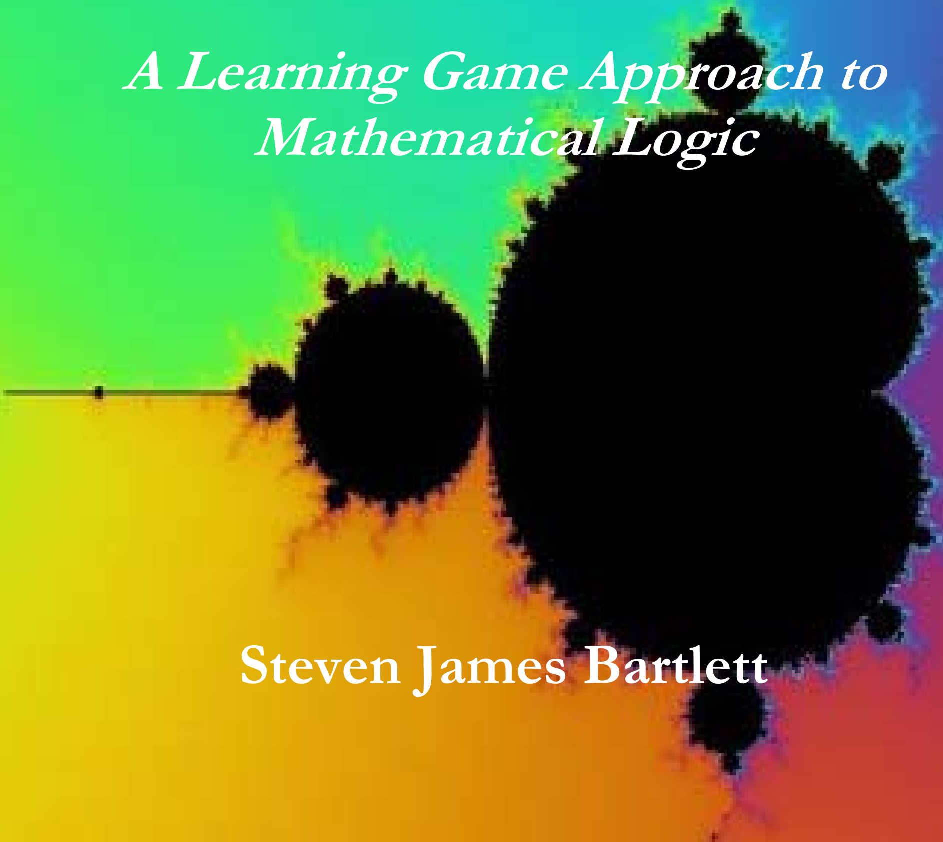
VALIDITY

A Learning Game Approach to Mathematical Logic



Steven James Bartlett

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ABOUT VALIDITY

VALIDITY is an "autotelic" learning game: it is designed to give students a *self-motivating* experience in creative problem-solving. I designed the game because I was interested in improving the teaching of mathematical logic at the university level, and wished to supplement the standard approach to mathematical logic that asks students to prove pre-formulated problems. The latter type of challenge is indisputably worthwhile despite the fact that someone other than the student has devised these problems as pre-formulated exercises in proof construction. But a pure diet consisting exclusively of constructing proofs which have already been anticipated by someone else shortchanges the student who does not experience the creative task of formulating his or her own problems, for which constructing proofs then becomes the challenge.

The text-manual for *VALIDITY* consists of a general introduction that describes earlier studies made of autotelic learning games, paying particular attention to work done at the Law School of Yale University, called the ALL Project (Accelerated Learning of Logic). Following the introduction, the game of *VALIDITY* is described, first with reference to the propositional calculus, and then in connection with the first-order predicate calculus with identity. Sections in the text are devoted to discussions of the various rules of derivation employed in both calculi. Three appendices follow the main text; these provide a catalogue of sequents and theorems that have been proved for the propositional calculus and for the predicate calculus, and include suggestions for the classroom use of *VALIDITY* in university-level courses in mathematical logic.

I have used *VALIDITY* with great success in numerous classes in mathematical logic. Its success has been attested by my students, and by my own observations of the skills in constructing proofs that I have seen student develop when playing the game. Those specific skills include: *improved ability and facility in constructing proofs*—which are the main goals of *VALIDITY*; *improved mental efficiency*—that is, the ability quickly to see directly through to an effective proof strategy; *improved mental anticipation and retention*— that is, increased ability to hold the whole anticipated proof in mind; and *improved cognitive flexibility*—that is, the ability to "re-group" and to re-formulate a proof strategy when the moves of other players change the framework within which a proof needs to be developed.

VALIDITY is not a parlor game. Although players become enthusiastic—indeed sometimes passionate—constructors of proofs, the game is serious and technically challenging.

The game, as I originally conceived and designed it, is intended to be used in conjunction with E. J. Lemmon's text, *Beginning Logic*, but the game can be adapted to other texts. I recognize that a professor's choice of logic text involves many factors, not least of which is the personal appeal of a certain approach to mathematical logic. There are no useful arguments, in my view, that can effectively persuade most mathematical logicians to adopt a system of logic to replace their own preferred system in courses they teach, and I shall not try to summon any. However, there is perhaps some value in sharing what it was and is that I find useful and attractive about Lemmon's particular system of natural deduction.

First, in his system there is an optimum number of natural deduction rules, adequate for both the propositional and the predicate calculus with identity. Many other systems of logic err in favor of an excessive number of rules, in order to maximize convenience and make proofs maximally short; other systems err in favor of mathematical elegance by admitting only a single rule, normally the rule of detachment, along with provisions for substitutivity.

I use the word 'err' advisedly in both contexts: If one is interested in encouraging students to develop and internalize logical skills that may spill over into other areas of their lives and studies, then a small set of rules, balancing convenience and elegance, and capable of being retained in the active memory of the average student, will turn out to be most desirable. Lemmon's system, furthermore, recommends itself through the use of a method of assumption annotation, a notational device permitting students to check their proofs quickly for correctness, and exhibiting, for each line a proof, exactly what is presupposed. Such an assumption annotation system has clear-cut advantages both formally and in the context of philosophical argument. Some other systems of natural deduction provide similar methods to keep track of what each line in a proof depends upon.

For these reasons, and others which relate to economy of statement and aesthetics of structure, I adapted *VALIDITY* to serve as a companion to Lemmon's book.

Any faculty member who is interested in incorporating a learning game approach within a rigorous course in mathematical logic will have to roll up his or her sleeves, for the decreased formality that results when students play an academic game in class means giving up some of the control and structure that a standard class in mathematical logic provides. Too, the open-ended and inherently flexible nature of *VALIDITY* game playing will require not only the students, but the teacher also, to learn some new skills. It can sometimes be challenging in such a context for the teacher to stay ahead of the brightest students.

I definitely do not advocate using any learning game to the exclusion of work with a text and pre-formulated exercises. For my purposes, I used *VALIDITY* perhaps one-third of class time: In a class meeting three times a week, one class meeting each week might be devoted to the use of *VALIDITY*; the other two class meetings consisted of lecture combined with students assigned to construct proofs.

Students were graded on their performance when playing *VALIDITY*—an expression of its non-parlor-game purpose. Appendix III of the *VALIDITY* text-manual describes how this was done. However, it is certainly not written in stone that other faculty should do as I did if they wish to make use of the game.

Teachers who prefer an axiomatic approach to mathematical logic and who wish to think of ways to adapt a *VALIDITY*-like approach to complement their existing method of teaching will, I would expect, be somewhat challenged to adapt *VALIDITY* to their own needs. But I believe this can, with perseverance and intelligent thought, be done. Teachers who use a natural deduction approach but who do not wish to use Lemmon's text will have an easier path in adapting *VALIDITY* to fit their interests.

But, clearly, whether it will be worthwhile to devote creative effort and time to adapt the basic approach of *VALIDITY* to your own teaching is a matter of your personal judgment.

VALIDITY has served hundreds of my students and their teacher very well. I hope the game as it exists or its underlying approach will be of value to others, and have decided to make *VALIDITY* freely available as an Open Access publication under the Creative Commons **Attribution-NonCommercial-NoDerivs** license.

HOW TO USE *VALIDITY*

VALIDITY was originally published in a boxed set consisting of the text that follows, three groups of playing squares, and three printed headings (NECESSARY, POSSIBLE, and REJECTED) below which the playing squares are placed during a game. In order to make *VALIDITY* available in electronic form, to play the game you will need to print the three pages that follow Appendix III in order to make the required number of playing squares, plus the three headings.

The NECESSARY playing squares and the NECESSARY heading were originally printed on white stiff plasticized stock with red-colored backing; the Possible playing squares and the POSSIBLE heading had light blue backing; and the Rejected playing squares and the REJECTED heading had dark blue backing. The discussion in the text refers to these colors.

You may therefore find it convenient to print the three pages of playing squares either on different colored sheets of paper or using different colors of type. (If you are able to use white stock that has a colored back, all of the logic symbols will have a white background, which makes for easier reading.) Once the three sheets are printed, cut the sheets to make the required number of individual playing squares and the three headings. This will produce the total number and kinds of required playing squares; these are also listed on Page 9 of the text that follows.

VALIDITY

A Learning Game Approach to Mathematical Logic

Modern mathematical logic is a highly useful aid to clear and precise thinking. Logic is basic to all quantitative disciplines, and it offers guidelines for many of the liberal fields of study. Recently it has been demonstrated that students who acquire skills of modern logic frequently raise their quantitative IQ scores significantly.* It is not therefore surprising that the logical, systematic thinker often is a more successful achiever in any field.

Unfortunately, the normal text-book approach to mathematical logic fails to appeal to many students, and also fails to foster good quantitative attitudes in many who do take a course in modern logic.

Certain helpful innovations in education have been suggested by the development of learning games, which attempt to encourage interest and motivation by making the learning experience exciting and pleasant.

The games of VALIDITY foster deductive skills desirable in a university-level introductory course in mathematical logic.

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* Cf. Layman E. Allen, Robin B.S. Brooks, James W. Dickhoff, Patricia James, "The ALL Project (Accelerated Learning of Logic)", American Mathematical Monthly (Vol. 68, No. 5, May 1961); L.E. Allen, R.W. Allen, James C. Miller, "Programmed Games and the Learning of Problem-Solving Skills: The WFF 'N PROOF Example", The J. of Educational Research (Vol. 60, No. 1, Sept. 1966); L.E. Allen, R.W. Allen, Joan Ross, "The Virtues of Nonsimulation Games", Simulation and Games (Sept. 1970).

V A L I D I T Y

A Learning Game Approach to Mathematical Logic

TABLE OF CONTENTS

page

V A L I D I T Y

A Learning Game Approach to Mathematical Logic

Introduction.

Educators have, perhaps somewhat belatedly, become increasingly aware of the importance of student motivation in learning. It has also become increasingly evident that many of the recognized approaches to teaching have become entrenched in the circular ruts of tradition, failing to capture the student's interest. The most significant conclusion to be drawn by theorists of learning and teaching is that what a student learns is of much less importance than the attitudes he develops toward learning in general. Competence in a subject matter, indispensable though this may be, is less valuable than a student's motivation further to educate himself once his formal education has ended.

The recent development of learning games is an attempt to provide a student with an enjoyable learning experience which can encourage him to continue learning on his own. In a related though secondary sense, learning games seek to establish a context for learning in which a specific skill can be developed through the student's own initiative. Such learning games have been called "autotelic" because a student's success in game play tends to sustain and to increase his interest, i.e., a game player's involvement becomes "self-motivating".

I am indebted to numerous persons who have contributed to the evolution of the learning game concept. In particular, in the field of mathematical logic, Professor Layman E. Allen's well-known game, WFF 'N PROOF, has been successful in motivating my own interest in innovative techniques for the teaching of formal logic. I have learned much from the use of the WFF 'N PROOF game in my introductory courses in mathematical logic at the University of Florida and at the University of Hartford. Reactions of my students to a learning game approach to logic have been consistently positive. My own teaching experiences utilizing WFF 'N PROOF in the university classroom have encouraged me to believe that learning games in general suggest a fruitful direction of educational development that should not be neglected. It is a shallow remark which is sometimes to be heard today - to the effect that students nowadays want only to play games in the university. The remark is shallow only because - _if_ the observation is correct - it dead-ends in the despair of the serious educator. The fact, perhaps the only _fact_ in this context, is that playing _some_ games is not the idle and worthless pastime it unreflectingly is often dismissed as being.

