Truth and Proof without Models: A Development and Justification of the Truth-valuational Approach

Hanoch Ben-Yami

ABSTRACT. I explain why model theory is unsatisfactory as a semantic theory and has drawbacks as a tool for proofs on logic systems. I then motivate and develop an alternative, the truth-valuational substitutional approach (TVS), and prove with it the soundness and completeness of the first order Predicate Calculus with identity and of Modal Propositional Calculus. Modal logic is developed without recourse to possible worlds. Along the way I answer a variety of difficulties that have been raised against TVS and show that, as applied to several central questions, model-theoretic semantics can be considered TVS in disguise. The conclusion is that the truth-valuational substitutional approach is an adequate tool for many of our logic inquiries, conceptually preferable over model-theoretic semantics. Another conclusion is that formal logic is independent of semantics, apart from its use of the notion of truth, but that even with respect to it its assumptions are minimal.

Table of Contents

One difficulty in the Fregean theory is the generality of the words 'concept' and 'object'. For even if you can count tables and tones and vibrations and thoughts, it is difficult to bracket them all together.
(Wittgenstein, PR 119; cf. PG 307 = BT 552)

Preface

I decided to make this work publicly available and not try to publish it in its current form, for several reasons.

I started working on the truth-valuational, substitutional approach (TVS) in late 2011, thinking it can provide a valuable alternative to the currently dominant model-theoretic semantics (MTS). At the time, I had, like most contemporary logicians, only a modest knowledge of the literature on the subject, published mainly around the seventies. I was writing this draft while gradually catching up with that literature, and consequently I occasionally reinvented the wheel in my proofs. In particular, it took me some time to obtain Hugues Leblanc's 1976 book, *Truth-value Semantics*. My university library ordered a used copy which I and some of my students read only after I had written and used in my classes a quite extensive version of this paper, and I think some students even suspected I took some things from Leblanc without due credit. The important thing, however, is that much of the formal work in this paper is contained in earlier publications, at least in its essentials. Accordingly, it repeats some results the way one expects from a textbook and not from an original research piece.

On the other hand, the philosophical or conceptual discussions contained here are, I believe, original. They both defend TVS against objections found in the literature and – unlike most of the literature on it – also claim it has conceptual *advantages* over MTS. Such material might be essential for the rehabilitation of TVS.

I have used TVS in some of my published works on the Quantified Argument Calculus (Quarc), as has since been done by others as well (Ben-Yami 2014, 2020; Pavlović & Gratzl 2019, 2023; Yin & Ben-Yami 2023; Ben-Yami & Pavlović 2023). In some of these works some of the mentioned objections are addressed. However, I thought it is worthwhile having them together at one place, as well as developing TVS independently of Quarc, applied to widely used calculi.

So, the purpose of this work is to provide an introduction and justification of the truth-valuational, substitutional approach. I think TVS should be better known and have much wider currency in logic than it currently has. I am also not familiar with any similar work in the literature. Leblanc wrote a survey forty years ago for Gabbay and Guenthner's *Handbook of Philosophical Logic* (Vol. I, 1983), which is still helpful, but it contains next to no conceptual discussion of the kind contained in this work, and our formal approaches are different in some important respects.

I didn't try to be *original* in this work; rather, I wanted to write a *useful* piece.

I shall probably update the paper now and then. So, if cited, please cite the full date of the version used, found in its footer. I also hope to transfer the work at some stage to a different processor, which will facilitate many formatting features.

My work on TVS has gained from teaching it, and from feedback and work of my students. I especially like to mention my supervisees, past and present, Edi Pavlović, Péter Susánszky, Simon Vonlanthen, and Hongkai Yin, who have also incorporated TVS in some of their work, which often goes beyond what is found here.

I. Reasons for Discontent with Model Theory, and Motivation for an Alternative Approach

Model Theory is currently used in the standard proofs on the system of predicate logic, modal logic and other logic systems, proofs that relate provability to validity. Namely, properties of the proof system, primarily soundness and completeness, are proved using Model Theory. The significance of Model Theory for logic, which goes beyond these proofs, is anchored in this role.

Validity concerns relations between possible truth values: an argument is valid just in case its conclusion is true if its premises are. Since methods of proof in formal logic address only the question of which sentences can be written after which in virtue of their form, irrespective of the truth values of these sentences, some theory that discusses truth is necessary in order to relate provability to validity. Model Theory does that by *providing a theory of truth*: it says what should be the case in the world, or in the domain our sentences are about, for these sentences to be true. By means of it we can determine the possible relations of the truth values of the premises to that of the conclusion, and in this way determine the validity of arguments. Since Model Theory relates truth to form, while proof theory relates provability to form, together they can relate provability to truth and, as a result, to validity.

Yet Model Theory is a problematic theory of truth. Let us consider a few valid arguments of the same form:

1. Socrates is a man
   All men are mortal
   Hence, Socrates is mortal

2. Courage is a virtue
   All virtues are rare
   Hence, courage is rare

3. The defeat of the Persians was a great victory
   All great victories are bought at a high price
   Hence, the defeat of the Persians was bought at a high price[1]

Due to their identical form, logic should explain the validity of these three arguments in the same way. Model Theory does that by describing in the same terms the situation in the world or in a domain for the three arguments or their formal translations; and by describing in the same terms the way the different sentences relate to the world or to the domain. According to it, 'Socrates' stands for a particular person, who is among the particulars of a domain; 'courage' stands for a particular character trait, which is among the particulars of a domain; and 'the defeat of the Persians' stands for a particular event, which is among the particulars of a domain. Particular persons and their names, particular character traits and their names, particular events and their names: all are treated in the same way by Model Theory. The name designates (refers to, stands for…) a person, a character trait, an event, or whatever; and the domain is constituted by particulars or objects of any sort. The interpretations or structures of Model Theory, writes Hodges, explain 'what *objects* some expressions *refer to*' (2009: §1; italics added).

[1] Unlike the former two arguments, this one uses the past tense in its conclusion and first premise, and not present tense in all three sentences. But this difference, whether or not it is significant for the logic of natural language, is irrelevant to the aspects of Model Theory I am considering here.

But surely the aspects of the use of 'courage' that make it mean what it means are quite unlike what makes 'Socrates' the name of Socrates, and neither are like what makes 'the defeat of the Persians' stand for what it stands for (and examples can easily be multiplied). There is no relation that 'reference', 'designation' or any other related term expresses in *all* these cases.

Moreover, to say that a domain may contain persons, character traits or events, or that all these are objects in some sense, is to impose a merely apparent uniformity where the differences are great and the affinity negligible. There is no sense of 'domain' in which it can be indifferently applied in all these cases, and no sense of 'objects' (or 'particulars' or 'things') in which Socrates, courage and the defeat of the Persians are all objects.

We *can* specify a meaningful notion of reference and of an object or particular; by 'reference' we may mean, say, the relation of a *person's name* to the person it names, and we can then consider people as 'objects' or particulars. But then our notion of reference would not apply, for instance, to the relation of names of character traits to what they name, and our notion of object or particular would similarly not apply to character traits. If we specify meaningful notions of reference and object, our explanation of the validity of inferences cannot apply, as it should, to all arguments of the same form.

But surely you won't deny that 'Socrates' refers *to Socrates, 'courage' to courage, and so on; so, there is a common notion of reference that applies to these terms.* – When we write this, the use of inverted commas misleads us. It makes us think that when the words occur without the inverted commas, we are using them to talk about courage, say, and so on. But in this case we in fact ignore what these words mean when making our assertion, which amounts only to saying that 'courage' is used the way it is used, and so on. The words are only mentioned, in either of their occurrences. Accordingly, no meaningful use of 'reference' was made here.

The affinity of all the arguments above is in the *grammar* of their sentences alone, and not in what their sentences are about or in what it takes for them to be about it. In all these cases we have, for instance, a singular term in the same grammatical role, although the ways these different terms acquire meaning are radically different. Accordingly, in order to account in the same way for the validity of these different arguments of the same form, we should rely only on their uniform grammar. We should not rely on any alleged uniformity in the way their sentences are endowed with meaning or in what these sentences say. Yet the latter is exactly what Model Theory does.

Independently of these rather philosophical reasons for discontent with Model Theory, let us consider an additional, formal reason for dissatisfaction with the way Model Theory relates validity and provability. Validity concerns a *relation* of truth values, the one holding between the possible truth values of the premises and those of the conclusion. But Model Theory basically provides something different: a theory of when a sentence is true, and not a theory of the relations between truth values of sentences. From a logical point of view, we get in this way more than we need. From a logical point of view, if a theory could provide an account of *truth value relations*, without committing us to a theory of *truth*, and be sufficient for our meta-logical needs, it would be more appropriate, as carrying no additional unnecessary theoretical commitments but being closely tailored to what is required. So, independently of the doubts above concerning Model Theory as a theory of meaning, we have a logico-mathematical reason for developing a different, more minimal theory (minimalism being a mathematical virtue), one directly relating the truth values of formulas, which should also be sufficient for relating provability and validity.

Such a theory is familiar from the Propositional Calculus. Truth tables do not say what should be the situation in the world or in some domain of discourse for any of the propositions or formulas to be true. Rather, they only specify what the *relation* is between the truth value of a complex formula and the truth values of the formulas that immediately compose it. For instance, $p \wedge q$ is true in case p is true and q is true, and false otherwise; and nothing is said about what should be the case in the world for p to be true, or on how p may come to acquire a truth value. And this truth table treatment is of course sufficient for proving the adequacy, namely the soundness and completeness of the Propositional Calculus with its standard proof systems.

I develop in this paper such an approach – the truth-valuational approach – to the first order Predicate Calculus with identity and to Modal Propositional Calculus. I shall prove the adequacy of these calculi by means of an account of the relations of truth values of formulas. This would establish that from this logical point of view, Model Theory is redundant for these calculi.

One might decide to call also an account as the one I am going to develop *a model*, as is occasionally done, and in this way have a non-problematic notion of model. In this paper, however, I restrict this notion to accounts that involve both a domain and a relation of designation of particulars, as is done by Model Theory.

Another, different sense of 'model' and 'interpretation' is that according to which we replace the meaningless symbols of our formulas by meaningful terms. Our formulas, before they are interpreted, are schemas which represent the logical structure of a variety of possible languages. When their symbols – *a*, *b*, *c*…, P, Q, R … – are replaced with meaningful words – say 1, 2, 3… *even*, *odd*, *prime* … – we get a meaningful language that has the logical properties we have been investigating. It is *not* that *a* designates 1, *b* 2, P the set of even numbers and so on, and in this way the formal language becomes meaningful; rather, those symbols are *replaced* by these meaningful terms. And how *these* terms get their meaning is irrelevant. Different possible terms might get their meaning in diverse ways, as was discussed above. We *presuppose* meaning, and do not provide a theory of meaning or a semantics.

With this sense of *model* or *interpretation*, there need not be any mentioning of a domain, and *interpretation* is not used in the sense of the interpretation function of model theory, which assigns objects and sets to names and predicates. Rather, the interpretation replaces a language-structure by a language. Even if we consider such a replacement a model, we should note that we then use this term not in the sense it is used by Model Theory. This model is not what we talk *about*, but a specific *talking*.

Models in this sense can serve to demonstrate various logical properties of a language with a given logical structure. For instance, if we wish to show that a linear order need not be dense, we consider a language with the two-place predicate 'greater than' and with the names for natural numbers as our constants, and show that the required properties hold for it. We can similarly show the independence of certain formulas, and more. But again, it would be a misunderstanding to say that in this case we found a domain in which to interpret our given formal language.

Additional, independent reasons for interest in Model Theory, say as a field of mathematical investigation, are possible. But they would not bestow on it the role in logic it currently has.

The question, which facts about the world and language's relation to it make this or that sentence true, is of course legitimate. Yet it need not be asked by formal logic, which should focus on the possible relations of truth values of sentences due to their forms, irrespective of their relations to the world in virtue of which they acquire these truth values.

II. Introduction of the Truth-valuational Substitutional Approach to Quantification

The basic idea of the truth-valuational approach is to provide rules that relate the truth value of a complex sentence to those of sentences from which it can be generated. In this sense, it is a theory of truth value relations, and not a theory of truth. A set of basic sentences are ascribed truth values, and the truth values of all other sentences are determined based on this assignment. We are familiar with such approach from the Propositional Calculus: the propositional variables form the set of basic sentences, all other sentences are recursively generated by means of sentential connectives, and there are rules relating the truth value of a generated sentence to those of the sentences from which it is generated. When considering other calculi, their specific properties should determine how this approach can be applied to them. As mentioned above, I shall develop it in this paper for the Predicate Calculus with identity and for Modal Propositional Calculus. These applications can serve as a model for additional ones. Such applications to other calculi are sometimes straightforward – see for instance the application to Quarc in (Ben-Yami 2014) and elsewhere – but successful applications in specific cases cannot guarantee general applicability. The limits of the approach should therefore be determined by further attempts to apply it to other calculi.

I start with considerations on the application of the truth-valuational approach to quantified sentences. The application to modal calculi will be considered in Section VII.

In the case of a quantified sentence, the truth-valuational rules relate its truth value to those of its substitution instances. Here is an illustration within natural language. The sentence

4. Every philosopher is mortal

is true just in case every substitution of a name of a philosopher for 'every philosopher' yields a true sentence. Namely, all sentences like

5. Socrates is mortal

should be true for sentence (4) to be true.

The corresponding rules for the relation of the truth value of a quantified sentence to those of its instances in the Predicate Calculus will be:

> **∃-Rule**: A sentence of the form $\exists x\varphi(x)$ is true just in case so is some sentence of the form $\varphi(a)$, where a is any name that has replaced all occurrences of x in $\varphi(x)$.
>
> **∀-Rule**: A sentence of the form $\forall x\varphi(x)$ is true just in case so is every sentence of the form $\varphi(a)$, with a as above.

Similar rules were suggested by Barcan Marcus (1962), Dunn and Belnap (1968), Leblanc (1968), Kripke (1976: 330) and others. I shall formulate these rules more rigorously once I introduce the formal language in the next section.

I assumed in the illustration from natural language above that every philosopher *has a name*. This assumption is necessary, for otherwise, in case there is an unnameable immortal philosopher, sentence (4) is false although it has no false substitution instance. However, while such an assumption might be justified when the subject is philosophers, certainly we cannot assume a similar nameability when we talk about all grains of sand in the world, say, or all real numbers. Isn't this a major difficulty with the substitutional approach?

Despite common claims to the contrary, going back at least to Quine,[2] this nameability assumption is *not* a disadvantage of the substitutional approach vis-à-vis Model Theory. Roughly speaking, the objectual-referential semantics of Model Theory has analogously to assume that all particulars in the domain *can be the values of interpretation functions*. More precisely, Model Theory assumes that for every interpretation function i, for every variable v and for every particular p in the domain, there is an interpretation function i^*, different from i at most in its value for v, for which $i^*(v) = p$ (variations on this assumption will face corresponding issues). This assumption is as problematic as that of the substitutional approach, this time with respect to interpretation functions instead of names. Like naming, interpretation is a human activity; and those who cannot name everything cannot provide all possible interpretations either. The substitutional approach's treatment of names is no more in need of justification than Model Theory's of interpretations.

But both approaches can resolve the issue in the same way. What the formal approach does is provide an abstraction or idealisation, for the purposes of a formal study, of a method in which the truth value of a quantified sentence is determined in practice. In both informal and formal contexts, we may determine it by substituting names for the relevant part of the sentence; and in formal contexts we may also do that by providing variable interpretations as described above. And when we consider an idealised formal system – idealised not in the sense of being *better* than actual languages or actual applications, but more in the sense of a frictionless limit – we abstract also from the limitations of actual practice.

The criticism of Model Theory in the former section was not for its being an idealisation of the kind just described. The main criticism there was that Model Theory tries to account for uniform logical relations between kinds of sentence by assuming uniform relations between the terms that such sentences contain and the world (designation of objects and of sets), relations that in fact obtain in any meaningful sense only in a limited group of cases. Namely, Model Theory looks at the wrong place (word–world relations) to account for these uniform logical relations. Because of this feature of the theory, it cannot be justified on the lines mentioned above.

In addition, in the standard Henkin proofs of the completeness of the calculus, witnessing constants are introduced so that every statement of the form $\exists x\varphi(x)$ is true just in case so is its substitution instance $\varphi(a)$, with a the Henkin witnessing constant for $\exists x\varphi(x)$.[3] The case is similar when the proof uses Lindenbaum's Lemma. Within the framework of Model Theory, this amounts to the assumption that if a quantified formula has instances, then at least some of them can be named. Accordingly, for the purposes of the acceptability of standard proofs on the system, Model Theory makes a nameability assumption close to that made by the substitutional approach.

