Retrieving the Mathematical Mission of the Continuum Concept from the Transfinitely Reductionist Debris of Cantor's Paradise
Extended Abstract

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To the memory of my spouse Eugenia.

À mes trois bienfaiteurs québécois:
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Abstract

What is so special and mysterious about the Continuum, this ancient, always topical, and alongside the concept of integers, most intuitively transparent and omnipresent conceptual and formal medium for mathematical constructions and the battle field of mathematical inquiries ? And why it resists the century long siege by best mathematical minds of all times committed to penetrate once and for all its set-theoretical enigma ?

The double-edged purpose of the present study is to save from the transfinite deadlock of higher set theory the jewel of mathematical Continuum – this genuine, even if mostly forgotten today *raison d'être* of all set-theoretical enterprises *to Infinity and beyond*, from Georg Cantor to W. Hugh Woodin to Buzz Lightyear, by simultaneously exhibiting the limits and pitfalls of all old and new reductionist foundational approaches to mathematical truth: be it Cantor's or post-Cantorian Idealism, Brouwer's or post-Brouwerian Constructivism, Hilbert's or post-Hilbertian Formalism, Gödel's or post-Gödelian Platonism.

In the spirit of Zeno's paradoxes, but with the enormous historical advantage of hindsight, we claim that *Cantor's set-theoretical methodology*, powerful and reach in proof-theoretic and similar applications as it might be, is inherently limited by its *epistemological framework of transfinite local causality*, and neither can be held accountable for the properties of the Continuum already acquired through geometrical, analytical, and arithmetical studies, nor can it be used for an adequate, conceptually sensible, operationally workable, and axiomatically sustainable re-creation of the Continuum.

From a strictly mathematical point of view, this *intrinsic limitation of the constative and explicative power of higher set theory* finds its explanation in the identified in this study *ultimate phenomenological obstacle to Cantor's transfinite construction*, similar to *topological obstacles* in homotopy theory and theoretical physics: *the entanglement capacity of the mathematical Continuum.*

Contents

I was beside the Master craftsman, delighting him day after day, ever at play in his presence, at play everywhere on his earth, delighted to be with the children of men.

Proverbs 8:30-31

1 Introduction: overview of the challenges, objectives, findings

The present study is carried out with a rather ambitious, if not unrealistic forethought to eventually meet all major challenges of the controversial, century long inquiry into the mathematical meaning of the Continuum. We leave to the reader the final judgement to what extent, if at all, did we succeed.

The below order of listing of the specific challenges and the corresponding objectives and findings of this study follows the common, mostly *top-down* logic of expository presentations, from most general and abstract considerations to more specific and technical problems.

2 Foundational discernment

We are finalizing here our elaborate and long on preparation discernment process [6]-[12], on the one hand, of the ***ontological and intuitive*** character and particular ***epistemic and technical*** implementations of the Continuum notion in modern «mainstream» mathematics. Here are some conclusions of this meticulous *winnowing of the mathematical grain from some transfinite set-theoretical chaff*:

Challenge 1 . Foundational deadlock. *What is the meaning of, and what might be the lessons to learn from, the unsuccessful and yet systematic and, in the case of higher set theory, both grandiose and intricate, century long* ***foundational efforts*** *to either* ***explain the Continuum away*** *by adopting one or another philosophy of the Infinite, or to* ***reclaim the Continuum from naught*** *by applying one or another mutually incompatible and manifestly* ***ad hoc accretion procedures****: finitist, constructivist, transfinite ?*

Objective 1 . What is Mathematical Truth ? *We feel the need to confront here the fact that all Cantorian and post-Cantorian programs of the transfinite interpretation of the Continuum, as well as their violent rejections by constructivist philosophies of mathematics are sharing paradoxically* *the same, central to every one of these systems article of faith, namely,one or another, mutually incompatible forms of* ***the absolutization of the existence of a specific, independent, and sovereign Mathematical Truth****, as in the following laconic, transparent, and conflicting affirmations by Erret Bishop and Kurt Gödel:*

«*Our point of view is to describe the mathematical operations that can be carried out by finite beings, man's mathematics for short. In contrast, classical mathematics concerns itself with operations that can be carried out by God.*» [20] (p. 9).

