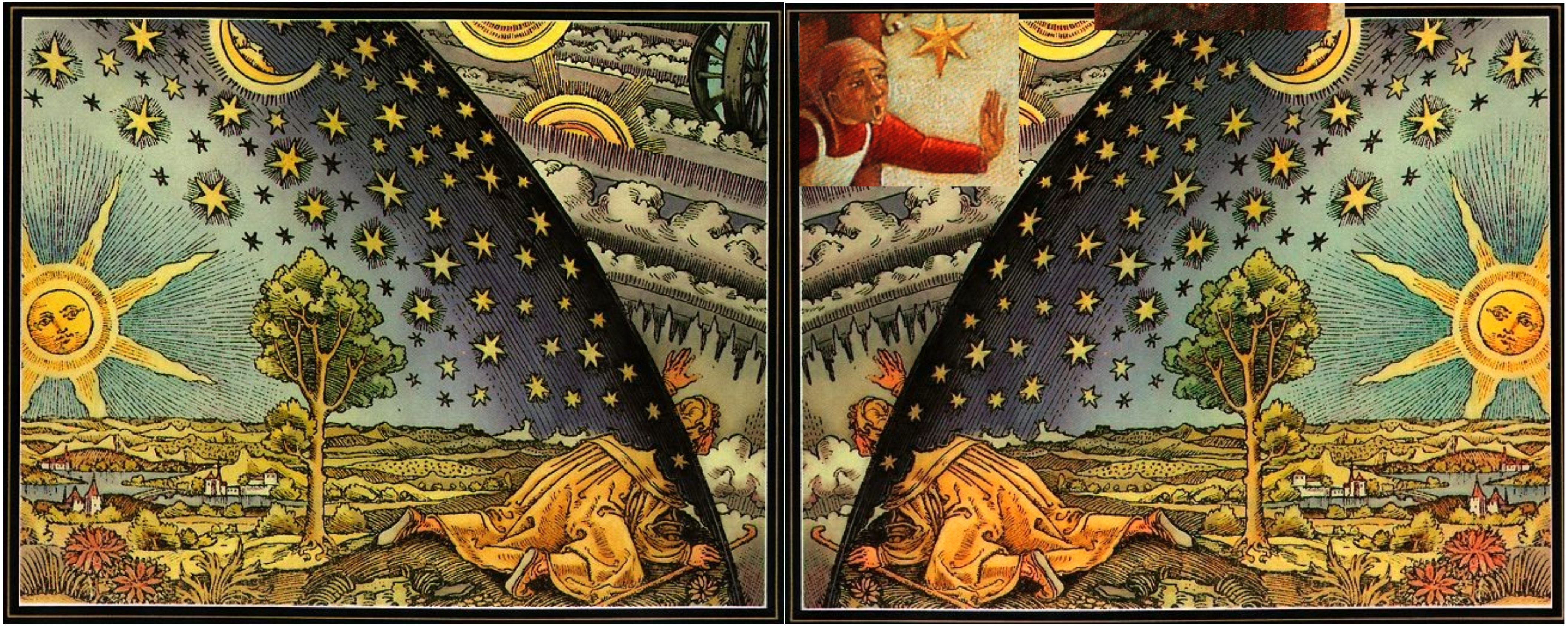

From Traditional Set Theory – that of Cantor, Hilbert, Gödel, Cohen – to Its Necessary Quantum Extension

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Plan of the talk

- Introduction.
- Review of traditional set theory.
- Summing up the traditional set theory's results.
- Starting anew.
- Philosophical context.
- New foundations of set theory.
- References.

Introduction: the main points of the talk

- The central theme of the talk.
- A brief, suggestive review of traditional set theory and its failure to find a proper place for the Continuum.
- Formalism for reviewing the veracity of traditional set theory.
- Metaphysical considerations, old and new.
- The role of "experimental physics" in processes of definition, understanding, study, and exploitation of the Continuum.
- What next ? Enters quantum mechanics of light.
- The original purpose of the present study, started with a preprint «On the Probable Failure of the Uncountable Power Set Axiom», 1988, is to save from the transfinite deadlock of higher set theory the jewel of mathematical Continuum — this genuine, even if mostly forgotten today raison d'être of all traditional set-theoretical enterprises to Infinity and beyond, from Georg Cantor to David Hilbert to Kurt Gödel to W. Hugh Woodin to Buzz Lightyear.