Peter van Inwagen has claimed that substitutional quantification, if it is not to mean what is meant by objectual-referential quantification, has not been given any sense (van Inwagen 1981; cf. van Inwagen 2004). Introducing the symbol Σ for the existential quantifier read substitutionally, and keeping $\exists$ for the same quantifier read Model-theoretically, he claims that the truth conditions of a quantified sentence using Σ are specified, for instance, as follows:

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[2] The earliest discussions of this issue of which I am aware are found in Quine's papers, 'Ontological Relativity' and 'Existence and Quantification', reprinted in (Quine 1969).

[3] This is not the place to present Henkin's proof, which however is developed in a modified form in Section V. Readers not familiar with the proof would then be able to assess the claim made in this paragraph.

'Σ*x* *x* is a dog' is true … iff ∃*x* *x* is a term and ⌜*x* is a dog⌝ is true

But, he continues, in addition it is said that 'Σ*x* *x* is a dog' does not *mean* the same as '∃*x* *x* is a term and ⌜*x* is a dog⌝ is true'. What it *does* mean, he adds, is not specified. Accordingly, no meaning has been specified for 'Σ*x* *x* is a dog'.

However, van Inwagen's specification of the truth conditions should be rejected: advocates of substitutional quantification do not consider themselves as using objectual-referential quantification in the metalanguage to explain substitutional quantification in the formal language. Rather, they consider its use in the constructed formal language as modelled on that in natural language, which is used to explain it. Van Inwagen writes that 'our understanding of the (objectual) quantifier-variable idiom resides in our ability to translate sentences couched in it into quantificational idioms of which we have a prior grasp, *viz.* those of ordinary language'. But this can be maintained by the advocate of substitutional quantification as well, who may also claim that ordinary language quantification is better explained substitutionally.

Lastly, van Inwagen is wrong to think that he has a clear understanding of quantification as objectual-referential. As we saw in the previous section, these concepts of object and of reference are hollow. There is no sense of 'object' which applies to human beings, fictional characters, numbers, events, character traits, 'tables and tones and vibrations and thoughts' – yet we quantify when we talk of all these things. An objectual-referential account of our general quantificational language is not an option.

An attempt to replace model-theoretic semantics with a truth-valuational approach should address an argument derived from Kreisel, which tries to establish that for first-order logic, intuitive validity is captured by model-theoretic validity.[4] Kreisel himself was interested in a different issue, the vindication of the intuitive notion of validity, but in the process he provided an argument for the above equivalence (1967: 89-91). Kreisel's 'squeezing' argument proceeds roughly as follows: from the time of Frege on, the derivation rules of the Predicate Calculus have been recognised as intuitively valid, so

Derivability → Intuitive Validity.

On the other hand, a Predicate Calculus argument for which there is a counter-model is intuitively invalid, so by contraposition:

Intuitive Validity → Model-Theoretic Validity

In addition, as Gödel and Henkin proved, the calculus with model-theoretic semantics is complete:

Model-Theoretic Validity → Derivability

Combining these entailments, we get,

Intuitive Validity ↔ Model-Theoretic Validity

And so, the model-theoretic notion captures our intuitive notion of Predicate Calculus validity.

[4] I am indebted to Ran Lanzet for drawing my attention to this argument and for helpful discussion.

What this line of thought shows is that, for the Predicate Calculus, intuitive validity and model-theoretic validity *coincide*; what it *does not* show is that the *meaning* of the former is captured by the latter (nor did Kreisel intend it to show that). If our models were restricted to domains consisting only of pebbles, as many as we wish, we would still have the equivalence between pebble model-theoretic validity and intuitive validity, yet the former obviously does not capture the latter. (In that case we would also have meaningful concepts of object and designation: object = pebble, designation = talk about pebbles.) Similarly, validity in Aristotelian logic coincides with validity according to Venn diagrams, yet the latter does not give the meaning of the former. Accordingly, what Kreisel's argument establishes is that to be faithful to our intuitive notion of validity, a truth-valuational approach should preserve the set of Predicate Calculus arguments that are model-theoretically valid (and we shall see that it does); it does not show that the model-theoretic concept of validity captures the intuitive notion.

I proceed to the formal part of the paper.

III. A Formal System, Validity and Soundness for the Predicate Calculus

Similar formal systems have been developed several times in the past, in most detail in (Leblanc 1976), and formally, much of what I do below is not new. I still develop such a system here for several reasons. First, the specific version below is not found in any earlier publication. This is partly because I make several choices which I find better justified conceptually than those found in the literature. The system which results is therefore, I think, better justified in this respect as a framework for logic than earlier ones. Some of my choices are not of this nature: for instance, the truth-valuational approach need not employ the specific proof method I use. The discussion below makes clear which features of the system I take to be essential to the approach and which could be replaced by others. Secondly, I address along the way conceptual issues and possible objections – this being one of the main objectives of this work – and these require familiarity with the formal system and reference to it. This makes the development of the system practically necessary, especially as many readers might not be familiar with earlier work. All the same, given the availability of earlier work, as well as the conceptual focus of this paper, I shall allow myself to be concise at places.

The Language

For reasons made clear below, when validity is discussed, TVS requires varying the set of individual constants of the language. Formally, that means that we shall define a language without specifying its set of individual constants.

Definition (Language). A *language*, $\mathfrak{L}$, of the Predicate Calculus (PC) consists of:

- Individual constants: a non-empty, denumerable set of symbols, disjoint from all other symbols listed below.
- for each $n > 0$, n-ary predicates: P_1^n, P_2^n, …
- connectives: ¬, ∧, ∨, → (but we shall use only ¬ and → in proofs)
- variables: x_1, x_2, …
- quantifiers: ∀, ∃
- parentheses: (,)

The set of individual constants can also be finite, as long as it is not empty, but making it denumerable simplifies some proofs. We shall use *a*, *b*, *c*, …, possibly with subscripts, for arbitrary individual constants; *x*, *y*, *z*, … for arbitrary variables; *P*, *R*, *S*, … for arbitrary predicates; φ, ψ, … for formulas. We shall usually use 'names' and not 'individual constants', apart from formal definitions. And occasionally, instead of talking of language $\mathfrak{L}_2$ different from $\mathfrak{L}_1$ in its set of names, we shall talk about adding or removing names from a language.

Definition (Formulas). Let $\mathfrak{L}$ be a PC language. The *formulas* of $\mathfrak{L}$ are:

- Atomic formulas: *Pa*, *Rab*, etc. (Atomic formulas do not contain variables.)
- Let φ and ψ be formulas, then so are ¬φ, (φ ∧ ψ), (φ ∨ ψ) and (φ → ψ).
- If φ(*a*/*x*) is a formula then so are ∃*x*φ(*x*) and ∀*x*φ(*x*). (φ(*a*/*x*), here and below, is the formula in which *a* replaced all occurrences of *x* in φ(*x*).)

Parentheses shall be omitted as standard. Formulas do not contain free variables. Since the introduction of identity, =, into the system requires some theoretical discussion, I postpone it to the next section.

Proofs

My method of writing proofs in Natural Deduction is based on the system introduced by Jaśkowski (1934) and further developed and streamlined by Fitch (1954), Lemmon (1978) and others.

Definition (Proof). A *proof* is a sequence of lines of the form <L, (i), φ>, where L is a possibly empty list of line numbers, written without repetition in ascending order; (i) is the *line number* in parenthesis; φ a PC formula; and the line is written according to the derivation rules specified below. We shall usually write to the right of the formula the name of the rule according to which the line is written, possibly followed by line numbers. (Proofs in which a line can be written according to more than a single rule, or according to the same rule applied to different lines, are possible, but we shall not encounter any in this work.) That rule is the *justification* of φ. φ is said to *depend* on the formulas listed in L. The formula in the last line of the proof is its *conclusion*. If there is a proof with the formula φ as conclusion, depending only of formulas from the set $\mathfrak{S}$, then φ is *provable* from $\mathfrak{S}$, or $\mathfrak{S} \vdash \varphi$. If there is a proof in which the last formula is φ and it does not depend on any formula, then φ is a *theorem* and we write ⊢ φ.

Before listing the derivation rules, I provide a simple, intuitive example of a proof. I first note that if a formula is written as a premise then it is taken to depend only on itself, and to the left of the line-number is written the line-number of that very line. The example also uses conjunction elimination and modus ponens:

1	(1)	p→q	Premise
2	(2)	p ∧ r	Premise
2	(3)	p	∧-Elimination 2
1, 2	(4)	q	Modus Ponens 1, 3

I proceed to the definition of the derivation rules. Since in the proofs on the system we shall use only ¬ and →, I don't provide rules for the other connectives. In lists of line numbers, 'L_1, L_2' stands for the list of all line numbers occurring in L_1 or L_2. 'L – j' stands for the line numbers occurring in L apart from j. The lines before a rule's last lines need not occur in the order they occur in the schema, the lines need not be consecutive or different from each other. When convenient, I use 'L' to refer to the formulas in lines L.

Definition (Derivation Rules). The *derivation rules* below specify all the ways for writing new lines in a proof, the last line in any of the rules.

Premise

i	(i)	φ	Premise

Negation Introduction, ¬I

i	(i)	φ	Premise
L_1	(j)	ψ	
L_2	(k)	$\neg\psi$	
$(L_1, L_2) - i$	(l)	$\neg\varphi$	¬I i, j, k

Negation Elimination, ¬E

L	(i)	$\neg\neg\varphi$	
L	(j)	φ	¬E i

→I

i	(i)	φ	Premise
L	(j)	ψ	
L – i	(k)	$(\varphi \to \psi)$	→I i, j

→E

L_1	(i)	$(\varphi \to \psi)$	
L_2	(j)	φ	
L_1, L_2	(k)	ψ	→E i, j

∃I

L	(i)	$\varphi(a/x)$	
L	(j)	$\exists x\varphi(x)$	∃I i

∃E

L_1	(i)	$\exists x\varphi(x)$	
j	(j)	$\varphi(a/x)$	Premise
L_2	(k)	ψ	
$L_1, (L_2 - j)$	(l)	ψ	∃E i, j, k

Constraint: *a* does not occur in $\exists x\varphi(x)$, ψ, any of L_1, and any of L_2 apart from $\varphi(a/x)$.

∀I

L	(i)	$\varphi(a/x)$	
L	(j)	$\forall x\varphi(x)$	∀I i

Constraint: *a* does not occur in any of L.

∀E

L	(i)	$\forall x\varphi(x)$	
L	(j)	$\varphi(a/x)$	∀E i

Truth Value Assignments

Definition (Valuation). An assignment of truth values will be called a *valuation*, a function from formulas of a language to truth values. Every valuation assigns to any formula in $\mathfrak{L}$ a single truth value, T or F, in which case we say that the formula is *true* or *false*, respectively, on that valuation. Truth values are assigned to formulas recursively, as follows. For any valuation $\mathcal{V}$, every atomic formula is either true or false (notice that we do not relate these formulas to any domain). If φ is a formula, then $\mathcal{V}[\neg(\varphi)] = T$ in case $\mathcal{V}[\varphi] = F$, and $\mathcal{V}[\neg(\varphi)] = F$ in case $\mathcal{V}[\varphi] = T$; similarly for all other sentential connectives. The rules for assignments of truth values to quantified sentences elaborate the rules given in Section II for the relation of the truth value of a quantified sentence to those of its instances:

> (**∃-valuation**) $\exists x\varphi(x)$ is true on a valuation in case $\varphi(a/x)$ is true on that valuation for some individual constant *a*, otherwise it is false
> (**∀-valuation**) $\forall x\varphi(x)$ is true on a valuation in case $\varphi(a/x)$ is true on that valuation for every individual constant *a*, otherwise it is false.

It is easy to see that every assignment of truth values to atomic formulas uniquely determines the assignment of truth values to all formulas.[5]

Validity

When we come to define validity, we should avoid the mistake of identifying universal quantification *in general* with applicability to a *given* set of individual constants or names, and similarly for existential quantification. When I talk of *specific* things, say the ten books on my shelf, then if a predicate P is true of b_1 to b_{10}, these being the titles of my books, the universal quantification over it, $\forall x\mathrm{P}(x)$, is then true as well; this fact justifies the substitutional approach to the *truth* of quantified sentences. But no fixed set of names yields such equivalence independently of subject matter, and therefore no fixed set of names can yield *validity*. We shouldn't therefore define validity while assuming a fixed set of names. We do that by allowing names to be added to our language and eliminated from it, and by defining validity relative to such different sets of names. Of course, when considering the validity of a specific argument, we consider only languages containing all the names occurring in it.

Definition (Validity). An argument whose premises are all and only the formulas in the set Ψ and whose conclusion is φ is *valid* just in case every valuation that makes all formulas in Ψ true makes φ true as well, in any language whose set of names contains all names occurring either in Ψ or φ. We then write, $\Psi \vDash \varphi$. We also say that Ψ *entails* φ. In case Ψ is $\{\psi_1, \ldots \psi_n\}$, we also write $\psi_1, \ldots \psi_n \vDash \varphi$. In case a formula φ is true on all valuations in any language whose names include all names occurring in φ, φ is called *a logical truth* or *tautology*, and we write $\vDash \varphi$.

Having defined validity, we may also define its dual notion, satisfiability. A formula φ is satisfiable if and only if ¬φ is not a logical truth. Accordingly, on the truth-valuational approach, a formula φ is defined as satisfiable just in case, in some language containing all its names, φ is true on some valuation. A set of formulas Ψ is satisfiable just in case, in

[5] This is proved in (Kripke 1976: 330-331) for a more general case.

some language containing all the names occurring in Ψ, all the formulas of Ψ are true on some valuation.

This definition of validity, which takes addition and elimination of names to and from the language into consideration, correctly renders arguments like the following invalid. Suppose the only name we had in our language was *a*; then any valuation that would make P(*a*) true would also make ∀*x*P(*x*) true. If we did not allow adding names to the language, then the argument with P(*a*) as its premise and ∀*x*P(*x*) as its conclusion would be valid. But since we may now add, say, the name *b*, this argument is not valid according to our definition: a valuation can make P(*a*) true, P(*b*) false and consequently ∀*x*P(*x*) false as well. And the case is the same with any fixed set of names, finite or not.

The problem with a definition of validity with a constant set of names mentioned in the previous paragraph was first noted by Dunn and Belnap (1968: 180). Although they primarily *criticise* substitutional quantification because of it, they add one paragraph, as if an afterthought, in which they write that the problem could be overcome if we use 'a notion of logical consequence which permits enrichment of the elementary language in question by the addition of new names'. They find this 'a good notion of logical consequence, for it immediately ensures […] that relations of logical consequence be preserved upon extension of a language' (183). I don't think this is a sufficient justification of the modified notion: this preservation of validity should be *derived* from what we understand by validity, and not stipulated into its definition. By contrast, I tried to provide an independent justification of my modified definition, based on the concept of universality. Moreover, these considerations brought us to allow not only adding names to the language, as Dunn and Belnap do, but also eliminating names from it. Consequently, this modified definition of validity doesn't have the ad hoc character, being tailored to overcome 'technical failings', that Dunn and Belnap's modification might seem to have.[6]

Technically, however, to get a satisfactory notion of validity it is enough if, with Dunn and Belnap, we only allow enriching the language with names, without eliminating names. This is because we do not add to the possible truth values of a set of formulas by eliminating from the language a name not occurring in any of them, as long as the language contains some names. Eliminating names cannot therefore make an argument invalid. This can be seen from the following considerations. Instead of eliminating a name *a*, we can assign to any atomic formula containing it the same truth value as that we assign to the formula in which another name, *b*, has replaced all occurrences of *a*. Informally, the formulas then say by *a* what they also say by *b*, and *a* doesn't contribute anything that *b* doesn't anyway contribute. Consequently – this is an immediate corollary of the lemma proved in the next subsection – the truth value of any formula not containing *a* will be the same as it would if *a* were eliminated from the language.

Despite this technical result, I preferred to define validity above by allowing also the elimination of names from the language, the language's names allowed to be any non-empty set of names that contains all names occurring in the argument's formulas, because this is what is justified by the idea of validity on the substitutional approach. Other, formally equivalent possibilities should be *proved.* As we shall see in Section VI, we can in fact allow adding a limited number of names per argument, determined by the number of quantifiers in the argument's formulas, and still get the same set of valid arguments. However, this result should also be *proved*, and not serve as part of the original definition of validity.

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[6] Dunn and Belnap also write there that once the modified definition of logical consequence is adopted, the calculus will be complete; but they seem not to have realised that this will require significant changes in the completeness proof, which they do not provide.

Leblanc (1968: §4) also noted the problem with the standard definition, but he adopted a different solution. Allowing denumerably many variables to his calculus (and as is standard, these function when free like our names), he relied on the compactness theorem and demanded that an infinite set of formulas Ψ will imply formula φ just in case some finite subset Ψ' of Ψ implies φ according to the standard definition, namely, every valuation that makes all formulas of Ψ' true makes φ true as well. But I don't think compactness is part of our understanding of validity, and indeed it does not generally hold in higher-order logics. I therefore find Leblanc's suggestion in his 1968 paper ad hoc and unjustified.

