Mereology

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Abstract

We use Isabelle/HOL to verify elementary theorems and alternative axiomatizations of classical extensional mereology.

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1 Introduction

In this paper, we use Isabelle/HOL to verify some elementary theorems and alternative axiomatizations of classical extensional mereology, as well as some of its weaker subtheories.[1] We mostly follow the presentations from [Simons, 1987], [Varzi, 1996] and [Casati and Varzi, 1999], with some important corrections from [Pontow, 2004] and [Hovda, 2009] as well as some detailed proofs adapted from [Pietruszczak, 2018].[2]

We will use the following notation throughout.[3]

typedecl i
consts $\mathit{part} :: i \Rightarrow i \Rightarrow \mathit{bool}$ (P)
consts $\mathit{overlap} :: i \Rightarrow i \Rightarrow \mathit{bool}$ (O)
consts $\mathit{proper\text{-}part} :: i \Rightarrow i \Rightarrow \mathit{bool}$ (PP)
consts $\mathit{sum} :: i \Rightarrow i \Rightarrow i$ (**infix** $\oplus$ *52*)
consts $\mathit{product} :: i \Rightarrow i \Rightarrow i$ (**infix** $\otimes$ *53*)
consts $\mathit{difference} :: i \Rightarrow i \Rightarrow i$ (**infix** $\ominus$ *51*)
consts $\mathit{complement}:: i \Rightarrow i$ $(-)$
consts $\mathit{universe} :: i$ (u)
consts $\mathit{general\text{-}sum} :: (i \Rightarrow \mathit{bool}) \Rightarrow i$ (**binder** σ *9*)
consts $\mathit{general\text{-}product} :: (i \Rightarrow \mathit{bool}) \Rightarrow i$ (**binder** π [*8*] *9*)

2 Premereology

The theory of *premereology* assumes parthood is reflexive and transitive.[4] In other words, parthood is assumed to be a partial ordering relation.[5] Overlap is defined as common parthood.[6]

locale *PM* =
  assumes *part-reflexivity*: $P\ x\ x$
  assumes *part-transitivity* : $P\ x\ y \Longrightarrow P\ y\ z \Longrightarrow P\ x\ z$

[1] For similar developments see [Sen, 2017] and [Bittner, 2018].
[2] For help with this project I am grateful to Zach Barnett, Sam Baron, Bob Beddor, Olivier Danvy, Mark Goh, Jeremiah Joven Joaquin, Wang-Yen Lee, Kee Wei Loo, Bruno Woltzenlogel Paleo, Michael Pelczar, Hsueh Qu, Abelard Podgorski, Divyanshu Sharma, Manikaran Singh, Neil Sinhababu, Weng-Hong Tang and Zhang Jiang.
[3] See [Simons, 1987] pp. 99-100 for a helpful comparison of alternative notations.
[4] For discussion of reflexivity see [Kearns, 2011]. For transitivity see [Varzi, 2006].
[5] Hence the name *premereology*, from [Parsons, 2014] p. 6.
[6] See [Simons, 1987] p. 28, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36.

assumes *overlap-eq*: $O\ x\ y \longleftrightarrow (\exists\ z.\ P\ z\ x \wedge P\ z\ y)$
begin

2.1 Parthood

lemma *identity-implies-part* : $x = y \Longrightarrow P\ x\ y$
proof –
  assume $x = y$
  moreover have $P\ x\ x$ **by** (*rule part-reflexivity*)
  ultimately show $P\ x\ y$ **by** (*rule subst*)
qed