History of traditional set theory

- Georg Cantor's set theory and its metaphysical ambition, the Continuum Hypothesis, CH: $2^{\aleph_0} = \aleph_1$ (1874-1877).
- David Hilbert's public recognition of the importance of Cantorian set theory: 1900.
- Bertrand Russel's paradox and the crisis of mathematics: 1901.
- Ernst Zermelo's set-theoretical axioms, ZFC: 1908.
- David Hilbert's program of recovering the consistency of math: 1922.
- Gerhard Gentzen proved the consistency of Peano axioms: 1936.
- Kurt Gödel, 1938: ZFC consistent ➔ there is no proof of $\neg$ CH from ZFC.
- Paul Cohen, 1963: ZFC consistent ➔ there is no proof of CH from ZFC.
- Emergence of large cardinals: from Felix Hausdorff's weakly inaccessible cardinals, 1908, to Stefan Banach's measurable cardinals, 1920, to William N. Reinhard's cardinal, 1967-74 (ZF set theory, not compatible with ZFC).
- W. Hugh Woodin's conjecture: 1999. Every set theory, compatible with the existence of large cardinals and preserving under forcing invariant properties of sets with hereditary cardinality at most $\aleph 1$, implies that the CH be false. Implication of Woodin's conjecture: $2^{\aleph_0} = \aleph_2$ (2010). Gödel had a similar idea: 1947.

Georg Cantor's set theory

Georg Cantor's discoveries:

- The idea of set: elements of a well-known mathematical structure (natural numbers, continuum, etc.), deprived of all their mathematical attributes and connections, are put in a "bag" called set.
- Mappings as the basic relationship between sets.
- The set of real numbers is "bigger" than the set of integers. The diagonal method used in this theorem implies that the set P(A) of all subsets of the set A is "bigger" than A.
- Transfinite numbers as the set-theoretical generalization of counting and the procedures of their infinite extension: power-set axiom, infinity, replacement

Georg Cantor's expectations:

- Ordinal transfinite numbers are called ordinals, cardinal numbers – cardinals:

$$\begin{array}{ccccccccc} 0,1,\ldots, & \omega, & \omega+1,\ldots,\omega+\omega,\ldots, & \omega_1, & \ldots, & \omega_2, & \ldots \\ & \| & & \| & & \| & \\ & \aleph_0 & & \aleph_1 & & \aleph_2 & \end{array}$$

- The "universe" Ω of transfinite numbers, which is not a set but a "class", covers all sets. In particular, the transfinite "counting" permits to reproduce the Continuum c.
- Where is the place of c in Ω ? Cantor Continuum Hypothesis (1877), CH: $c = \aleph_1$
($\aleph_1$ being the first uncountable cardinal).

Theme: the recovery of the Continuum

- What is so special and mysterious about the Continuum, this ancient, always topical, and alongside the concept of integers, most intuitively transparent and omnipresent conceptual and formal medium for mathematical constructions ?
- And why it resists the more than century long siege by best mathematical minds of all times committed to penetrate once and for all its (traditional) set-theoretical enigma ?
- The answer comes from the quantum reality, especially from its *quantum computing* experience and formalisms.
- The original purpose of the present study, started with a preprint «On the Probable Failure of the Uncountable Power Set Axiom», 1988, is to save from the transfinite deadlock of higher set theory the jewel of mathematical Continuum – this genuine, even if mostly forgotten today *raison d'être* of all traditional set-theoretical enterprises to Infinity and beyond, from Georg Cantor to David Hilbert to Kurt Gödel to W. Hugh Woodin to Buzz Lightyear.

David Hilbert's public recognition of the set theory: 1900

- David Hilbert formulated Cantor's Continuum Hypothesis, CH, at the International Congress of Mathematicians at Paris, in 1900: «Every system of infinitely many real numbers, i.e., every assemblage of numbers, or points, is either equivalent to the assemblage of natural integers, 1, 2, 3, ..., or to the assemblage of all real numbers and therefore to the Continuum, that is, to the points of a line; as regards equivalence there are, therefore, only two assemblages of numbers, the countable assemblage and the continuum.»
- David Hilbert's enthusiasm transformed the destiny of Cantorian set theory. He demanded Ernst Zermelo to construct its axiomatic: «This appears to me to be the most admirable flower of the mathematical intellect and in general one of the highest achievements of pure rational human activity. ... No one shall be able to drive us from the paradise that Cantor created for us.» David Hilbert 1925

Ernst Zermelo's set-theoretical axioms, ZFC: 1904-8

AXIOM I. Axiom of extensionality. Every set is determined by its elements.

