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Propositions as intentions

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Abstract I argue against the interpretation of propositions as intentions and proof-objects as fulfillments proposed by Heyting and defended by Tieszen and van Atten. The idea is already a frequent target of criticisms regarding the incompatibility of Brouwer's and Husserl's positions, mainly by Rosado Haddock and Hill. I raise a stronger objection in this paper. My claim is that even if we grant that the incompatibility can be properly dealt with, as van Atten believes it can, two fundamental issues indicate that the interpretation is unsustainable regardless: (1) it is hard to determine, without appealing to propositional intentions on pain of circularity, what intention a proof-object should be understood as a fulfillment of; (2) due to a difficult fulfillment dilemma, it is unclear, at best, what the object of an intention corresponding to a proposition is.

1 Introduction

In the intuitionistic tradition of Brouwer and Heyting, mathematical objects are conceived as having a mind-dependent existence as mental constructions given in intuition. The conception of intuition originally endorsed by Brouwer is borrowed from Kant's views on the intuition of time but motivated against the background of his own mystical ideologies. It has become clear from the unfounded claims found therein about the nature of intuition that his intuitionism is in desperate need of a tenable philosophical justification.[1]

This general dissatisfaction with Brouwer's philosophy has led to a growing interest in the support of intuitionism from a phenomenological approach over the last few decades.[2] The point of departure of this enterprise is the replacement of Brouwer's notion of intuition with Husserl's analysis of intuition as the experience of an object as it is intended by a subject, meaning that intuition now has to be understood in terms of fulfillment of intentions. As pointed out by Martin-Löf (1985), the roots of the idea can be traced mainly to Heyting's use

[1]Brouwer's philosophy itself lacks a foundation to its own satisfaction (van Atten, 2006, ch.5.5).

[2]See especially Tieszen (1989) and van Atten (2004, 2006, 2017). The phenomenological interpretation is also appealed to some extent in Sundholm (1983) and Martin-Löf (1985), figures prominently in Franchella (2007), and is assumed in the background of Martin-Löf (1993) and van der Schaar (2011).

of a Husserlian account of intuition, following a suggestion made by Becker, in the meaning explanations proposed for the intuitionistic logical constants in the 1930s.

More precisely, the explanation proposed by Heyting is that a mathematical proposition is an intention which is fulfilled with a proof-object of that proposition (Heyting, 1931, p.113).[3] The backbone of the phenomenological defense of intuitionism is therefore this interpretation of propositions and proof-objects as intentions and fulfillments, respectively. For brevity let us call it the 'propositions-as-intentions' interpretation. Its early formulation by Heyting leaves much to be desired and fails to meet today's standards of rigor, but the core ideas have been further developed by Tieszen (1989) and van Atten (2004, 2006, 2017).

In this paper I argue that the propositions-as-intentions interpretation is still misguided as a phenomenological defense of intuitionism despite its refinements. I must stress at the outset that the interpretation is a frequent target of criticisms regarding the incompatibility of Brouwer's and Husserl's positions, as in Rosado Haddock (2010) or Hill (2010). Their objection consists mainly in denying the interpretation its major premise. This is not exactly the direction I wish to take in this paper. My claim is rather that even if we grant that the incompatibility can be properly dealt with, as van Atten (2017) believes it can, two fundamental issues indicate that the interpretation is unsustainable regardless:

(1) Phenomenologically, to do justice to Brouwer's thesis that mathematical objects are mental constructions given in intuition, a proof-object a of a proposition A must be identified with an objectified fulfillment of a certain intention. But, on pain of circularity, this intention cannot be the one corresponding to the proposition A, for it appears its intentional object already needs to involve a to a certain degree. What other intention should a be an objectified fulfillment of and how to explain the fact that a is still expected to fulfill the intention corresponding to A since it is a proof-object of A?

(2) It is far from clear what the object of an intention corresponding to a proposition should be. Heyting's own suggestion is inadequate. The most plausible candidates for intentional objects seem to be sets of canonical proof-objects of the propositions, but this thesis immediately leads us to a difficult fulfillment dilemma: for Husserl, an intention is fulfilled when the intended object is genuinely presented to us in just the way it is intended; but here only one element of the set, not the set itself, can fulfill the intention.

My reservations against the propositions-as-intentions are stronger in that I embrace the premises of the interpretation but show that it leads to undesirable conclusions. Before my efforts can be directed to (1) and (2), I will introduce Husserl's theory of intentionality in Section 2 and briefly review some basic tenets of intuitionism with special emphasis on Heyting's meaning explanations and its refinements in Section 3. In Section 4, I turn to the charges of incompatibility against the propositions-as-intentions interpretation. Granting

[3] Roughly, a construction is a proof-object of a proposition if it satisfies certain conditions associated with it. For example, a proof-object of the conjunction $A \wedge B$ usually takes the form of a pair containing a proof-object of A and a proof-object of B (Troelstra and van Dalen, 1988).

that an intuitionistic rendering of Husserl's views is, in principle, possible, I discuss (1) and (2) in detail in Sections 5 and 6. Section 7 presents some concluding remarks.