«*Only someone who (like the intuitionist) denies that the concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not someone who believes them to describe some well-determined reality.*» [43] (p. 520)

Finding 1 . Separate set-theoretical controversies from the question of soundness of Classical Mathematics. *As the above* dialogue of the deaf *demonstrates, in their staunch defense of Georg Cantor's set theory, David Hilbert and Kurt Gödel have unfortunately and, as we intend to show in this study, mistakenly identified the problem of epistemic mathematical sustainability of Classical Mathematics with that of Cantorian set theory, and they did it by definitively deracinating the former from its rich ontological background.*

3 The Continuum in the light of Platonist and formalist delusions

The famous Platonist thesis from Gödel's philosophical testament of 1964 [44], quoted and contested below, will be followed (Section 5.1) by his two related and very profound arguments, which maintain even today all their visionary importance.

Challenge 2 . Platonist and formalist rupture. *The following statement of Kurt Gödel represents a quintessential Platonist rupture with the mathematical ontological practice: «In geometry, the question as to whether Euclid's fifth postulate is true retains its meaning if the primitive terms are taken . . . as referring to the behavior of rigid bodies, rays of light, etc. . . . On the other hand, the objects of transfinite set theory . . . clearly do not belong to the physical world, and even their indirect connection with physical experience is very loose. But, despite their remoteness from sense experience, we* *do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true.»* [44] (pp. 267-268)

Objective 2 . Inquiring into Gödel's juxtaposition. *We will show that by taking the initiative to clarify his characteristically Platonist point, by his questioning the difference between physical and intuitive mathematical perceptions, Kurt Gödel has opened the door for eventual refinements of this difference which definitely undermine Gödel's Platonist argument.*

As all founders of science in Antiquity and modern times, from Pythagorus to Francis Bacon to Johannes Kepler to Albert Einstein, we assume that the visible world is sustained by its intrinsic intelligence, with mathematical and natural sciences acquiring codified formal knowledge carefully distilled from our mental and material, contemplative, descriptive, and performative experiences. In particular:

Finding 2 . Intrinsic intelligence and extrinsic intelligibility of observable reality reflected in the Continuum notion. *The mathematical Continuum of modern times represents a* ***rich conceptual distillation*** *of centuries long contemplation and practical observations of the propagation of light, as well as the practice and theory of measurements and construction. We will defend in this study the core ontological basis of the Continuum as a phenomenologically elemental, axiomatically inscrutable insight into, or model of, what is palpably perceived as* ***spacial flow****, or* ***spacial unfolding****.*

4 Whither the transfinite Continuum ?

During the first half of the XXth century, Georg Cantor, David Hilbert, and later Kurt Gödel, to name just these three key figures of the transfinite revolution, have invested extraordinary intellectual and technical efforts, as well as their monumental personal moral and scientific reputations – and, at least in the case of Cantor and Gödel, their tragic destinies, – to impose on the mathematical community a novel, purely transfinite interpretation of the Continuum, reducing this richest and well-known since the time immorial mathematical habitat to a stair on the grandiose transfinite ladder *Ord*, «*set on the earth with its top reaching to heaven*», as in the biblical Jacob's dream.

Challenge 3 . And behold: *With all these high set theory specialists incessantly ascending and descending on this ladder during the last century, there is still no consensus, to put it mildly, as to the nature and viability* *of the transfinite interpretation of the Continuum. Besides, we still do not know, and probably will never know, to what exactly transfinite staircase step should we assign the Continuum: it is just another way to say that the famous Continuum Hypothesis of Georg Cantor, $\mathcal{CCH}$, is still hopelessly open.*

Objective 3 . Why the transfinite Continuum ? *Are there some substantial phenomenological and onto-epistemic underpinnings of this persisting, unheard in the history of mathematics crisis of both the philosophical and mathematical consensus concerning an important mathematical theory ?*

Finding 3 . The story of invention of the transfinite Continuum at glance. *We demonstrate the epistemically pure symbolic nature, stripped here of all technicalities, of the transfinite approach to the Continuum enigma: Cantor's symbolic* ***powerset*** *notation for the well-known machinery of Karl Weierstrass of approximation of reals by rational numbers and Cantor's transfinite invention, ontologically unrelated to this machinery, separately recursively auto-applied and then amalgamated, also symbolically, by Cantor's well-ordering. «**law of thought**».*

5 Axiomatic challenge

5.1 Kurt Gödel's profound onto-epistemic insights

During the last two millennia, Euclidean geometry has been universally accepted and admired as the unsurpassed in its efficaciousness and beauty paradigm of a perfectly disinterested and formal mathematical thinking [52]. Countless mathematical and, more general, scientific and philosophical theories have been created following this apparently absolutely objective, suggestive, extremely laconic, and formally irreproachable paradigm, with its creative potential demonstrated, for example, by the recent exhaustive classification of finite sporadic groups.

