EXTENSION AND SELF-CONNECTION

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Abstract. If two self-connected individuals are connected, it follows in classical extensional mereotopology that the sum of those individuals is self-connected too. Since mainland Europe and mainland Asia, for example, are both self-connected and connected to each other, mainland Eurasia is also self-connected. In contrast, in non-extensional mereotopologies, two individuals may have more than one sum, in which case it does not follow from their being self-connected and connected that *the* sum of those individuals is self-connected too. Nevertheless, one would still expect it to follow that *a* sum of connected self-connected individuals is self-connected too. In this paper, we present some surprising countermodels which show that this conjecture is incorrect.

1. Introduction

According to classical extensional mereology, for any things, there is exactly one thing they compose. In other words, classical extensional mereology combines two theses – *extensionalism*, according to which no things compose more than one thing, and *universalism*, according to which all things compose at least one thing.[1] So according to classical extensional mereology the North Island and the South Island, for example, compose New Zealand, in accordance with universalism, and nothing but New Zealand, in accordance with extensionalism.

Classical extensional mereology is famously unable to distinguish individuals which are scattered – such as New Zealand – from individuals which are self-connected – such as the Australian mainland. A standard solution

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[1]For this characterisation see, for example, Lewis 1991, p. 74 and Lando 2018.

is to introduce the relation of *connection* as an additional primitive, and to axiomatize its interaction with the parthood relation, thus leading to a corresponding mereo*topo*logy.[2] Then mainland Australia is self-connected whereas New Zealand is not, for example, because New Zealand can be divided into two disconnected parts – the South and North Islands – whereas mainland Australia cannot.

In the resulting theory, known as classical extensional mereotopology, this strategy works well. In particular, it's a theorem that if two individuals are self-connected, and connected to each other, then the sum of the two individuals is self-connected too.[3] Take, for example, mainland Asia and mainland Europe, which are both self-connected, and connected to each other. Then, as we shall see, it follows from universalism that mainland Eurasia exists and from extensionalism that it is self-connected.

However, both universalism and extensionalism are very controversial. Universalism is controversial, since it requires that *all* things compose at least one thing, no matter how scattered or different they are. But this entails the existence of many strange things – for example, that there is something composed of a trout and a turkey, or my hands and my laptop.[4] Denying that things which are scattered – like trouts and turkeys – compose does not undermine the theorem, but denying that connected things – like my hands and my laptop – compose does.

Nevertheless, it's unsurprising that in mereotopologies which reject universalism, some pairs of connected self-connected individuals may not have a unique self-connected sum, because they may have no sum at all. Although my laptop and I, for example, are both self-connected, and connected to each other (as my fingers are touching its keys right now), in the absence

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[2]See Varzi 1996, pp. 270–6 and Casati and Varzi 1999, pp. 52–62.

[3]See Varzi 1996, p. 271 and Casati and Varzi 1999, p. 58.

[4]The trout-turkey example is from Lewis 1991, pp. 79–80.

of universalism it does not follow that the sum of my laptop and I is self-connected, since it does not even follow that the sum of my laptop and I exists.

Extensionalism is controversial mainly because of the way mereology interacts with time and modality. Consider, for example, the United Kingdom, which includes Northern Island, and Great Britain, which does not. Intuitively, it's possible that Northern Ireland could leave the United Kingdom, in which case, intuitively, the United Kingdom and Great Britain would have all the same proper parts. Extensionalism would then predict that the United Kingdom and Great Britain are identical. But that would conflict with the necessity of distinctness.[5]

If extensionalism is false, then two individuals can have more than one sum – if Northern Ireland left the United Kingdom, for example, then Northern and Southern Britain, for example, would sum not only to Great Britain, but also to the United Kingdom. So it's unsurprising that in mereotopologies which reject extensionalism, some pairs of connected self-connected individuals may not have *unique* self-connected sums, not because any of their sums might not be self-connected, but simply because their self-connected sums might not be unique.

Nevertheless, one would expect that in mereotopologies which accept universalism and reject extensionalism, every pair of connected self-connected individuals have *at least* one self-connected sum, even if it is not unique. If Northern and Southern Britain, for example, were both to sum to the United Kingdom and to sum to Great Britain then, if Northern and Southern Britain are both self-connected and connected to each other, we would expect *at least one* of the United Kingdom and Great Britain to be self-connected too. (In fact, we would expect *both* the United Kingdom and Great Britain to be self-connected.)

[5] For a similar example see Lewis 1986, pp. 248–9.

In this paper, we show that this conjecture is not provable from standard axiomatizations of non-extensional mereology. *Very* roughly, the underlying problem is that because there are more sums in non-extensional mereology, there are also more ways to divide things into sums, and so more ways in which something may fail to be self-connected. If Belgium and the Netherlands, for example, had two sums, then Benelux may fail to be self-connected if one of these sums, but not the other, does not connect to Luxembourg.

There is one axiomatization of non-extensional mereology which escapes the problem. According to the *mutual parts* view, some things can compose more than one thing, as long as the things they compose are parts of one another.[6] If Northern Ireland left the United Kingdom, for example, then according to this view Great Britain and the United Kingdom would still be distinct, but in addition to Great Britain being part of the United Kingdom, the United Kingdom would also be part of Great Britain. In other words, the United Kingdom and Great Britain would both be distinct parts of each other, or mutual parts.

Sections 2-4 reviews the axioms of classical extensional mereology and its non-extensional rivals. Section 5 does the same for the relevant axioms and definitions of mereotopology, and proves that every pair of connected self-connected individuals has a self-connected sum in closed extensional mereotopology. Section 6 presents countermodels in closed mereotopology. Section 7 presents similar countermodels in general minimal mereotopology. Finally section 8 proves the theorem in general supplemented premereotopology, an axiomatization of the mutual parts view.

2. Extensional Mereology

In this section, we review the axioms of extensional mereology, and some of its weaker variations, especially minimal mereology and the mutual parts view. We begin by adopting a single primitive relation P, where Pxy is

[6] See, for example, Cotnoir 2010, 2016; Parsons n.d.; Thomson 1998, p. 155.

interpreted as meaning x is an (improper) part of y.[7] It's usually assumed that parthood is reflexive, antisymmetric and transitive:

(1) Pxx Reflexivity

(2) $Pxy \wedge Pyx \to x = y$ Antisymmetry

(3) $Pxy \wedge Pyz \to Pxz$ Transitivity

Axioms 1-3 constitute the theory known as *ground mereology*, abbreviated as M.[8] In the current context, reflexivity and transitivity are relatively uncontroversial.[9] However, antisymmetry is closely connected to extensionalism, since it entails that things with all and only the same parts are identical.[10] We will consider rejecting antisymmetry in section 8, when we discuss the mutual parts view.