WFF 'N PROOF unfortunately does possess numerous shortcomings that come to light in the university classroom. The effort which the student must expend to master the WFF 'N PROOF manual which is "programmed" for self-instruction is without any doubt excessive. The WFF 'N PROOF games can be simply taught and easily learned. The manual of instructions is far more difficult to follow than is the game to play. Secondly, the notation chosen for the WFF 'N PROOF game, the Polish notation of Łukasiewicz, is only infrequently to be found in Anglo-American works on mathematical logic. Although the Polish notation is unequaled for typographical convenience and conciseness, every student I have had has remarked upon the perspicuity to be gained in the structure of symbolic expressions through the use of one of the more standard notations, in spite of the requisite parenthesizing or other punctuation which the Polish notation renders unnecessary.

But an overly difficult instruction manual or the use of a notation that is at present of relatively little use to the student of logic, or the suggestion that structural clarity is perhaps better obtained in other notations, - none of these shortcomings is as important to the university teacher of logic as is the fact that the WFF 'N PROOF game is limited to the propositional calculus. Most university-level introductory courses in mathematical logic attempt to present to the student both propositional and predicate calculi. It is true that WFF 'N PROOF has never been offered as most appropriate for university-level courses in logic. It occurred to me that a learning game approach to mathematical logic, tailored to meet the needs of a university course, could perhaps provide a fruitful <u>supplement</u> to the standard text-book approach.

The game of VALIDITY was therefore developed with this end in view.

The notation employed is one basic to the majority of Anglo-American works in mathematical logic. The game is applicable both to the propositional calculus as well as to the first order predicate calculus with identity. The description of the game proceeds in a straight-forward manner, though the format of the instructions is not that of a stimulus-response "program" as in the WFF 'N PROOF manual. -Probably, my philosophical opposition to behaviorism is to blame for this. Somehow, in the mesh of stimulus and response, the behaviorist strainer fails to capture the creative impulse. All important work in mathematical logic, as in any area of human endeavor, has been creative.

VALIDITY is organized, as I have said, as a <u>supplement</u> to a text-book. The games of VALIDITY are intended to form a companion to one of the clearest introductory logic texts I have had the good fortune to come across, E.J. Lemmon's <u>Beginning Logic</u>.* The exposition there of the propositional calculus and of the first order predicate calculus is clear and concise. The rules of derivation are few in number, while Lemmon's format for proofs is incomparably more convenient in my opinion and in the opinion of most of my students than the method of subordinate proofs developed by Fitch**, and employed in WFF 'N PROOF.

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* Published by Thomas Nelson and Sons, Ltd., as a paperback in 1971.

** F.B. Fitch, <u>Symbolic Logic</u>: <u>An Introduction</u> (NY: The Ronald Press 1952).

Finally, the games of VALIDITY differ from WFF 'N PROOF in the deductive skills and strategies fostered. WFF 'N PROOF begins with the formulation of what is to be proved (the "Goal"), and proceeds to the construction of some premisses from which that conclusion can be derived. This approach tends to invert the thought-processes required in the solution of most problems in formal logic. One ordinarily does not begin with a conclusion and then wonder what premisses *might* imply that conclusion (although this is sometimes a very useful strategy, as Polya has pointed out*), but rather *begins* with some set of premisses**, and attempts generally to understand what these premisses deductively lead to, and, in particular, to determine whether they entail some conclusion.

Working "backwards" from conclusion to premisses can often aid in the construction of a proof - provided these premisses are somehow known in advance. It is the responsibility of the WFF 'N PROOF player to realize whether *some* set of premisses has been constructed which *could* entail the conclusion. He must, in other words, skillfully "run through" the ramifications of potential candidates for premisses.

It is questionable whether this skill is as significant to the logician as is the "forwards" approach, in which the interest is in what conclusions - if any - can validly be inferred from an initial set of premisses**. The deductive skills

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* G. Polya, *How to Solve It: A New Aspect of Mathematical Method* (NY: Doubleday 1957).

** In the case of theorems, this set is empty.

and strategies which the games of VALIDITY therefore attempt to foster are rather different from those developed by the WFF 'N PROOF player: they are different skills which are, I believe, much more useful to the mathematical logician.

The objective of VALIDITY.

The purpose of all the games of VALIDITY is to develop the player's logical skills of deduction. A deduction consists of one or more steps of inference, proceeding from zero, one, or more premisses (which are assumed to be true) by means of one or more rules of derivation, to a conclusion which follows validly based on those premisses (if there are any). Deductive skills enable a player to decide whether or not a given conclusion follows validly from what is assumed to be the case. It is this ability to decide by deductive means whether a conclusion does or does not follow validly based on what is assumed (if anything), which is fostered by all the games of VALIDITY.

The acquisition of deductive skills is of practical interest for two principal reasons: such skills facilitate an understanding of the logical ramifications that follow from the adoption of a set of beliefs or hypotheses. And, conversely, deductive skills make it possible to determine whether a given conclusion is in fact entailed by the set of premisses with which it is associated. In short, logical skills of deduction tend to increase a person's consciousness of the consequences that follow from the adoption of a point of view, and enable him to distinguish the valid from the invalid employment of deductive reasoning.

The games of VALIDITY.

The eight games of VALIDITY fall into two groups. Games in the first group develop deductive skills in connection with the propositional (or sentential) calculus. Games in the second group presuppose deductive skills acquired in earlier games and further help to develop these skills in connection with the first order predicate (or functional) calculus. The games in each group are arranged in order of increasing complexity, as additional rules of derivation are introduced into game play. Later games presuppose the mastery of earlier ones.

All the games of VALIDITY are played with plasticized squares on which various logical symbols have been printed. A complete game set consists of 210 squares, as shown on the next page.

In addition to the playing squares, each set of VALIDITY contains the present manual, three self-sealing storage and mixing bags, and three elongated rectangles on which the words 'Necessary', 'Possible', and 'Rejected' appear.

General rules for all games of VALIDITY.

There is a common set of rules for all games of VALIDITY.

Number of players. VALIDITY may be played by two or more players. Three is preferred for classroom use.*

The object of the games. The player's intent in all the games of VALIDITY is to

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* See Appendix III.

List of Logical Symbols by Color Code

Propositional calculus games:

Logical operators (12 of each):

-
&
v
$\rightarrow$
$\leftrightarrow$

} light blue

Propositional variables (12 of each):

x
y
z
w

Rule name squares (12 of each):

e
i

} dark blue

Blanks: 2 light blue
2 dark blue

Predicate calculus games:

Quantifiers and connectives (12 of each):

$\underline{\wedge}$
$\underline{\vee}$
=

Predicate letters (12 of each):

F
G
H

} red

Blanks: 2 red

Total: 210 squares

contribute to the gradual formation of a set of premisses (zero or more) and of rule names which must never together become sufficient to allow the deduction of a conclusion that can be represented, as described below, by means of playing squares. The player must, in other words, attempt to become conscious of consequences which follow logically based on the assumption of the premisses formulated at each stage of the game. And he must exercize caution that the available premisses and rule names formulated at each stage do not provide a basis sufficient to derive a symbolizable conclusion.

It is important to realize, then, that the player's main objective is to come as close as possible to, but without totally representing, a set of premisses and rule names from which a conclusion that can be symbolized with the remaining playing squares may be derived.

The skill of a good VALIDITY player involves two related abilities: the ability to know in advance what can be deduced from an evolving set of premisses and rules, and the capacity to continue playing with a less than minimal basis for a valid deduction. The development of this general skill facilitates a consciousness of the logical consequences of a set of assumptions, and is basic to an ability to determine the validity of deductive inferences.

The aim of the player of VALIDITY, then, is to participate in the construction of premisses (if there are to be any) and rule names up to the point that it is no longer possible to continue playing without completing a basis sufficient for a valid deduction.

Precisely what is meant by 'a basis sufficient for a valid deduction' will be defined shortly.

<u>Start of play</u>. The suggested numbers and types of playing squares for the particular game of VALIDITY to be played are first determined, by referring to the detailed description of that game. The three elongated rectangles bearing the words 'Necessary', 'Possible', and 'Rejected' are placed face up toward the top of a cleared playing area.

The playing squares are stored and mixed in three separate self-sealing bags, according to their color: all the light blue playing squares in one bag, all the dark blue playing squares in a second bag, etc.

An arbitrarily designated player shakes each bag to mix its contents throughly and then proceeds to draw (without looking at the contents of the bags, of course) the number of playing squares of each kind required for the game to be played. These, he turns face up and arranges: the operator symbols in a group, the variable symbols in a group, etc. These comprise the <u>set of available symbols</u> (the AS set).

A player is now chosen to begin the play. His first task is to determine whether the initial AS set contains a sufficient number of symbols of different kinds to represent <u>some</u> set of premisses and rule names from which a conclusion can both be deduced and then represented by means of the symbols remaining.

> EXAMPLE. The initial AS set contains playing squares of the following numbers and kinds:
>
> | | | | |
> |---|---|---|---|
> | 2 | - | 4 | x |
> | 3 | & | 2 | y |
> | 1 | v | 3 | z . |
> | 2 | → | | |
> | 1 | ↔ | | |
>
> Is this AS set adequate to play the first game of VALIDITY? -See instructions for that game, pp. 29-31.
>
> Since there are no i- or e-playing squares available, no rule names can be represented by means of playing squares taken from the AS set. No deduction is possible without recourse to rules of derivation. The above AS set is therefore <u>inadequate</u> for the purposes of the game to be played. (Note that there may be other reasons for the inadequacy of an AS set.)

<u>Provided</u> the initial AS set is adequate for the purposes of the game to be played, the first player is permitted to place one playing square below one of the headings 'Necessary', 'Possible', or 'Rejected'.

If the first player moves a playing square below one of the three headings, while another player notices that the initial AS set is inadequate in the sense described, then the first player is vulnerable to challenge, and will lose the game if challenged. Similarly, if the first player claims the initial AS set is inadequate, he will be vulnerable to challenge if he is mistaken.

A player's move may be challenged by any other player at any time.

If the first player claims the initial AS set is inadequate, and provided his fellow player(s) should agree with his claim, then the playing squares are returned to the bags and shaken, and another AS set is drawn.

Any playing square, once placed below a heading, cannot be moved below another heading.

A playing square placed in the <u>Necessary</u> category <u>must</u> be utilized either in the formation of a premiss or in the construction of a rule name subsequently to be used in the deduction. Playing squares placed in the <u>Possible</u> category may also be used to form premisses or rule names, but may be utilized in a deduction <u>only if their use renders possible a deduction not otherwise obtainable from those in the Necessary category</u>. These stipulations together result in the elimination of proofs involving - from their own standpoint - superfluous rules and premisses. These stipulations do not imply that a proof has to be the <u>shortest possible proof</u> of a particular conclusion. The only claim that is made of a particular set of premisses and rule names is that all these premisses and all the rule names can be employed in <u>some</u> proof in which they <u>are</u> indispensable.

EXAMPLE. Consider the following proof:

1	(1)	$(x \rightarrow y)\ \&\ (x\ \&\ z)$	A
1	(2)	$x \rightarrow y$	1 &e *
1	(3)	$x\ \&\ z$	1 &e
1	(4)	x	3 &e
1	(5)	y	2,4 →e
1	(6)	z	3 &e

In the above proof, lines (2), (4), and (5) are <u>inessential</u> to the derivation of z at line (6). A player who places '→e' under the Possible (or Necessary) heading is not therefore using the rule indispensably in the above proof.

The sequent proved at line (6) is

$$(x \rightarrow y)\ \&\ (x\ \&\ z) \vdash z\ ,$$

the proof of which need be only three lines long.