2.2 Overlap

lemma *overlap-intro*: $P\ z\ x \Longrightarrow P\ z\ y \Longrightarrow O\ x\ y$
proof–
  assume $P\ z\ x$
  moreover assume $P\ z\ y$
  ultimately have $P\ z\ x \wedge P\ z\ y$..
  hence $\exists\ z.\ P\ z\ x \wedge P\ z\ y$..
  with *overlap-eq* **show** $O\ x\ y$..
qed

lemma *part-implies-overlap*: $P\ x\ y \Longrightarrow O\ x\ y$
proof –
  assume $P\ x\ y$
  with *part-reflexivity* **have** $P\ x\ x \wedge P\ x\ y$..
  hence $\exists\ z.\ P\ z\ x \wedge P\ z\ y$..
  with *overlap-eq* **show** $O\ x\ y$..
qed

lemma *overlap-reflexivity*: $O\ x\ x$
proof –
  have $P\ x\ x \wedge P\ x\ x$ **using** *part-reflexivity part-reflexivity*..
  hence $\exists\ z.\ P\ z\ x \wedge P\ z\ x$..
  with *overlap-eq* **show** $O\ x\ x$..
qed

lemma *overlap-symmetry*: $O\ x\ y \Longrightarrow O\ y\ x$
proof–
  assume $O\ x\ y$
  with *overlap-eq* **have** $\exists\ z.\ P\ z\ x \wedge P\ z\ y$..
  hence $\exists\ z.\ P\ z\ y \wedge P\ z\ x$ **by** *auto*
  with *overlap-eq* **show** $O\ y\ x$..
qed

lemma *overlap-monotonicity*: $P\ x\ y \Longrightarrow O\ z\ x \Longrightarrow O\ z\ y$
proof –
  assume $P\ x\ y$
  assume $O\ z\ x$

with *overlap-eq* **have** $\exists\ v.\ P\ v\ z \wedge P\ v\ x$**..**
then obtain v **where** v: $P\ v\ z \wedge P\ v\ x$**..**
hence $P\ v\ z$**..**
moreover from v **have** $P\ v\ x$**..**
hence $P\ v\ y$ **using** ‹$P\ x\ y$› **by** (*rule part-transitivity*)
ultimately have $P\ v\ z \wedge P\ v\ y$**..**
hence $\exists\ v.\ P\ v\ z \wedge P\ v\ y$**..**
with *overlap-eq* **show** $O\ z\ y$**..**
qed

The next lemma is from [Hovda, 2009] p. 66.

lemma *overlap-lemma*: $\exists x. (P\ x\ y \wedge O\ z\ x) \longrightarrow O\ y\ z$
proof −
fix x
have $P\ x\ y \wedge O\ z\ x \longrightarrow O\ y\ z$
proof
assume *antecedent*: $P\ x\ y \wedge O\ z\ x$
hence $O\ z\ x$**..**
with *overlap-eq* **have** $\exists v.\ P\ v\ z \wedge P\ v\ x$**..**
then obtain v **where** v: $P\ v\ z \wedge P\ v\ x$**..**
hence $P\ v\ x$**..**
moreover from *antecedent* **have** $P\ x\ y$**..**
ultimately have $P\ v\ y$ **by** (*rule part-transitivity*)
moreover from v **have** $P\ v\ z$**..**
ultimately have $P\ v\ y \wedge P\ v\ z$**..**
hence $\exists v.\ P\ v\ y \wedge P\ v\ z$**..**
with *overlap-eq* **show** $O\ y\ z$**..**
qed
thus $\exists x. (P\ x\ y \wedge O\ z\ x) \longrightarrow O\ y\ z$**..**
qed

2.3 Disjointness

lemma *disjoint-implies-distinct*: $\neg\ O\ x\ y \Longrightarrow x \neq y$
proof −
assume $\neg\ O\ x\ y$
show $x \neq y$
proof
assume $x = y$
hence $\neg\ O\ y\ y$ **using** ‹$\neg\ O\ x\ y$› **by** (*rule subst*)
thus *False* **using** *overlap-reflexivity***..**
qed
qed