In terms of parthood, we define a proper part as a nonidentical part:

(4) $PPxy =_{def} (Pxy \wedge x \neq y)$ Proper Parthood

So Queensland is a proper part of Australia, for example, since Queensland is a part of Australia which is not identical to Australia.[11]

Likewise, overlap is defined as having a part in common:

(5) $Oxy \leftrightarrow (\exists z)(Pzx \wedge Pzy)$ Overlap

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[7]Following Varzi 1996, pp. 260–1 and Casati and Varzi 1999, p. 36.

[8]See Varzi 1996, p. 261 and Casati and Varzi 1999, p. 38.

[9]See Kearns 2011 for criticism of reflexivity. Varzi 2006 defends transitivity.

[10]See Cotnoir 2010, Cotnoir 2016, pp. 127–8 and Cotnoir 2013, pp. 837–9.

[11]For this definition see, for example, Leonard and Goodman 1940, p. 47 and Simons 1987, p. 11. Proper parthood can also be defined as non-mutual parthood, viz.: $PPxy =_{def} Pxy \wedge \neg Pyx$. See, for example, Varzi 1996, p. 261 and Casati and Varzi 1999, p. 36. It follows from antisymmetry that the two definitions are equivalent, but in non-extensional mereologies which reject antisymmetry, as the mutual-parts view does, the two definitions can come apart. See especially Cotnoir 2010, p. 398, Parsons 2014, pp. 6–7 and Cotnoir 2018.

For example, Egypt overlaps Asia, since there is something – namely, Sinai – which is part of Egypt and part of Asia.[12]

With these two definitions, we may state the axiom of weak supplementation, according to which if something is a proper part of another, there is some part of the latter which does not overlap the former:

(6) $PPxy \rightarrow (\exists z)(Pzy \wedge \neg Ozx)$ Weak Supplementation

Since Queensland is a proper part of Australia, for example, weak supplementation entails there is some part of Australia – such as, for example, Tasmania – which does not overlap Queensland.[13]

Axioms 1-3 together with weak supplementation constitute the theory known as *minimal mereology*, and abbreviated as MM (Casati and Varzi 1999, p. 39). The theory is so-called because weak supplementation is supposed to be analytic or, as Peter Simons writes, "constitutive of the meaning of 'proper part'" (Simons 1987, p. 116). Nevertheless, weak supplementation, in combination with reflexivity and transitivity, entails antisymmetry.[14] So it is rejected, amongst others, by proponents of the mutual parts view.[15] We will return to this issue in section 8.

[12]For this definition see, for example, Simons 1987, p. 28, Varzi 1996, p. 261 and Casati and Varzi 1999, p. 36.

[13]For weak supplementation see Simons 1987, p. 28 and Casati and Varzi 1999, p. 39.

[14]For suppose Pxy and Pyx and assume for reductio that $x \neq y$. Then from Definition 4, x is a proper part of y. From weak supplementation, it follows that there is z which is part of x but does not overlap y. Then since z is part of x and x is part of y, it follows from transitivity that z is part of y. But according to reflexivity, y is also part of y, so z and y overlap. But z and y don't overlap, which concludes the reductio. See also the proof in Pietruszczak 2018, p. 155 of his lemma 5.2(iii).

[15]See Cotnoir 2016, Cotnoir 2018 and Parsons n.d. However, note that if proper parthood is defined as non-mutual parthood instead of non-identical parthood, then the corresponding version of weak supplementation is innocuous even according to the mutual parts view. Weak supplementation is also rejected in mereotopologies inspired by Whitehead. See Whitehead 1929, pp. 345–53; Clarke 1981; Simons 1987, p. 98; Casati and Varzi 1999, p. 79 and Cotnoir 2018.

According to the axiom of strong supplementation, if something is not part of another, then there is some part of the former which does not overlap the latter:

(7) $\neg Pyx \rightarrow (\exists z)(Pzy \wedge \neg Ozx)$ Strong Supplementation

Since Turkey is not part of Asia, for example, strong supplementation entails there is some part of Turkey – such as, for example, Istanbul – which does not overlap Asia.[16]

Axioms 1-3 together with strong supplementation constitute the theory known as *extensional mereology*, and abbreviated as EM (Casati and Varzi 1999, p. 39).[17] In ground mereology, strong supplementation together with antisymmetry entails weak supplementation, so minimal mereology is a subtheory of extensional mereology (Simons 1987, p. 29; Casati and Varzi 1999, p. 39).[18] However, since proponents of the mutual parts view reject antisymmetry, they are willing and able to accept strong supplementation, while rejecting weak supplementation.[19] We will consider this possibility in more detail when we discuss the mutual parts view in section 8.

3. Closed Mereology

The axioms reviewed so far are mainly aimed at capturing extensionalism; in this section, we review axioms aimed at capturing universalism. To begin with, let us say that z is a *sum* of x and y just in case $(\forall w)(Owz \leftrightarrow$

[16]For strong supplementation see Simons 1987, p. 29, Varzi 1996, p. 262 and Casati and Varzi 1999, p. 39.

[17]In fact, axiom 1 is redundant in extensional mereology, since it is entailed by axioms 3 and 7. See the proof in Pietruszczak 2018, p. 157 of his lemma 5.7.

[18]For suppose y is a proper part of x. Then from the definition of proper parthood, y is part of x and $y \neq x$. Then suppose for reductio that x is part of y. Then y is part of x and x is part of y, so from antisymmetry $y = x$, contradicting that $y \neq x$. So x is not part of y. But then it follows from strong supplementation that there is z which is part of x and does not overlap y. See also the proof in Pietruszczak 2020, p. 31 of his lemma 2.3.5.

[19]See Cotnoir 2013, pp. 837–8, Cotnoir 2016, p. 127, and Parsons n.d.

$(Owx \vee Owy))$ or, in other words, all and only things which overlap z overlap x or y.[20] Then according to the axiom of sum closure, every pair has a sum:

(8) $(\exists z)(\forall w)(Owz \leftrightarrow (Owx \vee Owy))$ Sum Closure[21]

And if a pair of individuals have a unique sum, then we can define *the* sum of those individuals as the thing which is overlapped by all and only things which overlap either of them:

(9) $x + y = (\iota z)(\forall w)(Owz \leftrightarrow (Owx \vee Owy))$ Sum[22]

Eurasia is a sum of Europe and Asia, for example, because all and only things which overlap Eurasia either overlap Europe or else overlap Asia. And if Eurasia is the only thing overlapped by all and only things which overlap Europe or overlap Asia, then Eurasia is *the* sum of Europe and Asia.