EXAMPLE. Compare the following two proofs of the sequent

$$x \rightarrow w,\ -w \vdash -x\ .$$

1.

1	(1)	$x \rightarrow w$	A
2	(2)	$-w$	A
1,2	(3)	$-x$	1,2 ← e

2.

1	(1)	$x \rightarrow w$	A
2	(2)	$-w$	A
3	(3)	x	A
1,3	(4)	w	1,3 →e
1,2,3	(5)	$w\ \&\ -w$	2,4 &i
1,2	(6)	$-x$	3,5 -i

---

* Refer to the list of symbolizations of rule names in VALIDITY, p. 30.

Note that both proofs are correctly written and contain - each from its own standpoint - no inessential lines. A player having the longer second proof in mind would employ two additional rules in his proof without using these rules superfluously.

Playing squares moved to Rejected may not be employed at all in any deduction. Note that rejecting a playing square from the game serves only to reject that particular playing square, and not all those bearing the same symbol.

When a player places a playing square below a heading, he is not free to declare that the playing square he has moved is to be used as a premiss only, or as a rule name only. The evolution of the game will gradually make clear the function of the playing squares already moved.

EXAMPLE. A player places an '&' below the Possible heading. At this point it is indeterminate whether the '&' will be employed in the construction of a premiss or of a rule name. A subsequent player now places an 'e' to the right of the '&'. These juxtaposed playing squares (see below) from this point on can serve only one function - that of representing the &e-rule name.

Continuation of play. The play then moves to the left, to the second player. He must first, before moving a playing square from the AS set, decide whether he believes anything well-formed -- which can be represented by means of remaining playing squares in the AS set -- can be deduced from the first player's move. If, no matter how that first move is supplemented by other moves of playing squares from the AS set, no set of premisses and rule names can provide a sufficient basis for such a deduction, then the second player must challenge the move of the first.* His <u>failure</u> to challenge in such a case prior to moving leaves <u>him</u> vulnerable in turn to a decisive challenge from the other players, who are free to challenge, as has been indicated, out of turn at any time.

A challenger who is able to sustain his challenge wins; conversely, a player who shows his challenger wrong defeats the challenger and wins. It is clear, then, that a skillful player will seek to take note of mistakes made by his fellow players, and challenge <u>immediately</u> whenever it is possible for him to do so correctly.

Provided he feels a deduction is still possible, the second player, still prior to moving, must also decide whether the first player's move may itself have provided a <u>basis sufficient for a valid deduction</u> of a conclusion that can be represented by playing squares that remain in the AS set. Since no player is permitted to complete a basis sufficient to deduce a symbolizable conclusion, any player doing so should be challenged immediately. Failure to do so prior to moving, if detected, results in the loss

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* If there are three players, the third player may also challenge the move of the first.

of the game.

'A basis sufficient for a valid deduction' may now be defined as 'a representation, by means of playing squares that have been placed in the Necessary and Possible categories, of a set of premisses and rule names which are employed indispensably in the derivation of a conclusion representable by means of playing squares remaining in the AS set'.

EXAMPLE. If a player has in mind a proof of the sequent

$$x \rightarrow -y,\ y \vdash -x\ ,$$

a "basis sufficient for a deduction" will consist of playing squares placed in the Necessary and/or Possible categories representing the squared-off symbols in the following proof which he has in mind:

1	(1)	$\boxed{x \rightarrow -y}$	A
2	(2)	$\boxed{y}$	A
2	(3)	$--y$	2 DN
1,2	(4)	$-x$	1,3 $\boxed{\leftarrow e}$

For this proof to be feasible in a game of VALIDITY, there must be available playing squares to represent the two premisses, the one rule name which is not "free" (see page 25), and the conclusion.

Expressions on intermediate lines between the premisses and the conclusion are not to be represented by means of playing squares. In the example above, the expression on line (3), '--y', is not represented by means of playing squares.

For the proof in the example, the <u>initial</u> AS set must contain at least

2 x-squares, 2 y-squares, 2 --squares, 2 $\rightarrow$-squares, and 1 e-square. Should this be the case, and if, e.g., '$\leftarrow$e' and 'x$\rightarrow$-y' are represented by playing squares in the Necessary or Possible categories, then a player who places a y-playing square below the playing squares already appearing in either category should be challenged for completing a basis sufficient for a deduction.

EXAMPLE. Consider the following proof:

1	(1)	$\boxed{x \& z}$	A
2	(2)	w	A
1	(3)	z	1 $\boxed{\&e}$
1	(4)	w $\rightarrow$ z	2,3 $\boxed{\rightarrow i}$.

A player who places a w-playing square in the Possible or Necessary category to represent line (2) in the above proof which he has in mind has made a mistake. The proof as written is correct; however, the assumption at line (2) is discharged when the conclusion at line (4) is reached. By noting the assumption annotation number to the left of the last line, a player can recognize immediately that the conclusion rests only upon line (1) as premiss. The sequent proved is therefore

$$x \& z \vdash w \rightarrow z \ .$$

For the above proof, the initial AS set would have to contain sufficient playing squares to represent the single premiss, as well as the names of the two different rules employed. (Excepting the Rule of Assumptions, which is used in all games of VALIDITY along with the Rule of Double Negation without necessitating the representation of a rule name by means of playing squares -- again, see page 25.) No w-playing square is therefore needed to represent line (2).

Having determined that previous moves have not completed a sufficient basis for a deduction, but that it still remains possible to provide such a basis by supplementation from the AS set, the second player is <u>free to move</u> a playing square from the AS set to a position below one of the three headings.

<u>Juxtaposition of playing squares</u>. The second player (and all subsequent players), once he is free to move, may, if he chooses, place a logical symbol to the left or to the right of a symbol already placed in a category. When this is done, a premiss is partially or totally constructed, or a rule name is <u>totally</u> constructed (all rule names consist of <u>two</u> juxtaposed symbols).

Once a sequential order has been given to the playing squares in a category, that order cannot be changed, although the construction of a premiss can often be <u>added to</u> by supplementation from the AS set. A premiss may be added to from the left or from the right, <u>provided</u> that a well-formed formula will eventually result when suitably supplemented with playing squares remaining in the AS set. It is important to exercize caution in juxtaposing playing squares, since any logical expression, so long as it is not completed so as to be well-formed, <u>stands in need</u>

of some suitable supplementation with playing squares remaining in the AS set. If a premiss or rule name has not been completely formed by means of the juxtaposition of playing squares in the <u>Necessary</u> category, it <u>must</u> be added to until it stands in need of no further supplementation or at least until it lacks no more than a single playing square to be complete (in this case, the addition of this one playing square would complete a basis sufficient for a valid deduction). If a player adds to a playing square or sequence of playing squares by juxtaposition, but the resulting expression <u>cannot</u> be completed by supplementation (for reasons such as those illustrated below), then the player's move is inadmissible, and should be challenged.

> EXAMPLE. Suppose that game #1 of VALIDITY is being played (see the instructions for that game), and the following situation has developed: 'x → y' and 'e' appear, one below the other, under the Necessary heading, and 'x' appears under the Possible heading, while two z-squares and one w-square have been Rejected. The remaining AS set contains
>
> | | | | |
> |---|---|---|---|
> | 3 | - | 1 | y |
> | 3 | v | 1 | z . |
> | 1 | → | | |
> | 1 | ↔ | | |
>
> The sequent the players have in mind - this should be immediately apparent from the disposition of the game at this point - is
>
> $$x \rightarrow y, x \vdash y .$$
>
> Suppose that at this point in the game, a player moves the one remaining →-square below the x-square in the Possible category. A new premiss or

rule name is now being constructed. If the →-square is to be used in a premiss, at least two more variable playing squares would be needed before a wff would result -- one on the left and one on the right of the implication sign. Two variable playing squares remain in the AS set - a 'y' and a 'z'. But the 'y' is to be reserved for the conclusion of the deduction in progress. The '→' cannot then be used in a premiss. And, similarly, it cannot be used in the representation of a rule name, since no i- or e-playing squares remain in the AS set.

Finally, there is a consideration which could have brought these reflections to an abrupt end somewhat more quickly. The only basis sufficient for a valid deduction of y in the game in question would require that the →-playing square be placed to the left of the e-playing square already appearing under the Necessary heading. This e-playing square stands in need of supplementation; only the last available implication sign can serve the desired purpose -- and the essential →-square has been misused. The player moving the →-square to Possible should therefore be challenged.

Similarly, if a premiss or rule name has not been completely formed by means of juxtaposed playing squares in the Possible category, there must be sufficient playing squares in the AS set -- <u>which are not indispensable for other purposes in the deduction in progress</u> -- to make it <u>possible</u> to complete the formation of the premiss or rule name in question. In short, juxtapositions under the Necessary heading must actually bring a premiss or rule name at least a single playing square's "distance" from being completely represented, while those under the Possible heading - although they may require more extensive supplementation before they are completely formed - require that the needed supplementation be possible without interfering with the game being played.

It should be clear that inadmissible juxtapositions also include those which, no matter how parentheses are inserted, fail to yield expressions conforming to the formation rules of the calculus in question. Any move that results in an inadmissible juxtaposition is vulnerable to challenge.

Since the representation of any logical expression by means of playing squares is parenthesis-free, the symbolization of premisses by means of sequences of playing squares will often be ambiguous, and thus a variety of interpretations of a given premiss often will be possible.

EXAMPLE. Consider the following sequence of juxtaposed playing squares:

$$- x \vee - z \rightarrow w .$$

A number of different ways of grouping the playing squares suggest themselves:

$$(-x \vee -z) \rightarrow w$$
$$-(x \vee -z) \rightarrow w$$
$$-((x \vee -z) \rightarrow w)$$
$$-x \vee (-z \rightarrow w)$$
$$-x \vee -(z \rightarrow w)$$

$$-(x \vee (-z \rightarrow w))$$
$$-(x \vee -(z \rightarrow w)) .$$

Fortunately, only a very small number of such interpretations will usually suggest themselves as feasible within the context of an actual game!

This ambiguity encourages the player to conceive of all possible interpretations that may be given to a sequence of playing squares, any one of which interpretations expresses, without ambiguity, a premiss from which a symbolizable conclusion can be derived, using rules which can be represented by means of playing squares either appearing under Necessary and perhaps also under Possible, or remaining in the AS set. A skillful VALIDITY player will be aware of the different ways of reading a premiss, and develop his game strategy accordingly.

No sequential order is, of course, imposed upon playing squares in the Rejected category.

Should a player not wish to add by juxtaposition to a symbol or sequence of symbols already placed in a category, he may place a playing square <u>below</u> those already moved to that category. This indicates that a <u>new</u> premiss or rule name is being constructed.

The play continues to the left in the same fashion, with each player evaluating, prior to his own move, the legitimacy of previous moves.

End of game. The game ends either when a challenge is made, or when it is no longer possible to move without completing a basis sufficient for the deduction of a conclusion representable by playing squares remaining in the AS set. In the first case, a challenger unable to sustain his claim loses; a player who shows his challenger wrong wins. In the second case, the player finding it impossible to move any playing square remaining in the AS set without completing a basis sufficient for a deduction must claim this. If he is unchallenged in this claim, all players must write a correct proof* of a conclusion (a) which can be represented by means of remaining playing squares in the AS set, (b) using all the premisses and rule names under the Necessary heading, (c) employing any or all of the Possible premisses and rule names if their use renders possible a deduction - i.e., a proof - not otherwise obtainable, (d) not using any of the Rejected logical symbols, and (e) using one and only one additional playing square from the AS set. Failure to write a correct proof results in the loss of the game.**

Finally, it is important to note that there must be sufficient playing squares in the Necessary and Possible categories to represent, with the exception of a single needed logical symbol, (a) all premisses indispensable to the derivation of a conclusion, (b) all different rule

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* According to the standards given in Lemmon for a correctly written proof.

