kind of bad argument that can be made against the Zermelian Multiversist. Here, the Universist has noted that as a consequence of a certain kind of principle holding in $\mathcal{V} = V_\beta$, a model of set theory was generated: in this case both the relevant $(V_\kappa, \in, V_{\kappa+1})$ and $(V_\beta, \in, V_{\beta+1})$. Then, the Universist argues, that model was assumed to be there all along and hence the Zermelian Multiversist was assuming the very thing she was trying to prove. But the fact that a particular model was generated does not mean that it was assumed to exist all along! Indeed, in order for Welch's principle to hold for the Universist, the ordinal number sequence must have a minimal length and contain certain subsets at additional stages. A direct consequence of his argument that there are *proper class* many measurable Woodin cardinals is that there must be some V_α in *his* ontology that is numerically identical (from his perspective) to the V_β generated by the Zermelian Multiversist. Hence, the Zermelian assumed the existence of the required model no more than the Universist did. If we examine the steps in the Zermelian Multiversist's argument, we can see that at no point did she make mention of the *specific height* of the models in question, she just argued that the enriched concept of set implied that the PZ-universes are sufficiently structurally rich that there is one PZ-universe (whatever its ordinal height may be) with the collection of all its parts that is non-trivially elementarily embeddable into some other PZ-universe (whatever its ordinal height may be) with the collection of all its parts. There is no appeal to the specific height of the model anywhere, and the Zermelian Multiversist is in exactly the same position as the Universist with respect to motivating the existence of $\mathcal{V}$ in which there are good theories of the Iterative Conception (in the senses of varieties of ordinal unboundedness) and justifying the truth of PD.

So, if the Universist can motivate reflection principles *via* conceptual analysis of an enriched concept of set, then the Zermelian Multiversist is able to motivate the existence of PZ-universes satisfying $[RP3]$ and PZ-universes satisfying weak reflection. Hence (assuming that the Universist justification is legitimate) she (i) gives herself the means to interpret set-theoretic discourse involving weak reflection and $[RP3]$ as non-vacuous, and (ii) thereby ensures the truth of PD in every PZ-universe. It would make sense to stop here, the Zermelian Multiversist has secured the desirable theoretical consequences given by both PD and (in certain $\mathcal{V}$) ordinal unboundedness, and

also avoided issues of being unable to motivate the existence of PZ-universes needed for interpreting set-theorists. However, one further loose end needs to be tied up: why assert that the relevant reflection principles are *axioms* in the sense of being true in every CZ-universe?

Here, an important *asymmetry* between Universist and Zermelian Multiversist *is* highlighted. For any notion of reflection $[RP]$ satisfied by a particular $\mathcal{V}$ meeting the enriched conception of set, it is unclear that the next $\mathcal{V}'$ above $\mathcal{V}$ does *not* meet the enriched conception. For, $\mathcal{V}'$ can see a PZ-universe (namely $\mathcal{V}$) satisfying $[RP]$, but also contains *more* sets than $\mathcal{V}$. It is not usual, however, that such a $\mathcal{V}$ will satisfy $[RP]$.

To see this more clearly, consider $[RP3]$. Assuming that the Universist has motivated $[RP3]$ as true of $(V, \in, \mathcal{C})$, she will have justified its truth as an axiom: the principle is true in V, and V is the ultimate arbiter of truth. The same is *not obviously true* for the Zermelian, however. While she has justified the truth of $[RP3]$ (or weak reflection) as true relative to some $(\mathcal{V}', \in, \mathcal{C}')$, it is *not* yet clear that $[RP3]$ (or weak reflection) is true in every CZ-universe. For, her conceptual analysis of the enriched concept of set allowed her to maintain that the subject matter of set theory should be sufficiently rich that $[RP3]$ is satisfied by *some* $(\mathcal{V}', \in, \mathcal{C}')$. But it is as yet unclear that *every* CZ-universe will satisfy $[RP3]$. Indeed, if $(\mathcal{V}', \in, \mathcal{C}')$ satisfies $[RP3]$ then the next PZ-universe above $\mathcal{V}'$ will in general *not* satisfy $[RP3]$. To see this, note that any $\mathcal{V}' = V_\alpha$ satisfying $[RP3]$ is indexed by a limit of inaccessible cardinals (see, for example, [Kanamori, 2009], p312). However, the next PZ-universe above $\mathcal{V}'$ will be indexed by a successor inaccessible, and so cannot satisfy $[RP3]$. In the case of the Universist, the global richness they ascribe to the concept of set means that it is *the Universe* that satisfies $[RP3]$. For the Zermelian the enriched concept of set only justifies the existence of *at least one* (or possibly some) PZ-universe(s) of the required kind.

Despite this limitation, the non-vacuity of mathematics involving $[RP3]$ and weak reflection is assured, as is the truth of PD. The issue of ordinal unboundedness is somewhat more vexed. For a PZ-universe $\mathcal{V}$ satisfying a reflection principle $[RP]$ that implies ordinal unboundedness of kind Φ, the next PZ-universe above $\mathcal{V}$ will (i) not satisfy $[RP]$ (as it will be a garden variety inaccessible and thus not have ordinal unboundedness of kind Φ), and (ii) potentially still meet the enriched conception of set in virtue of seeing a PZ-universe satisfying $[RP]$. However, the possibility of analysing additional justifications (in particular those arising from the enriched concept of set) to limit the CZ-universes to just those PZ-universes satisfying reflec-

tion is open to the Zermelian. It is eminently plausible, for instance, that any PZ-universe meeting the enriched concept of set should be unbounded in the senses of §3. Such an argument requires a fuller defence than space permits here, but would guarantee that the ordinals are also unbounded in every CZ-universe.

We should, however, be mindful that, *modulo* the tenability of the Universist's justification from the enriched conception, the Zermelian is able to secure *some PZ*-universe satisfying the relevant reflection on the basis of the enriched conception of set. Of particular philosophical interest is that, in the case of $[RP3]$, she secures the truth of PD and the pleasing theoretical consequences it implies.

Conclusion.

Motivating reflection principles has often been seen as a substantial challenge for the Zermelian Multiversist: she does not countenance the existence of V which might lead one to believe that the project of intrinsic justification *via* reflection is hopeless on her view. The argument is, however, far too quick: for certain justifications provided by Universists, the Zermelian Multiversist may just as well appeal to the same underlying justification. While questions concerning the status of reflection principles as axioms remain open on the Zermelian picture, she is able to justify far more than previously thought.

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