Twenty years after the appearance of the transfinite theory of the Continuum, it has found its formalization in ***Zermelo-Fraenkel-Skolem axiomatic system***, $\mathcal{ZFC}$, a masterpiece of elegancy and frugality. Further investigations, the efforts to solve Cantor' Continuum Hypothesis, $\mathcal{CCH}$, in the first place, have produced a number of new powerful axiomatic additions to $\mathcal{ZFC}$, still characteristically insufficient for solution of the $\mathcal{CCH}$, whatever might be their transfinitely recursive power.

Kurt Gödel has defended the legitimacy of these set-theoretic axiomatic queries against the critics of the «*new axioms set up in the never-never land* *of large cardinals*» [69] and their «*disconcerting tendency to produce independence results rather than theorems in the usual sense*» [93] by two decisive for him and, in the context of the present study, profoundly suggestive arguments which could be roughly characterized as ***subjective*** and ***objective***:

1. «*The mere psychological fact of the existence of an intuition which is sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis.*» [44] (p. 268)

2. «*... even disregarding the intrinsic necessity of some new axioms, and even in case it has no intrinsic necessity at all, a probable decision about its truth is possible also in another way, ... namely* [by] *their fruitfulness in mathematics and, one may add, possibly also in physics.*» [44] (pp. 263, 269)

Gödel's second argument has been successfully implemented by the school of *necessary uses of the Transfinite in finite mathematics* [39], [40], and the first one is still makes sense when we are confronted by violent denials of the freedom of mathematical query:

«*At the beginning of this century a self-destructive democratic principle was advanced in mathematics (especially by Hilbert), according to which all axiom systems have equal right to be analyzed, and the value of mathematical achievement is determined, not by its significance and usefulness as in other sciences, but by its difficulty alone, as in mountaineering. This principle quickly led mathematicians to break from physics and to separate from all other sciences. In the eyes of all normal people, they were transformed into a sinister priestly caste of a dying religion, like Druids.*» [3] (pp. 7-8)

5.2 Post-Gödlian incompleteness principle

And yet, with the stagnation of inquiries into the transfinite mystery of the Continuum going into its second century, Gödel's profound subjective and objective insights are looking less and less satisfactory as possible *ultimate onto-epistemic arguments* in favor of the viability and relevancy of higher set theory for the Continuum research.

As to Gödel's first, *subjective argument*, it remains apparently viable even after being restated in a more generic form:

Thesis 1 . Gödelian thesis of the foundational legitimacy of mathematical intuition. *Any intuitive, systematic, and intellectually mature perception of a certain sufficiently rich system* $\mathfrak{S}$ *of abstract objects and their relationships, qualified to produce a relevant axiomatics, suffices to give meaning to the question of the truth or falsity of propositions concerning this system. We will call such systems* ***pre-axiomatizable****.*

The following immediate implication of the above thesis creates sort of a «geological fault line» where two Gödelian arguments meet, to allow the exposure of our contradictory argument:

Finding 4 . Proof-theoretical implication of the Gödelian thesis. *Any pre-axiomatizable system $\mathfrak{S}$, artifact assimilating the goodies of a sufficiently rich and sophisticated mathematical and conceptual environment similar to that which gave birth to higher set theory, allows a huge, super-exponentially expanding* ***«mathematical sci-fi novelization»****, i.e., creation of a multitude of* ***«fully consistent mathematical theories»*** *with unlimitedly extending axiomatics enriched by new axioms not only* ***«forced themselves upon us as being true»*** *(Gödel's aphorism* [44]*, p. 268), but being also intuitively compelling and esthetically attractive.*

Finding 5 . Post-Gödelian incompleteness principle. *And yet, such theories would have in their (more than) overwhelming majority, neither at this juncture nor whenever in future, no ontologically sound, objectively verifiable mathematical and/or notable extra-mathematical meaning outside their proper, self-absorbed scenes of formal deductions inside the corresponding system $\mathfrak{S}$ and its extensions.*