** In this single instance, it is conceivable that more than one player, and perhaps all the players, can lose the game.

names employed, with the exception of (c) the two "free rules", the Rule of Assumptions and the Rule of Double Negation. The first condition, (a), requires, in Lemmon's system of assumption annotation, that for a premiss to be considered indispensable, it must be indicated by an assumption annotation number to the left of the line number of the final line of the proof in which it appears. The second condition, (b), stipulates that once a name for a certain rule of derivation is constructed, the rule may be used one or more times in a deduction. To use a single rule a number of times, it is necessary to have sufficient playing squares to symbolize its name only once. The third condition, (c), indicates that the Rule of Assumptions and the Rule of Double Negation are "free rules", which may be employed without the construction of rule names by means of the playing squares.

In general, if the game does not end because of a challenge, there will be sufficient playing squares remaining in the AS set to represent a derivable conclusion, plus one playing square needed to provide a sufficient basis for the deduction of that conclusion.

<u>The blanks</u>. A blank of a certain color may be used to represent any <u>one</u> of the logical symbols with that color coding. E.g., a light blue blank may function as a '−', '&', 'v', '→', or '↔', but not as an 'x', 'y', 'z', 'w', 'i', or 'e'. A player moving a blank from the AS set to the Necessary or Possible category must

assign an appropriate symbol to the blank.*

Additional new variable letters and predicate letters may of course be generated by using the blank playing squares. E.g., a red playing square may be assigned the letter 'F', or another letter such as 'I'.

Sample strategies. The games of VALIDITY are designed to encourage the player to learn a variety of strategies, both proof strategies and game playing strategies. It is impossible to become a skillful game player without acquiring an understanding of techniques for constructing proofs; but a player, though he may be able to write and to recognize a correctly constructed proof, may fail to coordinate his deductive abilities within the context of a competitive game.

It would be neither possible nor desirable to give a complete enumeration of game playing strategies for the games of VALIDITY. First, VALIDITY is a learning game (as opposed to a parlor game) of considerable complexity; it is designed to be instructive, and is most successful when the player is capable of original, i.e., creative, thinking. Second, it is one of the purposes of the game to provide a context for a self-instructive experience. A detailed listing of game playing strategies would decrease the enjoyment and profit which come with some hard reflection.

Nevertheless, it does seem to be worthwhile to sketch a few of the most

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* Obviously, it is not necessary to assign a symbol to a blank playing square under 'Rejected'.

obvious and general strategies which may assist the beginning player to become involved in the game. Here, then, are a few suggestions:

The main way to win is by challenging correctly. The task of evaluating earlier moves is a competitive one: the first player to challenge correctly will win. In other words, exercize caution when challenging, but be alert and challenge quickly.

A player may Reject a playing square which he feels is essential to the proof his opponent has in mind. To do this effectively, the player should of course have an alternate proof in mind, should he be challenged.

Attempt to expand (by juxtaposition) a premiss formed by another player, thereby obstructing the proof he has in mind, while "rerouting" the game along the lines you wish.

Require that more complex rules be used by adding their rule names to the Necessary category, forcing the other players to modify their deductions accordingly.

Intentionally complete a basis sufficient for a valid deduction in order to challenge a player subsequently for his failure to notice before moving that a basis sufficient for a deduction had been completed. This strategy may backfire, should your opponent(s) notice your deception.

Place playing squares in the Possible category as a deceptive play. Remember, deceptions <u>do</u> frequently fail -- here, if, e.g., another player is successful in devising a proof which does make use of playing squares you have placed under Possible, and claims correctly that you have inadvertently completed a basis sufficient for a valid deduction.

Take advantage of ambiguity in one or more premisses to make possible a deduction that stands in need of few (or no) additional playing squares in the AS set.

Think.

Be calm and patient.

What you can find enjoyment in, will be that much easier for you to master.

Games of VALIDITY for the propositional calculus.

For the purposes of these games, 'x', 'y', 'z', and 'w' are to be considered as names of propositional variables.*

The table on the next page indicates how rule names found in Lemmon for the propositional calculus are to be represented by playing squares in games of VALIDITY.

Note: Proofs of the following form are not allowed in any of the games of VALIDITY for the propositional calculus:

$x \vdash x$, $--x \vdash x$, $x \vdash --x$, $x \vee y \vdash x \vee y$, &c.

This prohibition serves to disallow proofs employing the "free rules" of derivation alone.

All the games of VALIDITY for the propositional calculus are to be played with the light blue and the dark blue playing squares only.

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* It will be clear to the attentive reader why the usual symbols for propositional variables, 'P', 'Q', 'R', ..., would make the later games of VALIDITY excessively cumbersome. For this reason, this departure from convention seems justified.

Representation of Rule Names in VALIDITY for the Propositional Calculus

	Rule name in Lemmon	Symbolization in Lemmon	Representation in VALIDITY
(1)	Rule of Assumptions	A	none
(2)	Double Negation	DN	none
(3)	Modus Ponendo Ponens	MPP	→e
(4)	Modus Tollendo Tollens	MTT	←e
(5)	Conditional Proof	CP	→i
(6)	Conjunction Introduction	&I	&i
(7)	Conjunction Elimination	&E	&e
(8)	Disjunction Introduction	vI	vi
(9)	Disjunction Elimination	vE	ve
(10)	Reductio ad Absurdum	RAA	-i

<u>Game #1</u> [Corresponds to Lemmon, Chap. 1, §§ 1-2]

Rules (1) - (5) may be used.

An initial AS set of 9 light blue and 9 dark blue playing squares should be used, drawn from all of the light blue and dark blue playing squares that have been mixed in their respective storage bags.

<u>Game #2</u> [Corresponds to Lemmon, Chap. 1, § 3]

Rules (1) - (10) may be used.

Use an initial AS set of 12 light blue and 12 dark blue playing squares.

<u>Game #3</u> [Corresponds to Lemmon, Chap. 1, §§ 4-5]

Rules (1) - (10) may be used.

To these may be added the use of the definition of the biconditional (Df. $\leftrightarrow$), for which no representation by means of playing squares is necessary. (Df. $\leftrightarrow$ is not a rule, but a definition.)

Use an initial AS set of 12 light blue and 12 dark blue playing squares.

<u>Game #4</u> [Corresponds to Lemmon, Chap. 2, § 2]

Rules (1) - (10) may be used.

TI and SI (theorem introduction and sequent introduction, with substitution) may also be used. No representation of these rules by means of playing squares is required. A list of the sequents and theorems in Lemmon will be found in Appendix I.

Use an initial AS set of 12 light blue and 12 dark blue playing squares.

Sample games.*

I. Game #2 is to be played.
Initial AS set:

3 -	2 x
2 &	1 y
3 v	4 w
2 $\rightarrow$	3 i
2 $\leftrightarrow$	2 e

	Necessary	Possible	Rejected
Player 1	-		
2	-i		
3	Challenges		

Player 1 placed a '-' under Necessary. Player 2 placed an 'i' to the right of the '-', forming the -i-rule name (Reductio ad Absurdum). Player 3 notices that a basis sufficient for a valid deduction has been completed, and therefore challenges Player 2. The proof he has in mind is:

$$\vdash -(x \,\&\, -x)$$

$$\begin{array}{llll} 1 & (1) & x \,\&\, -x & A \\ & (2) & -(x \,\&\, -x) & 1,1\ \boxed{-i} \end{array} .$$

Note that the only needed basis sufficient for the derivation of -(x & -x) is the squared-off rule name above. Player 3 wins.

II. Game #2 is to be played.
Initial AS set:

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* The sample games given here will be most useful to the beginning player if he actually follows the games by means of playing squares.

4	-	2	x
1	&	2	y
3	v	3	w
3	→	3	i
1	↔	2	e

	Necessary	Possible	Rejected
Player 1	v		
2	vi		
3		&	
1		&i	
2			-
3	i		
1			w
2			w
3			v
1		→	
2		→ y	
3		w → y	
1			e
2		-w → y	
3			↔
1			→
2			→
3			y
1			e
2	Claims cannot move without completing basis sufficient for a valid deduction.		
3	Challenges		

Player 3 is unable to see how a proof can be constructed which will utilize the vi-rule name and the incomplete _i-rule name under Necessary, perhaps the &i-rule name and ambiguous premiss - w → y under Possible, and result in a conclusion that can be represented by means of the playing squares remaining in the AS set. Players 1 and 2 reply to his challenge by writing the following proof:

$\vdash$ x v -x

1	(1)	-(x v -x)	A
2	(2)	x	A
2	(3)	x v -x	2 $\boxed{\text{vi}}$
1,2	(4)	(x v -x) & -(x v -x)	1,3 $\boxed{\text{\&i}}$
1	(5)	-x	2,4 $\boxed{\text{-i}}$
1	(6)	x v -x	5 vi
1	(7)	(x v -x) & -(x v -x)	1,6 &i
	(8)	--(x v -x)	1,7 -i
	(9)	x v -x	8 DN.

To complete a basis sufficient for the deduction of x v -x in the above proof, playing squares are needed to represent the three squared-off rule names. (Recall that once a rule name has been represented, that rule may be used any number of times.) This basis sufficient for the deduction would be completed by placing a single --square from the AS set to the left of the 'i' appearing by itself under Necessary. The remaining playing squares in the AS set can then be used to represent the conclusion, x v -x.

Players 1 and 2 win, 3 loses.

III. Game #2 is to be played.
Initial AS set:

2	-	4	x
3	&	2	y
4	v	3	w
3	$\leftrightarrow$	1	i
		2	e

	Necessary	Possible	Rejected
Player 1			x
2			y

Player	Necessary	Possible	Rejected
3	e		
1		v	
2			v
3			↔
1		&	
2			&
3		&i	
1			↔
2		x v	
3		x v x	
1			-
2			-
3			v
1			↔
2		w	
3			e
1			&
2		w	
3			w
1			y
2	Claims cannot move without completing a basis sufficient for a valid deduction.		

Players 1 and 3 agree with the claim made by Player 2, and all write the following proof, placing the single 'v' remaining in the AS set to the left of the 'e' under Necessary, and by so doing complete a basis sufficient for a valid deduction:

x v x ⊢ x

1	(1)	$\boxed{\text{x v x}}$	A
2	(2)	x	A
1	(3)	x	1,2,2,2,2 $\boxed{\text{ve}}$.

The players draw.

<u>Summary of the rules of derivation for the propositional calculus</u>

<u>Assumption annotations</u>.

To the left of the line numbers of a proof will appear a column of numbers, called "assumption annotations". <u>Assumption annotations indicate, for any expression on a given line, the line numbers of expressions that are presupposed by that expression</u>. Any line in a proof may itself therefore be regarded as a representation of a sequent proved at that line.

EXAMPLE.	1	(1)	P	A
	2	(2)	Q	A
	1,2	(3)	P & Q	1,2 &I
		.....		

Line (1) expresses the sequent $P \vdash P$, line (2), the sequent $Q \vdash Q$, line (3), the sequent $P, Q \vdash P \& Q$. Etc.

A line to the left of whose number no assumption annotation appears may be thought of as representing a sequent proved at that line based on zero assumptions. An expression correctly annotated in this fashion is called a <u>theorem</u>.

The proof of any sequent is correct if and only if the conclusion of the proof is prefixed by assumption annotations which indicate the line numbers, and <u>only</u> the line numbers, of the expressions given in that sequent as premisses. If the assumption annotations to the left of the line number of the conclusion contain more than, or fewer than, the number of premisses given in the sequent to be proved, the sequent has not been proved.

Assumption annotations are written to the left of any line number in a proof in the following mechanical fashion: (a) If the expression on that line is justified by the Rule of Assumptions (A), then write the same number as the line number in the assumption annotation column. (b) If the expression is justified by a rule other than A, refer to the line number(s) appearing in the justification, and pool the assumption annotation(s) appearing to the left of the line(s) so numbered; should the justification in question cite a rule that requires that one or more assumptions be discharged, discharge those assumptions accordingly.

No proof is correctly written without a correctly written assumption annotation column.

Rule of Assumptions (A).

This rule authorizes the insertion of any well-formed formula at any stage in a proof. To the left of any line, the expression on which is justified by A, is written the same number as that line number. This indicates that the expression is proved based on itself as assumption.