2 Husserl's theory of intuition

Before I turn to the propositions-as-intentions interpretation, a brief exposition of Husserl's theory of intentionality will be of some value.[4] Intentionality is the intrinsic "directedness" of various mental acts like knowing or believing to some objects, including not only particulars but also universals and states of affairs among the things that count as intentional objects.

For Husserl, intentionality is not an objective relation between a subject and an intentional object existing in a reality independently of the thinking subject; otherwise, it would not be possible to intend non-existing objects such as a round square. But this does not mean that intentionality is regarded as a purely subjective relation between a subject and an intentional object existing as a mind-dependent entity; otherwise, it would not be possible to have distinct intentions to the same object presented in a different way.

Husserl's conception of intentionality provides an elegant solution to these problems with a tripartite distinction between act, content, and object (LI V §20). Following Tieszen (1989), the structure of an intention in Husserl's sense may be pictured as shown below:

$$\text{act}(\text{content}) \quad \rightarrowtail \quad [\text{object}]$$

Simply put, an intention is an act, such as thinking, believing or knowing, directed towards an object by way of the content of the act. Ordinary examples of intentions include the thought of the shining sun, belief that a cup of coffee is on the table, or knowledge that $5 + 7 = 12$. In the diagram above, the object of the act is "bracketed" in order to stress that we suspend any judgment about the existence of object. For instance, we may imagine Pegasus as a flying horse or think of the division of a certain real number by zero: in both cases the object of the act is nonexistent, but the intentions still have their contents in the representation of an object as a horse that flies or a real division of some numerator by zero.

2.1 Intuition as intentional fulfillment

Objects can be emptily intended or intuitively given depending on the absence or presence of the object just as intended by the subject. In an intentional fulfillment, the intended object is not merely thought about without presentation, as in an empty intention, but perceived in the strictest sense as "bodily present" (LI V §5). Husserl compares the distinction between empty intentions and fulfilled intentions with that of concept and intuition, well explored by Kant,

[4]Hereafter references to the *Logical Investigations* (Husserl, 1970) given in the text are in the form 'LI' followed by the investigation number. *Ideas* (Husserl, 1931) is referred to as 'Ideas' followed by the book number, *Formal and Transcendental Logic* Husserl (1969) as 'FTL', and *Experience and Judgment* (Husserl, 1973) as 'EJ'. These are sometimes followed by section numbers as well. The only exception is *Philosophy of Arithmetic* (Husserl, 2003), referred as 'PA', which instead is followed by the original pagination.

and adds that what we intuit stands before our eyes in perception or imagination exactly as we intended in our thinking (LI V §44). To give a concrete illustration, one can think of the shining sun or believe that a cup of coffee is on the table, but a fulfillment only occurs when one perceives the shining sun or a cup of coffee on the table just as they were intended.

Moreover, according to Husserl, each intention comes to be fulfilled, partially fulfilled, or frustrated within a horizon of possible experience. Intentions are 'one-sidedly' directed to objects only through their contents and consequently always incompletely determine the object intended no matter how far we go in our experience (Ideas I §44). The horizon of an intention further determines the object intended from its limited perspective because it includes all experiences that can be anticipated by a subject at a given time (EJ §8).

We intuit an object when an intention is fulfilled. Since Husserl maintains that a singular object given in perception is present in an immediate way, there is an agreement with Kant that in intuition a singular object is given to us immediately through sensibility. The difference is that for Husserl we can also fulfill intentions to aggregates, indefinite pluralities, totalities, numbers, disjunctions, predicates, and states of affairs. From this he concludes in direct opposition to Kant that our intuitions need not be singular and can be mediated by certain acts of the intellect that are founded on perception but not reducible to it (LI VI §48–52). This general and mediated kind of intuition is known as categorial intuition. The idea is used to explain the fulfillment of intentions involving categorial forms, that is, intentions expressed by terms such as 'the', 'a', 'some', 'many', 'few', 'two', 'is', 'not', 'which', 'and', 'or' etc. While objects of sense perception are given to us in one step, our apprehension of 'categorial objects' involves acts of the intellect such as abstraction, collection, and reflection.

3 The propositions-as-intentions interpretation

Brouwer (1913) famously claims that mathematics is a languageless activity of the mind and that it is based on the intuition of the movement of time commonly known as the twoity. It can be succinctly described as the experience of 'one thing which gives way to another' from which the natural numbers, sets, and choice sequences are given to us (Brouwer, 1947).

Every mathematical object is, according to Brouwer, a mental construction that is given to us by means of a 'self-unfolding' of the intuition of twoity. This is to say that mathematical objects have their existence as intuited mental constructions. The role of intuition in Brouwer's intuitionism is thus not merely epistemic but also ontological: it is used to delimit what is in the domain of mathematical objects. Therefore, like any constructions, the only way to ensure the existence of proof-objects is to determine how we may intuit them. To intuit a proof-object of a proposition is to know that the proposition is true. From the ontological role attributed to intuition it follows that all truths are experienced in time (Brouwer, 1948, p.1243). Yet, Brouwer has never bothered to describe how proof-objects are supposed to be given to us in accordance to the intuition of twoity, perhaps because he decided to leave Heyting in full charge of the elucidation of logic from the intuitionistic perspective.