lemma *disjoint-implies-not-part*: $\neg\ O\ x\ y \Longrightarrow \neg\ P\ x\ y$
proof −
assume $\neg\ O\ x\ y$
show $\neg\ P\ x\ y$
proof

```
    assume P x y
    hence O x y by (rule part-implies-overlap)
    with ‹¬ O x y› show False..
  qed
qed

lemma disjoint-symmetry: ¬ O x y ⟹ ¬ O y x
proof –
  assume ¬ O x y
  show ¬ O y x
  proof
    assume O y x
    hence O x y by (rule overlap-symmetry)
    with ‹¬ O x y› show False..
  qed
qed

lemma disjoint-demonotonicity: P x y ⟹ ¬ O z y ⟹ ¬ O z x
proof –
  assume P x y
  assume ¬ O z y
  show ¬ O z x
  proof
    assume O z x
    with ‹P x y› have O z y
      by (rule overlap-monotonicity)
    with ‹¬ O z y› show False..
  qed
qed

end
```

3 Ground Mereology

The theory of *ground mereology* adds to premereology the antisymmetry of parthood, and defines proper parthood as nonidentical parthood.[7] In other words, ground mereology assumes that parthood is a partial order.

```
locale M = PM +
  assumes part-antisymmetry: P x y ⟹ P y x ⟹ x = y
  assumes nip-eq: PP x y ⟷ P x y ∧ x ≠ y
begin
```

[7] For this axiomatization of ground mereology see, for example, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36. For discussion of the antisymmetry of parthood see, for example, [Cotnoir, 2010]. For the definition of proper parthood as nonidentical parthood, see for example, [Leonard and Goodman, 1940] p. 47.

3.1 Proper Parthood

lemma *proper-implies-part*: $PP\ x\ y \Longrightarrow P\ x\ y$
proof −
  assume $PP\ x\ y$
  with *nip-eq* **have** $P\ x\ y \wedge x \neq y$**..**
  thus $P\ x\ y$**..**
qed

lemma *proper-implies-distinct*: $PP\ x\ y \Longrightarrow x \neq y$
proof −
  assume $PP\ x\ y$
  with *nip-eq* **have** $P\ x\ y \wedge x \neq y$**..**
  thus $x \neq y$**..**
qed

lemma *proper-implies-not-part*: $PP\ x\ y \Longrightarrow \neg\ P\ y\ x$
proof −
  assume $PP\ x\ y$
  hence $P\ x\ y$ **by** (*rule proper-implies-part*)
  show $\neg\ P\ y\ x$
  proof
    from ‹$PP\ x\ y$› **have** $x \neq y$ **by** (*rule proper-implies-distinct*)
    moreover assume $P\ y\ x$
    with ‹$P\ x\ y$› **have** $x = y$ **by** (*rule part-antisymmetry*)
    ultimately show *False***..**
  qed
qed

lemma *proper-part-asymmetry*: $PP\ x\ y \Longrightarrow \neg\ PP\ y\ x$
proof −
  assume $PP\ x\ y$
  hence $P\ x\ y$ **by** (*rule proper-implies-part*)
  from ‹$PP\ x\ y$› **have** $x \neq y$ **by** (*rule proper-implies-distinct*)
  show $\neg\ PP\ y\ x$
  proof
    assume $PP\ y\ x$
    hence $P\ y\ x$ **by** (*rule proper-implies-part*)
    with ‹$P\ x\ y$› **have** $x = y$ **by** (*rule part-antisymmetry*)
    with ‹$x \neq y$› **show** *False***..**
  qed
qed

lemma *proper-implies-overlap*: $PP\ x\ y \Longrightarrow O\ x\ y$
proof −
  assume $PP\ x\ y$
  hence $P\ x\ y$ **by** (*rule proper-implies-part*)
  thus $O\ x\ y$ **by** (*rule part-implies-overlap*)
qed

end

The rest of this section compares four alternative axiomatizations of ground mereology, and verifies their equivalence.