In his later book (1976), Leblanc changed his definition of validity, following a suggestion of Hintikka's (p. 49). He defines validity through semantic consistency (p. 19). First, a non-empty set of formulas Ψ is defined as verifiable by means of a valuation if there is a valuation that makes all formulas of Ψ true. Next, a non-empty set Ψ is *semantically consistent in the truth value sense* if there is a verifiable set of formulas Ψ_1 to which Ψ is isomorphic. A set which is not semantically consistent in this sense is *inconsistent* in the truth value sense, and φ is a *semantic consequence* of Ψ in the truth value sense – i.e., $\Psi \vDash \varphi$ – if $\Psi \cup \{\neg\varphi\}$ is semantically inconsistent in that sense. On pages 49–51 Leblanc shows the equivalence of this definition with his former one and with Dunn and Belnap's, which is also essentially the one used in this work.

Leblanc does not explain, however, why this definition through isomorphism captures or at least approximates what we intuitively understand by validity. The formal adequacy appears in this way as a kind of trick, which does provide the results in which we were interested *for other reasons*, but does not yield any understanding of what validity is or why it should be defined in this way.[7] Indeed, Leblanc writes in his Preface that

> we have limited ourselves to technical matters. A philosophical appraisal of our semantics is very much in order, but this did not seem a suitable occasion for embarking upon it. (ix)

The formal adequacy of his definition was therefore sufficient for his purposes. But as the current work tries to justify the truth-valuational approach, Leblanc's definition is unsatisfactory for it.[8]

Later still, in his 1983 paper, Leblanc changed again his definition of validity, this time adopting (Dunn and Belnap 1968)'s term extensions definition, finding it 'handier and admittedly more natural' (p. 190). Being handier, however, is an insufficient justification for the use in the definition of validity: the justification for such definitions should explain why we get in this way an intuitive or correct concept of validity. And just stating that the term extensions definition is more natural leaves unspecified why we find it so – which is what I tried to explain above in my considerations on generality, validity, and name lists.

Soundness

Before we prove that the calculus is sound, we shall prove the following useful lemma:

[7] This is true also of Garson's definition of validity for the substitution interpretation of quantification in (Garson 2013, p. 215). Garson also adds there that valuations over different languages will entail 'a major complication'. If complexity in proofs is meant, it will be seen below that this is not the case.

[8] Relatedly, although Leblanc contributed more than any other logician to the formal research into the truth-valuational approach, primarily in his book, he does not criticise Model Theory as a semantic approach, a criticism central to this work. The only worry with Model Theory which he expresses in his 1983 paper, for instance, is that Model Theory's overt or covert use of sets might be unjudicious (259).

> **Lemma**. Suppose we add to a language a name c and extend a valuation $\mathcal{V}$ so that for some name b, if $\varphi(c)$ is any atomic formula containing c, then $\mathcal{V}[\varphi(c)] = \mathcal{V}[\varphi(b/c)]$, where b replaced all occurrences of c in $\varphi(c)$. Then (1) the truth value on $\mathcal{V}$ of any formula $\psi(c)$ is that of $\psi(b/c)$; and (2) the truth value on $\mathcal{V}$ of any formula not containing c has not changed.

The idea behind the Lemma is that if the 'basic truths' in which a new name is involved are the same as those involving an old one, then the new name does the same work as the old one: it is everywhere substitutable by the latter and it doesn't affect the truth values of any of the older formulas, because whatever is said with it has already been said with the old name.

> Proof. The proof of both parts is by induction on the stages in which a formula is generated from atomic formulas. We start with (1).
>
> The base case is that of atomic formulas, and for it, it is true by definition that $\mathcal{V}[\varphi(c)] = \mathcal{V}[\varphi(b/c)]$.
>
> Next, if $\varphi(c)$ is any formula for which $\mathcal{V}[\varphi(c)] = \mathcal{V}[\varphi(b/c)]$, then $\mathcal{V}[\neg\varphi(c)] = \mathrm{T}$ iff $\mathcal{V}[\varphi(c)] = \mathrm{F}$, namely only in case $\mathcal{V}[\varphi(b/c)] = \mathrm{F}$; but this is so iff $\mathcal{V}[\neg\varphi(b/c)] = \mathrm{T}$; so $\mathcal{V}[\neg\varphi(c)] = \mathrm{T}$ iff $\mathcal{V}[\neg\varphi(b/c)] = \mathrm{T}$. Similar proofs apply to all other sentential connectives.
>
> Now assume that $\varphi(a_1, \ldots a_m)$ is a formula generated in n steps, in which the a_i are occurrences of names, and that for any case in which one or more of the a's is c, $\mathcal{V}[\varphi(c)] = \mathcal{V}[\varphi(b/c)]$. Let us look at $\exists x\varphi(x, c)$, in which x replaced one or more of the a_i's and some of the other a_i's are c. $\mathcal{V}[\exists x\varphi(x, c)] = \mathrm{T}$ iff, for some d, $\mathcal{V}[\varphi(d/x, c)] = \mathrm{T}$. (i) If $d \neq c$, then according to our assumption $\mathcal{V}[\varphi(d/x, c)] = \mathrm{T}$ iff $\mathcal{V}[\varphi(d/x, b/c)] = \mathrm{T}$, and therefore for any such d, $\mathcal{V}[\exists x\varphi(x, b/c)] = \mathrm{T}$ as well. (ii) If $d = c$, then $\mathrm{T} = \mathcal{V}[\varphi(d/x, c)] = \mathcal{V}[\varphi(c/x, c)] = \mathcal{V}[\varphi(b/(c/x), b/c)] = \mathcal{V}[\varphi(b/x, b/c)]$, and therefore $\mathcal{V}[\exists x\varphi(x, b/c)] = \mathrm{T}$. So if $\mathcal{V}[\exists x\varphi(x, c)] = \mathrm{T}$, then also $\mathcal{V}[\exists x\varphi(x, b/c)] = \mathrm{T}$.
>
> To prove the other direction, assume that in $\varphi(x, c)$, x replaced one or more of the a_i's in $\varphi(a_1, \ldots a_m)$ and one or more of the remaining a_i's are c. Let us now assume that $\mathcal{V}[\exists x\varphi(x, b/c)] = \mathrm{T}$. This is so iff, for some d, $\mathcal{V}[\varphi(d/x, b/c)] = \mathrm{T}$. (i) If $d \neq c$, then according to our assumption $\mathcal{V}[\varphi(d/x, b/c)] = \mathrm{T}$ iff $\mathcal{V}[\varphi(d/x, c)] = \mathrm{T}$, and therefore for any such d, $\mathcal{V}[\exists x\varphi(x, c)] = \mathrm{T}$ as well. (ii) If $d = c$, then $\mathrm{T} = \mathcal{V}[\varphi(d/x, b/c)] = \mathcal{V}[\varphi(c/x, b/c)] = \mathcal{V}[\varphi(b/(c/x), b/c)] = \mathcal{V}[\varphi(b/x, b/c)] = \mathcal{V}[\varphi(b/x, c)]$, and so b provides an instance that makes $\exists x\varphi(x, c)$ true.
>
> A similar proof applies to the universal quantifier, and this completes our proof of part (1).
>
> We now have to prove, (2), that the truth value on $\mathcal{V}$ of any formula not containing c has not changed. This is true by definition for atomic formulas, which constitute the base case. The proof for sentential connectives is similar to that in part (1). We consider again the case of the existential quantifier. Assume that $\varphi(a_1, \ldots a_m)$ is a formula generated in n steps for which (2) holds for any combination of $a_1, \ldots a_m$ that does not include c, and consider $\exists x\varphi(x)$, in which x has replaced at least one of the a_i's, and which does not contain c. $\mathcal{V}[\exists x\varphi(x)] = \mathrm{T}$ iff for some d, $\mathcal{V}[\varphi(d/x)] = \mathrm{T}$. (i) If $d \neq c$, then $\mathcal{V}[\varphi(d/x)]$ did not change due to the introduction of c, and so also did not $\mathcal{V}[\exists x\varphi(x)]$. (ii) If $d=c$, then according to

(1), $\mathcal{V}[\varphi(c/x)] = \mathcal{V}[\varphi(b/x)]$, and since the latter did not change following the introduction of *c*, this also holds for $\mathcal{V}[\exists x\varphi(x)]$. A proof on similar lines applies to the universal quantifier. This concludes the proof.

I shall now prove that the calculus is sound, namely, that if φ is derivable from ψ_1 to ψ_n, then if the latter are true in a language on a valuation so is the former. That is, if $\psi_1, \ldots \psi_n \vdash \varphi$ then $\psi_1, \ldots \psi_n \vDash \varphi$. It immediately follows that if $\Psi \vdash \varphi$ then $\Psi \vDash \varphi$ even if Ψ contains infinitely many formulas. It can then also be shown that if $\vdash \varphi$ then $\vDash \varphi$. If a proof establishes that $\psi_1, \ldots \psi_n \vdash \varphi$, while it is also the case that $\psi_1, \ldots \psi_n \vDash \varphi$, we say that the proof preserves validity. I assume in this paper the soundness, and later the completeness, of the *Propositional* Calculus.

We prove soundness by induction on proof length. If a proof is one line long then its single line is a premise, since all other derivation rules mention earlier lines in order to write a new one. A one-line long proof is thus of the form

1	(1)	φ	Premise

and it shows that $\varphi \vdash \varphi$, where φ is any formula. But if φ is true on a valuation then φ is true on that valuation, and therefore $\varphi \vDash \varphi$. So any one-line proof preserves validity.

We now assume that any proof with up to n lines preserves validity, and show that any n+1-line long proof also does. The n+1 line can be a premise, and then validity is preserved for the same reason we have just specified. Or it can be written according to the derivation rules of the Propositional Calculus, which we assume preserve validity. We thus have to show that the four derivation rules for quantifiers preserve validity. We start with the ∃-Introduction rule. The n+1-line proof would then look like this:

L	(i)	$\varphi(a/x)$	
L	(n+1)	$\exists x\varphi(x)$	∃I i

Suppose L are true on a given valuation $\mathcal{V}$. Then according to the inductive hypothesis (IH), as $\varphi(a/x)$ is derived from L in a proof of at most n lines, since L are true on $\mathcal{V}$ so is $\varphi(a/x)$. But according to our rule for the assignment of truth value to $\exists x\varphi(x)$, this formula is true on a valuation just in case so is some sentence of the form $\varphi(a/x)$. So if L are true on $\mathcal{V}$ then so is $\exists x\varphi(x)$, and the n+1 long proof preserves validity.

We next prove that the rule of ∃-Elimination preserves validity. The n+1 lines proof would now look as follows:

L_1	(i)	$\exists x\varphi(x)$	
j	(j)	$\varphi(a/x)$	Premise
L_2	(k)	ψ	
L_1, (L_2 - j)	(n+1)	ψ	∃E i, j, k

And the constraint is that *a* does not occur in $\exists x\varphi(x)$, ψ, any of L_1, and any of L_2 apart from $\varphi(a/x)$. Now let us assume that all premises L_1, (L_2 – j) are true on a given valuation $\mathcal{V}$, and we have to show that so is ψ. Since L_1, the formulas on which line (i) depends, do not contain any occurrence of *a*, formula (j) is not one of them, and so they are all true on $\mathcal{V}$. From IH is follows that $\exists x\varphi(x)$ is true on $\mathcal{V}$. Consequently, according to the rule for the relation of the truth value of $\exists x\varphi(x)$ to those of its instances, there is some name b such that $\varphi(b/x)$ is true on $\mathcal{V}$.

If we could replace all occurrences of a in the proof by b, line (j) would then be true and so would all L_2 and consequently it would follow that ψ is true. However, such a replacement might create a problem: some formulas among L_1, L_2, $\exists x\varphi(x)$ and ψ might contain the name b, and then the derivation would not be according to the ∃E rule. Moreover, if ∀I was used anywhere in the proof, a similar problem might arise with it. We therefore must use a name that does not occur in the proof. For this purpose, we add a name c to the language, to do the work done by b, so to say. We extend $\mathcal{V}$ so that for any atomic formula $\varphi(c)$, $\mathcal{V}[\varphi(c)] = \mathcal{V}[\varphi(b/c)]$. According to the Lemma proved above, (1) the truth value on $\mathcal{V}$ of any formula $\varphi(c)$ is that of $\varphi(b/c)$, where b replaced all occurrences of c in $\varphi(c)$; and (2) the truth value on $\mathcal{V}$ of any formula not containing c has not changed.

Let us now replace all occurrences of a in the proof by c. Since the rules of inference depend on sameness and difference of symbols in different formulas but not on the identity of a symbol, the proof up to line n is still according to the derivation rules and therefore, given IH, preserves validity. Now since $\exists x\varphi(x)$ did not contain any occurrence of a, replacing all occurrences of x in $\varphi(x)$ by a and then replacing all occurrences of a in $\varphi(a/x)$ with c gives us $\varphi(c/x)$, and line (j) is therefore now $\varphi(c/x)$. According to the Lemma, $\mathcal{V}[\varphi(c/x)] = \mathcal{V}[\varphi(b/x)]$, and it is therefore true. In addition, since no formula of L_2 apart from (j) contained a, none of these formulas changed when a was replaced by c, and according to the Lemma none changed its truth value. Accordingly, *all* formulas L_2 are now true on $\mathcal{V}$ and therefore, according to IH, so is the formula in line (k), namely ψ. But ψ did not contain a, and therefore it did not change following the substitution of c for a. According to the Lemma, ψ was therefore true on $\mathcal{V}$ also before c was added to the language. And since ψ is also the formula in line (n+1), we have proved what we wished to prove.

Next, suppose the n+1-line was written according to the ∀-Introduction rule. It would then look like this:

L	(i)	$\varphi(a/x)$	
L	(n+1)	$\forall x\varphi(x)$	∀I i

And a does not occur in any of L. Suppose L are true on a given valuation $\mathcal{V}$. Then according to IH, as $\varphi(a/x)$ is derived from L in a proof of at most n lines, and since L are true on $\mathcal{V}$, so is $\varphi(a/x)$. But we should show that for *any* constant b in the language, $\varphi(b/x)$ is true. As in the soundness proof for ∃E, if we could replace all occurrences of a in the proof by b, the formula then in line (i), namely $\varphi(b/x)$, would be true according to the IH and the proof would be done, yet such a replacement might create a problem in case b already occurs in the proof.

We therefore again add a name c to the language, and extend $\mathcal{V}$ so that for any atomic formula $\varphi(c)$, $\mathcal{V}[\varphi(c)] = \mathcal{V}[\varphi(b/c)]$. According to the Lemma, (1) the truth value on $\mathcal{V}$ of any formula $\varphi(c)$ is that of $\varphi(b/c)$, where b replaced all occurrences of c in $\varphi(c)$; and (2) the truth value on $\mathcal{V}$ of any formula not containing c has not changed. We replace all occurrences of a in the proof by the fresh c, and the proof up to line n is still according to the derivation rules and therefore, given IH, preserves validity. Since formulas L did not contain a, they did not change by the c/b replacement, and thus remain true. Accordingly, by IH, the formula in line (i), namely $\varphi(c/x)$, is true. But $\mathcal{V}[\varphi(c)] = \mathcal{V}[\varphi(b/c)]$, and so, $\mathcal{V}[\varphi(b/x)]$ is also true, as we wished to prove.

I skip the proof for the rule ∀E, which is straightforward. It follows that every proof preserves validity, and therefore that the calculus is sound. QED

Accordingly, the proof system for the Predicate Calculus is sound on the truth-valuational substitutional approach.

IV. Identity

If we try to do without models in the Predicate Calculus, we should also specify and justify truth value assignment rules for identity. We cannot say that $a = b$ is true just in case both *a* and *b refer* to one and the same *particular* in the domain, for we do not make use of the notion of reference or of any related notion, nor of that of a particular, object or thing.

This is *not* a disadvantage of the truth-valuational approach. To revert to our opening critical discussion of Model Theory, there is *no* common notion of reference and none of object with which it is correct to say that each of the three following sentences is true just in case the two terms flanking the copula *refer* to the same *object*:

Cicero is Tully
Bravery is courage
The Great War is World War I

All the same, despite their semantic differences, these three sentences can be used in inferences of the same form, for instance $a = b$, $Pa \therefore Pb$:

Cicero is Tully
Tully was an important Roman statesman
Hence, Cicero was an important Roman statesman

Bravery is courage
Courage is a virtue
Hence, bravery is a virtue

The Great War is World War I
World War I was a tragic event
Hence, The Great War was a tragic event

What is common to these three inferences is not the way their sentences relate to the world but their structure. In addition, these inferences are valid for the same reason. Accordingly, logic, being the science of inference, should not concern itself with how identity statements relate to the world but with their function within language and the way their truth values relate to those of other statements. The notion of a model is logically redundant also when identity statements are concerned.