EXAMPLE. $P \vdash P$

1 (1) P A

Assumption annotations provide a continuous reminder throughout the course of a proof that certain expressions have been assumed.

Modus ponendo ponens (MPP).

MPP authorizes the inference from a conditional and the antecedent of the conditional as separate premisses, to the consequent of the conditional as conclusion.

EXAMPLE. $P, P \rightarrow R \vdash R$

1	(1)	P	A
2	(2)	$P \rightarrow R$	A
1,2	(3)	R	1,2 MPP

Note that there is no explicit or implicit stipulation made regarding the order of the two premisses; MPP is applicable whenever a conditional and the antecedent of that conditional appear separately on any lines in a proof.

Modus tollendo tollens (MTT).

MTT authorizes the inference from a conditional and the negation of the consequent of the conditional as separate premisses, to the negation of the antecedent of the conditional as conclusion.

EXAMPLE. $-R, P \rightarrow R \vdash -P$

1	(1)	$-R$	A
2	(2)	$P \rightarrow R$	A
1,2	(3)	$-P$	1,2 MTT

The same note applies as in MPP.

Double negation (DN).

DN authorizes the inference from a wff as premiss, to the double negation of that wff as conclusion, or from the double negation of a wff as premiss, to the wff as conclusion.

EXAMPLE.

--Q ⊢ Q				-Q ⊢ ---Q			
1	(1)	--Q	A	1	(1)	-Q	A
1	(2)	Q	1 DN	1	(2)	---Q	1 DN

Conditional proof (CP).

CP authorizes the inference from two expressions on separate lines in a proof, to the conditional resulting when the expression occurring earlier in the proof is placed in the antecedent position in the conditional, and when the expression occurring later in the proof is placed in the consequent position. When CP is applied, the assumptions, if any, presupposed by the earlier expression are normally (see caution below) discharged when the assumptions are annotated to the left of the resulting conditional.

EXAMPLE. Q ⊢ P → Q

1	(1)	P	A
2	(2)	Q	A
2	(3)	P → Q	1,2 CP

A rationale behind this discharge of assumptions is that redundancy of writing

1,2 (3) P → Q 1,2 CP

is avoided. (P is explicitly hypothesized in the conditional $P \rightarrow Q$; there is no need therefore to place a '1' in the assumption annotation column.)

Caution. In the case where both expressions - antecedent and consequent - are taken from lines having an identical assumption annotation, discharge whatever is presupposed by the antecedent only when the consequent-expression is actually derived from the antecedent-expression.

EXAMPLE.

1	(1)	$P \& Q$	A
1	(2)	P	1 &E
1	(3)	Q	1 &E
1	(4)	$P \rightarrow Q$	2,3 CP

Since line (3) is not obtained from line (2), the '1' in the assumption annotation column to the left of line (4) must appear, and cannot be discharged.

Conjunction introduction (&I).

Given two wffs as separate premisses, &I authorizes the derivation of their conjunction as conclusion.

Conjunction elimination (&E).

Given a conjunction of two wffs as premiss, &E authorizes the derivation of either wff as conclusion.

EXAMPLE. $P \& (Q \rightarrow R) \vdash Q \rightarrow R$

1	(1)	$P \& (Q \rightarrow R)$	A
1	(2)	$Q \rightarrow R$	1 &E

Disjunction introduction (vI).

Given a wff as premiss, vI authorizes the derivation of any disjunction in which the given wff occupies the position of one disjunct.

EXAMPLE. $P \rightarrow Q \vdash R \vee (P \rightarrow Q)$

1	(1)	$P \rightarrow Q$	A
1	(2)	$R \vee (P \rightarrow Q)$	1 vI

Note that there is no explicit or implicit stipulation made regarding the order of the disjuncts in the resulting expression. (Had the sequent to be proved been $P \rightarrow Q \vdash (P \rightarrow Q) \vee R$, then line (2) would have been written

1 (2) $(P \rightarrow Q) \vee R$ 1 vI .)

Disjunction elimination (vE).

Given

(a) a disjunction,
(b) a derivation of a certain conclusion based on the assumption of the first disjunct in (a), and
(c) a derivation of the same conclusion as in (b) based on the assumption of the second disjunct in (a),

then that conclusion may be derived by vE, based on the initial disjunction and on any assumptions employed in (b) and in (c), other than the assumption of the first disjunct in (b), and other than the assumption of the second disjunct in (c).

Five steps must be carried out in the application of vE: the initial disjunction must be written as a line, its first disjunct must be assumed, a certain conclusion must be derived based on the assumption of the first disjunct, the second disjunct must be assumed, and the same conclusion must be derived based on the assumption of the second disjunct. Then, and only then, can this conclusion be written as a line, based on the assumptions indicated above.

The five steps must be listed <u>in order</u> in the justification.

EXAMPLE. $P \vee Q \vdash Q \vee P$

1	(1)	$P \vee Q$	A
2	(2)	P	A
2	(3)	$Q \vee P$	2 vI
4	(4)	Q	A
4	(5)	$Q \vee P$	4 vI
1	(6)	$Q \vee P$	1,2,3,4,5 vE

<u>Reductio ad absurdum</u> (RAA).

Given a derivation of a contradiction based on a wff as premiss, RAA authorizes the derivation of the negation of that wff as conclusion. Two steps must precede the application of RAA: the wff in question must appear on a line, and a contradiction must be derived based on that wff as premiss. The negation of that wff may then be derived by RAA, and the assumptions presupposed by the original wff are then discharged. In the justification, the line numbers of the original wff and the contradiction derived from it are listed, in that order.

EXAMPLE. $P \rightarrow Q, P \rightarrow -Q \vdash -P$

1	(1)	$P \rightarrow Q$	A
2	(2)	$P \rightarrow -Q$	A
3	(3)	P	A
1,3	(4)	Q	1,3 MPP
2,3	(5)	$-Q$	2,3 MPP
1,2,3	(6)	$Q \& -Q$	4,5 &I
1,2	(7)	$-P$	3,6 RAA

Games of VALIDITY for the first order predicate calculus.

For the purposes of these games, 'x', 'y', 'z', and 'w' are to be considered as names of variables of individuals when prefixed by a predicate letter ('F', 'G', 'H', ...), as names of propositional variables when not so prefixed, and as symbols for arbitrary names when the x, y, z, and w playing squares are turned on their sides under the Necessary and Possible headings.

The table on the following page indicates how rule names found in Lemmon for the predicate calculus are to be represented by playing squares in games of VALIDITY.

Note: Proofs of the following form are not allowed in any of the games of VALIDITY for the predicate calculus:

$$x \vdash x, \; --Fx \vdash Fx, \; Fx \vdash --Fx, \; (\underline{\Delta}x)(Fx \vee Gx) \vdash (\underline{\Delta}x)(Fx \vee Gx), \text{ etc.}$$

All the games of VALIDITY for the predicate calculus are to be played with the light blue, dark blue, and red playing squares.

Representation of Rule Names in VALIDITY for the Predicate Calculus

Rule name in Lemmon			Symbolization in Lemmon	Representation in VALIDITY
(11)	Rule of	universal quantifier elimination	UE	$\underline{\wedge}$e
(12)	"	universal quantifier introduction	UI	$\underline{\wedge}$i
(13)	"	existential quantifier introduction	EI	$\underline{\vee}$i
(14)	"	existential quantifier elimination	EE	$\underline{\vee}$e
(15)	"	identity introduction	=I	=i
(16)	"	identity elimination	=E	=e

Game #5 [Corresponds to Lemmon, Chap. 3, §§ 1-2]

Rules (1) - (12) may be used.

An initial AS set of 8 light blue, 8 dark blue, and 8 red playing squares should be used, drawn from all of the light blue, dark blue, and red playing squares that have been mixed in their respective storage bags.

Game #6 [Corresponds to Lemmon, Chap. 3, § 3]

Rules (1) - (14) may be used.

Use an initial AS set of 10 light blue, 10 dark blue, and 10 red playing squares.

Game #7 [Corresponds to Lemmon, Chap. 3, §§ 4-5]

Rules (1) - (14) may be used.

TI and SI with substitution may also be used. A list of available sequents and theorems will be found in Appendixes I and II.

Use an initial AS set of 10 light blue, 10 dark blue, and 10 red playing squares.

Game #8 [Corresponds to Lemmon, Chap. 4, § 3]

Rules (1) - (16) may be used.

Use an initial AS set of 12 light blue, 12 dark blue, and 12 red playing squares.

Sample games.

I. Game #5 is to be played.
Initial AS set:

2	-	2	x	3	$\underline{\wedge}$
3	v	2	y	2	$\underline{V}$
2	&	1	w	1	F
1	↔	1	i	2	H
		2	e		

Player	Necessary	Possible	Rejected
Player 1	$\underline{\wedge}$		
2			$\underline{\wedge}$
3	e		
1			$\underline{V}$
2		y	
3			i
1			F
2			$\underline{V}$
3	&e		
1	&		
2	&y		
3	x&y		
1			v
2			v
3	Hx&y		
1	xHx&y		
2			↔
3			-
1		-y	
2			v
3	$\underline{\wedge}$xHx&y		
1	Claims cannot move without completing basis sufficient for a valid deduction.		

Players 2 and 3 agree with the claim made by Player 1, and all write the following proof, placing the single 'e' remaining in the AS set to the right of the '$\underline{\wedge}$' under Necessary, and by so doing complete a basis sufficient for a valid deduction. The single 'w' remaining in the AS set is turned on its side to represent the arbitrary name found in the conclusion. The players draw.

$\underline{\Lambda}xHx \;\&\; y \vdash Ha$

1	(1)	$\boxed{\underline{\Lambda}xHx \;\&\; y}$	A	
1	(2)	$\underline{\Lambda}xHx$	1	$\&e$
1	(3)	Ha	2	$\underline{\Lambda}e$

II. Game #7 is to be played.
Initial AS set:

3	-	2	x	3	$\underline{\Lambda}$
3	&	2	y	2	$\underline{V}$
3	→	3	w	2	$\underline{F}$
1	l.b.	1	i	2	G
	blank	2	e	1	H

	Necessary	Possible	Rejected
Player 1	F		
2	Fw		
3		&	
1		&i	
2			$\underline{\Lambda}$
3	$\underline{\Lambda}$		
1	$\underline{\Lambda}$e		
2	wFw		
3		-	
1		- x	
2		- x v*	
3			G
1			G
2		- x v y	
3	$\underline{\Lambda}$wFw		
1			$\underline{V}$

2 Challenges Player 1 for failing to notice that Player 3 had

* Blank

completed a basis sufficient for a deduction.

To support his claim, Player 2 produces the following proof:

$\underline{\Lambda}$xFw, -(x v y) ⊢ Fa & (-x & -y)

1	(1)	$\boxed{\underline{\Lambda}\text{wFw}}$	A
2	(2)	$\boxed{\text{-(x v y)}}$	A
1	(3)	Fa	1 $\boxed{\underline{\Lambda}\text{e}}$
2	(4)	-x & -y	2 SI(S) 1.5.1 (f-a)
1,2	(5)	Fa & (-x & -y)	3,4 $\boxed{\text{\&i}}$,

and explains that a w-playing square, turned on its side, would be employed to represent the arbitrary name in the conclusion.

III. Game #8 is to be played.
Initial AS set:

3	-	4	x	2	$\underline{\Lambda}$
3	v	2	y	4	$\underline{V}$
3	&	2	w	2	$\underline{F}$
1	→	1	i	2	H
2	↔	3	e	2	=

Player	Necessary	Possible	Rejected
Player 1	$\underline{\Lambda}$		
2	$\underline{\Lambda}$e		
3			↔
1			=
2		&	
3		&H	
1			-
2			-
3	$\underline{V}$		
1	$\underline{V}$i		

	Necessary	Possible	Rejected
Player 2	&		
3		&Hy	
1		x&Hy	
2			=
3		Fx&Hy	
1	$\underline{V}$		
2	$\underline{V}$e		
3			v
1			-
2		yFx&Hy	
3			v
1			v
2		$\underline{V}$yFx&Hy	
3		x$\underline{V}$yFx&Hy	
1			→
2			↔
3			F
1			w
2			w
3			&
1	&e		
2	Claims cannot move without completing a basis sufficient for a valid deduction.		