The first alternative axiomatization defines proper parthood as nonmutual instead of nonidentical parthood.[8] In the presence of antisymmetry, the two definitions of proper parthood are equivalent.[9]

locale *M1* = *PM* +
  assumes *nmp-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \wedge \neg\ P\ y\ x$
  assumes *part-antisymmetry*: $P\ x\ y \Longrightarrow P\ y\ x \Longrightarrow x = y$

sublocale $M \subseteq M1$
proof
  fix $x\ y$
  show *nmp-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \wedge \neg\ P\ y\ x$
  proof
    assume $PP\ x\ y$
    with *nip-eq* **have** *nip*: $P\ x\ y \wedge x \neq y$**..**
    hence $x \neq y$**..**
    from *nip* **have** $P\ x\ y$**..**
    moreover have $\neg\ P\ y\ x$
    proof
      assume $P\ y\ x$
      with ‹$P\ x\ y$› **have** $x = y$ **by** (*rule part-antisymmetry*)
      with ‹$x \neq y$› **show** *False***..**
    qed
    ultimately show $P\ x\ y \wedge \neg\ P\ y\ x$**..**
  next
    assume *nmp*: $P\ x\ y \wedge \neg\ P\ y\ x$
    hence $\neg\ P\ y\ x$**..**
    from *nmp* **have** $P\ x\ y$**..**
    moreover have $x \neq y$
    proof
      assume $x = y$
      hence $\neg\ P\ y\ y$ **using** ‹$\neg\ P\ y\ x$› **by** (*rule subst*)
      thus *False* **using** *part-reflexivity***..**
    qed
    ultimately have $P\ x\ y \wedge x \neq y$**..**
    with *nip-eq* **show** $PP\ x\ y$**..**
  qed
  show $P\ x\ y \Longrightarrow P\ y\ x \Longrightarrow x = y$ **using** *part-antisymmetry***.**
qed

[8] See, for example, [Varzi, 1996] p. 261 and [Casati and Varzi, 1999] p. 36. For the distinction between nonmutual and nonidentical parthood, see [Parsons, 2014] pp. 6-8.

[9] See [Cotnoir, 2010] p. 398, [Donnelly, 2011] p. 233, [Cotnoir and Bacon, 2012] p. 191, [Obojska, 2013] p. 344, [Cotnoir, 2016] p. 128 and [Cotnoir, 2018].

```
sublocale M1 ⊆ M
proof
  fix x y
  show nip-eq: PP x y ⟷ P x y ∧ x ≠ y
  proof
    assume PP x y
    with nmp-eq have nmp: P x y ∧ ¬ P y x..
    hence ¬ P y x..
    from nmp have P x y..
    moreover have x ≠ y
    proof
      assume x = y
      hence ¬ P y y using ‹¬ P y x› by (rule subst)
      thus False using part-reflexivity..
    qed
    ultimately show P x y ∧ x ≠ y..
  next
    assume nip: P x y ∧ x ≠ y
    hence x ≠ y..
    from nip have P x y..
    moreover have ¬ P y x
    proof
      assume P y x
      with ‹P x y› have x = y by (rule part-antisymmetry)
      with ‹x ≠ y› show False..
    qed
    ultimately have P x y ∧ ¬ P y x..
    with nmp-eq show PP x y..
  qed
  show  P x y ⟹ P y x ⟹ x = y using part-antisymmetry.
qed
```

Conversely, assuming the two definitions of proper parthood are equivalent entails the antisymmetry of parthood, leading to the second alternative axiomatization, which assumes both equivalencies.[10]

```
locale M2 = PM +
  assumes nip-eq: PP x y ⟷ P x y ∧ x ≠ y
  assumes nmp-eq: PP x y ⟷ P x y ∧ ¬ P y x

sublocale M ⊆ M2
proof
  fix x y
  show PP x y ⟷ P x y ∧ x ≠ y using nip-eq.
  show PP x y ⟷ P x y ∧ ¬ P y x using nmp-eq.
qed
```