The meaning explanations developed by Heyting (1930, 1931, 1934) aim to elucidate what is a proof-object of a compound proposition formed by the intuitionistic logical constants, so there is no mention of atomic propositions. Heyting (1931) stresses that proof-objects are mental constructions that can be themselves treated mathematically. Yet, curiously, there is no explicit mention of the intuition of twoity anywhere. The question of how we intuit proof-objects of a proposition under Brouwer's own account of intuition remains open with Heyting's writings and even today this is not a topic explored in the literature.

3.1 Heyting's original interpretation and its refinements

Although nothing is said about the intuition of twoity, Heyting's meaning explanations allude to the view of proof-objects as objects of intuition in a Husserlian sense with his interpretation of proof-objects in terms of fulfillments of intentions. This incorporation of Husserl's theory of intentional fulfillment is inspired by Husserl's student Oskar Becker, who tells Heyting that intuitionistic logic is a calculus of intention in an undated letter from September 1934:

> *Kolmogoroff's* interpretation of intuitionistic logic as a 'task calculus' is very important and also very valuable from a pedagogic point of view. It can be generalized to the *phenomenological* concept of 'intention' and its 'fulfillment' (corresponding to the solution to the task). 'Task' is a special case of 'intention'. [...] The intuitionistic logic is thus an 'intention calculus'. At first glance, this is apparently subjective-humane. But there is also an 'objective' (according to *Husserl*, 'noematic') intentionality, which one could connect with the classic distinction of potentia (δύναμις) and actus (ἐνέργεια) (van Atten, 2005, p.138).[5]

Unlike Becker, Heyting only had a passing knowledge of Husserl's phenomenology. But, as Martin-Löf (1985) notes, his appropriation of the terms 'intention' and 'fulfillment' in his meaning explanations seems adequate. To be exact, Heyting (1931) declares that the assertion of a sentence is the establishment of an empirical fact, namely, the fulfillment of the intention expressed by the sentence. That is, Heyting does not speak of propositions, only of what is expressed in declarative sentences. But since propositions are taken to be the meanings expressed by such sentences, we can see that propositions are explained as intentions in his meaning explanations. Heyting also briefly tells us how the object of an intention corresponding to a proposition must be construed:

> As already emphasized earlier, the intention is not to a state of affairs that is thought to exist independently of us, but to an experience that is thought to be possible, as is also clear from the above example. (Heyting, 1931, p.113)

Heyting refers above to his famous example that the sentence "Euler's constant C is rational" means the expectation that one can find two integers a and b such that $C = a/b$. Later Heyting is a bit more explicit on the subject, indicating that the object of such a propositional intention consists in a proof-object of the proposition:

[5] This and the following English translations are my own.

> [E]very statement stands for the intention to a mathematical construction that is supposed to satisfy certain conditions. A proof for a statement consists in the realization of the construction required in it. (Heyting, 1934, p.14)

In addition to that, Heyting (1931) views the intuitionistic logical constants as procedures to form new intentions from other given intentions. Curiously, Heyting (1931, 1934) limits his phenomenological analysis to only compound propositions formed by negation, disjunction, and implication in his original meaning explanations. Here is what he says:[6]

- $\neg A$ corresponds to the intention of a conflict connected with the original intention which A corresponds to;

- $A \vee B$ corresponds to the intention which is fulfilled exactly when one of the intentions which A or B correspond to is fulfilled;

- $A \supset B$ corresponds to the intention of a certain construction which from any proof-object of A leads to a proof-object of B.

Explanations of conjunctions and the quantifiers are found in Heyting (1934), but they are rather worded in Kolmogorov's jargon of tasks and their solutions. However, to fill the gap and make them fit the general intention–fulfillment pattern outlined above, it is natural to rephrase them as follows, as done by Tieszen (1989, ch.4.5) and van Atten (2004, p.21):

- $A \wedge B$ corresponds to the intention which is fulfilled exactly when both intentions which A and B correspond to are;

- $\forall(x \in D)B(x)$ corresponds to the intention of a general method of construction which, for any mathematical object a in the domain D, yields a proof-object of $B(a)$;

- $\exists(x \in D)B(x)$ corresponds to the intention of an object $a \in D$ such that the intention which $B(a)$ corresponds to is fulfilled.

The meaning explanations are given inductively and ultimately rest on the presupposition that the base case of atomic propositions can be explained. Intuitionistically, these are propositions that involve only natural numbers, sets, choice sequences, and their operations. Therefore, to even get the propositions-as-intentions interpretation off the ground we must first elucidate them all in terms of fulfillment of intentions. This has been done by Tieszen (1989) and van Atten (2006) to a certain extent. The former has proposed a Husserlian account of numeric and finite set intuition with Brouwerian overtones; the latter has justified the introduction of

[6] Except that instead of sentences I refer directly to propositions. Negation and disjunction are explained in (Heyting, 1931, pp.113–114) and implication in (Heyting, 1934, p.14). In his explanation of implication, Heyting actually uses the term 'proof' (*Beweis*).

choice sequences and the weak continuity principle. For reasons of space, we will have to leave a detailed investigation of these accounts out of this paper (see Section 7).