Players 1 and 3 agree with the claim made by Player 2, and write the following proof, placing the single '$\underline{\Lambda}$' remaining in the AS set to the left of the incompletely formed premiss under Possible, completing a basis sufficient for the deduction:

$$\underline{\Lambda}x\underline{V}y(Fx \,\&\, Hy) \vdash \underline{V}xHx$$

1	(1)	$\boxed{\underline{\Lambda}x\underline{V}y(Fx \,\&\, Hy)}$	A
1	(2)	$\underline{V}y(Fa \,\&\, Hy)$	1 $\boxed{\underline{\Lambda}e}$
3	(3)	Fa & Hb	A
3	(4)	Hb	3 $\boxed{\&e}$
3	(5)	$\underline{V}xHx$	4 $\boxed{\underline{V}i}$
1	(6)	$\underline{V}xHx$	2,3,5 $\boxed{\underline{V}e}$.

Summary of the rules of derivation for the predicate calculus with identity.

Symbols: for proper names: 'm','n','o',...
for arbitrary names: 'a','b','c', ...
for individual variables: 'x','y', 'z', ...
for predicate letters: 'F','G', 'H', ...

Terms are either proper names or arbitrary names.
Wffs include wffs of the propositional calculus, plus all wffs whose variables are quantified over. Note that expressions which fail to express propositions that are either true or false, of the form, e.g., 'Fx → Gx', are not wffs. Expressions of this form are called propositional functions.

Rules of derivation:

Universal quantifier elimination (UE).

Let A(v) be a propositional function in v, and t be a term; let A(t) be the result of replacing all and only occurrences of v in A(v) by t. Then, given (v)A(v), UE authorizes the derivation of A(t) as conclusion.

Existential quantifier introduction (EI).

Given A(t), EI authorizes the derivation of (∃v)A(v) as conclusion.

Universal quantifier introduction (UI).

Let A(e) be a wff containing the arbitrary name e, and v be a variable not occurring in A(e); let A(v) be the propositional function in v which results from replacing all and only occurrences of e in A(e) by v. Then, given A(e), UI authorizes the derivation of (v)A(v) as conclusion, provided that e occurs in no assumption on which A(e) rests. The conclusion rests on the same assumptions as the premiss.

Existential quantifier elimination (EE).

Given (∃v)A(v), together with a proof of some wff C from A(e) as assumption, EE authorizes the derivation of C as conclusion, provided that e does not occur in C or in any assumption used to derive C from A(e), apart from A(e) itself. The conclusion C rests on any assumptions on which (∃v)A(v) depends or which are used to derive C from A(e) (apart from A(e)).

[Intuitive re-statement:

UE authorizes the inference from the premiss that all things have a certain property, to the conclusion that any particular object has that property.

EI authorizes the inference from the premiss that a particular thing has a certain property, to the conclusion that something must have that property.

UI authorizes the inference from a proof that an arbitrarily selected object has a certain property, to the conclusion that everything must have that property. Restriction: before applying UI to '...a...', in order to obtain '(x)(...x ...)', check the assumptions on which

'...a...' rests to insure that 'a' nowhere appears in them. (Should '...a...' rest on certain <u>special</u> assumptions, i.e., any at all, about 'a', then 'a' has <u>not</u> been arbitrarily selected.)

If something has a certain property, EE authorizes the inference from a proof that <u>if</u> an arbitrarily selected object has that property, then some conclusion C follows, to C as conclusion. If something has the property, and an arbitrarily selected thing which has that property implies C, then C must hold. The conclusion C will rest on any assumptions on which the existential proposition rests, and on any assumptions used to derive C from the typical, arbitrarily selected object, apart from the assumption of such an object. <u>Restriction</u>: the arbitrary name in question must not appear either in the conclusion C or in the assumptions used to derive C from the arbitrarily selected object (though, of course, the arbitrary name will appear in the assumption of the arbitrarily selected object). (If this restriction is not met, from an existential premiss, a universal conclusion is derivable.)]

<u>Identity-introduction</u> (=I).

For any term t, =I authorizes the introduction of t = t at any stage in a proof, resting on no assumptions.

<u>Identity-elimination</u> (=E).

Let t and s be terms, and A(t) a wff containing occurrences of t; let A(s) be the result of replacing at least one occurrence, but not necessarily all, of t in A(t) by s; then, given the premiss t = s and A(t), =E authorizes the derivation of A(s) as conclusion, resting on whatever assumptions the premisses rest.

Appendix I. List of Sequents and Theorems for the Propositional Calculus

1 $P \rightarrow Q,\ P \vdash Q$

2 $-Q \rightarrow (-P \rightarrow Q),\ -Q\ -\ \vdash P \rightarrow Q$

3 $P \rightarrow Q,\ Q \rightarrow R,\ P \vdash R$

4 $P \rightarrow (Q \rightarrow R),\ P \rightarrow Q,\ P \vdash R$

5 $P \rightarrow Q,\ -Q \vdash -P$

6 $P \rightarrow (Q \rightarrow R),\ P,\ -R \vdash -Q$

7 $P \rightarrow -Q,\ Q \vdash -P$

8 $-P \rightarrow Q,\ -Q \vdash P$

9 $P \rightarrow Q \vdash -Q \rightarrow -P$

10 $P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)$

11 $Q \rightarrow R \vdash (-Q \rightarrow -P) \rightarrow (P \rightarrow R)$

1.2.1 (a) $P \rightarrow (P \rightarrow Q),\ P \vdash Q$

(b) $Q \rightarrow (P \rightarrow R),\ -R,\ Q \vdash -P$

(c) $P \rightarrow --Q,\ P \vdash Q$

(d) $--Q \rightarrow P,\ -P \vdash -Q$

(e) $-P \rightarrow -Q,\ Q \vdash P$

(f) $P \rightarrow -Q \vdash Q \rightarrow -P$

(g) $-P \rightarrow Q \vdash -Q \rightarrow P$

(h) $-P \rightarrow -Q \vdash Q \rightarrow P$

(i) $P \rightarrow Q,\ Q \rightarrow R \vdash P \rightarrow R$

(j) $P \rightarrow (Q \rightarrow R) \vdash (P \rightarrow Q) \rightarrow (P \rightarrow R)$

(k) $P \rightarrow (Q \rightarrow (R \rightarrow S)) \vdash R \rightarrow (P \rightarrow (Q \rightarrow S))$

(l) $P \rightarrow Q \vdash (Q \rightarrow R) \rightarrow (P \rightarrow R)$

(m) $P \vdash (P \rightarrow Q) \rightarrow Q$

(n) $P \vdash (-(Q \rightarrow R) \rightarrow -P) \rightarrow (-R \rightarrow -Q)$

12 $P, Q \vdash P \& Q$

13 $(P \& Q) \rightarrow R \vdash P \rightarrow (Q \rightarrow R)$

14 $P \& Q \vdash P$

15 $P \& Q \vdash Q$

16 $P \rightarrow (Q \rightarrow R) \vdash (P \& Q) \rightarrow R$

17 $P \& Q \vdash Q \& P$

18 $Q \rightarrow R \vdash (P \& Q) \rightarrow (P \& R)$

19 $P v Q \vdash Q v P$

20 $Q \rightarrow R \vdash (P v Q) \rightarrow (P v R)$

21 $P v (Q v R) \vdash Q v (P v R)$

22 $P \rightarrow Q,\ P \rightarrow -Q \vdash -P$

23 $P \rightarrow -P \vdash -P$

1.3.1 (a) $P \vdash Q \rightarrow (P \& Q)$

(b) $P \& (Q \& R) \vdash Q \& (P \& R)$

(c) $(P \rightarrow Q) \& (P \rightarrow R) \vdash P \rightarrow (Q \& R)$

(d) $Q \vdash P v Q$

1.3.1 (e) $P\&Q \vdash P\vee Q$

(f) $(P\rightarrow R)\&(Q\rightarrow R) \vdash (P\vee Q)\rightarrow R$

(g) $P\rightarrow Q,\ R\rightarrow S \vdash (P\&R)\rightarrow(Q\&S)$

(h) $P\rightarrow Q,\ R\rightarrow S \vdash (P\vee R)\rightarrow(Q\vee S)$

(i) $P\rightarrow(Q\&R) \vdash (P\rightarrow Q)\&(P\rightarrow R)$

(j) $-P\rightarrow P \vdash P$

24 $P\leftrightarrow Q \vdash Q\leftrightarrow P$

25 $P,\ P\leftrightarrow Q \vdash Q$

26 $P\leftrightarrow Q,\ Q\leftrightarrow R \vdash P\leftrightarrow R$

27 $(P\&Q)\leftrightarrow P \vdash P\rightarrow Q$

28 $P\&(P\leftrightarrow Q) \vdash P\&Q$

29 $P \vdash P$

30 $(P\&Q)\rightarrow R \dashv\vdash P\rightarrow(Q\rightarrow R)$

31 $P\&(P\vee Q) \dashv\vdash P$

32 $P\vee(P\&Q) \dashv\vdash P$

33 $P\vee P \dashv\vdash P$

34 $P,\ -(P\&Q) \vdash -Q$

35 $P\rightarrow Q \dashv\vdash -(P\&-Q)$

36 $P\vee Q \dashv\vdash -(-P\&-Q)$

1.5.1 (a) $P\vee Q \vdash P\vee Q$

(b) $P\&P \dashv\vdash P$

(c) $P\&(Q\vee R) \dashv\vdash (P\&Q)\vee(P\&R)$

(d) $P\vee(Q\&R) \dashv\vdash (P\vee Q)\&(P\vee R)$

(e) $P\&Q \dashv\vdash -(P\rightarrow -Q)$

(f) $-(P\vee Q) \dashv\vdash -P\&-Q$

(g) $-(P\&Q) \dashv\vdash -P\vee -Q$

(h) $P\&Q \dashv\vdash -(-P\vee -Q)$

(i) $P\rightarrow Q \vdash -P\vee Q$

(j) $-P\rightarrow Q \vdash P\vee Q$

37 $\vdash -(P\&-P)$

38 $\vdash P\rightarrow P$

39 $\vdash P\rightarrow --P$

40 $\vdash --P\rightarrow P$

41 $\vdash P\&Q\rightarrow P$

42 $\vdash (P\rightarrow Q)\rightarrow(-Q\rightarrow -P)$

43 $\vdash (P\rightarrow(Q\rightarrow R))\rightarrow((P\rightarrow Q)\rightarrow(P\rightarrow R))$

44 $\vdash P\vee -P$

45 $P \vdash (P\&Q)\vee(P\&-Q)$

46 $P\rightarrow Q \vdash P\&Q\leftrightarrow P$

47 $P\&Q\leftrightarrow P \dashv\vdash P\rightarrow Q$

48 $-P\vee Q \vdash P\rightarrow Q$

49 $P\rightarrow Q \dashv\vdash -P\vee Q$

50 $P \vdash Q\rightarrow P$

51 $-P \vdash P\rightarrow Q$

52 $-P,\ P\vee Q \vdash Q$

53 $-Q,\ P\vee Q \vdash P$

54 $\vdash (P\rightarrow Q)\vee(Q\rightarrow P)$

55 $P\rightarrow Q,\ -Q \vdash -P$

2.2.1 (a) $\vdash (Q\rightarrow R)\rightarrow((P\rightarrow Q)\rightarrow(P\rightarrow R))$

(b) $\vdash P\rightarrow(Q\rightarrow(P\&Q))$

(c) $\vdash (P\rightarrow R)\rightarrow((Q\rightarrow R)\rightarrow(P\vee Q\rightarrow R$

2.2.1 (d) $\vdash (P \rightarrow Q\&-Q) \rightarrow -P$
(e) $\vdash (-P \rightarrow P) \rightarrow P$
.4 (a) $P\&Q \dashv \vdash P\&(P \leftrightarrow Q)$
.5 (a) $\vdash Pv(P \rightarrow Q)$
(b) $\vdash (P \rightarrow Q)v(Q \rightarrow R)$
(c) $\vdash ((P \rightarrow Q) \rightarrow P) \rightarrow P$
(d) $-Q \vdash P \rightarrow (Q \rightarrow R)$
(e) $P, -P \vdash Q$
(f) $PvQ \dashv \vdash -P \rightarrow Q$
(g) $-(P \rightarrow Q) \dashv \vdash P\&-Q$
(h) $(P \rightarrow Q) \rightarrow Q \dashv \vdash PvQ$
(i) $(P \rightarrow Q)v(P \rightarrow R) \dashv \vdash P \rightarrow QvR$
(j) $P \rightarrow Q \dashv \vdash (P \leftrightarrow Q)vQ$
(k) $Q \vdash P\&Q \leftrightarrow P$
(l) $-Q \vdash PvQ \leftrightarrow P$
2.4.1 (a) $P\&Q \vdash P \rightarrow Q$
(b) $-P\&Q \vdash P \rightarrow Q$
(c) $-P\&-Q \vdash P \rightarrow Q$
(d) $P\&-Q \vdash -(P\&Q)$
(e) $-P\&Q \vdash -(P\&Q)$
(f) $-P\&-Q \vdash -(P\&Q)$
(g) $P\&-Q \vdash PvQ$
(h) $-P\&Q \vdash PvQ$
(i) $P\&Q \vdash P \leftrightarrow Q$
(j) $P\&-Q \vdash -(P \leftrightarrow Q)$
(k) $-P\&Q \vdash -(P \leftrightarrow Q)$
(l) $-P\&-Q \vdash P \leftrightarrow Q$

Appendix II. List of Sequents and Theorems for the First Order Predicate Calculus.