Likewise, let us say that z is a *product* of x and y just in case $(\forall w)(Pwz \leftrightarrow (Pwx \wedge Pwy))$ or, in other words, if and only if all and only parts of z are part of both x and y. Then according to the axiom of product closure, every overlapping pair has a product:

[20]In closed extensional mereology and stronger theories, it would be equivalent to say that z is a sum of x and y just in case $Pxz \wedge Pyz \wedge (\forall w)(Pwz \rightarrow (Owx \vee Owy))$ or, in other words, x is part of z, y is part of z, and every part of z overlaps x or y. But in weaker mereological theories, these two formulations of sum may come apart, in ways which matter to the controversy over whether universalism entails extensionalism. See, for example, Varzi 2009, p. 601.

[21]For this version of sum closure, see Masolo and Vieu 1999, p. 238. Varzi 1996, p. 263 and Casati and Varzi 1999, p. 43 prefer a slightly weaker version, according to which two individuals have a sum if they underlap (in other words, if there is some individual they are both a part of). Corresponding to the stronger definition of sum in footnote 20, one might prefer a stronger version of sum closure according to which $(\exists z)(Pxz \wedge Pyz \wedge (\forall w)(Pwz \rightarrow (Owx \vee Owy)))$.

[22]For this definition see, for example, Leonard and Goodman 1940, p. 48, Simons 1987, p. 13, Varzi 1996, p. 263 and Casati and Varzi 1999, p. 43. We follow Casati and Varzi 1999, p. 204 in adopting the Russellian treatment of definite descriptions (Whitehead and Russell 1925, pp. 173–86).

(10) $Oxy \to (\exists z)(\forall w)(Pwz \leftrightarrow (Pwx \wedge Pwy))$ Product Closure

And if a pair of individuals has a unique product, then we can define *the* product of those individuals as the thing all and only parts of which are parts of both of them:

(11) $x \times y = (\iota z)(\forall w)(Pwz \leftrightarrow (Pwx \wedge Pwy))$ Product

Sinai is a product of Egypt and Asia, for example, because all and only parts of Sinai are parts of both Egypt and Asia. And if Sinai is the only thing all and only parts of which are parts of both Egypt and Asia, then Sinai is *the* product of Egypt and Asia.

Ground mereology together with sum and product closure constitutes the theory known as *closed mereology*, abbreviated as CM. Similarly, closed mereology together with minimal mereology constitutes the theory known as *closed minimal mereology*, abbreviated as CMM. And closed mereology together with extensional mereology constitutes the theory known as *closed extensional mereology*, abbreviated as CEM.[23]

Note that in closed extensional mereology, if an individual overlaps two others, then its product distributes over their sum:

(12) $(Oxy \wedge Oxz) \to x \times (y + z) = (x \times y) + (x \times z)$ Distributivity

Since Turkey overlaps Europe and Asia, for example, the product of Turkey with Eurasia is the sum of the product of Turkey with Europe and the product of Turkey with Asia.[24]

It's a surprising but well known fact that closed minimal mereology and closed extensional mereology are equivalent, because weak supplementation in combination with product closure entails strong supplementation.[25] This provides the seed of an argument from universalism, expressed in terms

[23]See Varzi 1996, p. 263 and Casati and Varzi 1999, p. 43.

[24]For proofs see Pietruszczak 2018, pp. 102–4 and Pietruszczak 2020, pp. 81–2.

[25]See Simons 1987, p. 31 and Casati and Varzi 1999, p. 44

of sum and product closure, to extensionalism, expressed in terms of anti-symmetry and strong supplementation. However, we shall see in the next section that this argument from universalism to extensionalism is far from straightforward.

4. General Mereology

Recall that according to universalism, all things compose at least one thing. To capture this idea in full generality, let us say for any predicate $\phi(x)$ that x is a *general sum* of the individuals satisfying $\phi(x)$ just in case $(\forall y)(Oyx \leftrightarrow (\exists z)(\phi(z) \wedge Oyz))$ or, in other words, all and only things which overlap z overlap something satisfying $\phi(x)$.[26] Then according to the axiom schema of fusion, if anything satisfies $\phi(x)$, there is a general sum of the things satisfying $\phi(x)$:

(13) $(\exists x)\phi(x) \rightarrow (\exists x)(\forall y)(Oyx \leftrightarrow (\exists z)(\phi(z) \wedge Oyz))$ Fusion

If there is an ocean, for example, then there is something – viz., the Ocean – which is overlapped by all and only things which overlap an ocean.

Ground mereology together with fusion constitutes the theory known as *general mereology*, abbreviated as GM. Similarly, general mereology together with minimal mereology constitutes the theory known as *general minimal mereology*, abbreviated as GMM. And general mereology together with extensional mereology constitutes the theory known as *general extension mereology*, abbreviated as GEM.[27] GEM is otherwise known as classical extensional mereology, the paradigmatic mereological theory.

[26] In classical extensional mereology, it would be equivalent to say that x is a sum of the individuals satisfying $\phi(x)$ just in case $(\forall y)(\phi(y) \rightarrow Pyx) \wedge (\forall y)(Pyx \rightarrow (\exists z)(\phi(z) \wedge Oyz))$ or, in other words, everything satisfying $\phi(x)$ is part of x and every part of x overlaps something satisfying $\phi(x)$. But in weaker mereological theories, these formulations may come apart. See Hovda 2009, pp. 57–9 and Varzi 2009, p. 601. We consider a version of the axiom schema of fusion based on this formulation in section 8.

[27] See Varzi 1996, p. 265 and Casati and Varzi 1999, p. 46.

Since closed minimal mereology is equivalent to closed extensional mereology, it would be natural to conclude that general minimal mereology is also equivalent to general extensional mereology.[28] But it is now well known that this is not the case, due to the countermodel illustrated by the Hasse diagram in figure 1.[29] In this countermodel, weak supplementation is satisfied but strong supplementation is not, since although, for example, b is not part of d, there is no part of d which does not overlap b (since all three parts of d – viz., a, c and d – all have parts in common with b).