Finally, it should be noted that Heyting's discussion of proof-objects is lacking. One key distinction to the proper understanding of the meaning explanations is that between non-canonical and canonical proof-objects. It was made explicit by Dummett (1975) although it can be argued that the idea is present in Brouwer's proof of the bar theorem.[7] Roughly, a canonical proof-object is given in a form through which one can directly check that it belongs to or 'proves' a particular proposition (Martin-Löf, 1985). When given a non-canonical proof-object one can only check that it proves a certain proposition indirectly by bringing it to its canonical form through one of more calculation steps. Heyting's original meaning explanations just explain what are the canonical proof-objects of the proposition. For completeness, we must add the condition that a proposition has a non-canonical proof-object when it can be calculated to a canonical proof-object of it.

The propositions-as-intentions interpretation needs to reflect this amendment. The idea is advanced as an intuitionistic generalization of Husserl's thesis in LI VI §18 that in the intuition of a complex number such as $(5^3)^4$ the act of fulfillment comprises each individual step in its definitory chain that leads from $(5^3)^4$ to a sum of ones. To be precise, it is here said that the evaluation of a non-canonical proof-object into a canonical one consists in an analysis of the intentional structure of the former and that, as fulfillments, non-canonical proof-objects comprise all the steps involved in the evaluation (Tieszen, 1989, ch.6.3, van Atten, 2004, p.17).

4 The intuitionistic rendering of Husserl's views

There are at least two ways in which one can argue against the propositions-as-intentions. First, one can deny the interpretation its major premise that there is a way in which Brouwer's notion of intuition can be replaced with Husserl's. Second, one can assume the major premise for the sake of argument and show that it leads to undesirable conclusions. In this paper I take the latter direction, which, it is worth stressing, results in a stronger objection that indicates that the interpretation falls short in its own terms.

Until now objections have only taken the former direction. Indeed, the most obvious challenge to the propositions-as-intentions interpretation is the reconciliation of the strong Platonist tendencies of Husserl's thought with the distinctively anti-realist intuitionistic standpoint the whole approach requires. This tension has been thoroughly documented in Rosado Haddock (2010) and Hill (2010). Da Silva (2017) rejects the constructive reading of Husserl's ideas as well, but with little mention of the propositions-as-intentions. Recently, Berghofer (2020) has put forward new arguments that show just how much the basic tenets of intuitionism are opposed to a Husserlian philosophy of mathematics.

More could be said about the tension between Brouwer's and Husserl's views, but let me just summarize a few points of relevance. Contrary to the intuitionistic attitude, Husserl argues

[7] See e.g. (Sundholm and van Atten, 2008, §6).

in the Prolegomena §24 against the temporal dependence of logical truths and in LI II §8 he speaks of mathematical objects as having genuine existence. The emphasis on pure-logical grammatical laws and its connections with laws of thinking in LI VI §62–64 is in contrast with Brouwer's dismissive attitude about language in mathematics. Even after his turn to transcendental phenomenology, Husserl still firmly defends Platonism against the thesis that mathematical objects are merely mental constructions in Ideas I §22, and explicitly embraces the law of the excluded middle in FTL §77.[8] So, with all that said, it is hard to tell what made Heyting (1931, 1934) think that Husserl's views could be assimilated into Brouwer's intuitionism or even support a constructivist kind of logical reasoning. But, in his defense, the propositions-as-intentions interpretation is, for better or worse, completely left out in his later and most mature presentation of intuitionism (Heyting, 1956).

Tieszen (1984) and van Atten (2006, 2017) admit that Husserl's reluctance to revise any parts of classical mathematics is indisputable, but maintain that certain elements of his phenomenology provide scope for an intuitionistic conception of mathematics. Tieszen contends that Husserl should have endorsed a constructive epistemology of mathematics. Da Silva (2017, p.90) at least concedes that Husserl denies the validity of the law of excluded middle for the logic of experience. Van Atten (2006, ch.5.2–5.3) goes even further and asserts that Husserl actually relates mathematical epistemology and ontology by means of an implied thesis according to which a mathematical object exists if and only if it is constituted according to the laws of categorial formation. Van Atten (2017, 12.2.3) then argues that Brouwer's intuitionism is the only philosophy of mathematics that should be considered part of Husserl's transcendental-phenomenological foundations of pure mathematics.

4.1 Two major challenges

For van Atten the tension with Brouwer's views is only apparent, and any claims made by Husserl in support of classical mathematics are in fact wrong by his own standards. It is not my purpose here to investigate to what extent this is correct. For the sake of argument, I will concede that Husserl's views are amenable to an intuitionistic conception of mathematics. The obvious question is then: which claims originally made by Husserl must be rejected and how they must be rectified to accommodate the propositions-as-intentions?

The reservations (1) and (2) previously outlined in the introduction are direct implications of what I take to be the two major challenges involved in the answer to this question:

- Husserl regards fulfillments as acts, but proof-objects must be mathematical objects, so it is imperative that fulfillments be objectified. Moreover, the only way to reaffirm the Brouwerian treatment of proof-objects as constructions phenomenologically is to maintain that a proof-object has its existence as an objectified fulfillment of a certain intention. Yet, it is far from obvious which intention to choose.