From the formal system's point of view, an identity sentence $a = b$ is true on a truth value assignment just in case both names function in the same way on that assignment or valuation. In this way, the formal system does not commit itself to any fact about the relation of identity statements to the world.[9] Since, given the rules listed in the

[9] This view of identity answers the difficulty Garson thought truth value semantics faces when trying to account for identity (2006: 266–8). Garson thought that $a = b$ should be explained as true in case the terms *refer to the same thing*, and truth value semantics obviously lacks such resources. He did not consider the explanation provided here, relating to the terms' identical functioning in the language on a valuation. Apart from this interpretative aspect, my formal treatment of identity is close to that of Garson.

previous section, a valuation is determined by the truth values it assigns to atomic formulas, this means that if a valuation makes $a = b$ true, the truth value it assigns to an atomic formula in which b has replaced a should not be affected by this substitution.

With this understanding of identity in mind, I proceed to introduce it into the calculus in this section, before proving in the next one the completeness of the Predicate Calculus with identity. The formation rules for identity are the standard ones: for any two names a and b, not necessarily different, $a = b$ is a formula of the language. A formula of the form $a = b$ is considered an atomic formula, '=' being a two-place predicate. The rules for valuations that relate to identity are as follows:

> (**Law of Identity, LoI**) $a = a$ is true on all valuations.
>
> (**Indiscernibility of Identicals, IoI**) If $a = b$ is true on a valuation $\mathcal{V}$ and $P\ldots a\ldots$ is an atomic formula containing a, then if $P\ldots a\ldots$ is true on $\mathcal{V}$ then so is $P\ldots b/a\ldots$, where b has replaced all or some occurrences of a in $P\ldots a\ldots$.

(A more cautious IoI rule would allow b to replace only a single occurrence of a, and derive the rule for the case in which b replaces several occurrences of a. Since the rule as formalised above follows immediately from this more cautious formulation, I use it to save space below.) The Indiscernibility of Identicals rule is a constraint on valuations of atomic formulas, and once introduced the truth values of different atomic formulas cannot generally be assigned independently of each other. For instance, a valuation that makes $a = b$ and Pa true but Pb false is unacceptable.

I did not stipulate in IoI that also if P(… a …) is *false* then so is P(… b/a …), although this is part of what one would expect from indiscernibility. I didn't do that since it follows from the two rules above. First, assume $a = b$ is true on a valuation $\mathcal{V}$. Then, since $a = a$ is true on $\mathcal{V}$ (LoI), replace in this formula the first occurrence of a by b, and we get that $b = a$ is true on $\mathcal{V}$ (IoI). So if $a = b$ is true on $\mathcal{V}$, then so is $b = a$. Accordingly, if $a = b$ is true on $\mathcal{V}$, then since $b = a$ is also true, it follows that if $P\ldots b/a\ldots$ is true on $\mathcal{V}$ then so is $P\ldots a\ldots$ (IoI). So if $a = b$ is true on $\mathcal{V}$ and $P\ldots a\ldots$ false, $P\ldots b/a\ldots$ is also false.

The Indiscernibility of Identicals rule *has* to be formulated for atomic formulas, and not for any formula, since we have rules that determine the truth values of non-atomic formulas according to those of atomic formulas. However, it can be proved that the Indiscernibility of Identicals generalises, namely, that if $a = b$ is true on a valuation, then the truth value of $\varphi(a)$ is the same as that of $\varphi(b/a)$ for any formula φ. I shall now provide a sketch of the proof (see Mendelson 1996: 96).

We need to show that if $a = b$ is true on $\mathcal{V}$, then the truth value of $\varphi(a)$ is the same as that of $\varphi(b/a)$ for any formula φ, where b replaced all or some occurrences of a in $\varphi(a)$. We do that by induction on formula complexity, C(φ).

Definition (Complexity). We define the *complexity* C of an atomic formula as zero, that of $\neg\varphi$ as $1+C(\varphi)$, that of $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$ and $(\varphi \rightarrow \psi)$ as $1+\text{Max}\{C(\varphi), C(\psi)\}$, and that of $\exists x\varphi(x)$ and $\forall x\varphi(x)$ as $1+C(\varphi(a/x))$. (Note that substituting one name for another in a formula does not change its complexity.)

Since IoI was defined for all formulas of complexity 0, and given its generalisation to the case of falsity which we have just proved, the claim holds for the induction base. Suppose it holds for all formulas of complexity n or less, and let us prove it for formulas of complexity n+1. If $\varphi(a)$ is of the form $\neg\psi(a)$, with $\psi(a)$ having complexity n, then the truth value on a valuation of $\varphi(b/a)$ is true (false) just in case that of $\psi(b/a)$ is false (true);

but the truth value of the latter, being of complexity n, is identical to that of $\psi(a)$; and the truth value of $\psi(a)$ is false (true) just in case that of $\varphi(a)$ is true (false); so the truth value of $\varphi(b/a)$ is the same as that of $\varphi(a)$. Similar proofs apply to the other connectives. Suppose next that $\varphi(a)$ is of the form $\exists x\psi(x, a)$, with $\psi(c/x, a)$ of complexity n; so $\varphi(b/a)$, namely $\exists x\psi(x, b/a)$, is true on a valuation just in case there is some name c such that $\psi(c/x, b/a)$ is true; but any formula $\psi(c/x, b/a)$ is of complexity n, and therefore is true just in case so is $\psi(c/x, a)$; so some formula of the form $\psi(c/x, b/a)$ is true just in case so is some formula of the form $\psi(c/x, a)$; but some formula of the form $\psi(c/x, a)$ is true just in case $\exists x\psi(x, a)$, namely $\varphi(a)$, is true; as we wished to prove. Similarly for the case in which $\varphi(a)$ is of the form $\forall x\psi(x, a)$, which completes our proof.

I next add two derivation rules involving identity, =I and =E:

=I

	(i)	$a = a$	=I

=E

L_1	(i)	$a = b$	
L_2	(j)	$P\ldots a\ldots$	
L_1, L_2	(k)	$P\ldots b/a\ldots$	=E i, j

Unlike the standard rule for elimination of identity, I limited the =E rule to atomic formulas alone, to make the derivation rules and valuation ones for identity match each other. However, it can be proved by induction on formula complexity that the rule generalises to any formula φ, namely that $a = b, \varphi(a) \vdash \varphi(b/a)$. I shall now sketch the proof.

The induction basis is complexity zero, namely atomic formulas, where this is simply an application of the rule =E.

Suppose the claim is true for any formula of complexity not greater than n, and let us prove it for any formula of complexity n+1. If the formula of complexity n+1 is of the form $\neg\varphi(a)$, then $\varphi(a)$ is of complexity n, and IH holds for it, and also for $b = a, \varphi(b/a) \vdash \varphi(a)$. We proceed as follows:

1	(1)	$a = b$	Premise
2	(2)	$\neg\varphi(a)$	Premise
3	(3)	$\varphi(b/a)$	Premise
	(4)	$a = a$	=I
1	(5)	$b = a$	=E 1, 4
1, 3	(i)	$\varphi(a)$	provable from (5) and (3) according to IH
1, 2	(i+1)	$\neg\varphi(b/a)$	¬I 3, i, 2.

Similar proofs can be provided for the other sentential connectives. I shall now prove the claim for the case in which the formula of complexity n+1 is of the form $\forall x\varphi(x, a)$, where φ is of complexity n and so IH holds for it.

1	(1)	$a = b$	Premise
2	(2)	$\forall x\varphi(x, a)$	Premise
2	(3)	$\varphi(c/x, a)$	∀E 2; c is not b and does not occur in $\forall x\varphi(x, a)$.
1, 2	(i)	$\varphi(c/x, b/a)$	Provable from (1) and (3) according to IH
1, 2	(i+1)	$\forall x\varphi(x, \mathrm{b}/a)$	∀I i.

A similar proof can be provided for the existential quantifier. This completes the proof.

Given our valuation rules it is simple to check that these rules preserve validity and that the proof system remains sound when they are added to it. Notice also that now a one-line proof can be written not only according to the premise rule but also according to the =I rule. Since according to LoI every valuation makes $a = a$ true, it follows that $\vDash a = a$, and any one-line proof still preserves validity.

Having introduced identity into the system, we can answer a difficulty which was raised against the substitutional approach by Harry A. Lewis (1985).[10] We have defined the relation between the truth value of a quantified sentence and those of its instances for the universal and existential quantifiers – the 'standard' quantifiers used in logic – but when we come to extend the approach to other, 'nonstandard' quantifiers (the terms are Lewis's) – for instance, 'two', 'many' or 'most' – as we obviously should, a simple generalisation would not work. To illustrate from natural language, we cannot say that

6. Two men married Olivia Langdon

is true because its following two substitution instances are true,

7. Mark Twain married Olivia Langdon
8. Samuel Langhorne Clemens married Olivia Langdon.

This is because the two substituted names name the same person.

To overcome this difficulty, one might have suggested to stipulate that no two names designate the same person or, more generally, object. This stipulation, however, is ad hoc and unacceptably restrictive as far as name use is concerned; it would make the only identity sentences that are true those of the form $a = a$ and thus trivialise identity; and it uses the very concepts we wanted to avoid, namely the overly general 'designate' and 'object'. This stipulation shouldn't therefore be adopted.[11]

The way to resolve the difficulty is different. All these nonstandard quantifiers presuppose the concept of identity: in order to know whether we counted two items, many items, or most items we need to know whether we counted *the same* item more than once.

This is seen in the case of numerical quantifiers by the possibility of defining them by means of standard quantifiers *and identity*. The dual quantifier $\exists_2$, for instance, is defined as follows:

$$\exists_2 x\varphi(x) \leftrightarrow \exists x\exists y(\neg x = y \wedge \varphi(x) \wedge \varphi(y)).$$

Once defined this way, it is clear that $\exists_2 x\varphi(x)$ is true just in case it has two substitution instances a and b for which $a = b$ is false.

Generally, for quantifiers that presuppose the concept of identity, we should use only names for which $a = b$ is false. For instance, if we introduce a quantifier M for 'most' into the calculus,[12] then $\mathrm{M}x\varphi(x)$ is true just in case most substitution instances of the following kind are: for all substitution instances a and b, $a = b$ is false; and there's no c that hasn't been substituted for x in $\varphi(x)$ for which $a = c$ is false for all names a that have been so substituted. We can call the set of names generated in this way, a *maximal substitution set*. Since all these nonstandard quantifiers involve the notion of identity, the

[10] I have addressed this issue in (Ben-Yami 2022: §2.7.2).

[11] This stipulation is suggested and rejected, for some of these reasons, by Lewis.

[12] As is well-known, this quantifier cannot capture natural language's 'most', but this is a different issue.

suggested resolution of the difficulty is not ad hoc. And unlike the 'one name per object' stipulation, neither is it restrictive as far as name use is concerned, nor does it trivialise identity or use the concepts of designation or object.

V. The Completeness of the Predicate Calculus[13]

The completeness proof below is based on Henkin's standard proof (Henkin 1949). As will be seen, it is somewhat simpler than the standard proof; the reasons for this will be discussed later.

Henkin's proof consists of a few stages: adding witnessing constants; defining Henkin's theory; proving the elimination theorem; defining the Henkin construction; and some final steps. The proof developed below departs from standard proofs in its replacement of the Henkin construction, which is the step in the standard proof in which models are introduced, by what I shall call the Henkin *assignment*, and in the final steps that depend on it. For this reason I allow myself to be concise in the presentation of the elimination theorem (the first two stages are mainly definitional). Also, to simplify the proof, I use only the existential quantifier: the universal one can be introduced as a defined symbol, $\forall x\varphi(x) \equiv \neg\exists x\neg\varphi(x)$, and then its valuation and derivation rules are derived rules of the system.

Adding witnessing constants

Let $\mathfrak{L}$ be a PC-language. For any formula $\exists x\varphi(x)$ of $\mathfrak{L}$ we define the *witnessing constant* $c_{\varphi(x)}$. Adding these witnessing constants to the language $\mathfrak{L}$ enriches it into language $\mathfrak{L}_1$, which also has some *new* formulas of the form $\exists x\varphi(x)$. We repeat this process for these new formulas of $\mathfrak{L}_1$, defining new witnessing constants and in this way generating $\mathfrak{L}_2$. Iterating this process, we get the Henkin language $\mathfrak{L}_H$, which adds to $\mathfrak{L}$ all symbols each of which belongs to some language $\mathfrak{L}_n$, with n any natural number. The stage of formation of a witnessing constant is called its *date of birth*. No witnessing constant occurs in a formula that belongs to a language formed earlier than its date of birth.

The Henkin Theory

This is a set of formulas of $\mathfrak{L}_H$, called *the Henkin axioms*, comprising all formulas of the following forms:

H1 $\exists x\varphi(x) \rightarrow \varphi(c_{\varphi(x)}/x)$ (the Henkin *witnessing axioms*)
H2 $\varphi(c/x) \rightarrow \exists x\varphi(x)$
H3 $c = c$
H4 $c = d \rightarrow (P\ldots c\ldots \rightarrow \mathrm{P}\ldots d/c\ldots)$

If c_1 and c_2 are two witnessing constants, and the date of birth of c_2 is not earlier than that of c_1, then c_2 does not occur in the witnessing axiom of c_1. All formulas H2 to H4 are theorems of $\mathfrak{L}$, derivable by means of the derivation rules listed above.

[13] The formal work in this section overlaps with that found in publications by Hugues Leblanc, but to the best of my knowledge nowhere in his work does he prove exactly what is proved here. As noted above, Leblanc adopted Dunn and Belnap's term extension approach, which is close but not identical to that followed here, only in his 1983 paper. However, in that paper he does not discuss identity; he occasionally refers for proofs to his earlier work that follows a different approach; and the discussion is done by comparing and handling a wide variety of semantic approaches, so that the parts of the proof which overlap with the one provided here are scattered along his paper.

The Elimination Theorem

I first list a few propositions and lemmas used to prove the theorem. $\mathfrak{T}$ is a set of formulas of our language, and φ, ψ, and ξ are formulas.

Proposition 1 (Deduction Theorem). If $\mathfrak{T}, \varphi \vdash \psi$ then $\mathfrak{T} \vdash \varphi \rightarrow \psi$

Proposition 2. If $\mathfrak{T}, \varphi_1, \ldots \varphi_n \vdash \psi$, and for each i = 1, … n, $\mathfrak{T} \vdash \varphi_i$, then $\mathfrak{T} \vdash \psi$. (This is an immediate consequence of the Deduction Theorem.)

Lemma 3.1. If $\mathfrak{T} \vdash \varphi \rightarrow \psi$ and $\mathfrak{T} \vdash \neg\varphi \rightarrow \psi$ then $\mathfrak{T} \vdash \psi$.

Lemma 3.2. If $\mathfrak{T} \vdash (\varphi \rightarrow \psi) \rightarrow \xi$, then $\mathfrak{T} \vdash \neg\varphi \rightarrow \xi$ and $\mathfrak{T} \vdash \psi \rightarrow \xi$.

Lemma 4. Suppose c is an individual constant that does not occur in $\mathfrak{T}$, $\exists x\varphi(x)$ or ψ. Then if $\mathfrak{T} \vdash \varphi(c/x) \rightarrow \psi$, then $\mathfrak{T} \vdash \exists x\varphi(x) \rightarrow \psi$. (This is provable with the ∃E rule.)

Lemma 5 (Eliminating witnessing axioms). Suppose c is an individual constant that does not occur in $\mathfrak{T}$, $\exists x\varphi(x)$ or ψ. Then if $\mathfrak{T}, \exists x\varphi(x) \rightarrow \varphi(c/x) \vdash \psi$, then $\mathfrak{T} \vdash \psi$. (This is proved by applying first the Deduction Theorem, then Lemma 3.2, then Lemma 4 on the second result of Lemma 3.2, and then Lemma 3.1.)

We can now prove the Elimination Theorem:

Proposition 6 (Elimination Theorem). If ψ is a formula of $\mathfrak{L}$ derivable from formulas $\varphi_1, \ldots \varphi_n$ of $\mathfrak{L}$ together with formulas $h_1, \ldots h_k$ from the Henkin Theory, then ψ is derivable from $\varphi_1, \ldots \varphi_n$ alone; namely, if $\varphi_1, \ldots \varphi_n, h_1, \ldots h_k \vdash \psi$ then $\varphi_1, \ldots \varphi_n \vdash \psi$.