AXIOM II. Axiom of elementary sets. There exists a set, the null set, ∅, that contains no element at all. If a is any object of the domain, there exists a set {a} containing a and only a as element. If a and b are any two such objects, there exists a set {a, b}.

AXIOM III. Axiom of separation. If the propositional function –(x) is definite for all elements of a set M, M possesses a subset M′ containing precisely those elements x of M for which –(x) is true.

AXIOM IV. Axiom of the power set. To every set T there corresponds a set T' , the power set of T, that contains as elements precisely all subsets of T.

AXIOM V. Axiom of the union. To every set T there corresponds a set ∪T, the union of T, that contains as elements precisely all elements of the elements of T.

AXIOM VI. Axiom of choice. If T is a set whose elements all are sets that are different from ∅ and mutually disjoint, its union ∪T includes at least one subset S having one and only one element in common with each element of T.

AXIOM VII. Axiom of infinity. There exists in the domain at least one set Z that contains the null set as an element and is so constituted that to each of its elements a there corresponds a further element of the form {a}, in other words, that with each of its elements a it also contains the corresponding set {a} as element".

… Ernst Zermelo’s set-theoretical axioms, ZFC: 1904-8

- Compared, say, to Euclidian axiomatic, Zermelo’s one and the soon to follow, large cardinals definitions postulate “actions of somebody” infinitely more powerful than human beings. This was recognized by all participants:
- «Let us say that the assertion of a large cardinal property is a strong axiom of infinity. The adaptation of strong axioms of infinity is thus a theological venture, involving basic questions of belief concerning what is true about the universe.» (Akihiro Kanamori, Menachem Magidor 1978)
- «Our point of view is to describe the mathematical operations that can be carried out by finite beings, man's mathematics for short. In contrast, classical mathematics concerns itself with operations that can be carried out by God.» (Errett Bishop 1985)
- «I came to the conclusion some years ago that CH is an inherently vague problem. This was based partly on the results from the meta-theory of set theory showing that CH is independent of all remotely plausible axioms extending ZFC, including all large cardinal axioms that have been proposed so far. In fact it is consistent with all such axioms (if they are consistent at all) that the cardinal number of the continuum can be "anything it ought to be", i.e. anything which is not excluded by König's theorem. The other basis for my view is philosophical: I believe there is no independent platonic reality that gives determinate meaning to the language of set theory in general, and to the supposed totality of arbitrary subsets of the natural numbers in particular, and hence not to its cardinal number.» (Solomon Feferman 2007)

David Hilbert and the foundations of math: 1922

- Facing Russel’s paradox and the war on foundations, Hilbert's program had the goal to provide secure finitistic foundations for all mathematics: (1) A formalization of all mathematics. (2) Completeness. (3) Consistency: a “finitistic” proof that no contradiction can be obtained in the formalism of mathematics. (4) Conservation: a proof that any result about "real objects" obtained using reasoning about "ideal objects”, such as uncountable sets, can be proved without using ideal objects. (5) Decidability: there should be an algorithm for deciding the truth or falsity of any mathematical statement.
- **Ten years later Hilbert’s program has been considerably weakened:**
  With his incompleteness theorem, Kurt Gödel showed (1931) that most of the goals of Hilbert's program were impossible to achieve, already for Peano arithmetic: (1) It is not possible to formalize all of mathematics, as any attempt at such a formalism will omit some true mathematical statements. (2) There is no complete consistent extension of even Peano arithmetic with a recursively enumerable set of axioms. (3) A theory such as Peano arithmetic cannot even prove its own consistency, so a restricted "finitistic" subset of it certainly cannot prove the consistency of more powerful theories such as set theory. (4) There is no algorithm to decide the truth or provability of statements in any consistent extension of Peano arithmetic.