[8]In an unpublished manuscript on set theory dated from 1920 in the Husserl Archives in Cologne, Husserl seriously considered constructivism as a way to avoid the threat of paradoxes, possibly under the influence of Weyl (see (Rosado Haddock, 2010, pp.28–30)). There are no signs of constructivism left in FTL, his conclusive treatise on logic and mathematics.

- Husserl's indication that only intentions to states of affairs deserve the treatment of propositions must be discarded completely to make room for a new account of intentional objects that is consistent with the interpretation of the proof-objects of propositions as the fulfillments of intentions that correspond to them.

To my knowledge, these issues have never been discussed in the literature until now. I will elaborate on them, in turn, in the two following sections. In the spirit of Heyting's meaning explanations, I will focus solely on compound propositions and the logical constants forming them, simply because we already have a well-established description of how their canonical proof-objects looks like (Troelstra and van Dalen, 1988). There will be no loss of generality in our discussion, given that all problems pointed out along the way should be easily transposable to the context of atomic propositions as well.

5 What are propositions and proof-objects phenomenologically?

So far I have been loosely speaking of propositions as intentions and proof-objects as their fulfillments without investigating what exactly this view amounts to intuitionistically. But the basic premise of the propositions-as-intentions interpretation is the replacement of Brouwer's notion intuition with that Husserl's in a defense of intuitionism. So it should at least preserve the basic tenet that mathematics is a mental activity rooted in intuition. It turns out that its vindication will require a phenomenological understanding of propositions and their proof-objects that is far more sophisticated than anything proposed so far.

5.1 Propositions as contents of intentions

Propositions cannot be interpreted as intentions, strictly speaking. This would be incorrect because from Husserl's tripartite distinction it follows that intentions may be directed to the same object and have the same content but still differ in the act. Therefore, if we believe, know, or remember that A, for a proposition A, then the act changes but the content remains the same in those different intentions. Tieszen (1989, ch.2.2) thinks that we should rather take a proposition to be an expression of the content of an intention. I can only agree that the focus should be on contents and not the intentions themselves, but his qualification of propositions as expressions seems to be uncalled for from a Brouwerian perspective.

Propositions should be languageless mind-dependent entities and thus cannot be treated as expressions in the same way an utterance is said to 'express' a thought. They can only be regarded as expressions in the sense of meanings of sentences. Perhaps Tieszen is mislead to view propositions as expressions of intentional contents because, as we saw, in his original meaning explanations Heyting (1931) chooses to speak of intentions as expressions of sentences and does not identify intentions and propositions explicitly.

But once we observe that the content of an intention is the mode or way in which the mind is directed to an object, and therefore that, as often noted, contents have a semantic aspect similar to Frege's senses, it is only fitting to take propositions to be contents. I will, nonetheless, often speak of propositions as intentions for the sake of briefness, always bearing in mind that what is meant is not the intention itself but its content.

5.2 Proof-objects as fulfillments of intentions

The fulfillment of an intention is treated as a mental process in Husserl's writings, but in intuitionism proof-objects demand the status of mathematical objects. If fulfillments of intentions are not seen as objects, clearly the propositions-as-intentions interpretation would be making a serious category mistake. Inspired by Sundholm's (1983) distinction between proofs as acts and objects in constructivism, Tieszen comes to the conclusion that fulfillments in the propositions-as-intentions sense must be understood as the output of the process.

Tieszen (1989, pp.86–87) argues that the phenomenological view of a mental process as a mind-dependent object is as consequence of an act of reflection, a higher-order intentional act directed at one's own intentional experiences. This strategy is actually supported by Husserl's own remark in Ideas I §77 that any mental process which is not an object of regard can become the object of an intention by means of a reflection act. Sundholm and van Atten (2008) stress that Brouwer himself also appeals to reflection in his work, specifically in the construction of bars and determination of their properties in his proof of the bar theorem.

I will take for granted that mental processes may be turned into objects by reflection and accept that proof-objects are not fulfillments but 'objectified fulfillments' in this sense. It should be pointed out that van der Schaar (2011, p.404) also treats proof-objects as objectified fulfillments, although no indication is given as to how the objectification is achieved.

The identification of proof-objects with objectified fulfillments is the only way to do justice phenomenologically to Brouwer's thesis that mathematical objects only exist as mind-dependent constructions given in intuition (Brouwer, 1948, p.1237). To substantiate this claim in phenomenological terms an ontological reduction of proof-objects to objectified fulfillments is imperative: the ontological status of a proof-object must be exactly that of an objectified fulfillment of an intention. At this point one may begin to wonder whether a proof-object of a proposition has its existence as a fulfillment of the intention corresponding to the proposition in question. The answer must be negative.

Say a is a proof-object of some proposition A. My claim is that it would be circular to take a to be an objectification of a fulfillment of the intention corresponding to A. The argument already touches on the problem of intentional objects to be discussed in the next section. But for now I just need to say, in agreement with Heyting (1934), that the object that an intention corresponding to a given proposition is directed to consists of its proof-objects. That is, the very existence of the intentional object depends on that of the proof-objects. Now, recall from Section 2 that, for Husserl, first we have an intention to an object and, only then, depending on the presence of the object exactly as intended, the intention can be fulfilled. The object of an intention is what determines its fulfillment. If a is an objectified fulfillment of the intention corresponding to A, then a only exists after reflecting on the fulfillment of this intention, despite the fact that, as anticipated above, the intended object here already has to involve the proof-object a of A. In sum, we are forced to contradict Husserl's theory of intentional fulfillment by accepting that intentions corresponding to propositions are directed to objects whose existence are conditional to their own fulfillments.