The proof is by induction on the number k of formulas from the Henkin theory from which ψ is derivable. If ψ is derivable from no formulas from the Henkin Theory (k = 0), $\varphi_1, \ldots \varphi_n \vdash \psi$, then there are no Henkin axioms to eliminate. Suppose next that the claim is true for any formula derivable from at most k Henkin axioms, and let us prove it for a formula ψ derivable from k+1 Henkin axioms. We distinguish two cases. First Case: If one of the Henkin axioms is of the form H2 to H4, then it is a theorem, and therefore derivable from $\varphi_1, \ldots \varphi_n$ and the other k Henkin axioms. It follows from Proposition 2 that ψ is derivable from $\varphi_1, \ldots \varphi_n$ and the other k Henkin axioms, and therefore, by IH, that it is derivable from $\varphi_1, \ldots \varphi_n$ alone. Second Case: all the Henkin axioms fall under H1, namely, they are of the form $\exists x\varphi(x) \rightarrow \varphi(c_{\varphi(x)}/x)$. Let us choose a Henkin axiom whose witnessing constant c has a date of birth which is greater than or equal to that of any of the witnessing constants of the other Henkin axioms (suppose it is h_{k+1}). Accordingly, c does no occur in any of the other Henkin axioms, and obviously not in any formula that belongs to $\mathfrak{L}$, namely none of $\varphi_1, \ldots \varphi_n$ and ψ. We accordingly have, $\varphi_1, \ldots \varphi_n, h_1, \ldots h_k, \exists x\varphi(x) \rightarrow \varphi(c_{\varphi(x)}/x) \vdash \psi$. By Lemma 5, $\exists x\varphi(x) \rightarrow \varphi(c_{\varphi(x)}/x)$ can be eliminated, and we accordingly have, $\varphi_1, \ldots \varphi_n, h_1, \ldots h_k \vdash \psi$. By IH, the k Henkin axioms can also be eliminated, and $\varphi_1, \ldots \varphi_n \vdash \psi$. QED

The Henkin Assignment (Proposition 7)

Suppose $\mathcal{V}$ is a valuation that assigns truth values to the formulas of $\mathfrak{L}_H$ while respecting the valuation rules for the connectives of the *Propositional Calculus*, and that $\mathcal{V}$ also makes all the Henkin axioms true. Then $\mathcal{V}$ also respects the valuation rules for quantified sentences and the valuation rules that involve identity.

Proof: First, the existential quantifier. The rule for the relation of the truth value of a formula of the form $\exists x\varphi(x)$ to those of its instances is that this formula is true just in case so is some formula of the form $\varphi(a/x)$, where a is any name that has replaced all occurrences of x in $\varphi(x)$. Given our assumption that $\mathcal{V}$ makes all the Henkin axioms true and respects the valuation rules for the connectives of the Propositional Calculus, and by the Henkin witnessing axioms H1, namely, $\exists x\varphi(x) \rightarrow \varphi(c_{\varphi(x)}/x)$, it follows that if $\exists x\varphi(x)$ is true then so is $\varphi(c_{\varphi(x)}/x)$, which is of the form $\varphi(a/x)$. On the other hand, if for some name a, $\varphi(a/x)$ is true, then since on our assumption the Henkin axioms H2, namely $\varphi(c/x) \rightarrow \exists x\varphi(x)$, are true and $\mathcal{V}$ respects the valuation rules for the connectives of the Propositional Calculus, so is $\exists x\varphi(x)$. So $\mathcal{V}$ respects the truth value assignment rule for existentially quantified sentences.

Secondly, identity. We have two valuation rules that involve identity: the Law of Identity and the Law of the Indiscernibility of Identicals. The Law of Identity says that all formulas of the form $c = c$ are true. Since $\mathcal{V}$ makes all Henkin axioms true, and in particular H3, namely $c = c$, the Law of Identity is respected by $\mathcal{V}$. The Law of the Indiscernibility of Identicals says that if $a = b$ is true then if $P\ldots a\ldots$ is true, then so is $P\ldots b/a\ldots$. Since $\mathcal{V}$ makes true the Henkin axioms H4, namely $c = d \rightarrow (P\ldots c\ldots \rightarrow P\ldots d/c\ldots)$, and since $\mathcal{V}$ respects the valuation rules for the connectives of the Propositional Calculus, it follows that if $a = b$ and $P\ldots a\ldots$ are true, then so is $P\ldots b/a\ldots$. So $\mathcal{V}$ respects the Law of the Indiscernibility of Identicals. QED

Final Steps

Assume now that formulas $\mathfrak{T}$ and formula φ all belong to $\mathfrak{L}$ (namely, they contain no witnessing constant). Suppose also that $\mathfrak{T} \vDash \varphi$, that is, even if we add names to our language $\mathfrak{L}$ and eliminate names from it, any valuation that respects all assignment rules of the sentential connectives of the Propositional Calculus, of the existential quantifier and of identity, and which makes $\mathfrak{T}$ true, makes φ true as well. Since $\mathfrak{L}_H$ differs from $\mathfrak{L}$ only by having additional names (the witnessing constants), and as even if we add names to $\mathfrak{L}$ any assignment that makes $\mathfrak{T}$ true makes φ true as well, $\mathfrak{T} \vDash \varphi$ also in $\mathfrak{L}_H$. We now need to prove that $\mathfrak{T} \vdash \varphi$. By Proposition 7 (the Henkin Assignment), any valuation that assigns truth values to the formulas of $\mathfrak{L}_H$ while respecting the valuation rules for the connectives of the Propositional Calculus and makes all the Henkin axioms true, respects the assignment rules of the existential quantifier and identity as well; accordingly, any such assignment that makes $\mathfrak{T}$ true, makes φ true as well. Let us designate by H the Henkin Theory. That means that *in the Propositional Calculus*, H, $\mathfrak{T} \vDash \varphi$. But since the Propositional Calculus is complete, φ is derivable in it, namely, H, $\mathfrak{T} \vdash \varphi$. But now (since any proof depends on a finite number of premises) it follows from the Elimination Theorem (Proposition 6) that $\mathfrak{T} \vdash \varphi$. QED

We have thus proved, on the truth-valuational substitutional approach and without the use of models, that the Predicate Calculus with its standard proof system is complete.

Those familiar with standard Henkin completeness proofs that use Model Theory would have noticed that the proof above is somewhat simpler than them, especially in its treatment of identity. The reason for this is as follows. What in fact matters for completeness is the relations between possible truth values of sentences in the language, namely, the thing the truth-valuational approach claims to be relevant. Henkin had therefore to find a way to discuss these, although his semantic rules forced him to discuss models. He consequently made language its own domain and the basis of his model, and in this way managed to return to language by making a detour through models. This indicates that in fact models are not what is essential for questions of completeness, since completeness expresses a property of language irrespective of its relation to the world. And indeed, the proof above addresses directly language and the possible relations of truth values between its sentences, and in this way avoids Henkin's detour.

Model Theory is thus unnecessary for the basic proofs on the system of the Predicate Calculus – a result that should be welcomed, given the reasons for discontent with this semantic theory that were discussed in the first section. The successful application of the truth-valuational approach to other major systems of logic would show that Model Theory is generally inessential for these fundamentals of logic. Below I shall apply the approach to Modal Propositional Calculus.

VI. Model Theory: Working Parts and Useless Wheels

The arguments and proofs so far support the claim that Model Theory is conceptually problematic and, at least as regards the Predicate Calculus, formally unnecessary for formulating some central logic concepts and proving its adequacy, since the truth-valuational approach can replace it as providing a framework for the study of truth in that calculus while not being beset by its conceptual problems.

We can, however, push our criticism even further. We have seen that the adequacy of the formal system is independent of the way its sentences relate to the world. Accordingly, if the conclusions of Model Theory about the soundness and completeness of the system are to have full generality, its mentioning of objects in a domain and of a designation relation between words and objects should in fact be idle, the actual work being done only by reference to language structure. This is because any content these concepts of object and designation might have would limit the application of language in an unjustified way. The fact that they are indeed idle was intimated, I suggested, by Henkin's original completeness proof, in which he used language as its own model, in this way making the reference to anything beyond the structure of language in fact otiose. I shall now try to show that this is even more fundamentally the case, namely, as regards Model Theory's concepts of truth and validity as well.

What I shall try to show is that the general talk of a model for our language is equivalent to a talk about a language that contains an image of the atomic formulas of the Predicate Calculus, and possibly additional sentences of the same structure. We assign truth values to the sentences of this potentially richer language, and in this way assign truth values to all sentences of the Predicate Calculus as well. And these assignments are according to the truth-valuational approach. The functioning part of Model Theory emerges as a way of inquiring into language which is in fact equivalent in a sense to the truth-valuational approach and which is not committed to any specific way in which language relates to the world.

I shall provide only a sketch of the correspondence with the truth-valuational approach, as the full technical details are not necessary for appreciating the idea.

Suppose we have a standard Predicate Calculus language $\mathfrak{L}$ and we discuss its relation to a model $\langle D, i\rangle$, with D a domain and *i* an interpretation function, assigning to each name *a* a particular α of D and to each n-place predicate a subset of the n-ary Cartesian product set of the domain. Suppose *P* is a one-place predicate and that $i(P) = \Pi \subset D$. We now say that *Pa* is true in the model just in case $i(a) \in i(P)$, or $\alpha \in \Pi$, and that otherwise it is false, and similarly for many-place predicates. I shall write below, instead of *i*(*a*) and *i*(*P*), a^i and P^i.

The work that the symbol ∈ has done so far amounts to providing us with a binary relation, enabling us to say that either α is related to Π or not. If ∈ has any additional content in set theory, this content has not played any role yet. I therefore suggest a typographical variation: instead of writing, '$\alpha \in \Pi$', I shall write, 'Πα is true'. Even if one thinks that this is *not* what ∈ *generally* means – and I am not contesting this here – any additional meaning it has hasn't contributed so far to our use of model talk. So far, we can think of D and its subsets as constituting a *language*, $\mathfrak{D}$, with Πα etc. as its formulas.

Accordingly, in the model, *Pa* is true just in case $P^i a^i$ is true. Similarly, *Pa* is false if and only if it is not the case that $P^i a^i$ is true (namely, it's not the case that $a^i \in P^i$); we may then say, $P^i a^i$ *is false*. By contrast, ¬*Pa* is false in a model in case $P^i a^i$ is true, and true otherwise, namely if $P^i a^i$ is false. $Pa \wedge Qb$ is true just in case $P^i a^i$ is true and $Q^i b^i$ is true, and similarly for other sentential connectives. Although we haven't defined for our description of the relations in $\mathfrak{D}$ a 'sentence' paralleling ¬*Pa* or $Pa \wedge Qb$ but only sentences paralleling $\mathfrak{L}$'s atomic formulas (i.e. Πα), the truth values of these non-atomic $\mathfrak{L}$ sentences are determined by those of $\mathfrak{D}$'s atomic ones.

I continue with the existential quantifier. There are several ways to define the truth value of the formula $\exists x\varphi(x)$ in a model <D, *i*>. Since in my definition of a model, *i* has as arguments names but not variables, I shall define it as follows. The formula $\exists x\varphi(x)$ is true in a model <D, *i*> just in case there is an interpretation *i**, different from *i* at most in its value for a name *a* which does not occur in φ(*x*), for which $i^*(\varphi(a/x))$ is true. For instance, $\exists xPx$ is true just in case $P^i a^{i^*}$ is true for some such *i**. Namely, we think of $\exists xPx$ as true not necessarily in case *P* has an instance *a* for which *Pa* is true, but in case P^i has an instance α in the sense that $P^i\alpha$ is true. That is, we apply the TVS valuation rule for ∃ to the language $\mathfrak{D}$. We think of $\mathfrak{D}$ as a possibly richer language than $\mathfrak{L}$, containing more names, and the truth values of the sentences in $\mathfrak{L}$ are determined indirectly according to the TVS rules.

So far, the model with its P^i, a^i, and so on has functioned as a language with predicates and individual constants, without any constraint on the latter's number or cardinality – D need not even contain just denumerably many items or 'names'. In this way we can change the set of names of the languages $\mathfrak{D}$ we use to determine truth values of the language $\mathfrak{L}$, which is effectively like changing the set of names of $\mathfrak{L}$ on the truth-valuational approach.

i need not be an injection, and in case $\mathfrak{L}$ contains, in a model, a true sentence of the form '*a* = *b*' with two different names, it cannot be an injection: $a^i =^i b^i$ is true just in case a^i *is* b^i. ('$a^i =^i b^i$ is true' is our way of writing, $<a^i, b^i> \in =^i$.) We therefore cannot generally speak of $\mathfrak{D}$ as containing under *i* an isomorphic image of $\mathfrak{L}$'s atomic sentences. In this respect, we treat $\mathfrak{L}$ and $\mathfrak{D}$ differently. But this does not amount to treating $\mathfrak{D}$ as anything beyond a language. (We could also draw $\mathfrak{D}$ closer to $\mathfrak{L}$ by having $=^i$ not as the identity relation, for which each α is identical only to itself, but as an equivalence relation, and imposing the additional condition, if $\alpha =^i \beta$, then if Πα then Πβ. *i* could then be an injection from $\mathfrak{L}$ to D. But I rather not deviate here from the standard way of construing models.)

Finally, a formula is a logical truth just in case it is true in all models. Interpreting $\mathfrak{D}$ as a Language, which contains an image of the atomic sentences of $\mathfrak{L}$ with the reservation just noted about identity and injection, and which can have as its items or names a non-empty set of any cardinality, we see that according to Model Theory, validity amounts to the following. A formula φ of $\mathfrak{L}$ is a logical truth just in case on any assignment of truth values to any language $\mathfrak{D}$ to which $\mathfrak{L}$ is correlated as above, which determines the truth values of sentences of $\mathfrak{L}$ according to the valuation rules of the truth-valuational approach, φ is true. Namely, Model Theory can be reinterpreted as a covert truth-valuational approach.

One might object that although $\mathfrak{D}$ *can* be such a language, it *need not be*. Since we put no limitations on the entities that the domain D may contain, linguistic items may constitute it as well, and then my interpretation above is valid. But the domain can also be something else, not a language, and therefore my reductive interpretation of models is unjustified.

However, what I was trying to show is not that a language can *also* serve as a model, but that our *general* talk of a model is equivalent to our talk of a language with truth values as specified above. Our use of the symbol ∈ creates the impression that we are doing something different, but this is unjustified, for the reasons mentioned. What the equivalence *does* show, however, is that the talk of *truth* commits us to very little about that concept – I'll say more on this at the end.

I claimed in the first section that Model Theory cannot in its generality offer an alternative conception of validity, since its ideas of a domain with its particulars and of a designation relation between language and reality, conceived in their intended generality, are empty. And we indeed now see that these ideas do no real work in the theory, which in fact develops a truth-valuational substitutional approach under a notational veil, which creates the illusion that something else is doing the work while in fact it reconsiders a language under an alternative notation.

Model-theoretic semantics works not because it provides a good theory of truth, but because it provides all the possible truth value relations between sentences of a language that the truth-valuational approach requires.

VII. The Truth-valuational Approach to Modal Propositional Calculus

If philosophers have taken themselves to be considering, by doing model-theoretic semantics, the relation of language to the world, they have thought that when applied to modal logic it brings them even further, into an inquiry of all possible worlds. Despite the logician's refrain, that *anything* can count as a possible world, the technic and terminology of possible world semantics suggested that we are dealing with something with significant ontological import. Much metaphysics has been provoked in this way.

By contrast, applying the approach developed above to the relevant calculi offers a different perspective on what modal logic is committed to: we again look at language alone, without any reference either to reality, to possible reality, or to what language can be about; and we consider the possibilities open to specific truth value calculi. Modal logic turns out to be a way of reflecting within language possibilities for assigning truth values to its own formulas. Moreover, as we shall see, the truth-valuational approach suggests along the way different choices than those prevalent in possible world semantics.

Whether applied to Modal Propositional Calculus or to Modal Predicate Calculus, the principles of the application of the truth-valuational approach are the same. The proofs of soundness and completeness, on the other hand, and other formal characteristics of the system, are more complex when applied to Modal Predicate

Calculus. This additional complexity, however, is not a result of the truth-valuational principles. I shall therefore discuss in detail and prove soundness and completeness only of Modal *Propositional* Calculus, which should suffice to demonstrate the way the truth-valuational approach works. All the same, the truth-valuational approach has some interesting immediate results when applied to Modal Predicate Calculus, and I discuss some of these in Section IX.

Language and Proof System

The language I use is the standard language of the Propositional Calculus, with denumerably many propositional variables p_1, p_2 … and negation and implication as sentential operators, enriched by the necessity operator box, □, and the possibility operator diamond, ◊. In proofs on the system, we shall consider the possibility operator as defined, ◊φ being another way of writing, ¬□¬φ.