5.3 Transfinite axiomatic deadlock and its anticipated Euclidean redemption

Gödel's intricate arguments were meant to philosophically supplement his and many others more direct mathematical argument in support of the viability of higher set theory, of its transfinite axiomatics, and of associated with it *pro and contra* consistency results [27]:

« *The proof that the axiom of parallels does not follow from the other Euclid axioms did not close geometry, but made the emergence of non-Euclidian geometries possible, and opened the question of recognizing, among all possible geometries, the most relevant for describing the physical world. Likewise, Gödel's and Cohen's results show that several universes are possible from $\mathcal{ZFC}$, and, therefore, they open the study of the various possible universes.*»

Objective 4 . Whither the mathematical Continuum ? *Without contesting this popular vision of* ***various possible set-theoretical universes****, we will demonstrate in this study that the mathematical Continuum can be found in neither of them.*

We will do it in several steps, starting with the analysis of the distinctive characteristics of Euclidean and Zemelo-Fraenkel-Skolem axiomatic enterprises.

5.4 Euclidean constative axiomatics *versus* set-theoretical generative and performative axiomatics

Challenge 4 . What is the difference between Euclidean and set-theoretical axioms, and if exists, does it matter ? *Our query concerns the nature of axioms of Set Theory, from $\mathcal{ZF}$ to $\mathcal{ZFC}$ to* ***Axioms of Determinacy*** *to* ***Axioms of Large Cardinals****, which are typically* ***transfinite recursive principles and procedures of emergence*** *of set-theoretical entities acting on swaths of transfinite ordinals and cardinals* [55].

We are advancing in the present study the understanding of the axiomatic enterprise as a conditional and conditioned search for a ***formalization of an objectively sustainable truth***. The enterprise which is a part and parcel, even if rarely recognized as such, of the contemporaneous to it general human cultural and, even more general, existential endeavor shouldering its way through the intellectual challenges of its time and building its axiomatic instruments of research with due respect for, and professional knowledge of the cultural and scientific heritage:

Objective 5 . Axiomatic commitment as a cultural challenge: reductionist principles and the new phenomenology of light. (1) *We demonstrate in this study that the fundamental vision materialized by Cantorian set theory is distorted by uncritical reductionist universalization of finite recursive procedures.* **(2)** *On the positive side, we are intending to enrich the phenomenological background of inquiries into the mathematical Continuum, extracted from the centuries long observations of light, by distilling into abstract mathematical concepts the quantum mechanical phenomenon of entanglement of light.* **(3)** *This phenomenological acquisition will expose the* ***locally causal character*** *of transfinite universes of higher set theory and exhibit* ***the ultimate phenomenological obstacle*** *to Cantor's transfinite construction, similar to* ***topological obstacles*** *in homotopy theory and theoretical physics, namely:* ***the entanglement capacity of the mathematical Continuum****.*

Let us start with the first task, leaving the second one to the below Section 6:

Finding 6 . Axiomatization modes: constative *versus* performative. *We will demonstrate that, from a pure formal metamathematical point of view, one needs to differentiate between the* ***contemplative and constative character*** *of the Euclidean axiomatics, or for that matter, of group theoretical or any similar «****classical axiomatics****», and the* ***hazardously***

innovative, performative, recursive character *of set-theoretical axioms, with our* ***post-Turingean halting principle*** *sealing the inescapably hypothetical and conditional nature of the Cantorian* ***fundamental laws of thought*** *and the post-Cantorian* ***large cardinal principles*** [25].

5.5 Post-Turingean halting principle

The undecidability of the Halting Problem, an algorithmic projection of Gödel's incompleteness results, is well-known: there is no «***halting-test Turing machine***» capable of distinguish between halting and non-halting programs [100]. In other words, given a program, the only way to discover whether it halts or not is to run it, possibly, indefinitely.

Transfinite recursive definitions and principles of higher set theory are metamathematical and set-theoretical analogues of Turing machines: they «calculate» new transfinite totalities. Hence, the following post-Turingean metamathematical *Halting Principle*:

Finding 7 . Post-Turing Halting Barrier for performative, transfinitely recursive set-theoretical axioms. *There exists neither general metamathematical principle, nor logical criterion, nor verifiably terminating computational procedure to establish the objective and substantial «**truth**» of performative set-theoretical axioms of recursive nature postulating the existence of transfinite objects outside the already existing transfinite scale – otherwise that is that «**to run**» the theory completed with new axioms until it would be discovered some «**independent necessary uses**» of the object in question.*

Gerhard Gentzen's pioneering work [42] has laid down the foundations of the method of *gauging the recursive proof-theoretical strength of finite mathematical theories and combinatorial problems by ordinals.* With all known uncontroversial cases involving only countable ordinals: from Cantor's $\epsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$ [45], [86] to the first non-predicative ordinal Γ_0 [34], [40].