Gerhard Gentzen and the Peano arithmetic: 1936

- Gerhard Gentzen's pioneering work (1936) has laid down the foundations of the method of *gauging the recursive proof-theoretical strength of finite mathematical theories and combinatorial problems by ordinals*, starting with Peano arithmetics. With all uncontroversial cases involving only countable ordinals: from Cantor's ordinal ε_0,

$$(1, 2, 3, \ldots \omega, \omega + 1, \ldots \omega \cdot 2, \ldots, \omega^{\omega}, \ldots \epsilon_0 = \omega^{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}})$$

  to the first non-predicative Feferman-Schutte ordinal Γ_0 (1965).
- In a similar vein, Reuben L. Goldstein (1944) has constructed a truly elementary function whose arithmetic structure mimics Cantor's transfinite hierarchy up to ε_0, and whose iterates ultimately terminate at 0 for any n. However, with n growing, it takes them to arrive at 0 so long indeed, that any proof of this fact necessary uses a mathematical induction through transfinite numbers up to to ε_0.

- Then an original general interpretation of (explicitly defined denumerable) ordinals as succinct symbolic notations for algorithmic structures with multiple loops has been given by Alan Turing (1949), and his approach has substantially contributed to the development of the modern theory of program verification.
- And Harvey Friedman discovered a remarkably transparent, finitistic version, called FFF, of Kruskal's theorem concerning infinite sequences of finite trees. The proof of FFF demonstrably requires mathematical induction up to the first impredicative denumerable ordinal Γ_0 (1986). The impredicativity of Γ_0 signifies, in particular, that there exists no explicit transfinitely recursive formula for it, similar to those of

$$\varepsilon_0,\ \varepsilon_1,\ \ldots\ \varepsilon_{\varepsilon_0}.$$

Kurt Gödel returns to the CH: 1938

- After refuting Hilbert's idea of mathematics as pure syntax, Kurt Gödel turned his attention to the Continuum Conjecture, CH, the reality and truth verification point of the transfinite brunch of traditional set theory:
  Gödel proved that, provided ZFC is not contradictory, there is no proof of ¬CH from ZFC.

- From this moment on, Gödel has decided to stay with Cantorian set theory. Here is one of his explanation in what he believes after Paul Cohen's 1963 discovery that CH is as independent of ZFC as ¬CH:

«But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true.» (Kurt Gödel 1964)

Paul Cohen: There is no proof of CH from ZFC

- In 1963, Cohen proved that, similarly to Gödel's result, ZFC doesn't imply CH. His method of *forcing* became a powerful tool in investigation of set-theoretic axiomatic systems. Here is his appreciation of CH:

«A point of view which the author [Cohen] feels may eventually come to be accepted is that CH is obviously false. The main reason one accepts the Axiom of Infinity is probably that we feel it absurd to think that the process of adding only one set at a time can exhaust the entire universe. Similarly with the higher axioms of infinity. Now aleph1 is the set of countable ordinals and this is merely a special and the simplest way of generating a higher cardinal. The set C is, in contrast, generated by a totally new and more powerful principle, namely the Power Set Axiom. It is unreasonable to expect that any description of a larger cardinal which attempts to build up that cardinal from ideas deriving from the Replacement Axiom can ever reach C. Thus C is greater than aleph-n, aleph-w, aleph-a, where a=aleph-w, etc. This point of view regards C as an incredibly rich set given to us by one bold new axiom, which can never be approached by any piecemeal process of construction. Perhaps later generations will see the problem more clearly and express themselves more eloquently.» Paul Cohen 1966

Large cardinal axioms

- «Large cardinal axioms are generalizations of the axioms of extent of ZFC—Infinity and Replacement—in that they assert that there are large levels of the universe of sets. Examples of such axioms are those asserting the existence of strongly inaccessible cardinals, Woodin cardinals, and supercompact cardinals. Axioms of definable determinacy can also be seen as generalizations of a principle inherent in ZFC, namely, Borel determinacy, which was shown to be a theorem of ZFC by Martin. Examples of such axioms are PD (the statement that all projective sets are determined) and ADL(R) (the statement that all sets of reals in L(R) are determined). Large cardinal axioms and axioms of definable determinacy are intrinsically plausible but the strongest case for their justification comes through their fruitful consequences, their connections with each other, and an intricate network of theorems relating them to other axioms.» Peter Koellner and W. Hugh Wooding (2009)