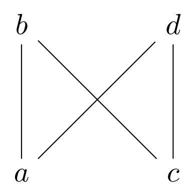

Figure 1. Countermodel in GMM

The reason the argument that closed minimal mereology is equivalent to closed extensional mereology does not extend to show that general minimal mereology is not equivalent to general extensional mereology is that in the absence of strong strong supplementation, the fusion axiom fails to entail product closure.[30] Of course, if two individuals overlap, then they have a common part, so it follows from the fusion axiom that there is a general sum of their common parts, so we have the following lemma:

(14) $Oxy \rightarrow (\exists z)(\forall v)(Ovz \leftrightarrow (\exists w)((Pwx \wedge Pwy) \wedge Ovw))$

However, the countermodel in figure 1 shows that lemma 14 does not entail product closure, since although b and d overlap, and both b and d are general

[28]This mistake is in Simons 1987, p. 37 and Casati and Varzi 1999, p. 46.

[29]Simons 1987, p. 28 discusses this model in connection with weak supplementation. The mistake is corrected in Pietruszczak 2000, translated in Pietruszczak 2005, pp. 228–9, as well as Pontow 2004 and Hovda 2009. See also Varzi 2009, 2019.

[30]See Pontow 2004, p. 205 and Hovda 2009, pp. 64–5.

sums of a and c, which are the common parts of b and d, neither b nor d are a product of b and d.

This point threatens to undermine the line of argument from universalism to extensionalism considered in the last section. We will reconsider whether this line of argument can be salvaged in section 8, when we consider a stronger version of the axiom schema of fusion which does entail product closure and so, in combination with weak supplementation, does entail strong supplementation. We will then consider the possibility of resisting this argument by accepting strong supplementation, but nevertheless avoiding extensionalism by denying the antisymmetry of parthood.

5. Mereotopology

In this section, we consider mereotopological theories which combine the primitive relation P, still interpreted as improper parthood, with an additional primitive relation C, where Cxy is interpreted as meaning x is connected to y.[31] It's assumed that connection is reflexive and symmetric:

(15) Cxx Reflexivity

(16) $Cxy \rightarrow Cyx$ Symmetry

In addition, it's assumed that if something is part of another, then everything connected to the former connects to the latter:

(17) $Pxy \rightarrow (\forall z)(Czx \rightarrow Czy)$ Monotonicity

Since Queensland is part of Australia, for example, everything connected to Queensland is connected to Australia.

Ground mereology together with axioms 15-17 constitutes the theory known as *ground mereotopology*, abbreviated as MT.[32] Then each mereological theory X combined with MT constitutes a mereo*topo*logical theory

[31] Following Varzi 1996, p. 268 and Casati and Varzi 1999, p. 52.

[32] See Varzi 1996, p. 268 and Casati and Varzi 1999, p. 54.

XT. For our purposes the most important of these are closed mereotopology or CMT, closed extensional mereotopology or $CEMT$, general minimal mereotopology or $GMMT$ and general extensional mereotopology or $GEMT$.[33]

As we mentioned in the introduction, mereology alone cannot distinguish between objects which are scattered and objects which are self-connected. But in mereotopology, an individual can be defined as self-connected if and only if for every way of dividing it into a sum, its summands are connected:

(18) $SCz =_{def} (\forall x)(\forall y)((\forall w)(Owz \leftrightarrow (Owx \vee Owy)) \rightarrow Cxy)$

New Zealand, for example, is not self-connected, since New Zealand is a sum of the South Island and the North Island, but the South Island is not connected to the North Island.[34]

In closed extensional mereotopology, the existence and uniqueness of sums is guaranteed, so Definition 18 is equivalent to the following lemma:

(19) $SCz \leftrightarrow (\forall x)(\forall y)(z = x + y \rightarrow Cxy)$

New Zealand, for example, is not self-connected, since New Zealand is *the* sum of the South Island and the North Island, but the South Island and the North Island are not connected.[35]

So it is a theorem of closed extensional mereotopology that if two self-connected individuals are connected, their sum is self-connected too:

(20) $(Cxy \wedge SCx \wedge SCy) \rightarrow SC(x + y)$

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[33]See Varzi 1996 and Casati and Varzi 1999, p. 57. Casati and Varzi's formulation of CMT includes an additional axiom according to which connected individuals underlap (Casati and Varzi 1999, p. 57). This axiom is not needed here as it is entailed by our stronger version of the sum closure axiom.

[34]Given the characterisation of sums in footnote 20, the definition of self-connection would instead be $SCz =_{def} (\forall x)(\forall y)(Pxz \wedge Pyz \wedge (\forall w)(Pwz \rightarrow (Owx \vee Owy)) \rightarrow Cxy)$.

[35]For this point see Casati and Varzi 1999, p. 58. Varzi 1996, p. 271 uses lemma 19 as the definition of self-connection.

Proof. Suppose that the antecedent is true. Then to show the consequent, suppose that $x + y = v + w$. Then we have to show that Cvw. There are four cases. In the first case, v and w both overlap x. But then since x is part of $x + y$, it follows x is part of $v + w$, so $x = x \times (v + w)$. But from the distributivity of product over sum, $x \times (v + w) = (x \times v) + (x \times w)$, so $x = (x \times v) + (x \times w)$. Then since x is self-connected, $C(x \times v)(x \times w)$, and so Cvw. In the second case, v and w both overlap y, and Cvw follows from the same reasoning as in the first case except with x replaced by y. In the third case, x does not overlap v and y does not overlap w. Then $x = w$ and $y = v$, and Cvw follows because x and y are connected. Likewise in the fourth case x does not overlap w and y does not overlap v, so $x = v$ and $y = w$, and Cvw follows from the same reasoning as in the third case.[36] □

Via the definition of sums, it's an obvious corollary, which we call *self-connected sum closure*, that if two self-connected individuals are connected, then they have a self-connected sum:

(21) $(Cxy \wedge SCx \wedge SCy) \rightarrow (\exists z)(SCz \wedge (\forall w)(Owz \leftrightarrow (Owx \vee Owy)))$

There is something self-connected which is overlapped by all and only overlappers of Europe or Asia, for example, because Eurasia is overlapped by all and only overlappers of Europe or Asia.[37]

It is not surprising that neither lemma 19, theorem 20 nor corollary 21 are provable in extensional mereotopology or weaker, since in the absence of sum closure, connected self-connected individuals may not have *any* sum, self-connected or not. Similarly, it is not surprising that lemma 19 and theorem 20 are not provable in closed mereotopology or weaker, since in the absence of strong supplementation, connected self-connected individuals may not have a unique sum, and so the description abbreviated by $x + y$ may not be uniquely satisfied.

[36]For the statement (without proof) of this theorem see Casati and Varzi 1999, p. 58.

[37]Given the characterisation of sums in footnote 20, self-connected sum closure would be written $(Cxy \wedge SCx \wedge SCy) \rightarrow (\exists z)(SCz \wedge Pxz \wedge Pyz \wedge (\forall w)(Pwz \leftrightarrow (Owx \vee Owy)))$.