One simple way to circumvent this difficulty is to abandon Brouwer's thesis and become open to the possibility that proof-objects may not be mental constructions. As a consequence, proof-objects need not be identified with objectified fulfillments, although it may still be maintained that they are intuited in such way. This path is followed by Tieszen (1989, ch.8.4), but it is unclear if it is taken as a way to overcome the circular reasoning described here, or even if he is aware of the circularity, since it is not mentioned anywhere. Without Brouwer's thesis a Platonist account of proof-objects as mind-independent objects is arguably compatible with a phenomenological analysis of the intuition of such objects. But we are led to a denial of the ontological role that Brouwer originally attributed to intuition. Brouwer used intuition to delimit what belongs to mathematics or not. Intuition had strong ontological implications, because only what is intuited is recognized as a mathematical object. Once Brouwer's thesis is left behind the resulting position is hardly recognizable as a form of intuitionism.

As we seek to retain Brouwer's thesis, to my mind the only way out of this difficulty is to identify the proof-object a with an objectified fulfillment of an intention to a second object a' that must necessarily exist independently of a. But what could a' be? I see two alternatives:

(a) We allow a' to be a physical or premathematical kind of abstract object, immediately committing ourselves to explaining how a acquires the status of a mathematical object considering that its existence is derived from the presence of the intentional object a';

(b) We determine that a' is some mathematical object, but here instead we end up with a puzzling regress in between an analogous bifurcation. From Brouwer's thesis, similar concerns now arise surrounding a', for its mathematical existence has to be non-circularly derived from a second fulfillment associated with yet another third intentional object a'' existing independently of a'. Once we ask ourselves what a'' is:

  (a') We follow, *mutatis mutandis*, (a), replacing a' with a'';

  (b') We follow, *mutatis mutandis*, (b), introducing a new intentional object a''' in place of a'', but, as a result, getting caught in another bifurcation;

We may say that something is a primitive intuitionistic mathematical object if its intuition, and consequently its constitution, in light of Brouwer's thesis, is independent of other mathematical objects. The above argument shows that proof-objects need either be primitive intuitionistic mathematical objects or somehow reducible to a sequence of fulfillments that end with primitive intuitionistic mathematical objects as the intentional objects.

In the setting of his own notion of intuition, the twoity, Brouwer takes natural numbers as primitive intuitionistic mathematical objects by constructing them in terms of the pure intuition of units and pairs that are abstracted from our perception of temporal change and sensory perception (Brouwer, 1981, p.91). This is a result of the first act of intuitionism. Brouwer also recognizes sets and choice sequences as primitive and claims that they are an immediate consequence of the self-unfolding of the twoity, although, unlike the natural numbers, he never really attempts to show how sets and choice sequences are constructed by means of in the intuition of units and pairs (Brouwer, 1981, p.93, ft.†). There is no indication in Brouwer's and Heyting's writings that proof-objects are taken as primitive as well, so, they must be, it seems, definable in terms of natural numbers, sets, and choice sequences.

It is reasonable to conclude, due to Brouwer's views on primitive intuitionistic mathematical objects, that proof-objects must be in some way analyzed based on objectified fulfillments of intentions involving natural numbers, sets, or choice sequences. Unfortunately, as far as I know, there have been no research efforts in this direction in the literature. Tieszen (1989) and van Atten (2006) shed some light on the intuition of numbers, sets, and choice sequences phemonenologically, as mentioned in Section 3.1, but the problem of determining the intentions proof-objects are objectified fulfillments of is never addressed.

What is far worse is that, regardless of whether we give a satisfactory explanation of the intention which a proof-object a proving A is an objectified fulfillment of, we are left with an even more pressing concern: given that this intention cannot be the intention corresponding to A, why should one expect a to fulfill it too, as the propositions-as-intentions interpretation demands? To address this issue we must first be able to tell what the objects of the intentions that correspond to propositions are. This brings us to my second reservation.

6 What are the objects of propositional intentions?

Husserl seems prepared to say that the intentions that correspond to propositions are directed to states of affairs as their intentional objects, for example, in LI IV §11 and LI V §17. Such intentions are then fulfilled when the states of affairs associated with the propositions are given to the subject exactly as intended by means of categorial intuition. However, intuitionism does not admit the existence of a mathematical reality independent of the mind, so, following Heyting (1931, 1934), states of affairs must be replaced with proof-objects. Unfortunately, Heyting is too brief on this point and, as we saw on Section 3.1, only vaguely mentions that a propositional intention should be directed to a possible experience and to a proof-object of the proposition. How should this claim be understood?

In the following I will attempt to make precise Heyting's suggestion in three ways by proposing different candidates for the intuitionistic intentional objects, namely, canonical proof-objects, horizons, and sets of canonical proof-objects. I conclude that the third is the less problematic choice among the three but is susceptible to a fulfillment dilemma that questions the very feasibility of the propositions-as-intentions interpretation.