100 $Fm, (\bigwedge\underline{x})(Fx \rightarrow Gx) \vdash Gm$

101 $Fm, (\bigwedge\underline{x})(Fx \rightarrow -Gx) \vdash -Gm$

102 $(\bigwedge\underline{x})(Fx \rightarrow Gx), (\bigwedge\underline{x})(Gx \rightarrow Hx) \vdash (\bigwedge\underline{x})(Fx \rightarrow Hx)$

103 $(\bigwedge\underline{x})(Fx \rightarrow Gx), (\bigwedge\underline{x})Fx \vdash (\bigwedge x)Gx$

3.2.2 (a) $(\bigwedge\underline{x})(Fx \rightarrow Gx), (\bigwedge\underline{x})(Gx \rightarrow -Hx) \vdash (\bigwedge\underline{x})(Fx \rightarrow -Hx)$

(b) $(\bigwedge\underline{x})(Fx \rightarrow -Gx), (\bigwedge\underline{x})(Hx \rightarrow Gx) \vdash (\bigwedge\underline{x})(Fx \rightarrow -Hx)$

(c) $(\bigwedge\underline{x})(Fx \rightarrow Gx), (\bigwedge\underline{x})(Hx \rightarrow -Gx) \vdash (\bigwedge\underline{x})(Fx \rightarrow -Hx)$

(d) $(\bigwedge\underline{x})(Gx \rightarrow -Fx), (\bigwedge\underline{x})(Hx \rightarrow Gx) \vdash (\bigwedge\underline{x})(Fx \rightarrow -Hx)$

(e) $(\bigwedge\underline{x})(Fx \rightarrow Gx) \vdash (\bigwedge\underline{x})Fx \rightarrow (\bigwedge\underline{x})Gx$

(f) $(\bigwedge\underline{x})(Fx \vee Gx \rightarrow Hx), (\bigwedge\underline{x})-Hx \vdash (\bigwedge\underline{x})-Fx$

104 $(\bigwedge\underline{x})Fx \vdash (\bigvee\underline{x})Fx$

105 $(\bigwedge\underline{x})(Fx \rightarrow Gx), (\bigvee\underline{x})Fx \vdash (\bigvee\underline{x})Gx$

106 $(\bigwedge\underline{x})(Gx \rightarrow Hx), (\bigvee\underline{x})(Fx \& Gx) \vdash (\bigvee\underline{x})(Fx \& Hx)$

3.3.1 (a) $(\bigwedge\underline{x})(Fx \rightarrow Gx), (\bigvee\underline{x})-Gx \vdash (\bigvee\underline{x})-Fx$

(b) $(\bigwedge\underline{x})(Fx \rightarrow Gx \& Hx), (\bigvee\underline{x})Fx \vdash (\bigvee\underline{x})Hx$

(c) $(\bigwedge\underline{x})(Fx \vee Gx \rightarrow Hx), (\bigvee\underline{x})-Hx \vdash (\bigvee\underline{x})-Fx$

.2 (a) $(\bigwedge\underline{x})(Gx \rightarrow -Hx), (\bigvee\underline{x})(Fx \& Gx) \vdash (\bigvee\underline{x})(Fx \& -Hx)$

(b) $(\bigwedge\underline{x})(Hx \rightarrow Gx), (\bigvee\underline{x})(Fx \& -Gx) \vdash (\bigvee\underline{x})(Fx \& -Hx)$

(c) $(\bigwedge\underline{x})(Hx \rightarrow -Gx), (\bigvee\underline{x})(Fx \& Gx) \vdash (\bigvee\underline{x})(Fx \& -Hx)$

(d) $(\bigwedge\underline{x})(Gx \rightarrow Hx), (\bigvee\underline{x})(Gx \& Fx) \vdash (\bigvee\underline{x})(Fx \& Hx)$

(e) $(\bigvee\underline{x})(Gx \& Hx), (\bigwedge\underline{x})(Gx \rightarrow Fx) \vdash (\bigvee\underline{x})(Fx \& Hx)$

(f) $(\bigwedge\underline{x})(Gx \rightarrow -Hx), (\bigvee\underline{x})(Gx \& Fx) \vdash (\bigvee\underline{x})(Fx \& -Hx)$

3.3.2 (g) $(\underline{\bigvee x})(Gx \,\&\, -Hx),\ (\underline{\bigwedge x})(Gx \rightarrow Fx) \vdash (\underline{\bigvee x})(Fx \,\&\, -Hx)$

107 $(\underline{\bigwedge x})(Fx \rightarrow Gx) \vdash (\underline{\bigwedge x})Fx \rightarrow (\underline{\bigwedge x})Gx$

108 $(\underline{\bigwedge x})(Fx \rightarrow Gx) \vdash (\underline{\bigvee x})Fx \rightarrow (\underline{\bigvee x})Gx$

109 $(\underline{\bigwedge x})(Fx \,\&\, Gx) \dashv\vdash (\underline{\bigwedge x})Fx \,\&\, (\underline{\bigwedge x})Gx$

110 $(\underline{\bigvee x})(Fx \vee Gx) \dashv\vdash (\underline{\bigvee x})Fx \vee (\underline{\bigvee x})Gx$

111 $(\underline{\bigvee x})(Fx \,\&\, Gx) \vdash (\underline{\bigvee x})Fx \,\&\, (\underline{\bigvee x})Gx$

112 $(\underline{\bigwedge x})Fx \vee (\underline{\bigwedge x})Gx \vdash (\underline{\bigwedge x})(Fx \vee Gx)$

113 $(\underline{\bigvee x})Fx \dashv\vdash -(\underline{\bigwedge x})-Fx$

114 $(\underline{\bigwedge x})Fx \dashv\vdash -(\underline{\bigvee x})-Fx$

115 $(\underline{\bigwedge x})Fx \dashv\vdash (\underline{\bigwedge y})Fy$

116 $(\underline{\bigvee x})Fx \dashv\vdash (\underline{\bigvee y})Fy$

117 $(\underline{\bigwedge x})(Fx \rightarrow Gx) \dashv\vdash -(\underline{\bigvee x})(Fx \,\&\, -Gx)$

118 $(\underline{\bigwedge x})(Fx \rightarrow P) \dashv\vdash (\underline{\bigvee x})Fx \rightarrow P$

119 $(\underline{\bigvee x})(P \rightarrow Fx) \dashv\vdash P \rightarrow (\underline{\bigvee x})Fx$

3.4.1 (a) $(\underline{\bigwedge x})(Fx \rightarrow Gx) \vdash (\underline{\bigwedge x})-Gx \rightarrow (\underline{\bigwedge x})-Fx$

(b) $(\underline{\bigwedge x})(Fx \rightarrow Gx) \vdash (\underline{\bigvee x})-Gx \rightarrow (\underline{\bigvee x})-Fx$

(c) $(\underline{\bigvee x})-Fx \dashv\vdash -(\underline{\bigwedge x})Fx$

(d) $(\underline{\bigwedge x})-Fx \dashv\vdash -(\underline{\bigvee x})Fx$

(e) $(\underline{\bigwedge x})(Fx \rightarrow -Gx) \dashv\vdash -(\underline{\bigvee x})(Fx \,\&\, Gx)$

(f) $(\underline{\bigwedge x})(Fx \leftrightarrow Gx) \dashv\vdash (\underline{\bigwedge x})(Fx \rightarrow Gx) \,\&\, (\underline{\bigwedge x})(Gx \rightarrow Fx)$

(g) $(\underline{\bigwedge x})(Fx \leftrightarrow Gx) \vdash (\underline{\bigwedge x})Fx \leftrightarrow (\underline{\bigwedge x})Gx$

(h) $(\underline{\bigwedge x})(Fx \leftrightarrow Gx) \vdash (\underline{\bigvee x})Fx \leftrightarrow (\underline{\bigvee x})Gx$

.3 (a) $(\underline{\bigwedge x})(P \rightarrow Fx) \dashv\vdash P \rightarrow (\underline{\bigwedge x})Fx$

(b) $(\underline{\bigwedge x})(P \,\&\, Fx) \dashv\vdash P \,\&\, (\underline{\bigwedge x})Fx$

(c) $(\underline{\bigvee x})(P \,\&\, Fx) \dashv\vdash P \,\&\, (\underline{\bigvee x})Fx$

3.4.3 (d) $(\underline{\Lambda}x)(P\vee Fx) \dashv\vdash P\vee(\underline{\Lambda}x)Fx$

(e) $(\underline{V}x)(P\vee Fx) \dashv\vdash P\vee(\underline{V}x)Fx$

(f) $(\underline{V}x)(Fx\rightarrow P) \dashv\vdash (\underline{\Lambda}x)Fx\rightarrow P$

120 $(\underline{\Lambda}x)(\underline{\Lambda}y)Fxy \dashv\vdash (\underline{\Lambda}y)(\underline{\Lambda}x)Fxy$

121 $(\underline{V}x)(\underline{V}y)Fxy \dashv\vdash (\underline{V}y)(\underline{V}x)Fxy$

122 $(\underline{V}x)(\underline{\Lambda}y)Fxy \vdash (\underline{\Lambda}y)(\underline{V}x)Fxy$

123 $(\underline{\Lambda}x)(Fx\rightarrow Gx) \vdash (\underline{\Lambda}x)[(\underline{V}y)(Fy\&Hxy)\rightarrow(\underline{V}y)(Gy\&Hxy)]$

124 $(\underline{V}x)(Fx\&(\underline{\Lambda}y)(Gy\rightarrow Hxy)),\ (\underline{\Lambda}x)(Fx\rightarrow(\underline{\Lambda}y)(By\rightarrow -Hxy)) \vdash (\underline{\Lambda}x)(Gx\rightarrow -Bx)$

125 $(\underline{V}x)(Fx\&Gx),\ (\underline{V}x)(Fx\&(\underline{\Lambda}y)(Gy\rightarrow -Hxy)) \vdash (\underline{V}x)(Fx\&-(\underline{\Lambda}y)(Fy\rightarrow Hyx))$

126 $(\underline{\Lambda}x)(\underline{\Lambda}y)(Fxy\rightarrow(\underline{V}z)Izxy),\ (\underline{\Lambda}x)(\underline{\Lambda}y)(\underline{\Lambda}z)(Izxy\rightarrow Gzx\&Gzy),\ (\underline{\Lambda}x)Fxm \vdash$
$(\underline{\Lambda}x)(\underline{V}y)(Iyxm\&Gym)$