However, we would still expect self-connected sum closure to be provable in closed mereotopology and general minimal mereotopology, since sum closure ensures that connected self-connected individuals have at least one sum, which one would expect to be self-connected according to the original definition of self-connection, since it does not require a unique sum.[38] The following two sections present surprising countermodels to demonstrate that this conjecture is false, and self-connected sum closure is not a theorem either of closed mereotopology or general minimal mereotopology.

6. Closed Mereotopology

In this section, we present a countermodel to show that self-connected sum closure is not a theorem of closed mereotopology. Suppose there are just four individuals, a, b, c and d. And suppose that a and c are proper parts of b, that d is also a proper part of c, and that a is not part of c or d, as illustrated by the Hasse diagram in figure 2a. Moreover, suppose that a is connected to c, but not connected to d, as illustrated by the graph in figure 2b. Then a is self-connected, c is self-connected, and a is connected to c, so the antecedent of self-connected sum closure is satisfied by a and c.

(a) Parthood (b) Connection

Figure 2. Countermodel in closed mereotopology

Nevertheless, a and c do not satisfy the consequent of self-connected sum closure. Of course, a and c do have a sum in the model, since something overlaps b if and only if it overlaps a or overlaps c. In other words, b is a

[38]For the (incorrect) claim that self-connected sum closure is a theorem of closed mereotopology see Casati and Varzi 1999, p. 58.

sum of a and c. The problem is that b is not self-connected, because b is also a sum of a and d – in other words, something overlaps b if and only if it overlaps a or overlaps d – but a is not connected to d. Moreover, nothing other than b is a sum of a and c, so nothing else satisfies the consequent of self-connected sum closure either.[39]

There are at least two counterintuitive features of the countermodel. First, it violates weak supplementation, because d is a proper part of c, but there is no part of c which does not overlap d. Adding weak supplementation to closed mereotopology obtains closed minimal mereotopology, which is equivalent to closed extensional mereotopology, and so would be strong enough to entail self-connected sum closure. However, in section 7, we will show that in general minimal mereotopology there are countermodels to self-connected sum closure that do satisfy weak supplementation.

Second, it's counterintuitive that a is connected to c, without being connected to d, the only proper part of c. This suggests we might rule out the countermodel by adding an axiom, which we call *demonotonicity*, according to which if an individual is connected to a complex, then it is connected to a proper part of that complex, viz.:

(22) $(\exists z)PPzx \rightarrow (\forall y)(Cyx \rightarrow (\exists z)(PPzx \wedge Cyz))$ Demonotonicity

Suppose, for example, that an ax has two proper parts – its handle and its blade. Then nothing can be connected to the ax without being connected to either the handle or the blade.[40]

However, the addition of axiom 22 is still too weak to entail self-connected sum closure. For suppose there are countably many individuals a, b, c_0, c_1,

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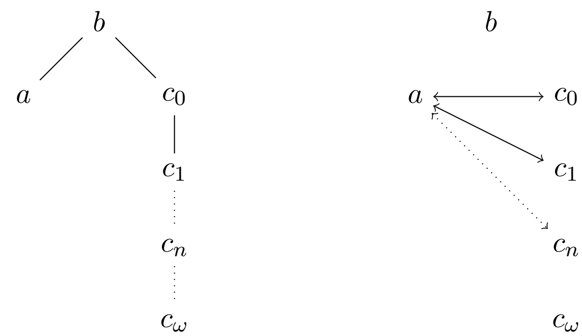
[39] Note that the countermodel still works given the characterisation of sum in footnote 20. In this case b is still the (unique) sum of a and c because a is part of b, c is part of b, and everything which is part of b overlaps a or c. But b is also still a sum of a and d, since a is part of b, d is part of b and every part of b overlaps a or overlaps d. And so b still fails to be self-connected, because a and d are not connected.

[40] See Blumson and Singh 2020, p. 123 for discussion of this axiom.

c_2, ..., c_ω. And suppose again that a and c are proper parts of b, that a is not part of any c_n, but that each c_{n+1} is an immediate proper part of each c_n, and suppose c_ω is a proper part of every c_n, as illustrated by the Hasse diagram in figure 3a. Moreover, let us say that two individuals are *externally connected* just in case they are connected but not overlapping.[41] And suppose that a is externally connected to every c_n, but not externally connected to c_ω, and that nothing else is externally connected, as illustrated by the graph in figure 3b.[42]



(a) Parthood (b) External Connection

FIGURE 3. Countermodel in closed mereotopology with 22

In this countermodel a is self-connected, c_0 is self-connected, and a is connected to c_0, so the antecedent of self-connected sum closure is satisfied by a and c_0. Nevertheless, a and c_0 do not satisfy the consequent of self-connected sum closure. Of course, a and c_0 do have a sum in the model, namely b. The problem is that b is not self-connected, because b is also a sum of a and c_ω, but a is not connected to c_ω. Moreover, nothing other than b is a sum of a and c_0, so nothing else satisfies the consequent of self-connected sum closure either.[43]

[41]See Varzi 1996, p. 268 and Casati and Varzi 1999, pp. 54–5.

[42]Cotnoir 2016, p. 126 suggests the purpose of supplementation principles is partly to rule out models of this kind.

[43]Note that the countermodel still works if we characterize sums as in footnote 20.

In order to rule out this countermodel, we could adopt as an axiom a stronger version of demonotonicity, which we call *atomic demonotonicity*, according to which if something is connected to another, then the former is connected to a part of the latter which has no proper parts:

(23) $Cxy \rightarrow (\exists z)(Pzy \wedge \neg(\exists v)PPvz \wedge Cxz)$ Atomic Demonotonicity

The addition of this axiom would rule out the counterexample in figure 3, since in order for a to be connected to c_0, it would follow from atomic demonotonicity that it is also connected to c_ω, and then from the monotonicity of connection that a is also connected to each c_n.[44]

In general, the addition of atomic demonotonicity as an axiom of closed mereotopology would entail self-connected sum closure via the following lemma, which we call *coincidence implies connection*, according to which if two individuals have all and only the same overlappers, then they are also connected to all and only the same individuals:

(24) $(\forall z)(Ozx \leftrightarrow Ozy) \rightarrow (\forall z)(Czx \leftrightarrow Czy)$

Proof. Suppose all and only things which overlap x overlap y. And suppose z is connected to x. Then from Axiom 23, z is connected to a part v of x which has no proper parts. Since v is part of x, and parthood implies overlap, v overlaps x. So v also overlaps y. From the definition of overlap, something is part of both v and y. But since v has no proper parts, v itself is part of y. Then since z is connected to v, it follows from monotonicity that z is connected to y. Mutatis mutandis, if z is connected to y, then z is connected to x. So all and only things connected to x connect to y. □