6.1 Canonical proof-objects

Recall that Heyting's meaning explanations mainly tell us what the canonical proof-objects of a compound proposition formed by one of the intuitionistic logical constants are. In this setting, a proof-object for Heyting simply means what we now call a canonical proof-object. When we only have a non-canonical proof-object at our disposal we must first bring them to their canonical form to check whether they prove the proposition. This process may take some time but we are always guaranteed to find a canonical proof-object as a value.

Thus, considering that Heyting (1931) views a proposition as an intention that is fulfilled with a proof-object, we are inclined to take canonical proof-objects as the objects of the intentions whose contents are propositions. There is at least a clear sense in which we can think of canonical proof-objects as 'possible experiences': as mental constructions, they are experiences derived from intuition; as experiences they are thought to be possible in that they have the canonical form that constructions can be brought to.

Allow me to use an example to better illustrate the proposal and its flaws. For proof-objects a and b of respectively the propositions A and B, we may define the conjunction $A \wedge B$ by prescribing the ordered pair $\mathsf{pair}(a, b)$ as its canonical proof-object. One may then think that the corresponding intention is directed to the object $\mathsf{pair}(a, b)$. But this would still be a rather superficial picture of what the intentional object is. In general, $A \wedge B$ has not just one but several different canonical proof-objects of the form $\mathsf{pair}(a, b)$, that is, one for each two proof-objects a and b of A and B, which need not even be canonical. The intention corresponding to $A \wedge B$ needs to have a single intentional object, but we cannot just arbitrarily choose one particular canonical proof-object and leave all others out of consideration.

If it were determined that, say, $\mathsf{pair}(a, b)$ is the object intended, then the intention would be fulfilled just in case that very object it is presented to us. But assuming that $\mathsf{pair}(c, d)$ is a proof-object of $A \wedge B$ we have no means to explain how it fulfills the intention, unless $a = c$ and $b = d$. This is the main reason why Heyting's (1931) indication that a proposition corresponds to an intention to a proof-object of it cannot be understood in such a way. It needs further refinement: the intention cannot be to just one particular proof-object in general.

What other options do we have? One might want to ignore all this plurality and say that the intended object is the one form $\mathsf{pair}(-, -)$ all canonical proof-objects share. Of course, we cannot make sense of this suggestion without an account of abstraction of some sort. But let us not worry about this, since this approach is flawed for at least two different reasons.

First, it leaves out any propositions that admit canonical proof-objects of different forms. A case in point is the disjunction $A \vee B$, defined by prescribing the left and right injections $\mathsf{inl}(a)$ and $\mathsf{inr}(b)$ as canonical forms of proof-objects, where, again, a and b are supposed to be proof-objects of A and B, respectively. The corresponding intention cannot be to both forms $\mathsf{inl}(-)$ and $\mathsf{inr}(-)$, but only to one of them. Yet we cannot choose one over another as the intentional object either, for we have no means to know in advance which disjunct A or B, if any, is provable with a proof-object. That is why intuitionists reject the general validity of the law of excluded middle, which states that $A \vee \neg A$ always has a proof-object for any A.

Second, intentions are fulfilled when the intentional object is genuinely present to the subject, and, because a proof-object a of A is meant to fulfill the intention corresponding to A, the intention must be directed to a proof-object not to its form. To say that $\mathsf{inl}(-)$ fulfills the intention which $A \vee B$ corresponds to is the same as to suggest that one knows that the left disjunct, A, is provable without being able to tell what is a proof-object of it.

6.2 Horizons and finished totalities of all proof-objects

The main lesson to be drawn from the previous discussion is that, since propositions in general have several canonical proof-objects of possibly different forms, and only proof-objects should fulfill propositional intentions, intentional objects must somehow encompass multiple proof-objects of a proposition (and not their forms) but still be regarded as a single object.

Given that the horizon of an intention describes the possible experience for its fulfillment, one idea that naturally comes to mind is to have it as the intentional object, especially given how Heyting (1931) explicitly speaks of possible experience. But closer examination reveals that this seemingly 'natural' answer is unsustainable philosophically. The horizon of a propositional intention must comprise all the possible canonical and non-canonical ways in which a proposition may have a proof-object. For example, the intention $A \wedge B$ corresponds to would be directed to all canonical and non-canonical proofs of the conjunction.

Surely, horizons are not objects but possible experiences, but, as we saw in Section 4.1, neither are fulfillments strictly speaking. So, perhaps horizons could be regarded as objects through reflection, following the same objectifying strategy taken in Tieszen (1989, pp.86–87). But reflection on the horizon of an intention corresponding to a proposition would require an introspective examination of all proof-objects of a proposition, which is impossible.

Intuitionistically, when we introduce a new proposition we define it by postulating new canonical proof-objects and attributing them to the proposition. Every time a new proposition is introduced we thus extend the totality of mental constructions intuited so far. Brouwer (1907, pp.148–149) states quite clearly that this totality has an open-ended nature and can never be fully circumscribed. Yet, if we could run through all proof-objects and exhaustively prescribe in advance whether or not they prove a proposition, that is, reflecting on the horizon as a whole, we could never introduce new propositions. For that we would have to introduce new canonical proof-objects in the supposedly finished totality.