Finding 8 . Corollary: the first uncountable ordinal ω_1 is a class, not a set. *According to post-Turingean halting principle, the set-theoretical existence of ω_1 cannot be assured. The following proof-theoretical interpretation of ω_1 lends credit to this claim: Cantor's first uncountable ordinal ω_1 is the class of all countable ordinals, which eventually gauge the recursive proof-theoretical strength of finite mathematical theories and combinatorial problems; thus, ω_1 represents the adequate, universal, ever emerging and never completed ordinal scale of the power and sophistication of past and future iterative logical arguments.*

6 New phenomenology bearing on the evolving Continuum notion

The above conclusions are buttressed by two outwardly independent and yet intrinsically related mathematical and physical developments of the last century: the emergence of Quantum Mechanics with its modern hands-on branch of Quantum Information Processing and progress in Mathematical Analysis, Diophantine Approximations, Celestial Mechanics.

Consequently, we claim that both the novel insights into arithmetical and analytical enigmas of the Continuum and the counterintuitive quantum mechanical entanglement phenomena, implying the demise of the universal local causality dogma, become crucial for all eventual progress of our understanding of *what is the Continuum*:

Finding 9 . Entanglement obstacle to the transfinite enterprise. *We demonstrate the existence in the Continuum case of an* ***intrinsic limitation of the explicative and performative power of higher set theory****, finding its explanation in the identified in this study* ***ultimate phenomenological obstacle to Cantor's transfinite construction****, similar to* ***topological obstacles*** *in homotopy theory and theoretical physics. In our case, it is* ***the entanglement capacity of the mathematical Continuum****.*

7 Higher set theory Continuum brainteaser

Finally, let us consider, in the light of the findings of the present study, what remains of the status of higher set theory, only recently fully in charge of the Continuum research.

Challenge 5 . So what about higher set theory ? *So what about higher set theory, this vast, complex, technically intricate and conceptually mostly incomprehensible and insecure for a mainstream mathematician field of mathematical sciences ? The field conceived and, during the last hundred years, matured as* ***the core constituent*** *of the foundations for mathematics – to finally surrender its importance: because today, «among mathematicians, there is a widespread view that ongoing current mathematics on the whole is more reliable than any of the philosophically motivated programs that have been proposed to replace it, and that the only foundation that need be considered (if any at all) is organizational.»* [35]*. Worse still: the field reputed for its «inherently vague open problems»* [33]*, unsettling «new axioms set up* *in the never-never land of large cardinals»* [69]*, «disconcerting tendency to produce independence results rather than theorems in the usual sense»* [93]*. And all this, in the intimidating environment of «interminable foundational controversies»* [35]*.*

Objective 6 . Higher set theory Continuum brainteaser. *We recognize the need to be as clear about what are* ***not*** *the intended implications of the present inquiry for higher set theory, as we were precise above about what are its stated purposes. In particular, we see no reasons to share the popular doubts* [3],[69]-[72] *of the* ***legitimacy of higher set theory*** *as mathematical domain. And we reject as mathematically particularly ill-considered all recent finitist critiques* [111] *of proof theory results gauging the strength of an axiomatically defined theory by an appropriate transfinite ordinal* [87], [88]*.*

Finding 10 . Higher set theory as programming language of transfinite design. *In the opposite tack, we conclude that, as the conceptual and mathematical Continuum enigma concerned, the trouble of higher set theory is hidden exactly where its pride resides: in its most advanced, at the limits of human imagination and beyond,* ***built-in iterative mechanisms of set generation****. Accordingly, this theory is steadily gaining in its performative power on the expense of its contemplative, constative, descriptive, and explicative abilities, to become finally a sophisticated* ***programming language of transfinite design****, successfully and, as in its applications to proof theory, usefully mimicking recursive aspects of Mathematical Infinity, without being able nevertheless to advance new valuable insights into our understanding of the nature of the Continuum.*

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