W. Hugh Woodin’s conjecture: 1999

In his papers and books, from 1999 to 2010, Woodin developed the methods and proved theorems, approaching, as he believes, an eventual solution of the Continuum Hypothesis:

«There is a tendency to claim that the Continuum Hypothesis is inherently vague and that this is simply the end of the story. But any legitimate claim that CH is inherently vague must have a mathematical basis, at the very least a theorem or a collection of theorems. My own view is that the independence of CH from ZFC, and from ZFC together with large cardinal axioms, does not provide this basis. I would hope this is the minimum meta-mathematical assessment of the solution to CH that I have presented. Indeed, for me , the independence results for CH simply show that CH is a difficult problem.» W. Hugh Woodin 2001.

Summing up

- Traditional set theory had an enormous and uniquely inspirational impact on the development of mathematics, mathematical logic, proof theory, computer science, etc. However, it didn't and, today, doesn't succeed to unite or even to fruitfully relate its transfinite counting procedure (important by itself as interesting mathematics, but also in proof theory, computer science, etc.) with traditional and new mathematics on, and related to the Continuum.
- However, it is even more and really disheartening that the different mathematical schools prefer to ignore the challenge of such a fundamental failure, with the traditional set-theoretical schools slowly losing their erstwhile importance (Clay Mathematics Institute Millennium Prizes forgot about CH) and with the "outside" mathematics just (and sometimes happily) ignoring the problem.
- The only serious and systematic plan to face this challenge was that of Harvey Friedman (with his paper "Necessary Uses of Abstract Set Theory in Finite Mathematics", 1986) and, later, Stephen G. Simpson, with their "Reverse Mathematics and Degrees of Unsolvability" program (1986-2005). Unfortunately, this didn't alter the general picture.

Starting anew

- New spiritual, intellectual, and social context:
  «Some years ago word reached me concerning your proficiency, of which everybody constantly spoke. At that time I began to have a very high regard for you... For I had learned that you had not merely mastered the discoveries of the ancient astronomers uncommonly well but had also formulated a new cosmology. In it you maintain that the earth moves; that the sun occupies the lowest, and thus the central, place in the universe... Therefore with the utmost earnestness I entreat you, most learned sir, unless I inconvenience you, to communicate this discovery of yours to scholars, and at the earliest possible moment to send me your writings on the sphere of the universe together with the tables and whatever else you have that is relevant to this subject …» Cardinal Nikolaus von Schönberg, Archbishop of Capua, to Nicolas Copernicus, 1536.
- New working philosophy of world and science:
  Georg Cantor's philosophical and religious motivations were formed in the XIXth century context, with the radical mode of founding scientific disciplines on "humanly comprehensible" measures and methods of "creation", from Pierre—Simon de Laplace's mechanical universe to Charles Darwin's natural selection as the motor of the evolution, to Karl Marx' socio-political theory of Marxism and dialectical materialism whose practical applications are driven by class struggles, to Sigismund Freud's psychological theory and its unique libido motor, etc.:
  «We must not be surprised, therefore, that, so to speak, all physicists of the last [XIXth] century saw in classical mechanics a firm and final foundation for all physics, yes, indeed, for all natural science, and that they never grew tired in their attempts to base Maxwell's theory of electromagnetism, which, in the meantime, was slowly beginning to win out, upon mechanics as well.» Albert Einstein, 1970

Metaphysical context ...

- «A lot of our knowledge derives from sense experience: from what we see, hear, touch, etc. Other things we know — about mathematics and logic, for example — seem quite independent of sense experience: we know them simply by thinking — about, for example, the definition of a triangle or the meaning of the term "implies." Philosophers have long recognized this fact by a distinction between knowledge that is a posterori (derived from sense experience) and knowledge that is a priori (derived from mere thinking, independent of sense experience). This distinction concerns the ways in which we know.» Gary Gutting, 2009.

- «There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivist way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory.» Kurt Gödel 1947

… Metaphysical context …

«Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap. The Jacobi identity (which forces the heights of a triangle to cross at one point) is an experimental fact in the same way as that the Earth is round (that is, homeomorphic to a ball). But it can be discovered with less expense. In the middle of the twentieth century it was attempted to divide physics and mathematics. The consequences turned out to be catastrophic. Whole generations of mathematicians grew up without knowing half of their science and, of course, in total ignorance of any other sciences. They first began teaching their ugly scholastic pseudo-mathematics to their students, then to schoolchildren — forgetting Hardy's warning that ugly mathematics has no permanent place under the Sun.»