For this reason the propositions-as-intentions interpretation must assume an open horizon whenever intentions corresponding to propositions are concerned. To ensure open-endedness it should not be possible to effectively determine the sort of experience required to fulfill an intention. This also shows that Tieszen (1989, pp.53–54) has to be wrong in his claim that the horizon associated with an intention prescribes in advance a rule-governed process for carrying out the fulfillment of the intended object. The open-ended reading of the horizon is, however, endorsed in van Atten (2006, p.98), but for an unrelated, different reason, since his justification of choice sequences needs to make use of an open horizon to explain the free choices involved in a non-lawlike sequence.

6.3 Sets of canonical proof-objects

Although an intention cannot be directed to the horizon of a propositional intention, perhaps we may have the set containing only every canonical proof-object of the proposition as the intentional object instead. We are thus led to the thesis that an intention corresponding to A is directed to the set of canonical proof-objects of A. So in the case of conjunctions $A \wedge B$, that includes every such $\mathsf{pair}(a, b)$, and, for disjunctions $A \vee B$, each $\mathsf{inl}(a)$ and $\mathsf{inr}(b)$.

Notice that this proposal remains somewhat close to our guess about horizons, since such a set can be thought of as a more restrictive kind of 'sub-horizon' that comprises only canonical ways to fulfill the intention with a proof-object. As expected, we must overcome a similar problem because sets for Husserl originate from mental processes, such as acts of collection, and therefore need to be objectified through reflection as indicated in PA 18–21.[9] Husserl writes:

> Wherever we are presented with a particular class of wholes, the concept of that class can only have originated through reflection upon a well-distinguished manner of combining parts, a manner which is identical in each whole belonging to that class. (PA 20)

We cannot run through all (possibly infinitely many) canonical proof-objects of a proposition. This was the problem we ran into with horizons and it reappears in this setting. But at least, unlike the horizon of the intention, the set intended must be effectively prescribed: otherwise, if from the meaning explanations of a proposition no rule tells us how to obtain its canonical proof-objects, then what we have is not a proposition after all. If this can be characterized as a 'well-distinguished manner of combining' proof-objects, then it is perhaps plausible to take sets of canonical proof-objects of a proposition as objects of reflection.

6.4 The fulfillment dilemma

Even if we grant that sets of canonical proof-objects can be objects of reflection, such sets still could not be the objects of propositional intentions. If that were the case, the intention would only be fulfilled when the set in its totality is presented to the subject. But, to give an example, the intention which $A \wedge B$ corresponds to is not in any sense fulfilled by the presence of the set of every such canonical proof-object $\mathsf{pair}(a, b)$. Only a single element of the set, a canonical proof-object of the conjunction, is a fulfillment of the intention, not the set itself. This contrast is clearest in the case of the falsum $\perp$, which is to say, the proposition defined by no canonical proof-objects. If the corresponding intention were directed to its set of canonical proof-objects it would be fulfilled whenever $\emptyset$ is intuited. But it cannot be fulfilled, as $\perp$ has no proof-object.

We find ourselves caught in what I would be tempted to describe as the 'fulfillment dilemma': on the one hand, the objects of intentions corresponding to propositions must not be canonical proof-objects but sets of them, as already seen; yet, at the same time, only one element of the intended set should be understood as a fulfillment of the intention. Note that Husserl's account of states of affairs as intentional objects is not prone to this dilemma. Their fulfillment

[9] This theory of sets is, however, abandoned by Husserl shortly after the publication of *Philosophy of Arithmetic* Husserl (2003). Regardless, I want to examine the issues in connection with this particular view because it is closer to the theory of sets advocated by (Tieszen, 1989, p.143).

simply consist in the experience of the state of affairs precisely as the subject intended. The object intended is precisely the one that needs to be experienced for the fulfillment.

It is far from clear in what sense a canonical proof-object of a proposition should be viewed as a fulfillment of an intention that corresponds to it, if we admit, as it seems we must, that the intention is directed to the set of canonical proof-objects of the proposition. The situation is in fact much more complicated for non-canonical proof-objects. They are not even elements of the intended set but are supposed to fulfill propositional intentions too.

7 Concluding remarks

Objections to the propositions-as-intentions interpretation so far have denied from the start that Husserl's views are amenable to an intuitionistic reading. In this paper, I grant this as a possibility and provide a critique of the interpretation in its own terms. I argue that (1) without appealing to propositional intentions, on pain of circularity, it is hard to determine what intention a proof-object should be an understood as a fulfillment of; (2) it is unclear, due to the fulfillment dilemma, what the object of an intention corresponding to a proposition is. The success of the propositions-as-intentions interpretation depends on a solution to these difficulties.

One interesting direction for further research would be to turn to atomic propositions and investigate the phenomenological interpretation of natural numbers, sets, and choice sequences, and their operations. In particular, Tieszen (1989) leaves out the intuition of infinite sets, so, his account is insufficient to explain the status sets of canonical proof-objects supposedly have as mathematical objects.

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