The different modal systems are distinguished in the proof system by the different derivation rules they admit. All systems include all derivation rules of the Propositional Calculus. In addition, all standard systems have the following derivation rule, NEC:

NEC
– (i) φ
– (j) □φ NEC i

φ in line (i) does not depend on any other formula. This is indicated by the dash, –, which is not part of the language and is used here only for clarity. In addition, we shall consider only *normal* modal system, namely, those in which the K-rule, or (K), holds. In our Lemmon-style natural deduction system, (K) is formulated as follows:

K
L_1 (i) □(φ → ψ)
L_2 (j) □φ
L_1, L_2 (k) □ψ K i, j

When inquiring into a stronger modal system, we shall consider T, which contains in addition to NEC and (K) also (T):

T
L (i) □φ
L (j) φ T i

(T) is a rule also of systems B, S4 and S5. We shall not generally discuss B, S4, S5 or other systems below: I am mainly interested in developing the *principles* of the truth-valuational approach as applied to modal systems, and the additional complexities accruing as we inquire into more and more complex systems do not contribute to the clarification of the principles or of the way they are applied. This leaves open the possibility that inquiries into modal systems on the truth-valuational approach will reveal some interesting facts about these systems as incorporated in this approach. I shall list the additional derivation rules of B, S4 and S5:

B (included in systems B and S5)
L (i) φ
L (j) □◊φ B i

4 (included in systems S4 and S5)
L (i) $\Box\varphi$
L (j) $\Box\Box\varphi$ 4 i

5 (included in S5)
L (i) $\Diamond\varphi$
L (j) $\Box\Diamond\varphi$ 5 i

If we include the last, (5) rule in the modal system S5, we don't need to include in it (B) and (4). This is not specific to the truth-valuational approach and I shall not discuss it any further here.

Since we have several proof systems, derivability will be relative to proof system. Since all our systems contain NEC and (K), we shall write $\Psi \vdash \varphi$ if φ can be proved from a finite number of $\psi1, \psi2, \ldots \psi n \in \Psi$ by means of the derivation rules of the Propositional Calculus, NEC and (K). If it can be similarly proved by these rules and (T), we write $\Psi \vdash_T \varphi$; if from the above, (T) and (B), we write $\Psi \vdash_B \varphi$; etc.

Truth Value Assignments and Validity
The approach developed in this paragraph is close to that suggested in (Leblanc 1973, p. 11) and later developed in (Leblanc 1976, §8.3).

Definition (Valuation). Any truth value assignment or *valuation* $\mathcal{V}$ assigns to each atomic formula either truth or falsity, and the truth values of all other formulas are determined as follows. With any valuation $\mathcal{V}$ we associate a set of valuations $S_{\mathcal{V}}$, $\mathcal{V}$'s *valuation set*. The following assignment rules hold:

- $\neg\varphi$ is true on $\mathcal{V}$ in case φ is false on $\mathcal{V}$, and false otherwise.
- $\varphi \rightarrow \psi$ is true on $\mathcal{V}$ in case φ is false or ψ true on $\mathcal{V}$, otherwise it is false.
- $\Box\varphi$ is true on $\mathcal{V}$ in case φ is true on every valuation $\mathcal{W} \in S_{\mathcal{V}}$; otherwise it is false.
- $\Diamond\varphi$ is true on $\mathcal{V}$ in case φ is true on some valuation $\mathcal{W} \in S_{\mathcal{V}}$; false otherwise.

The last point introduced $\Diamond\varphi$ not as another way of writing $\neg\Box\neg\varphi$ but treated $\Diamond$ as an independently defined symbol. We shall see below that the two formulas are equivalent in their truth values.

For simplicity, we assume below that for any $\mathcal{V}$, $S_{\mathcal{V}}$ isn't empty. This assumption makes $\Box\varphi \rightarrow \Diamond\varphi$ into a logical truth. If eliminated, there are several ways to proceed, including three-valued logics. None of these will be explored here.

As can be seen from the definition of valuations, we always consider a collection of valuations, such that each valuation in the collection has part of the collection as its valuation set. (I use here 'collection' and not 'set' only to distinguish verbally the collection from the valuation set.) We call each such collection a *frame*.

Truth values are assigned recursively as follows. First, on any valuation in the frame, we assign truth values to atomic formulas (in our case, propositional variables), and then, according to the rules for sentential connectives, to all other formulas that do not contain any modal operator. Next, we assign on every valuation truth values according to the modal rules to all formulas of the form $\Box\varphi$ or $\Diamond\varphi$, where φ is a formula to which a truth value has already been assigned. We then again assign truth values to any formula which is formed only by means of sentential connectives from formulas to which we have already assigned a truth value and which hasn't yet been assigned a truth value. We repeat this process denumerably many times, until on any valuation every formula has been assigned a truth value.

Each formula is assigned on all valuations in a frame a truth value at the same stage. Notice that two different valuations $\mathcal{V}$ and $\mathcal{W}$ may coincide on the truth values they assign to atomic formulas but differ because they have different valuation sets $S_{\mathcal{V}}$ and $S_{\mathcal{W}}$ and consequently they do not coincide on the truth values they assign to formulas containing modal operators.

However, valuations are functions, and these are determined by the values they assign to all items in their domain. The question then arises, can we have the same valuation with different assignment sets? – The answer depends on the specific language and kinds of frame we are using in our calculus.

Suppose we do not allow *any* logical operator apart from □ and ◊, and that our language has only two propositional variables, *p* and *q*. We consider the following valuations, defined on atomic formulas:

$\mathcal{V}_1$: $\mathcal{V}_1$(p)= *true*, $\mathcal{V}_1$(q) = *false*.
$\mathcal{V}_2$: $\mathcal{V}_2$(p)= *true*, $\mathcal{V}_2$(q) = *false*.
$\mathcal{W}_1$: $\mathcal{W}_1$(p)= *true*, $\mathcal{W}_1$(q) = *true*.
$\mathcal{W}_2$: $\mathcal{W}_2$(p)= *false*, $\mathcal{W}_2$(q) = *false*.
$\mathcal{W}_3$: $\mathcal{W}_3$(p)= *true*, $\mathcal{W}_3$(q) = *false*.
$\mathcal{W}_4$: $\mathcal{W}_4$(p)= *false*, $\mathcal{W}_4$(q) = *true*.

Although $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{W}_3$ are identical in the value they assign to *p* and in that they assign to *q*, they shall have different valuation sets. Consider now the following frame:

Valuations: $\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4\}$; $S_{\mathcal{V}_1} = \{\mathcal{W}_1, \mathcal{W}_2\}$, $S_{\mathcal{V}_2} = \{\mathcal{W}_3, \mathcal{W}_4\}$,
i = 1, 2, 3, 4 $S_{\mathcal{W}_i} = \{\mathcal{W}_2\}$

Although $S_{\mathcal{V}_1}$ is different from $S_{\mathcal{V}_2}$, $\mathcal{V}_1$ is identical to $\mathcal{V}_2$. A simple check shows that for i = 1, 2:

$\mathcal{V}_i$(p) = *true*, $\mathcal{V}_i$(q) = *false*, $\mathcal{V}_i$(◊p) = $\mathcal{V}_i$(◊q) = *true*, $\mathcal{V}_i$(□p) = $\mathcal{V}_i$(□q) = *false*.

And since the assignment set of all $\mathcal{W}_i$ is $\mathcal{W}_2$, any formula with more than a single modal operator is assigned the same truth value on $\mathcal{V}_1$ and $\mathcal{V}_2$. Accordingly, $\mathcal{V}_1 = \mathcal{V}_2$. (We shall get the same identity result if we also allow negation, ¬.)

By contrast, in case we allow all standards connectives to our language, including ¬ and ∧ or equivalents, and any valuation set contains a *finite* number of valuations, then any two valuations different in their valuation sets are different from each other. Let us prove that.

We prove for such a case that if $S_{\mathcal{V}} \neq S_{\mathcal{W}}$, then $\mathcal{V} \neq \mathcal{W}$. Since $S_{\mathcal{V}} \neq S_{\mathcal{W}}$, at least one of the two sets contains a valuation different from all those of the other; say it's $\mathcal{V}_1 \in S_{\mathcal{V}}$. Accordingly, any $\mathcal{W}_i \in S_{\mathcal{W}}$ assigns to some formula φ_i a truth value different from $\mathcal{V}_1(\varphi_i)$. Since our system contains ¬, we can assume that $\mathcal{V}_1(\varphi_i)$ = *true* and $\mathcal{W}_i(\varphi_i)$ = *false*. Suppose $S_{\mathcal{W}}$ contains *n* valuations, and consider the formula, $\Diamond(\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n)$. Since on $\mathcal{V}_1$, $\mathcal{V}_1(\varphi_i)$ = *true* for each *i*, $\mathcal{V}_1(\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n)$ = *true*, and therefore $\mathcal{V}(\Diamond(\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n))$ = *true*. However, given our construction, φ_i is *false* on $\mathcal{W}_i$, and therefore on each $\mathcal{W}_j$, $\mathcal{W}_j(\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n)$ = *false*, and thus $\mathcal{W}(\Diamond(\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_n))$ = *false*. Accordingly, $\mathcal{V} \neq \mathcal{W}$.

However, if the cardinality of the valuation set is allowed to be infinite, the construction above cannot work. Moreover, cardinality considerations show that there must be identical valuations with different valuation sets: if the cardinality of a language's formulas is $\aleph_0$, the cardinality of its valuation set is $2^{\aleph_0} = \aleph$, and that of its sets of valuation sets $2^{\aleph}$. It follows that there is no bijection of valuations to valuation sets. (I owe this proof to Hongkai Yin.)

Accordingly, it seems that unless we wish to force ad hoc constraints on frames, we should allow frames to have several *occurrences* of the same valuation, each occurrence with different valuation sets. Two occurrences of the same valuation will count as different if either their valuation sets contain different valuations (not just different *occurrences*), or their valuation sets contain different occurrences of the same valuation. The latter clause makes the definition recursive, but since the former shall provide the base case, the definition requires that different occurrences of the same valuation involve, at some stage down the chain of valuation sets, different valuations. It blocks, for instance, frames that contain only two occurrences of the same valuation, each occurrence being the only member of the valuation set of the other occurrence.

Validity is defined relative to frames. A set Ψ of formulas *entails* a formula φ relative to a frame F, or the argument with Ψ as premises and φ as conclusion is *valid* relative to F, or $\Psi \vDash_F \varphi$, iff φ is true on any valuation of the frame on which all formulas of Ψ are true. In case a formula φ is true on all valuations in a frame F, φ is *a logical truth* or *tautology* relative to F, and we write $\vDash_F \varphi$. If SF is a set of frames, then a set Ψ of formulas entails a formula φ relative to the set of frames SF, or $\Psi \vDash_{SF} \varphi$, iff $\Psi \vDash_F \varphi$ for every F ∈ SF. Similarly for $\vDash_{SF} \varphi$.

On these rules, □φ is equivalent with ¬◊¬φ, namely, for each set of frames, □φ entails ¬◊¬φ and ¬◊¬φ entails □φ. Let us see that. On any valuation $\mathcal{V}$, ¬◊¬φ is true iff ◊¬φ is false. Thus, on any valuation $\mathcal{V}$, ¬◊¬φ is true iff on no valuation $\mathcal{W} \in S\mathcal{V}$ is ¬φ true. So, on any valuation $\mathcal{V}$, ¬◊¬φ is true iff on every valuation $\mathcal{W} \in S\mathcal{V}$, φ is true. Accordingly, on any valuation, ¬◊¬φ is true iff □φ is.

In possible world semantics, different sets of frames are characterised by conditions on an *accessibility* relation between 'worlds'. The analogue of this on the truth-valuational approach developed here would be to put various constrains on the membership relation in a valuation set. For instance, we can require that it be 'reflexive', each valuation being a member of its valuation set, $\mathcal{V} \in S\mathcal{V}$. This, however, is *not* the approach we shall adopt here.

It is more natural to characterise features of a truth theory by the concept of truth or related ones. Moreover, constraints on the membership relation, or the accessibility relation, have no obvious logical status. Accordingly, *we shall characterise sets of frames by the formulas which they make logical truths.*

we characterise the set of frames T, B, S4 and S5 by the following schemata:

- T: □φ → φ
- B: φ → □◊φ
- 4: □φ → □□φ
- 5: ◊φ → □◊φ

And the characterisation is then standard:

- The set of frames T is that containing all and only frames in which all instances of schema (T) are logical truths.
- The set of frames B: similarly for schemas (T) and (B).
- Set of frames S4: similarly for schemas (T) and (4).
- Set of frames S5: similarly for schemas (T), (B) and (4); or (T) and (5).

If all instances of a schema are logical truths, we say that the schema is a logical truth.

We don't need to similarly characterise a set of frames for the schema (K):

- K: $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

This is because all instances of schema (K) are logical truths on all frames. Let F be a frame, $\mathcal{V}\in$F a valuation, and φ and ψ be any two formulas. For $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ to be false on $\mathcal{V}$, $\mathcal{V}[(\Box(\varphi \rightarrow \psi)]$ and $\mathcal{V}[\Box\varphi]$ should be true and $\mathcal{V}[\Box\psi]$ false. But the former are true just in case $\mathcal{W}[\varphi \rightarrow \psi]$ and $\mathcal{W}[\varphi]$ are true on any $\mathcal{W}\in S\nu$. But then $\mathcal{W}[\psi]$ = T for any $\mathcal{W}\in S\nu$, and therefore $\mathcal{V}[\Box\psi]$ = T as well. Accordingly, the set of all frames on which schema (K) is a logical truth is the set of all frames.

What kind of fact about the truth value assignment rules for modal operators *does* schema (K) then represent? It is at least *related* to the condition that *valuation sets are not a function of the formula being assigned a truth value*. We could, in principle, have the valuation set of a valuation $\mathcal{V}$ change from one formula to another, and so, for the two formulas φ and ψ, $S\nu(\varphi)$ need not have been identical to $S\nu(\psi)$. I in fact proved above that schema (K) is a logical truth in case $S\nu(\varphi) = S\nu(\psi)$ for any valuation $\mathcal{V}$ in the frame and any two formulas φ and ψ. So the condition was shown to be sufficient.

I now prove that if we don't put that constraint, namely, if valuation sets can be a function of the formula being assigned a truth value, then a frame in which schema (K) isn't a logical truth can be constructed. Consider the valuation $\mathcal{V}$ on which $\mathcal{V}$(p) = T and $\mathcal{V}$(q) = T, and let the valuation $\mathcal{W}$ be such that $\mathcal{W}$(p) = T and $\mathcal{W}$(q) = F. Further, make $S\nu(p \rightarrow q) = S\nu(p) = \{\mathcal{V}\}$, while $S\nu(q) = \{\mathcal{W}\}$. Let us now determine the truth value that $\mathcal{V}$ assigns to $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$. Since both p → q and p are true on all valuations in their valuation sets, namely on $\mathcal{V}$, it follows that both $\Box(p \rightarrow q)$ and □p are true on $\mathcal{V}$. Since q is false on the valuation in its valuation set, namely $\mathcal{W}$, it follows that $\mathcal{V}(\Box q)$ = F. Accordingly, $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ is false on $\mathcal{V}$, and schema (K) isn't a logical truth.

Although this shows that there is a frame to which the constraint does not apply and in which schema (K) isn't a logical truth, it does not show that in *any* frame to which the constraint does not apply, schema (K) isn't a logical truth. I doubt that that's the case, but I leave the question open.

Characterising the above sets of frames by the formulas which are logical truths in them is *not* always equivalent to characterising them by constraints on the relation of membership in valuation sets. We shall show that for the T set of frames.

A *sufficient condition* on the membership relation in a frame for the schema $\Box\varphi \rightarrow \varphi$ to be a logical truth on that frame is that for any valuation $\mathcal{V}$, $\mathcal{V}\in S\nu$: On any valuation $\mathcal{V}$, any formula □φ is either true or false. If it is false, then $\Box\varphi \rightarrow \varphi$ is true on $\mathcal{V}$. If □φ is true on $\mathcal{V}$, then by definition, φ is true on all valuations in $S\nu$. Since $\mathcal{V}\in S\nu$, φ is true on $\mathcal{V}$ as well, and so is $\Box\varphi \rightarrow \varphi$. So if $\mathcal{V}\in S\nu$ then $\Box\varphi \rightarrow \varphi$ is true on $\mathcal{V}$ for any φ. And if for any $\mathcal{V}\in$F, $\mathcal{V}\in S\nu$, the schema $\Box\varphi \rightarrow \varphi$ is a logical truth on that frame.

However, we shall show that this condition is not *necessary* for the schema $\Box\varphi \rightarrow \varphi$ to be a logical truth on a frame.

In possible world semantics, the reflexivity of the accessibility relation can be shown not to be a necessary condition on a frame for $\Box\varphi \rightarrow \varphi$ being a logical truth as follows. The frame can contain exactly two, identical 'worlds', in the sense that any propositional variable true at the one is true at the other as well, and have each world accessible only to the other (so accessibility isn't reflexive). In this way, every formula has identical truth values in both worlds. However, the truth-valuational approach developed here cannot adapt this example, for this would require that each of the two occurrences of the same valuation is the only member of the valuation set of the other occurrence. But, since no different *valuations* (not only occurrences) are members of this frame, they would *not* count as different occurrences of the same valuation. We should therefore provide a different counterexample for the necessity of reflexivity.