Vladimir I. Arnold. Address on teaching of mathematics Palais de Découverte, Paris, 1997.

… Metaphysical context

- Thomas Aquinas sought to make a distinction between philosophy (which at his time was also including the knowledge referred today as science) and theology: «The theologian and the philosopher consider creatures differently. The philosopher considers what belongs to their proper natures, while the theologian considers only what is true of creatures insofar as they are related to God.» (*Summa contra gentiles,* Book II, chap. 4)
- Johannes Kepler, 350 years later, was a scientist theologically motivated and inspired: he believed that the number and trajectories of planets were chosen by God according to His principles. Kepler has tried three of them, known to him: music, perfect three-dimensional bodies, and then, correctly, some properties of the trajectories of planets. Here are the final words of his scientific treaty *Harmonice Mundi*: «If I have been allured into brashness by the wonderful beauty of Thy works, or if I have loved my own glory among men, while advancing in work destined for Thy glory, gently and mercifully pardon me: and finally, deign graciously to cause that these demonstrations may lead to Thy glory and to the salvation of souls, and nowhere be an obstacle to that. Amen»

New foundations of set theory …

- We accept and highly appreciate Zermelo's axiomatic ZFC. For us, it does not concern the Continuum – with its rich theories of transcendent numbers (from Joseph Liouville, 1844, to David Hilbert, 1900, to Alexandr Gelfond, 1934, to Barbara Margolis, 2003), or diophanitine approximations, or analysis, etc. – but it lays ground for the theory of transitive numbers, with beautiful and effective applications in proof theory, program verification, etc.
- The reason of our prudence is that Zermelo's most important axioms are discrete and performative, compared to Euclidian axioms – which are continuously infinite and contemplative. This is why we feel necessary to attach to Zermelo's axiomatic the post-Turingian halting barrier, modeled on the well-known halting problem phenomenon : *There exists neither general metamathematical principle, nor logical criterion, nor verifiably terminating computational procedure to establish the objective and substantial "truth" of a performative set-theoretical axiom of iterative nature which postulates the existence of a transfinite object outside the already existing (say, ZF-based) transfinite scale – otherwise that is than "to run" the theory completed with the new axiom until it would be discovered some independent "necessary uses" of the object in question.*

- Example: «The first uncountable ordinal ω_1. It is the smallest ordinal number that, considered as a set, is uncountable. It is the supremum of all countable ordinals. The elements of ω_1 are the countable ordinals, of which there are uncountably many. Like any ordinal number, ω_1 is a well-ordered set, with set membership ("$\in$") serving as the order relation. ω_1 is a limit ordinal, i.e. there is no ordinal α with $\alpha + 1 = \omega_1$. The cardinality of the set $\omega 1$ is the first uncountable cardinal number, $\aleph_1$ (aleph-one). The ordinal ω_1 is thus the initial ordinal of $\aleph_1$. Indeed, in most constructions ω_1 and $\aleph_1$ are equal as sets. To generalize: if α is an arbitrary ordinal we define $\omega\alpha$ as the initial ordinal of the cardinal $\aleph_\alpha$.» (Wiki)

- *The existence of ω_1 is implied by Zermelo's axioms. The post-Turingian halting barrier guarantees only that ω_1 is a class.*

... New foundations of set theory ...

- Our dissatisfaction with Zermelo's axiomatic in the context of the reality of the Continuum is rooted in the fundamental *sequentiality* of its constructions, the sequentiality which implies the *sequential causality* of all what could be said about transfinites. The advanced principles of transfinite set theory are designed to overcome this sequentiality obstruction, but they cannot eliminate it.
- Ultimately, all modern transfinite set theory is represents only a well designed fantasy founded on Zermelo's axiomatic, the fantasy which pushes to their limits the rich constructionist faculties of this system. All adaptations of these fantasies to even very modest aspects of the Continuum realities remain absolutely unsatisfactory.
- This is because, as we claim, the origins of the Continuum are outside all set-theoretical "explanations".