3.5.1 (a) $(\underline{\Lambda}x)(\underline{\Lambda}y)(\underline{\Lambda}z)Fxyz \vdash (\underline{\Lambda}z)(\underline{\Lambda}y)(\underline{\Lambda}x)Fxyz$

(b) $(\underline{\Lambda}z)(\underline{V}y)(\underline{\Lambda}z)Fxyz \vdash (\underline{\Lambda}x)(\underline{\Lambda}z)(\underline{V}y)Fxyz$

(c) $(\underline{V}x)(\underline{V}y)(\underline{\Lambda}z) \vdash (\underline{\Lambda}z)(\underline{V}y)(\underline{V}x)Fxyz$

127 $\vdash (\underline{\Lambda}x)(Fx\rightarrow Gx)\rightarrow((\underline{\Lambda}x)Fx\rightarrow(\underline{\Lambda}x)Gx)$

128 $\vdash (\underline{\Lambda}x)(Fx\rightarrow Gx)\rightarrow((\underline{V}x)Fx\rightarrow(\underline{V}x)Gx)$

129 $\vdash (\underline{\Lambda}x)-(Fx\&-Fx)$

130 $\vdash (\underline{\Lambda}x)(Fx\rightarrow Fx)$

131 $\vdash (\underline{\Lambda}x)(Fx\vee -Fx)$

132 $(\underline{\Lambda}x)Fx \vdash (\underline{\Lambda}x)(Gx\rightarrow Fx)$

133 $(\underline{\Lambda}x)-Fx \vdash (\underline{\Lambda}x)(Fx\rightarrow Gx)$

134 $(\underline{V}x)Fx\rightarrow(\underline{V}x)Gx \vdash (\underline{V}x)(Fx\rightarrow Gx)$

4.2.2 (a) $(\underline{\Lambda}x)Fx\&(\underline{V}x)Gx \vdash (\underline{V}x)(Fx\&Gx)$

(b) $(\underline{\Lambda}x)Fx\vee(\underline{V}x)Gx \vdash (\underline{V}x)(Fx\vee Gx)$

(c) $(\underline{V}x)Fx\rightarrow(\underline{\Lambda}x)Gx \vdash (\underline{\Lambda}x)(Fx\rightarrow Gx)$

4.2.2 (d) $(\underline{\Lambda}x)(Fx \vee Gx) \vdash (\underline{\Lambda}x)Fx \vee (\underline{V}x)Gx$

(e) $(\underline{\Lambda}x)Fx \rightarrow (\underline{\Lambda}x)Gx \vdash (\underline{V}x)(Fx \rightarrow Gx)$

(f) $(\underline{V}x)(Fx \rightarrow Gx) \dashv\vdash (\underline{\Lambda}x)Fx \rightarrow (\underline{V}x)Gx$

.3 (a) $\vdash (\underline{\Lambda}y)((\underline{\Lambda}z)Fx \rightarrow Fy)$

(b) $\vdash (\underline{\Lambda}y)(Fy \rightarrow (\underline{V}x)Fx)$

(c) $\vdash (\underline{V}y)(Fy \rightarrow (\underline{\Lambda}x)Fx)$

(d) $\vdash (\underline{V}y)((\underline{V}x)Fx \rightarrow Fy)$

135 $a=b \vdash b=a$

136 $a=b \;\&\; b=c \vdash a=c$

137 $Fa \dashv\vdash (\underline{V}x)(x=a \;\&\; Fx)$

138 $\vdash (\underline{\Lambda}x)(x=x)$

139 $\vdash (\underline{\Lambda}x)(\underline{\Lambda}y)(x=y \rightarrow y=x)$

140 $\vdash (\underline{\Lambda}x)(\underline{\Lambda}y)(\underline{\Lambda}z)(x=y \;\&\; y=z \rightarrow x=z)$

141 $(\underline{\Lambda}x)(Kx \rightarrow x=m \vee x=n),\ (\underline{V}x)(Kx \& Sx) \vdash Sm \vee Sn$

142 $(\underline{V}x)Fx \vdash (\underline{V}x)(\underline{V}y)(Fx \& Fy)$

4.3.1 (a) $Fa \dashv\vdash (\underline{\Lambda}x)(x=a \rightarrow Fx)$

(b) $\vdash (\underline{\Lambda}x)(\underline{\Lambda}y)(Fx \;\&\; x=y \rightarrow Fy)$

(c) $b=a,\ c=a \vdash b=c$

(d) $a=b \vdash Fa \leftrightarrow Fb$

(e) $a=b \vdash c=a \leftrightarrow c=b$

(f) $\vdash (\underline{V}x)(x=a)$

143 $(\underline{V}x)(Fx \& Gx) \vee (\underline{V}x)(Fx \& -Gx) \dashv\vdash (\underline{V}x)Fx$

144 $\vdash -((\underline{\Lambda}x)(Fx \rightarrow Gx) \leftrightarrow (\underline{V}x)(Fx \& -Gx))$

145 $\vdash -((\underline{\Lambda}x)(Fx \rightarrow -Gx) \leftrightarrow (\underline{V}x)(Fx \& Gx))$

146 $(\underline{V}x)Fx \vdash -((\underline{\Lambda}x)(Fx \rightarrow Gx) \& (\underline{\Lambda}x)(Fx \rightarrow -Gx))$

147 $(\underline{V}x)Fx \vdash (\underline{\Lambda}x)(Fx \rightarrow Gx) \rightarrow (\underline{V}x)(Fx \& Gx)$

148 $(\underline{\bigvee}x)Fx \vdash (\underline{\bigwedge}x)(Fx \rightarrow -Gx) \rightarrow (\underline{\bigvee}x)(Fx \& -Gx)$
149 $\vdash -((\underline{\bigwedge}x)Fx \leftrightarrow (\bigvee x)-Fx)$
150 $\vdash -((\overline{\bigwedge}x)-Fx \leftrightarrow (\bigvee x)Fx)$
151 $\vdash -((\overline{\bigwedge}x)Fx \& (\bigwedge x)-Fx)$
152 $\vdash (\underline{\bigvee}x)Fx \vee (\underline{\bigvee}x)-Fx$
153 $\vdash (\overline{\bigwedge}x)Fx \rightarrow (\underline{\bigvee}x)Fx$
154 $\vdash (\overline{\bigwedge}x)-Fx \rightarrow (\underline{\bigvee}x)-Fx$
155 $(\underline{\bigvee}x)Hx,\ (\underline{\bigwedge}x)(Hx \rightarrow Gx),\ (\bigwedge x)(Gx \rightarrow Fx) \vdash (\underline{\bigvee}x)(Fx \& Hx)$
156 $-(\underline{\bigvee}x)(\underline{\bigvee}y)Rxy \vdash (\overline{\bigwedge}x)(\bigwedge y)(Rxy \rightarrow Ryx)$
157 $-(\overline{\bigvee}x)(\overline{\bigvee}y)Rxy \vdash (\overline{\bigwedge}x)(\bigwedge y)(Rxy \rightarrow -Ryx)$
158 $(\underline{\bigvee}x)(\underline{\bigvee}y)Rxy \vdash -((\underline{\bigwedge}x)(\underline{\bigwedge}y)(Rxy \rightarrow Ryx) \& (\underline{\bigwedge}x)(\underline{\bigwedge}y)(Rxy \rightarrow -Ryx)$
159 $(\overline{\bigwedge}x)(\overline{\bigwedge}y)(Rxy \rightarrow -Ryx) \vdash (\underline{\bigwedge}x)(\underline{\bigwedge}y)(x \neq y\ \&\ Rxy \rightarrow -Ryx)$
160 $-(\underline{\bigvee}x)(\underline{\bigvee}y)Rxy \vdash (\underline{\bigwedge}x)-Rxx$
161 $-(\overline{\bigvee}x)(\overline{\bigvee}y)Rxy \vdash -(\underline{\bigwedge}x)Rxx$
162 $-(\overline{\bigvee}x)(\overline{\bigvee}y)Rxy \vdash -((\underline{\bigvee}x)Rxx \& (\underline{\bigvee}x)-Rxx)$
163 $(\bigwedge x)(\underline{\bigwedge}y)(Rxy \rightarrow -Ryx) \vdash (\underline{\bigwedge}x)-Rxx$
164 $(\overline{\bigwedge}x)-Rxx,\ (\bigwedge x)(\underline{\bigwedge}y)(\underline{\bigwedge}z)(Rxy \& Ryz \rightarrow Rxz) \vdash (\underline{\bigwedge}x)(\underline{\bigwedge}y)(Rxy \rightarrow -Ryx)$
165 $(\overline{\bigwedge}x)(\underline{\bigwedge}y)(\underline{\bigwedge}z)(Rxy \& Ryz) \vdash (\underline{\bigwedge}x)-Rxx$

Appendix III. Suggestions for Classroom Use.

The students are first familiarized with the notation, the formation rules, and the general rules for the games of VALIDITY. Players are then grouped in threes, with no more than three permitted in a group. An "odd" player, when added to a group of three, forces that group to divide into two groups, each with two members. In this way, all the students will be grouped in threes, with perhaps one or two groups of two.

An initial set of "run off" games is played by all players until each group has a winner, a loser, and, if there are three players, a "middle-man". The winners of all the different playing groups are now considered to be members of the "3" team, the losers, members of the "1" team, and those who neither won nor lost, members of the "2" team. From this point on, similarly organized groups of players - in twos and threes - compete with members of their own teams. A "3" team player competes against other "3" team players, etc.

A player who <u>wins</u> at the "2" or "1" level is promoted to membership in the next higher team, and must play other players at that level. A player who <u>loses</u> at the "3" or "2" level is demoted to membership in the next lower team. Middle-men remain at their given team level until they win or lose. Individual players of a game ending in a draw remain at that level, but must, after two consecutive draws, "relocate" in other playing groups containing different players. This serves to break up groups of players which have become excessively stable to the point that there is little element of challenge. Winners at the "3" team level and losers at the "1" team level remain at those levels until they lose or win, respectively.

It is the responsibility of the individual members of a playing group to report the outcome of each game to the instructor. The average frequency of a player's membership in the three teams determines a corresponding game playing grade - an "A", "B", "C", "D", or "F" - which comprises some fraction of the student's final course grade.

The instructor may find it useful to require that a certain minimal number of games be played by all playing groups during a single class hour. This will automatically insure a reasonably fluidity of game play, and will provide the same computational basis for the game playing grade of all students.

I am grateful for certain suggestions made by Robert W. Allen in his <u>Teacher's Guide to EQUATIONS</u> (WFF 'N PROOF Publishers 1965).

[When the following three pages are printed they will produce all of the playing squares needed for *VALIDITY*. If color-backed paper is available, print the page with the heading "NECESSARY" on red-backed paper (or use red-colored type); the page with the heading "POSSIBLE" on light-blue-backed paper (or use type in this color); and the page with the heading "REJECTED" on dark-blue-backed paper (or use type in this color).

The three headings—NECESSARY, POSSIBLE, and REJECTED—that are included at the top of each of these pages should be cut out so they can be placed on the surface of a table where the game is to be played.]

Necessary

F F F F F F F F F F

F F G G G G G G G G

G G G G H H H H H H

H H H H H H = = = =

= = = = = = = = V V

V V V V V V V V V V

Λ Λ Λ Λ Λ Λ Λ Λ Λ Λ

Λ Λ

Possible

→	→	→	→	→	→	→	→	→	→
→	→	↔	↔	↔	↔	↔	↔	↔	↔
↔	↔	↔	↔	–	–	–	–	–	–
–	–	–	–	–	–	v	v	v	v
v	v	v	v	v	v	v	v	&	&
&	&	&	&	&	&	&	&	&	&

Rejected

i i i i i i i i i i

i i e e e e e e e e

e e e e w w w w w w

w w w w w w x̱ x̱ x̱ x̱

x̱ x̱ x̱ x̱ x̱ x̱ x̱ x̱ y y

y y y y y y y y y y

z z z z z z z z z z

z z