First, we show that if for a valuation $\mathcal{V}$ in a frame F, $\mathcal{V} \notin S_{\mathcal{V}}$, then if $S_{\mathcal{V}}$ contains a *finite* number of valuations, $\mathcal{V}_1$ to $\mathcal{V}_n$, then $\Box\varphi \rightarrow \varphi$ is false on $\mathcal{V}$ for some formula φ, which we construct. Since $\mathcal{V}$ is different from each $\mathcal{V}_i$, then for every $\mathcal{V}_i$, for some ψ_i, $\mathcal{V}_i(\psi_i) \neq \mathcal{V}(\psi_i)$. If $\mathcal{V}_i(\psi_i) = T$, and therefore $\mathcal{V}(\psi_i) = F$, we set $\varphi_i = \psi_i$; if $\mathcal{V}_i(\psi_i) = F$, we set $\varphi_i = \neg\psi_i$. We now define $\varphi = \varphi_1 \vee \varphi_2 \vee \ldots \varphi_n$. Given our construction, $\mathcal{V}_i(\varphi_i) = T$, and therefore $\mathcal{V}_i(\varphi) = T$; it follows that $\mathcal{V}(\Box\varphi) = T$. On the other and, since $\mathcal{V}(\varphi_i) = F$, it follows that $\mathcal{V}(\varphi) = F$. Accordingly, $\mathcal{V}(\Box\varphi \rightarrow \varphi) = F$. It follows that, to show reflexivity isn't necessary for the validity of the T schema in a frame, each valuation in the frame which is not in its valuation set should have infinitely many valuations in its valuation set.

We construct the frame as follows. Suppose the language $\mathfrak{L}$ has denumerably many propositional variables, $p_1, p_2, p_3 \ldots$. We consider a valuation $\mathcal{V}_0$, which assigns to these variables the truth values $\mathcal{V}_0(p_1), \mathcal{V}_0(p_2), \mathcal{V}_0(p_3) \ldots$ at random. We now construct the valuations that form $\mathcal{V}_0$'s valuation set, $\mathcal{V}_{1,1}, \mathcal{V}_{1,2}, \mathcal{V}_{1,3} \ldots$. For every *i*, $\mathcal{V}_{1,i}$ coincides with $\mathcal{V}_1$ on the truth values it assigns to all propositional variables up to p_i, and differs from it on all the rest. For instance, $\mathcal{V}_{1,2}(p_1) = \mathcal{V}_0(p_1)$, $\mathcal{V}_{1,2}(p_2) = \mathcal{V}_0(p_2)$, and for every $i > 2$, $\mathcal{V}_{1,2}(p_i) \neq \mathcal{V}_0(p_i)$. This construction guarantees that any valuation in $\mathcal{V}_0$'s valuation set is different from $\mathcal{V}_0$, and therefore $\mathcal{V}_0 \notin S_{\mathcal{V}_0}$. Next, for any valuation $\mathcal{V}_{1,i}$, we construct its valuation set $\mathcal{V}_{2,i,1}, \mathcal{V}_{2,i,2}, \mathcal{V}_{2,i,3} \ldots$, repeating the previous method of construction. Namely, for each j, $\mathcal{V}_{2,i,j}(p_k) = \mathcal{V}_{2,i}(p_k)$ if $k \leq j$, and $\mathcal{V}_{2,i,j}(p_k) \neq \mathcal{V}_{2,i}(p_k)$ otherwise. We repeat this process denumerably many times. In this way our frame F is constructed.

The idea behind this construction is guided by the fact that any formula φ contains a finite number of propositional variables. Suppose the last one from the list $p_1, p_2, p_3 \ldots$ is p_m. Consider any valuation $\mathcal{V}$ in the frame F. The *m*-th valuation in its valuation set is identical to $\mathcal{V}$ in the values it assigns to $p_1, p_2 \ldots p_m$, and as *its* valuation set was constructed in the same way as $\mathcal{V}$'s, it is therefore identical to $\mathcal{V}$ in the values it assigns to all formulas whose only propositional variables are from $p_1, p_2 \ldots p_m$. The valuation set of $\mathcal{V}$ is thus 'as good as' reflexive for the formula φ.

To prove that $\Box\varphi \rightarrow \varphi$ is a logical truth on F, we rely on the following *Fact*, which we won't prove: *the truth value on a valuation of a formula φ does not depend on the truth value on any valuation of a propositional variable that does not occur in φ*. (The proof would be by induction on formula complexity. We extend the above definition of complexity to the Modal Propositional Calculus by defining, $C(\Box\varphi) = C(\Diamond\varphi) = 1 + C(\varphi)$.) We need consider only the cases in which $\Box\varphi$ is true. Moreover, since all valuation sets are constructed in the same way, it is enough that we prove that $\Box\varphi \rightarrow \varphi$ is true on $\mathcal{V}_0$ for any φ. Suppose that for some formula φ, $\mathcal{V}_0(\Box\varphi) = T$, and that its propositional variable with largest index is p_m. Since $\mathcal{V}_0(\Box\varphi) = T$, $\mathcal{V}_{1,i}(\varphi) = T$ for any *i*; in particular, $\mathcal{V}_{1,m}(\varphi) = T$. Now, given our construction, $\mathcal{V}_0(p_i) = \mathcal{V}_{1,m}(p_i)$ for any $i \leq m$. Moreover, given our construction, $\mathcal{V}_{1,j}(p_i) = \mathcal{V}_{2,m,j}(p_i)$ too, for any $i \leq m$ and for any *j*. And so on for any corresponding pair of valuations. Namely, the whole structures of the two valuations $\mathcal{V}_0$ and $\mathcal{V}_{1,m}$, their valuation sets, *their* valuation sets and so on for the whole hierarchy of the valuations, coincide on the truth values it assigns to any p_i, for any $i \leq m$. Given *Fact* – the truth value on a valuation of a formula does not depend on the truth value on any valuation of a propositional variable that does not occur in it – it follows that $\mathcal{V}_0(\varphi) = \mathcal{V}_{1,m}(\varphi) = T$. But then $\mathcal{V}_0(\Box\varphi \rightarrow \varphi) = T$. QED

In the construction above, some valuations coincide. For instance, $\mathcal{V}_0$ and $\mathcal{V}_{2,1,1}$ coincide on the values they assign to all propositional variables, and also on all the valuation in the hierarchy of valuation sets proceeding from them. They therefore coincide on the values they assign to any formula. This means that $\mathcal{V}_0 = \mathcal{V}_{2,1,1}$. Similarly, since any two valuations in our construction that are identical in the values they assign to all propositional variables also have the same structure of valuations proceeding from them, any such two valuations coincide on the truth value they assign to any formula and they are one and the same valuation. We therefore do not have two valuations that coincide on the truth values they assign to any formula yet have different valuation sets, and no two occurrences of the same valuation.

We shall not try to provide proofs for parallel results for the other schemata. Namely,

B, $\varphi \rightarrow \Box\Diamond\varphi$: Symmetry. $\mathcal{W} \in S\mathcal{v} \Rightarrow \mathcal{V} \in S\mathcal{w}$.
4, $\Box\varphi \rightarrow \Box\Box\varphi$: Transitivity. $\mathcal{W} \in S\mathcal{v}$ & $\mathcal{U} \in S\mathcal{w} \Rightarrow \mathcal{U} \in S\mathcal{v}$.
5, $\Diamond\varphi \rightarrow \Box\Diamond\varphi$: Euclidean Relation. $\mathcal{W} \in S\mathcal{v}$ & $\mathcal{U} \in S\mathcal{v} \Rightarrow \mathcal{U} \in S\mathcal{w}$.

We leave it an open question whether these conditions are also sufficient, necessary in the finite case, but not necessary in the infinite one.

Soundness

Soundness is understood as follows. We need to show that if $\Psi \vdash \varphi$, then $\Psi \vDash \varphi$, where the provability is by the derivation rules of the Propositional Calculus, NEC and K, and validity is according to *any* set of frames. We need similarly to show that if $\Psi \vdash_T \varphi$, then $\Psi \vDash_T \varphi$, where provability is by the above and (T), and as set of frames we allow all and only those in which schema T, $\Box\varphi \rightarrow \varphi$, is a logical truth. Likewise for the other modal systems.

The soundness of the different systems or sets of frames is proved by induction on proof length as follows. We should show that if all premises listed on a line of a proof are true on a valuation in an appropriate frame, then the conclusion is also true on that valuation. By *appropriate frame* is meant: any frame for the rule (K); any frame in which $\Box\varphi \rightarrow \varphi$ is a logical truth for the rule (T); and so on.

First, any one-line proof is a premise, say φ, and then the one-line long proof shows that $\varphi \vdash \varphi$; but for any formula, $\varphi \vDash \varphi$ on any frame, namely, on any valuation in any frame, if φ is true then φ is true. So any one-line proof preserves validity.

Next, suppose that for a given n, any n-line long proof preserves validity on a given frame of the set of frames we consider, and let us prove this for n+1. The n+1 line is (1) either again a premise; or (2) derived from earlier lines by derivation rules of the Propositional Calculus; or (3) derived from earlier lines by (a) NEC, (b) (K), or if we are considering a stronger modal system, by the additional modal derivation rules it allows – we shall consider only (c) the set of frames T with the (T) derivation rule.

Case (1) has just been discussed, and for case (2) we again assume that the Propositional Calculus is sound.

For case (3a), if the formula in the n+1 line was derived by NEC, then it is of the form □φ and depends on no other formula, and there is an earlier line k with the formula φ, which also depends on no other formula. The inductive hypothesis (IH) entails that since ⊢ φ, ⊨ φ as well, namely φ is true on all valuations in each relevant frame. In particular, φ is also true on all valuation in the valuation set of any valuation, and therefore that □φ is true on any valuation of each relevant frame. Accordingly, ⊨ □φ and the n+1-line proof preserves validity.

For case (3b), if the formula in the n+1 line was derived by (K), then it is of the form □ψ, and the premises Γ on which it depends include the premises Γ1 on which a formula □(φ → ψ) occurring in an earlier line depends, and those Γ2 on which a formula □φ occurring in another earlier line depend. If all formulas Γ are true on a valuation $\mathcal{V}$, then so are all formulas Γ1 and Γ2, and according to IH, also □(φ → ψ) and □φ are true on $\mathcal{V}$. But we have shown earlier that the schema K, □(φ → ψ) → (□φ → □ψ), is a logical truth, and therefore it is true on $\mathcal{V}$. Accordingly, if □(φ → ψ) and □φ are true on $\mathcal{V}$ then □ψ is true on $\mathcal{V}$ as well, and (K) preserves validity.

For case (3c), we are considering the set of frames T, which is constituted by all frames in which the schema □φ → φ is a logical truth, and which contains in addition to the rules considered so far also (T). Assume the formula in the n+1 line was derived by (T), then it is of the form φ, depending on the premises Γ, and on these premises depends also a formula □φ occurring in an earlier line. If Γ are true on a valuation $\mathcal{V}$, then, according to IH, so is □φ. Since we are considering only frames in which □φ → φ is a logical truth, then, since $\mathcal{V}$(□φ → φ) = true and $\mathcal{V}$(□φ) = true, it follows that $\mathcal{V}$(φ) = true too, and the T-rule preserves validity in the T set of frames.

Accordingly, for any m, if $\psi_1, \dots \psi_m \vdash \varphi$ then $\psi_1, \dots \psi_m \vDash \varphi$ and if $\psi_1, \dots \psi_m \vdash_T \varphi$ then $\psi_1, \dots \psi_m \vDash_T \varphi$. Since any proof uses a finite number of premises, the result follows immediately for any set of formulas Ψ: if Ψ ⊢ φ then Ψ ⊨ φ and if $\Psi \vdash_T \varphi$ then $\Psi \vDash_T \varphi$. The soundness proof for the other modal systems we mentioned is similar. QED

VIII. Completeness of the Modal Propositional Calculus

We show that if Ψ ⊨ φ then Ψ ⊢ φ, where validity is according to any set of frames and provability is by the derivation rules of the Propositional Calculus, NEC, and K. We similarly show that if $\Psi \vDash_T \varphi$, then $\Psi \vdash_T \varphi$, where the set of frames consists of all those in which schema T, □φ → φ, is a logical truth, and provability is by the above and (T). The proof for other modal systems is similar.

I start by constructing maximal consistent sets of formulas and stating some of their properties. Most of these properties are proved in the Propositional Calculus, and most of these, not pertaining specifically to modal logic, will not be proved below. I then show that a valuation that makes all and only formulas in a maximal consistent set true, and that is associated with an appropriate valuation set and frame, observes the rules for the assignment of truth values to formulas. A few final steps then yield completeness. I do not presuppose the completeness of the Propositional Calculus: if the parts specific to modal logic are dropped from the proof, then one has a completeness proof for the Propositional Calculus.

I use only negation, implication, and the necessity operator as sentential operators. I start with a few definitions and lemmas.

Definition 1: Derivability. A formula ψ is *derivable* from a set of formulas Φ iff for some finite number of formulas $\varphi_1, \dots \varphi_n \in \Phi$, $\varphi_1, \dots \varphi_n \vdash \psi$. We then write, $\Phi \vdash \psi$.

Definition 2: Inconsistency. A set of formulas Φ is *inconsistent* iff some formula ψ and its negation are both derivable from Φ, namely, $\Phi \vdash \psi$ and $\Phi \vdash \neg\psi$.

Definition 3: Consistency. A set of formulas Φ is *consistent* iff it is not inconsistent.

Definition 4: A *maximal consistent set* is a consistent set that would be made inconsistent if any formula which it does not contain is added to it.

Lemma 5: If Φ is a consistent set and $\Phi \vdash \psi$, then adding ψ to Φ leaves it consistent. We shall not prove this lemma.

Lemma 6: Any consistent set is contained in some maximal consistent set. We shall not prove this lemma either.

Lemma 7: If Φ is a maximal consistent set and $\Phi \vdash \psi$, then ψ belongs to Φ. This follows from Lemma 5 and the fact that Φ is maximal.

Lemma 8: If Φ is a maximal consistent set and ψ a formula, then exactly one of ψ and $\neg\psi$ is contained in Φ.
Proof: Φ does not contain both ψ and its negation because it is consistent. If Φ does not contain ψ, then since it is a maximal consistent set, adding ψ to it would render it inconsistent. So from a finite set of Φ formulas, which must contain ψ, one can prove, say, both η and $\neg\eta$: $\varphi_1, \dots \varphi_n, \psi \vdash \eta, \neg\eta$. By negation introduction, $\varphi_1, \dots \varphi_n \vdash \neg\psi$. From lemma 7 it follows that Φ contains $\neg\psi$. Φ therefore either contains ψ or contains $\neg\psi$. QED

Lemma 9: If Φ is a maximal consistent set then $\varphi \to \psi \in \Phi$ iff either $\neg\varphi \in \Phi$ or $\psi \in \Phi$. We shall not prove this lemma here.

Lemma 10, Deduction Theorem: If $\Phi \cup \{\varphi\} \vdash \psi$, then $\Phi \vdash \varphi \to \psi$. We shall not prove this theorem.

Consider now a valuation that makes all and only the formulas of some maximal consistent set Φ true, all others false. Let us call such a valuation a *canonical* valuation, and designate it $\mathcal{V}_\Phi$. We next associate with it a valuation set. We start by defining $\Box^-(\Phi)$ as the set to which a formula φ belongs just in case $\Box\varphi$ belongs to Φ. We now consider all maximal consistent sets that contain $\Box^-(\Phi)$ (Φ might be among them). Each of these has its canonical valuation: let the set of these canonical valuations, $S\mathcal{V}_\Phi$, be the one associated with $\mathcal{V}_\Phi$. (If a system contains the T derivation rule, which allows the inference from $\Box\varphi$ of φ, then if Φ is consistent so is $\Box^-(\Phi)$, which is a subset of Φ. This is not true of a system not containing (T), in which case there might not be any maximal consistent set containing $\Box^-(\Phi)$ and $S\mathcal{V}_\Phi$ is empty.) We also need to associate a valuation set with each of the canonical valuations of $S\mathcal{V}_\Phi$. We do that by iterating the process: with any canonical valuation $\mathcal{W}$ of a maximal consistent set Ψ that contains $\Box^-(\Phi)$, we associate all the canonical valuations of the maximal consistent sets that contain $\Box^{-}(\Psi)$; and so on. The set of all these valuations constitutes our frame.

We now prove the following theorem:

Theorem 11: If Φ is a maximal consistent set containing the formula $\neg\Box\varphi$, then some maximal consistent set contains both $\neg\varphi$ and $\Box^{-}(\Phi)$.