… New foundations of set theory …

- More generally, the rich theory of *large cardinals and beyond* cannot be assumed on the basis of today's criteria, for example, those of the new monograph of S. Hugh Woodin (2010): «The starting point for this monograph is the previously unknown connection between the Continuum Hypothesis and the saturation of the non-stationary ideal on ω_1; and the principle result of this monograph is the identification of a canonical model in which the Continuum Hypothesis is false.»

- This is because of the following *Post-Gödelian Incompleteness thesis*:
  Any conceptually sufficiently rich and logically/axiomatically sufficiently sophisticated mathematical system allows a huge, super-exponentially expanding "mathematical sci-fi novelization" – the creation of a multitude of "mathematical sci-fi novels", i.e., fully consistent mathematical theories with unlimitedly extending axiomatic bases – "forced themselves upon us", as it were, not only "as being true", according to Kurt Gödel, but being also intellectually compelling and esthetically attractive – and yet which do not have in their (more than) overwhelming majority, either at this juncture or whenever in future, any objectively verifiable mathematical and/or substantial extra-mathematical meaning outside the proper, self-absorbed scene of formal deductions inside the system in question.

… New foundations of set theory …

- Armed with these novel non-transfinite insights and arguments, we shall abandon for good both the framework of the classical set-theoretical reasoning firmly rooted, all protestations to the contrary notwithstanding, in the XIXth century's reductionist paradigm of mechanistic causality and the closely related to it – in fact, implicitly underlying, even if postdating it – classical theory of computation, formalized by Alfonso Church and Alan Turing:
- «Only the first few levels of the cumulative hierarchy bear any resemblance to external reality. The rest are a huge extrapolation based on a crude model of abstract thought processes. Gödel himself comes close to admitting as much.» Stephen G. Simpson, 1988.
- «When modern set theory is applied to conventional mathematical problems, it has a disconcerting tendency to produce independence results rather than theorems in the usual sense. The resulting preoccupation with "consistency" rather than "truth" may be felt to give the subject an air of unreality.» Saharon Shelah, 1992.

The Continuum: analysis of traditional backgrounds.

— In general, the insufficiency of both traditional foundations and motivated by it unnecessary complexity and bizarreness of mathematical theory is a well-known phenomenon in the history of our science. In some cases, such difficulties are resolved immediately or even before they become observed (Albert Einstein's relativity theory). Sometimes, however, the difficulty persists, as in the case of Ptolemy astronomy: the main obstacle to a correct understanding could be rooted in metaphysical, if not religious perceptions.

— Traditional understanding and usage of the Continuum comes from geometry, Euclidian and before.

— The abstractness and logical perfection of Euclidian axioms is coming as the result of many hundred years of observations and occupations with light and land and house measurements.

— In other words, abstract Euclidian geometry and the related perception, definition, and treatment of the Continuum are *physically founded and motivated* abstractions.

The Continuum: new foundations.

— About hundred fifty years before Euclid (300 BC), the Greek philosopher Zeno (450 BC) exposed the the cleavage between the discrete, or Pythagoreanas, as Ronald Jensen has chosen to call it (1995) and, continuous, or Newtonian according to Jensen, accounts of the world: Achilles and Tortoise paradox. Two thousand five hundred years later, the paradox remains actual: Henri Bergson 1959, Belaga 2009.