[10] For this point see especially [Parsons, 2014] pp. 9-10.

```
sublocale M2 ⊆ M
proof
  fix x y
  show PP x y ⟷ P x y ∧ x ≠ y using nip-eq.
  show P x y ⟹ P y x ⟹ x = y
  proof −
    assume P x y
    assume P y x
    show x = y
    proof (rule ccontr)
      assume x ≠ y
      with ‹P x y› have P x y ∧ x ≠ y..
      with nip-eq have PP x y..
      with nmp-eq have P x y ∧ ¬ P y x..
      hence ¬ P y x..
      thus False using ‹P y x›..
    qed
  qed
qed
```

In the context of the other axioms, antisymmetry is equivalent to the extensionality of parthood, which gives the third alternative axiomatization.[11]

```
locale M3 = PM +
  assumes nip-eq: PP x y ⟷ P x y ∧ x ≠ y
  assumes part-extensionality: x = y ⟷ (∀ z. P z x ⟷ P z y)

sublocale M ⊆ M3
proof
  fix x y
  show PP x y ⟷ P x y ∧ x ≠ y using nip-eq.
  show part-extensionality: x = y ⟷ (∀ z. P z x ⟷ P z y)
  proof
    assume x = y
    moreover have ∀ z. P z x ⟷ P z x by simp
    ultimately show ∀ z. P z x ⟷ P z y by (rule subst)
  next
    assume z: ∀ z. P z x ⟷ P z y
    show x = y
    proof (rule part-antisymmetry)
      from z have P y x ⟷ P y y..
      moreover have P y y by (rule part-reflexivity)
      ultimately show P y x..
    next
      from z have P x x ⟷ P x y..
      moreover have P x x by (rule part-reflexivity)
```

[11] For this point see [Cotnoir, 2010] p. 401 and [Cotnoir and Bacon, 2012] p. 191-2.

```
      ultimately show P x y..
    qed
  qed
qed

sublocale M3 ⊆ M
proof
  fix x y
  show PP x y ⟷ P x y ∧ x ≠ y using nip-eq.
  show part-antisymmetry: P x y ⟹ P y x ⟹ x = y
  proof −
    assume P x y
    assume P y x
    have ∀ z. P z x ⟷ P z y
    proof
      fix z
      show P z x ⟷ P z y
      proof
        assume P z x
        thus P z y using ‹P x y› by (rule part-transitivity)
      next
        assume P z y
        thus P z x using ‹P y x› by (rule part-transitivity)
      qed
    qed
    with part-extensionality show x = y..
  qed
qed
```

The fourth axiomatization adopts proper parthood as primitive.[12] Improper parthood is defined as proper parthood or identity.

```
locale M4 =
  assumes part-eq: P x y ⟷ PP x y ∨ x = y
  assumes  overlap-eq: O x y ⟷ (∃ z. P z x ∧ P z y)
  assumes proper-part-asymmetry: PP x y ⟹ ¬ PP y x
  assumes proper-part-transitivity: PP x y ⟹ PP y z ⟹ PP x z
begin

lemma proper-part-irreflexivity: ¬ PP x x
proof
  assume PP x x
  hence ¬ PP x x by (rule proper-part-asymmetry)
  thus False using ‹PP x x›..
qed

end
```

[12] See, for example, [Simons, 1987], p. 26 and [Casati and Varzi, 1999] p. 37.