From coincidence implies connection, we can prove self-connected sum closure by an argument similar to the proof of theorem 20 in closed extensional mereotopology, except without assuming that sums are unique. For suppose x is self-connected, y is self-connected and x is connected to y. Then from sum closure, there is a sum z of x and y. To show that z is self-connected,

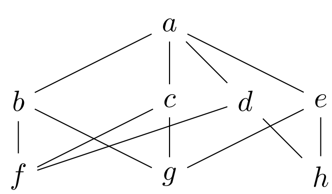
[44]For discussion of atomic demonotonicity see Blumson and Singh 2020, p. 123.

suppose that z is a sum of v and w. Then there are four cases. In the first case, v and w both overlap x. Then x is a sum of $x \times v$ and $x \times w$. Since x is self-connected, $x \times v$ is connected to $x \times w$, so from the monotonicity of parthood v is connected to w. In the second case, v and w both overlap y, and the reasoning is the same as in the first case except with x replaced by y. In the third case, x does not overlap v and y does not overlap w. Then all and only things which overlap x overlap v and all and only things which overlap y overlap w. So from lemma 24, coincidence implies connection, all and only things connected to x are connected to v and all and only things connected to y are connected to w. But then since x is connected to y, v is connected to w. Finally, in the fourth case x does not overlap w and y does not overlap v, and the reasoning is as in the third case except that x and y are interchanged.

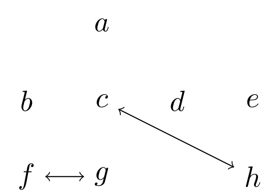
7. General Minimal Mereotopology

The previous countermodels in closed mereotopology are excluded in general minimal mereotopology by the axiom of weak supplementation. And since the sum closure axiom follows from the fusion axiom, every reason we had to expect self-connected sum closure to be a theorem of closed mereotopology, we also had as a reason to expect it to be a theorem of general minimal mereotopology too. Nevertheless, for similar reasons, self-connected sum closure is not a theorem of general minimal mereotopology either, as the following countermodel shows.

Suppose that there are three simple individuals, f, g and h. And suppose that f and h uniquely compose d, that g and h uniquely compose e, but that f and g compose *two* individuals, b and c, and that all the individuals taken together compose a, as illustrated by the Hasse diagram in figure 4a. Moreover, suppose f is externally connected to g and h is externally connected to c, but no other individuals are externally connected, as illustrated by the graph in figure 4b. Then c is self-connected and h is self-connected and c is connected to h, so c and h satisfy the antecedent of self-connected sum closure.



(a) Parthood



(b) External Connection

FIGURE 4. Countermodel in general minimal mereotopology

Nevertheless, c and h do not satisfy the consequent of self-connected sum closure. Of course, c and h do have a sum in the model, since a is overlapped by all and only overlappers of c or overlappers of h. In other words, a is a sum of c and h. The problem is that a is not self-connected, because a is also a sum of b and h – in other words, something overlaps a if and only if it overlaps b or overlaps h – but b is not connected to h. Moreover, nothing else in the model is a sum of c and h, so nothing else satisfies the consequent of self-connected sum closure either.[45]

As in the previous section, there are at least two counterintuitive features of the countermodel. First, it violates strong supplementation – although b is not a part of c, for example, there is no part of b that does not overlap c. This violation of strong supplementation allows f and g to have two sums,

[45]Note that given the characterisation of sum in footnote 20, a is still the (unique) sum of c and h because c is part of a, h is part of a, and everything which is part of a overlaps c or h. But a is also still a sum of b and h, since b is part of a, h is part of a and every part of a overlaps b or overlaps h. And so a still fails to be self-connected, because b and h are not connected. However, notice that not everything in the model has a sum in the sense of footnote 20. In particular, although b and c are both part of a, not every part of a overlaps b or overlaps c, since h is part of a but does not overlap b or c (in the weaker sense of sum b and c have two sums, b and c themselves). So this countermodel is ruled out by axiom 26, which we discuss in section 8.

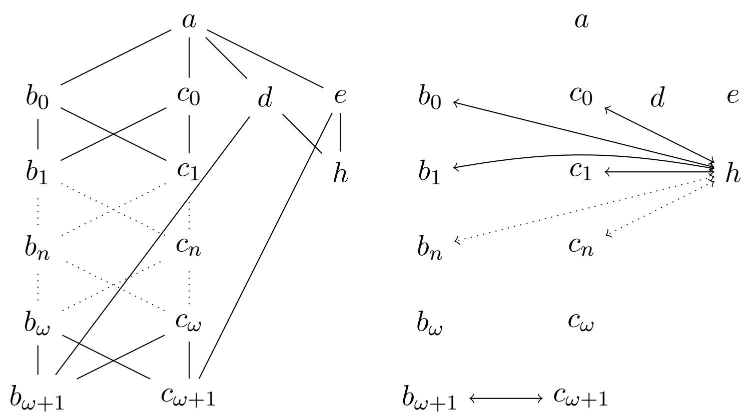
which is in turn what allows c to be be connected to h without a being self-connected. In contrast, we will show in section 8 that on the mutual parts view, which rejects extensionalism but accepts strong supplementation, self-connected sum closure *is* provable.

Second, it's counterintuitive that h is connected to c, without being connected to f or g, the proper parts of c. This suggests again that we might rule out the countermodel by adding axiom 22, demonotonicity, according to which an individual is connected to a complex only if it connects to one of its proper parts, as an additional axiom of general minimal mereotopology. However, for the same reason as in the previous section, the addition of demonotonicity as an axiom is too weak to entail self-connected sum closure, as the following countermodel shows.

Suppose there are countably many individuals a, b_0, b_1, ..., b_n, ..., b_ω, $b_{\omega+1}$, c_0, c_1, ..., c_n, ..., c_ω, $c_{\omega+1}$, d, e and h, such that each b_{n+1} is an immediate part of each b_n and c_n, $b_{\omega+1}$ is an immediate part of b_ω, d and c_ω, each c_{n+1} is an immediate part of each c_n and b_n, $c_{\omega+1}$ is an immediate part of c_ω, b_ω and e, h is an immediate part of d and e, b_0, c_0, d and e are immediate parts of a, and b_ω and c_ω are part of each c_n and b_n, as illustrated in the Hasse diagram in figure 5a. And suppose $b_{\omega+1}$ is externally connected to $c_{\omega+1}$, and h is externally connected to each b_n and c_n, but nothing else is externally connected, as illustrated by the graph in figure 5b.