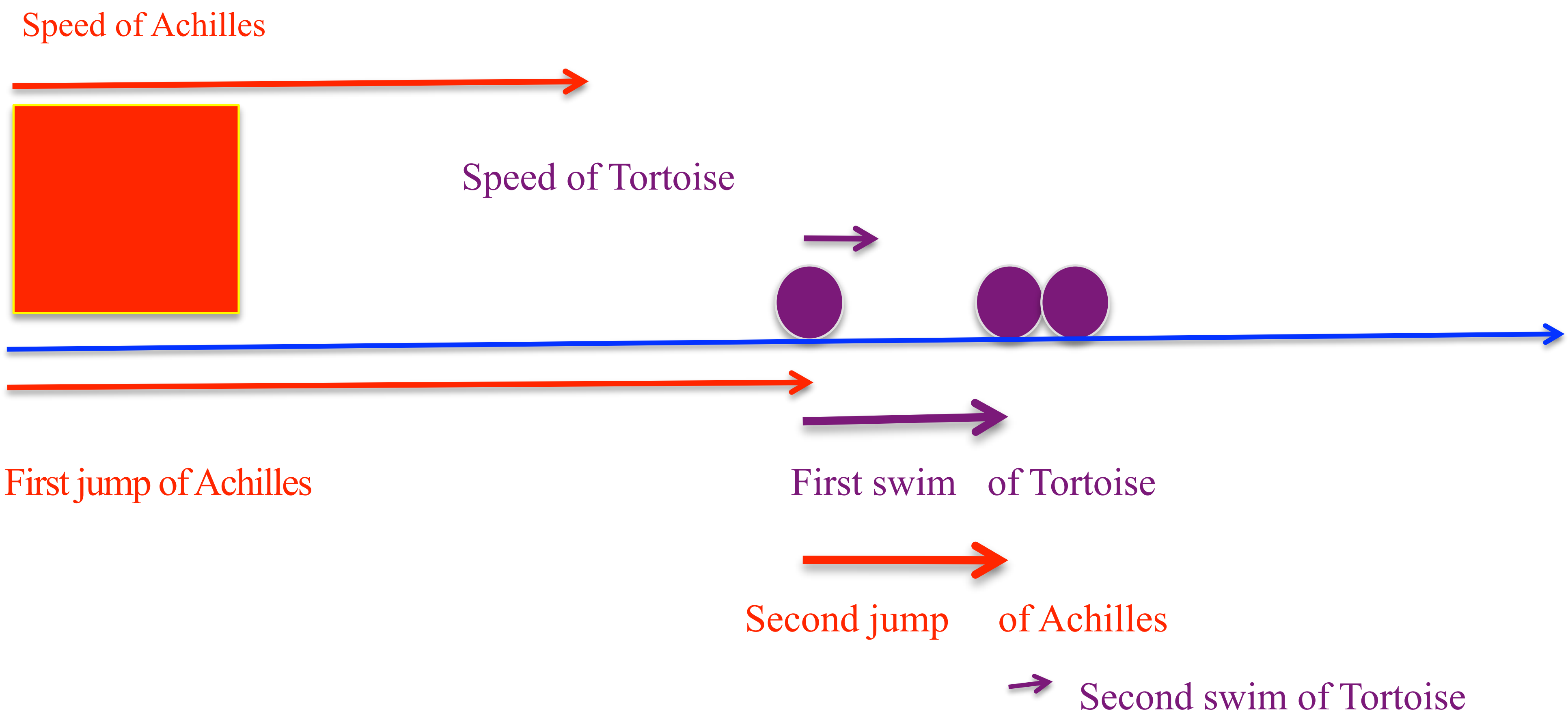
— However, it was only quantum mechanics that opened really new windows of opportunity in a better understanding of the Continuum:

(1) on the one hand, the uncertainty principle of Werner Heisenberg, 1927, could be seen as a sort of physical re-interpretation of Zeno's paradox;

(2) on the other hand, the *entaglement phenomenon* involving light (or electrons), discovered by Albert Einstein, 1935, John Stewart Bell, 1964, and Alain Aspect, 1982, demonstrates that the *Continuum of light* is interiorly somehow "*tightly interlaced"*, so that the distance, even enormous, between its "*entangled*" points becomes unimportant;

(3) this signifies that the Continuum is not a set, a "bag of points", but that the points on it appear as the consequence of *our activities*.

Zeno's Achilles and Tortoise paradox:



… New foundations of set theory

- Outline of the metaphysical foundations of a new axiomatic:
  — Axiomatically, quantum mechanical Hilbertian space H is at least as important as the Continuum. It is, in fact, more fundamental and primary: human beings observe and act in *their Hilbertian spaces*, with the *parameter field*, theoretically, an arbitrary filed K, added in the process.
  — The field of real numbers R epitomizes the Continuum.
  — Thus, the primary notion of the new axiomatic, similar to the notion of *line* in Euclidian geometry, should be not the Continuum but a Hilbertian space, at the beginning — without a precise qualification of the accompanying field.
- The Continuum C (or the field R) appears as a *numerical approximation* to a complex reality of observations.
  — It is not *a set* in the original sense of traditional set theory. In particular, the *power set axiom* cannot be applied to it.
  — More generally, the *power set operation* acts on finite "sets" (this might be formulated as *axiom*). But its application to the countable set ω which gives an approximation to the Continuum, is a *uniquely observable phenomenon*.

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