sublocale $M \subseteq M4$
proof
  fix $x\ y\ z$
  show *part-eq*: $P\ x\ y \longleftrightarrow (PP\ x\ y \vee x = y)$
  proof
    assume $P\ x\ y$
    show $PP\ x\ y \vee x = y$
    proof *cases*
      assume $x = y$
      thus $PP\ x\ y \vee x = y$**..**
    next
      assume $x \neq y$
      with ‹$P\ x\ y$› **have** $P\ x\ y \wedge x \neq y$**..**
      with *nip-eq* **have** $PP\ x\ y$**..**
      thus $PP\ x\ y \vee x = y$**..**
    qed
  next
    assume $PP\ x\ y \vee x = y$
    thus $P\ x\ y$
    proof
      assume $PP\ x\ y$
      thus $P\ x\ y$ **by** (*rule proper-implies-part*)
    next
      assume $x = y$
      thus $P\ x\ y$ **by** (*rule identity-implies-part*)
    qed
  qed
  show $O\ x\ y \longleftrightarrow (\exists\ z.\ P\ z\ x \wedge P\ z\ y)$ **using** *overlap-eq*.
  show $PP\ x\ y \Longrightarrow \neg\ PP\ y\ x$ **using** *proper-part-asymmetry*.
  show *proper-part-transitivity*: $PP\ x\ y \Longrightarrow PP\ y\ z \Longrightarrow PP\ x\ z$
  proof −
    assume $PP\ x\ y$
    assume $PP\ y\ z$
    have $P\ x\ z \wedge x \neq z$
    proof
      from ‹$PP\ x\ y$› **have** $P\ x\ y$ **by** (*rule proper-implies-part*)
      moreover from ‹$PP\ y\ z$› **have** $P\ y\ z$ **by** (*rule proper-implies-part*)
      ultimately show $P\ x\ z$ **by** (*rule part-transitivity*)
    next
      show $x \neq z$
      proof
        assume $x = z$
        hence $PP\ y\ x$ **using** ‹$PP\ y\ z$› **by** (*rule ssubst*)
        hence $\neg\ PP\ x\ y$ **by** (*rule proper-part-asymmetry*)
        thus *False* **using** ‹$PP\ x\ y$›**..**
      qed
    qed
    with *nip-eq* **show** $PP\ x\ z$**..**

```
  qed
qed

sublocale M4 ⊆ M
proof
  fix x y z
  show proper-part-eq: PP x y ⟷ P x y ∧ x ≠ y
  proof
    assume PP x y
    hence PP x y ∨ x = y..
    with part-eq have P x y..
    moreover have x ≠ y
    proof
      assume x = y
      hence PP y y using ‹PP x y› by (rule subst)
      with proper-part-irreflexivity show False..
    qed
    ultimately show P x y ∧ x ≠ y..
  next
    assume rhs: P x y ∧ x ≠ y
    hence x ≠ y..
    from rhs have P x y..
    with part-eq have PP x y ∨ x = y..
    thus PP x y
    proof
      assume PP x y
      thus PP x y.
    next
      assume x = y
      with ‹x ≠ y› show PP x y..
    qed
  qed
  show P x x
  proof -
    have x = x by (rule refl)
    hence PP x x ∨ x = x..
    with part-eq show P x x..
  qed
  show O x y ⟷ (∃ z. P z x ∧ P z y) using overlap-eq.
  show P x y ⟹ P y x ⟹ x = y
  proof -
    assume P x y
    assume P y x
    from part-eq have PP x y ∨ x = y using ‹P x y›..
    thus x = y
    proof
      assume PP x y
      hence ¬ PP y x by (rule proper-part-asymmetry)
      from part-eq have PP y x ∨ y = x using ‹P y x›..
```

```
        thus x = y
        proof
          assume PP y x
          with ‹¬ PP y x› show x = y..
        next
          assume y = x
          thus x = y..
        qed
      qed
    qed
    show P x y ⟹ P y z ⟹ P x z
    proof –
      assume P x y
      assume P y z
      with part-eq have PP y z ∨ y = z..
      hence PP x z ∨ x = z
      proof
        assume PP y z
        from part-eq have PP x y ∨ x = y using ‹P x y›..
        hence PP x z
        proof
          assume PP x y
          thus PP x z using ‹PP y z› by (rule proper-part-transitivity)
        next
          assume x = y
          thus PP x z using ‹PP y z› by (rule ssubst)
        qed
        thus PP x z ∨ x = z..
      next
        assume y = z
        moreover from part-eq have PP x y ∨ x = y using ‹P x y›..
        ultimately show PP x z ∨ x = z by (rule subst)
      qed
      with part-eq show P x z..
    qed
  qed
```