Then c_0 is self-connected and h is self-connected, and c_0 is connected to h, so c_0 and h satisfy the antecedent of self-connected sum closure. Moreover, c_0 and h have a sum a. However, a is not self-connected, because it is also a sum of b_ω and h, and b_ω is not connected to h. Since c_0 and h do not have any other sum apart from a, the consequent of self connected sum-closure is not satisfied, and so self-connected sum closure is not a theorem of general minimal mereotopology, even with demonotonicity as an additional axiom.

In order to rule out this countermodel, we could add to general minimal mereotopology the stronger axiom 23, atomic demonotonicity. This would



(a) Parthood (b) External Connection

FIGURE 5. Countermodel in GMM with axiom 22

rule out the counterexample, since it would require that in order to be connected to each c_n, h must be connected to $b_{\omega+1}$ or $c_{\omega+1}$, and thus by the monotonicity of connection to b_ω and c_ω as well. In general, the addition of atomic demonotonicity to general minimal mereotopology would entail lemma 24, coincidence implies connection, by the same proof as in closed mereotopology.

Then from coincidence implies connection, we could prove self-connected sum closure by an argument similar to the proof of theorem 20 in closed extensional mereotopology. However, as well as being unable to assume the uniqueness of sums, we cannot assume the existence of products, even of overlapping individuals. But, if two individuals overlap, we can assume the existence of a general sum of their common parts, in accordance with lemma 14. Then from atomic demonotonicity we can prove that if an individual is connected to a general sum of the common parts of x and y, then it is also connected to x (as well as to y), in accordance with the following lemma:

(25) $Csz \wedge (\forall v)(Ovz \leftrightarrow (\exists w)((Pwx \wedge Pwy) \wedge Ovw))) \rightarrow Csx$

Proof. Suppose the antecedent. Since s is connected to z, it follows from atomic demonotonicity that there is some a with no proper parts which is part of z and connected to s. Since a is part of z, a overlaps z. So there is a common part w of x and y which overlaps a. Since a overlaps w, a and w have a common part. But since a has no proper parts, this common part must be a itself, and so a is part of w. But w is part of x, so from the transitivity of parthood a is part of x. Moreover, since s is connected to a, it follows from the monotonicity of connection that s is connected to x. □

Finally, to complete the proof of self-connected sum closure in general minimal mereotopology with atomic demonotonicity, suppose again that x is self-connected, y is self-connected and x is connected to y. From sum closure, there is a sum z of x and y. To show z is self-connected, suppose that z is a sum of v and w. Then there are four cases. In the first case, v and w both overlap x. From lemma 14 there is a general sum s of the common parts of x and v and a general sum t of the common parts of x and w. Moreover x is a sum of s and t, so since x is self-connected s is connected to t. From two applications of lemma 25, it follows that v is connected to w. In the second case, v and w both overlap y, and the reasoning is the same as in the first case except with x replaced by y. And the third and fourth cases are the same as in section 6.

8. The Mutual Parts View

Recall from section 4 that the argument from universalism to extensionalism fails in general minimal mereology, since in the absence of strong supplementation the axiom schema of fusion does not entail product closure. That suggests adopting the following stronger form of the axiom schema of fusion:

(26) $(\exists x)\phi(x) \rightarrow (\exists x)((\forall y)(\phi(y) \rightarrow Pyx) \wedge (\forall y)(Pyx \rightarrow (\exists z)(\phi(z) \wedge Oyz)))$

If there is an ocean, for example, there is something, viz. the Ocean, such that every ocean is part of it, and such that every part of it overlaps an ocean.[46]

Together with the axioms of minimal mereology, this stronger version of the fusion axiom schema entails both strong supplementation and product closure, so that the theory resulting from adding axiom 26 to minimal mereology is equivalent to general extensional mereology.[47] So by adopting axiom schema 26 in place of axiom schema 13, proponents of extensional mereology may be able to salvage the argument from universalism to extensionalism mooted in section 3. How should proponents of non-extensional mereology respond to this new version of the argument?

According to the mutual parts view, instead of resisting adopting axiom schema 26, non-extensionalists should accept both it and strong supplementation, but reject weak supplementation and antisymmetry.[48] To understand the mutual parts view, it's helpful to contrast the model illustrated by the Hasse diagram in figure 1 with that illustrated by the directed graph of the parthood relation in figure 6.[49] In figure 1, strong supplementation was not satisfied because although b was not a part of d, there wasn't any part of b which didn't overlap d (and vice versa).

But in figure 6, strong supplementation is satisfied since although there isn't any part of b which doesn't overlap d, b *is* part of d (and vice versa). Nevertheless, by denying the antisymmetry of parthood, the model avoids extensionalism, since although b and d have all and only the same parts, and overlap all and only the same things, b and d are not identical. In other

[46]For this version of the fusion axiom see, for example, Lesniewski 1992; Tarski 1983, p. 25; Lewis 1991, pp. 73–4 and Hovda 2009, p. 62. This version of the fusion axiom corresponds to the formulation of general sums in footnote 26.

[47]For proofs and detailed discussion see Hovda 2009, pp. 65–67.

[48]See Cotnoir 2010, 2016, 2018 and Parsons n.d.

[49]See Cotnoir 2010, p. 399 and Cotnoir 2016, p. 127.

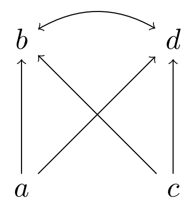


Figure 6. The Mutual Parts View

words, extensionalism is false in the model since a and c compose more than two things, viz. b and d.

Axiomatically, axiom 1 reflexivity, axiom 3 transitivity, axiom 7 strong supplementation and axiom schema 13 fusion constitute the theory of *general supplemented premereology*, which we abbreviate as $GSPM$.[50] Since axiom schema 26 is a theorem of $GSPM$, and entails axiom schema 13, an equivalent axiomatization could be obtained by swapping axiom schema 13 for axiom schema 26. $GSPM$ combined with axiom 15 reflexivity, axiom 16 symmetry and axiom 17 monotonicity of connection constitutes *general supplemented premereotopology*, which we abbreviate $GSPMT$.

Both sum closure and product closure are theorems of $GSPM$. But neither sums nor products are unique. In the model illustrated in figure 6, for example, a and c have two sums b and d. And b and d have two products, b and d. Nevertheless, self-connected sum closure is still derivable in general supplemented premereotopology via lemma 24, coincidence implies connection. Moreover coincidence implies connection can be proved in general supplemented premereotopology without any additional axiom, so neither demonotonicity nor atomic demonotonicity need be adopted in order to prove self-connected sum closure within the mutual parts view.