Proof: Let Φ be a maximal consistent set containing the formula $\neg\Box\varphi$. We first show that $\Box^{-}(\Phi) \cup \{\neg\varphi\}$ is consistent. Suppose for reductio that it is not, so for some η, both it and its negation are derivable from $\Box^{-}(\Phi) \cup \{\neg\varphi\}$. That is, for some $\psi_1, \ldots \psi_n \in \Box^{-}(\Phi)$,

$$\psi_1, \ldots \psi_n, \neg\varphi \vdash \eta, \neg\eta.$$

(η and $\neg\eta$ might be derivable without $\neg\varphi$, but then they are derivable with $\neg\varphi$ as well. In case $\{\neg\varphi\}$ is already inconsistent, some simple changes are needed in the proof below.) By negation introduction and then negation elimination, we get

$$\psi_1, \ldots \psi_n \vdash \varphi.$$

By iteration of Lemma 9, the Deduction Theorem, we now get

$$\vdash \psi 1 \rightarrow (\psi 2 \rightarrow \ldots (\psi_n \rightarrow \varphi)\ldots).$$

We now apply NEC and get

$$\vdash \Box(\psi 1 \rightarrow (\psi 2 \rightarrow \ldots (\psi_n \rightarrow \varphi)\ldots).$$

Next, by (K), we can prove as a theorem any instance of the K schema,

$$\vdash \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$$

We therefore have by appropriate substitution:

$$\vdash \{\Box(\psi 1 \rightarrow (\psi 2 \rightarrow \ldots (\psi_n \rightarrow \varphi)\ldots)\} \rightarrow \{\Box\psi 1 \rightarrow \Box(\psi 2 \rightarrow \ldots (\psi_n \rightarrow \varphi)\ldots) \}.$$

Applying Modus Ponens we get:

$$\vdash \Box\psi 1 \rightarrow \Box(\psi 2 \rightarrow \ldots (\psi_n \rightarrow \varphi)\ldots).$$

Iterating this inference, and since if $\vdash p \rightarrow q$ and $\vdash q \rightarrow r$ then $\vdash p \rightarrow r$, we then get:

$$\vdash \Box\psi 1 \rightarrow (\Box\psi 2 \rightarrow (\ldots (\Box\psi_n \rightarrow \Box\varphi)\ldots).$$

Accordingly,

$$\Box\psi 1 , \Box\psi 2 , \ldots \Box\psi_n \vdash \Box\varphi$$

And since $\Box\psi_1 , \Box\psi_2 , \ldots \Box \psi_n$ belong to Φ, which is a maximal consistent set, it follows from Lemma 7 that $\Box\varphi$ also belongs to Φ. But Φ also contains $\neg\Box\varphi$, and it is therefore inconsistent, which is a contradiction. So our assumption that $\Box^{-}(\Phi) \cup \{\neg\varphi\}$ is inconsistent was wrong.

Since $\Box^{-}(\Phi) \cup \{\neg\varphi\}$ is consistent, it follows from Lemma 6 that it is contained in some maximal consistent set.

Since the only derivation rules we used are those of the Propositional Calculus, NEC and (K), which are included among the rules of all modal proof systems we are considering, the result holds for all these systems. QED

Suppose now that Φ is a maximal consistent set, $\mathcal{V}_\Phi$ its canonical valuation, and that the set of valuations associated with $\mathcal{V}_\Phi$ is $S\mathcal{V}_\Phi$, the set of the canonical valuations of all maximal consistent sets that contain $\Box^{-}(\Phi)$. We show that $\mathcal{V}_\Phi$ assigns truth values in accordance with our rules for truth value assignments. Namely, we should show that each atomic formula is assigned a unique truth value, and that the rules for negation, implication and necessitation are observed.

Proof: First, each atomic formula is either in Φ, in which case $\mathcal{V}_\Phi$ assigns to it the truth value T, or not in Φ, in which case it is assigned the truth value F. So each atomic formula is assigned a unique truth value.

We proceed with negation. We should show that if $\mathcal{V}_\Phi$ assigns T to a formula φ it assigns F to its negation, $\neg\varphi$, and if it assigns F to φ it assigns T to $\neg\varphi$. Although this proof belongs to the Propositional Calculus, I provide it here.

First, if $\mathcal{V}_\Phi(\varphi) = T$, then φ is in Φ, in which case its negation is not in Φ, for Φ is consistent. So $\mathcal{V}_\Phi(\neg\varphi) = F$. Secondly, if $\mathcal{V}_\Phi(\varphi) = F$, then φ is not in Φ. But since Φ is a maximal consistent set, it follows from Lemma 8 that $\neg\varphi$ is in Φ. Accordingly, $\mathcal{V}_\Phi(\neg\varphi) = T$. So $\mathcal{V}_\Phi$ assigns truth conditions according to the rules for negation.

I shall not provide a parallel detailed proof for implication. One should show that $\mathcal{V}_\Phi$ assigns T to $\varphi \rightarrow \psi$ just in case it assigns F to φ or T to ψ, namely, just in case φ is not in Φ or ψ is in Φ. This follows immediately from Lemma 9.

We proceed with necessitation. We should consider the case in which $\Box\varphi$ is true and then the case in which it is false.

If $\mathcal{V}_\Phi(\Box\varphi) = T$, then $\Box\varphi$ is in Φ. Since each associated valuation of $\mathcal{V}_\Phi$ is one that assigns truth to all formulas of some maximal consistent set that contains the formulas of $\Box^-(\Phi)$, and one of these formulas is φ, each associated valuation assigns truth to φ, which is according to the rules for truth value assignment to necessitation.

Next, if $\mathcal{V}_\Phi(\Box\varphi) = F$, then $\Box\varphi$ is not in Φ. Since Φ is a maximal consistent set, it follows from Lemma 8 that $\neg\Box\varphi$ is in Φ. Now, according to Theorem 11, some maximal consistent set contains both $\neg\varphi$ and $\Box^-(\Phi)$. This maximal consistent set does not contain φ. The canonical valuation of this maximal consistent set thus assigns F to φ, while since this maximal consistent set contains $\Box^-(\Phi)$, its canonical valuation is associated with $\mathcal{V}_\Phi$. So this case is again according to the rules for truth value assignment to necessitation. QED

Since any valuation in $\mathcal{V}_\Phi$'s valuation set is also a canonical valuation of a maximal consistent set, and so are the valuation in its valuation set, and so on for the whole frame, all these valuations assign truth values in accordance with our rules for truth value assignments. All of them are therefore among the valuations of Modal Propositional Calculus, and the frame constituted in this way is a frame of valuations of that calculus.

We also need to prove that in case our proof system is T, then the frame generated in this way is a T set of valuations, namely, one in which the schema $\Box\varphi \rightarrow \varphi$ is a logical truth. Any instance of this schema can be proved by (T):

1	(1)	$\Box\varphi$	Premise
1	(2)	φ	T 1
	(3)	$\Box\varphi \rightarrow \varphi$	$\rightarrow$I 1, 2

Accordingly, all maximal consistent sets contain all instances of schema T, and therefore all canonical valuations in a frame assign truth to all instances of the schema, and the schema is a logical truth on any of the frames we are considering.

We can now prove the completeness of the calculus, namely, that if Ξ is a set of formulas and $\Xi \vDash \varphi$, then $\Xi \vdash \varphi$; and likewise, if $\Xi \vDash_T \varphi$, then $\Xi \vdash_T \varphi$.

Proof: Suppose first that Ξ is inconsistent. Then some formula η and its negation, $\neg\eta$, are both derivable from Ξ, and, therefore, also from $\Xi \cup \{\neg\varphi\}$.

Namely, $\Xi \cup \{\neg\varphi\} \vdash \eta, \neg\eta$. By negation introduction, $\Xi \vdash \neg\neg\varphi$, and thus by negation elimination, $\Xi \vdash \varphi$. This holds for any modal system.

Suppose next that Ξ is consistent. Suppose also that $\Xi \cup \{\neg\varphi\}$ is consistent. Then, according to Lemma 6, some maximal consistent set contains Ξ and $\neg\varphi$. But then, the canonical valuation that makes the sentences of this maximal consistent set true makes $\neg\varphi$ true; while because $\Xi \vDash \varphi$ and a canonical valuation is according to the rules for truth value assignment, it also makes φ true. But this is impossible. So it is not the case that $\Xi \cup \{\neg\varphi\}$ is consistent. So some formula η and its negation, $\neg\eta$, are derivable from $\Xi \cup \{\neg\varphi\}$, and as we saw in the previous paragraph, it follows that $\Xi \vdash \varphi$. This proof again applies to all modal system considered here. QED

We noted above that since valuations are functions, a frame can contain two occurrences of the same valuation only if each occurrence has different valuation sets, and to be different these should contain different valuations or different occurrences of the same valuation. We should make sure that this holds for the construction above of the frame of canonical valuations. We do that as follows. The construction of the frame started with a maximal consistent set Φ, and the canonical valuation assigned truth exactly to its formulas. Next, the valuation set of Φ consisted of the canonical valuations of all maximal consistent sets that contain $\Box^{-}(\Phi)$, and we noted that Φ might be among them. If this happens, we consider this the same occurrence of Φ, and its canonical valuation is identified with the initial one. We identify in this way all identical canonical valuations that might occur in the construction. We thus have no two different occurrences of the same valuation in the construction.

We have seen that the truth-valuational approach is sufficient to account for the adequacy of Modal Propositional Calculus. Moreover, it is also straightforward to generalise it to standard Modal Predicate Calculus, and adequacy proofs can then be provided as well (Leblanc 1976; but is should be checked whether the deviations from Leblanc in this work affect his results). Accordingly, the questions on modal logic that had been answered by means of possible world semantics (Kripke 1959, 1963), and in this way were responsible for the rise of the philosophical interest in this semantics, do not require it. Whether there is any other reason for a philosophical interest in possible world semantics remains to be seen.

IX. Applying the Truth-valuational Approach to Modal Predicate Calculus: Contingent Identity

In this short section I mainly show why it is natural to have contingent identity on the truth-valuational approach to modal logic, namely, why it is natural *not* to have $a = b \rightarrow \Box(a = b)$ as a logical truth.

The rules for truth value assignments for identity and those for modal operators were formulated independently, as they should: we have modal systems without identity (Modal Propositional Calculus) as well as systems with identity without modal operators (the Predicate Calculus with identity). When we have a system with both identity and modality, the obvious option is to combine both sets of rules without modifying either of them: such a modification would amount to a change of meaning of the symbol whose rules have been modified, and we are not interested in that. Perhaps new kind of formula might be added in such a merge, and additional rules might be needed for them, but this is not to indicate an inadequacy in the original rules. However, combining these truth value assignment rules for identity and modality would not force us to make $a = b$ true on all valuations in S$\mathcal{V}$ in case it is true on a valuation $\mathcal{V}$, and thus while $a = b$ is true, $\Box(a = b)$ need not be so.

Moreover, as mentioned above, modality on the truth-valuational approach can be seen as representing the possibilities of language within language itself. In that case, since language does not give us any reason for keeping the truth value of $a = b$ fixed across different valuations, we have an additional reason for allowing $a = b$ to be true on $\mathcal{V}$ but not on all valuations in S$\mathcal{V}$.

I turn to the deductive system. Our derivation rules for identity were:

=I

	(i)	$a = a$	=I

=E

L_1	(i)	$a = b$	
L_2	(j)	$P\ldots a\ldots$	
L_1, L_2	(k)	$P\ldots b/a\ldots$	=E i, j

$P\ldots a\ldots$ is an atomic sentence with *a* as one of its arguments; in $P\ldots b/a\ldots$, *b* replaced all or some of the occurrences of *a* in $P\ldots a\ldots$.

We saw above that it can be proved, for any formula $\varphi(a)$ *that does not contain modal operators*, that $a = b, \varphi(a) \vdash \varphi(b/a)$. However, this is *not* generally the case if $\varphi(a)$ contains modal operators. For instance, $a = b, \Box Pa \nvdash \Box Pb$. This can be proved on the basis of the soundness of Modal Predicate Calculus. The proof system is thus in harmony with the truth-valuational one, and identity is contingent on both. As far as logic is concerned, Hesperus might not have been Phosphorus.

Quine (1953: 156), and later others, proved the necessity of identity by relying on the universal form of Leibniz's Law of substitutivity of identicals, $a = b, \varphi(a) \vdash \varphi(b/a)$ for *any* $\varphi(a)$. Although the necessity of identity generated much discussion and controversy, the *universal* form of Leibniz's Law on which its proof relies has rarely been criticised. Burgess (2014), in a careful study of the history of the issue, while critically considering many assumptions made along the way, never suggests that it can be questioned. However, from the point of view of the truth-valuational approach, this law cannot be a basic postulate, for the reasons discussed above, and there is no logical reason to add it to the list of truth value assignment rules. From this point of view, Leibniz's Law should be postulated *only* for atomic formulas. The model-theoretic approach to truth prevented logicians from realising that the universal form of the law involves a non-trivial assumption and that from a logical point of view this assumption, and with it the necessity of identity, are unjustified. However, once the possibility and way of making identity logically contingent is realised, it can also be adopted by model theory (see Ben-Yami 2018).

Another related observation is that rigidity cannot be defined on the truth-valuational approach. In model theory, a name is rigid if the interpretation assigns to it the same object at every world, this assignment being a formal version of reference. According to Kripke, 'a designator rigidly designates a certain object if it designates that object wherever the object exists' (1980: 48–49). However, valuations do not assign objects to names but only truth values to formulas. Nothing in the formal system can therefore reflect the concept of reference or designation, and therefore of rigidity. Rigidity is consequently not a notion of *logic*. This leaves it possible that some notion akin to rigidity can be defined elsewhere in semantics (see Ben-Yami 2014a).

X. Conclusion

This work raised arguments against model-theoretic semantics and developed the truth-valuational approach as an alternative. I argued that the truth-valuational approach is preferable from a conceptual point of view, and we saw that it also provides logic with a satisfactory framework for considering soundness, completeness, satisfiability, and related notions. We saw it applied to the Propositional Calculus, Predicate Calculus, and both calculi extended to modal logic. As mentioned above, the approach has also been applied in several publication to the Quantified Argument Calculus. The way to extend the truth-valuational approach to second-order logic is straightforward, as is the way to apply it to many-valued systems.[14] I am not familiar with a logic system to which Model Theory but not the truth-valuational approach is applicable. I therefore think that from a logical point of view, Model Theory as a semantics is both conceptually problematic and does not provide the only or best formal framework.

Although the truth-valuational approach provides the necessary framework for addressing soundness, completeness, satisfaction, and other related concepts, not all concepts that have interested people working with model-theoretic semantics have a parallel on the truth-valuational approach. We noted in the previous section that *rigidity* cannot be defined on this approach. Another important concept might be that of *categoricity*: a theory is said to be categorical in case all its *models* are isomorphic (Button and Walsh 2018: 139). Once the truth-valuational approach is adopted and models are eliminated, is there any substitute *in logic* for the concept of categoricity? We can still ask, whether a language we use to characterise a structure, a structure which is not a logical entity, characterises it up to isomorphism. I haven't investigated whether this change in question amounts to any significant change in the research into categoricity.

As for semantics, it might gain by adopting the truth-valuational approach. Model Theory, as an instance of a 'Fido'–Fido theory of meaning, might actually have been an *obstacle* to semantics, offering a merely apparent account of the meaning of words and creating an illusion of an explanation, making any further investigation into meaning seem redundant. The uses of words are far too varied for a semantics to rest content with a 'names refer to objects' idea, yet as long as Model Theory seems necessary to formal logic, with no satisfactory alternative available, the pressure to use it as a basis for the understanding of meaning is considerable. The truth-valuational approach should dissolve this pressure. We might then look afresh at the relations of language to the world.

The truth-valuational approach cannot be considered a *semantic* theory: it does not try to provide a theory of *meaning*, only one of truth value relations. How language relates to the world does not interest it. It takes logic to be the study of bivalent and other calculi, irrespective of their manner of application – while meaning is generated exactly through this application.

The concept of *truth*, however, *is* a semantic concept: what we say is true if things are as we say they are. But even this is misleading, for this concept commits us to very little when it comes to meaning. What it takes for 'The cat is on the mat' to be true is quite unlike what it takes for Fermat's last theorem to be so, and both are unlike what the truth of 'Had it rained I would have stayed home' amounts to. Language is used in different ways in different areas, and the criteria for something being true vary with the variation of domains. Once we admit such alethic pluralism, we have an additional argument against model-theoretic semantics and in support of the truth-valuational approach, which is not committed to any theory of truth but only to one of truth relations.

[14] Much valuable formal work on such systems is found in (Leblanc 1976).

Accordingly, logic does not involve a semantics. Philosophers have thought that by means of Model Theory language looks out into the world, but in fact it has been staring at itself all along, as we saw in Section VI. The truth-valuational approach makes this explicit. Logic is the study of the possible truth value relations of a language; it is not committed to any theory of meaning and it cannot serve as a basis for any metaphysics.

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