4 Minimal Mereology

Minimal mereology adds to ground mereology the axiom of weak supplementation.[13]

locale *MM* = *M* +

[13]See [Varzi, 1996] and [Casati and Varzi, 1999] p. 39. The name *minimal mereology* reflects the, controversial, idea that weak supplementation is analytic. See, for example, [Simons, 1987] p. 116, [Varzi, 2008] p. 110-1, and [Cotnoir, 2018]. For general discussion of weak supplementation see, for example [Smith, 2009] pp. 507 and [Donnelly, 2011].

assumes *weak-supplementation*: $PP\ y\ x \Longrightarrow (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$

The rest of this section considers three alternative axiomatizations of minimal mereology. The first alternative axiomatization replaces improper with proper parthood in the consequent of weak supplementation.[14]

locale $MM1 = M +$
  assumes *proper-weak-supplementation*:
    $PP\ y\ x \Longrightarrow (\exists\ z.\ PP\ z\ x \wedge \neg\ O\ z\ y)$

sublocale $MM \subseteq MM1$
proof
  fix $x\ y$
  show $PP\ y\ x \Longrightarrow (\exists\ z.\ PP\ z\ x \wedge \neg\ O\ z\ y)$
  proof –
    assume $PP\ y\ x$
    hence $\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y$ **by** (*rule weak-supplementation*)
    then obtain z **where** z: $P\ z\ x \wedge \neg\ O\ z\ y$**..**
    hence $\neg\ O\ z\ y$**..**
    from z **have** $P\ z\ x$**..**
    hence $P\ z\ x \wedge z \neq x$
    proof
      show $z \neq x$
      proof
        assume $z = x$
        hence $PP\ y\ z$
          using $\langle PP\ y\ x\rangle$ **by** (*rule ssubst*)
        hence $O\ y\ z$ **by** (*rule proper-implies-overlap*)
        hence $O\ z\ y$ **by** (*rule overlap-symmetry*)
        with $\langle\neg\ O\ z\ y\rangle$ **show** *False***..**
      qed
    qed
    with *nip-eq* **have** $PP\ z\ x$**..**
    hence $PP\ z\ x \wedge \neg\ O\ z\ y$
      using $\langle\neg\ O\ z\ y\rangle$**..**
    thus $\exists\ z.\ PP\ z\ x \wedge \neg\ O\ z\ y$**..**
  qed
qed

sublocale $MM1 \subseteq MM$
proof
  fix $x\ y$
  show *weak-supplementation*: $PP\ y\ x \Longrightarrow (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
  proof –
    assume $PP\ y\ x$
    hence $\exists\ z.\ PP\ z\ x \wedge \neg\ O\ z\ y$ **by** (*rule proper-weak-supplementation*)
    then obtain z **where** z: $PP\ z\ x \wedge \neg\ O\ z\ y$**..**

[14] See [Simons, 1987] p. 28.

hence $PP\ z\ x$..
hence $P\ z\ x$ **by** (*rule proper-implies-part*)
moreover from z **have** $\neg\ O\ z\ y$..
ultimately have $P\ z\ x \wedge \neg\ O\ z\ y$..
thus $\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y$..
qed
qed