To prove coincidence implies connection, suppose that all and only individuals overlapping x overlap y. It follows that x is part of y. For suppose for reductio x is not part of y. Then from strong supplementation, there is

[50]Parsons n.d., p. 8 names this theory *sum-complete supplemented preordering*, which he abbreviates as $SSPO$. See also Cotnoir 2016, p. 129 for this axiomatization.

some part of x which does not overlap y, contradicting the assumption that all individuals overlapping x overlap y. Mutatis mutandis, y is part of x. Now to show that all and only individuals connected to x are connected to y, suppose z is connected to x. Since x is part of y, it follows from demonotonicity that z is connected to y. Mutatis mutandis, if z is connected to y, z is connected to x.

Finally, to prove self-connected sum closure in general supplemented premereotopology, suppose again that x is self-connected, y is self-connected and x is connected to y. From sum closure, there is a sum z of x and y. To show z is self-connected, suppose that z is a sum of v and w. Then there are four cases. In the first case, v and w both overlap x. From product closure there is a product of x and v and a product of x and w. Moreover, x is a self-connected sum of these products, and so they are connected to each other, and it follows from monotonicity that v is connected to w. In the second case, v and w both overlap y, and the reasoning is the same as in the first case except with x replaced by y. And the third and fourth cases are the same as in sections 6 and 7.[51]

References

Blumson, Ben and Manikaran Singh (2020). “Whitehead’s Principle”. *Thought: A Journal of Philosophy* 9.2, pp. 115–127. DOI: 10.1002/tht3.450.

Casati, Roberto and Achille C. Varzi (1999). *Parts and Places. The Structures of Spatial Representation.* Cambridge, Mass.: MIT Press. 856–863.

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Clarke, Bowman L. (1981). "A Calculus of Individuals Based on "Connection"." *Notre Dame Journal of Formal Logic* 22.3, pp. 204–218. DOI: 10.1305/ndjfl/1093883455.

Cotnoir, Aaron J. (2010). "Anti-Symmetry and Non-Extensional Mereology". *The Philosophical Quarterly* 60.239, pp. 396–405. DOI: 10.1111/j.1467-9213.2009.649.x.

– (2013). "Strange Parts: The Metaphysics of Non-Classical Mereologies". *Philosophy Compass* 8.9, pp. 834–845. DOI: 10.1111/phc3.12061.

– (2016). "Does Universalism Entail Extensionalism?" *Noûs* 50.1, pp. 121–132. DOI: 10.1111/nous.12063.

– (2018). "Is Weak Supplementation Analytic?" *Synthese*. DOI: 10.1007/s11229-018-02066-9.

Hovda, Paul (2009). "What Is Classical Mereology?" *Journal of Philosophical Logic* 38.1, pp. 55–82. DOI: 10.1007/s10992-008-9092-4.

Kearns, Stephen (2011). "Can a Thing Be Part of Itself?" *American Philosophical Quarterly* 48.1, pp. 87–93. JSTOR: 23025076.

Lando, Giorgio (2018). *Mereology: A Philosophical Introduction*. London: Bloomsbury.

Leonard, Henry S. and Nelson Goodman (1940). "The Calculus of Individuals and Its Uses". *The Journal of Symbolic Logic* 5.2, pp. 45–55. DOI: 10.2307/2266169.

Lesniewski, Stanislaw (1992). *Collected Works*. Ed. by S. J. Surma et al. Nijhoff International Philosophy Series. Springer Netherlands.

Lewis, David (1986). *On the Plurality of Worlds*. Oxford: Blackwell.

– (1991). *Parts of Classes*. Blackwell.

Masolo, Claudio and Laure Vieu (1999). "Atomicity vs. Infinite Divisibility of Space". *Spatial Information Theory. Cognitive and Computational Foundations of Geographic Information Science*. Lecture Notes in Computer Science. Berlin: Springer, pp. 235–250. DOI: 10.1007/3-540-48384-5_16.

Parsons, Josh (2014). "The Many Primitives of Mereology". *Mereology and Location*. Oxford: Oxford University Press.

– (n.d.). "An Extensionalist's Guide to Non-Extensional Mereology".

Pietruszczak, Andrzej (2000). "Kawalki Mereologii". *Logika & Filozofia Logiczna*. Ed. by Andrzej Pietruszczak and Jerzy Perzanowski. Toruń: Wydawnictwo Naukowe Uniwersytetu Mikołaja Kopernika, pp. 357–374.

– (2005). "Pieces of Mereology". *Logic and Logical Philosophy* 14.2 (2), pp. 211–234. DOI: 10.12775/LLP.2005.014.

– (2018). *Metamereology*. Torun: Nicolaus Copernicus University Scientific Publishing House.

– (2020). *Foundations of the Theory of Parthood: A Study of Mereology*. Vol. 54. Trends in Logic. Cham: Springer International Publishing. DOI: 10.1007/978-3-030-36533-2.

Pontow, Carsten (2004). "A Note on the Axiomatics of Theories in Parthood". *Data & Knowledge Engineering* 50.2, pp. 195–213. DOI: 10.1016/j.datak.2003.12.002.

Simons, Peter (1987). *Parts: A Study in Ontology*. Oxford: Oxford University Press. 416 pp.

Tarski, Alfred (1983). "Foundations of the Geometry of Solids". *Logic, Semantics, Metamathematics*. Indianapolis: Hackett Publishing, pp. 24–29.

Thomson, Judith Jarvis (1998). "The Statue and the Clay". *Nous* 32.2, pp. 149–173. JSTOR: 2671962.

Varzi, Achille C. (1996). "Parts, Wholes, and Part-Whole Relations: The Prospects of Mereotopology". *Data & Knowledge Engineering*. Modeling Parts and Wholes 20.3, pp. 259–286. DOI: 10.1016/S0169-023X(96)00017-1.

– (2006). "A Note on the Transitivity of Parthood". *Applied Ontology* 1.2, pp. 141–146.

– (2009). "Universalism Entails Extensionalism". *Analysis* 69.4, pp. 599–604. DOI: 10.1093/analys/anp102.

– (2019). "On Three Axiom Systems for Classical Mereology". *Logic and Logical Philosophy* 28.2 (2), pp. 203–207. DOI: 10.12775/LLP.2018.014.

Whitehead, Alfred North (1929). *Process and Reality: An Essay in Cosmology.* New York: Macmillan.

Whitehead, Alfred North and Bertrand Russell (1925). *Principia Mathematica.* 2nd ed. Vol. 1. 3 vols. Cambridge: Cambridge University Press.