The following two corollaries are sometimes found in the literature.[15]

context *MM*
begin

corollary *weak-company*: $PP\ y\ x \Longrightarrow (\exists\ z.\ PP\ z\ x \wedge z \neq y)$
proof –
assume $PP\ y\ x$
hence $\exists\ z.\ PP\ z\ x \wedge \neg\ O\ z\ y$ **by** (*rule proper-weak-supplementation*)
then obtain z **where** z: $PP\ z\ x \wedge \neg\ O\ z\ y$..
hence $PP\ z\ x$..
from z **have** $\neg\ O\ z\ y$..
hence $z \neq y$ **by** (*rule disjoint-implies-distinct*)
with ‹$PP\ z\ x$› **have** $PP\ z\ x \wedge z \neq y$..
thus $\exists\ z.\ PP\ z\ x \wedge z \neq y$..
qed

corollary *strong-company*: $PP\ y\ x \Longrightarrow (\exists\ z.\ PP\ z\ x \wedge \neg\ P\ z\ y)$
proof –
assume $PP\ y\ x$
hence $\exists z.\ PP\ z\ x \wedge \neg\ O\ z\ y$ **by** (*rule proper-weak-supplementation*)
then obtain z **where** z: $PP\ z\ x \wedge \neg\ O\ z\ y$..
hence $PP\ z\ x$..
from z **have** $\neg\ O\ z\ y$..
hence $\neg\ P\ z\ y$ **by** (*rule disjoint-implies-not-part*)
with ‹$PP\ z\ x$› **have** $PP\ z\ x \wedge \neg\ P\ z\ y$..
thus $\exists\ z.\ PP\ z\ x \wedge \neg\ P\ z\ y$..
qed

end

If weak supplementation is formulated in terms of nonidentical parthood, then the antisymmetry of parthood is redundant, and we have the second alternative axiomatization of minimal mereology.[16]

locale *MM2* = *PM* +

[15] See [Simons, 1987] p. 27. For the names *weak company* and *strong company* see [Cotnoir and Bacon, 2012] p. 192-3 and [Varzi, 2016].
[16] See [Cotnoir, 2010] p. 399, [Donnelly, 2011] p. 232, [Cotnoir and Bacon, 2012] p. 193 and [Obojska, 2013] pp. 235-6.

assumes *nip-eq*: $PP\ x\ y \longleftrightarrow P\ x\ y \wedge x \neq y$
assumes *weak-supplementation*: $PP\ y\ x \Longrightarrow (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$

sublocale *MM2* $\subseteq$ *MM*
proof
  fix $x\ y$
  show $PP\ x\ y \longleftrightarrow P\ x\ y \wedge x \neq y$ **using** *nip-eq*.
  show *part-antisymmetry*: $P\ x\ y \Longrightarrow P\ y\ x \Longrightarrow x = y$
  proof –
    assume $P\ x\ y$
    assume $P\ y\ x$
    show $x = y$
    proof (*rule ccontr*)
      assume $x \neq y$
      with ‹$P\ x\ y$› **have** $P\ x\ y \wedge x \neq y$..
      with *nip-eq* **have** $PP\ x\ y$..
      hence $\exists\ z.\ P\ z\ y \wedge \neg\ O\ z\ x$ **by** (*rule weak-supplementation*)
      then obtain z **where** z: $P\ z\ y \wedge \neg\ O\ z\ x$..
      hence $\neg\ O\ z\ x$..
      hence $\neg\ P\ z\ x$ **by** (*rule disjoint-implies-not-part*)
      from z **have** $P\ z\ y$..
      hence $P\ z\ x$ **using** ‹$P\ y\ x$› **by** (*rule part-transitivity*)
      with ‹$\neg\ P\ z\ x$› **show** *False*..
    qed
  qed
  show $PP\ y\ x \Longrightarrow \exists z.\ P\ z\ x \wedge \neg\ O\ z\ y$ **using** *weak-supplementation*.
qed

sublocale *MM* $\subseteq$ *MM2*
proof
  fix $x\ y$
  show $PP\ x\ y \longleftrightarrow (P\ x\ y \wedge x \neq y)$ **using** *nip-eq*.
  show $PP\ y\ x \Longrightarrow \exists z.\ P\ z\ x \wedge \neg\ O\ z\ y$ **using** *weak-supplementation*.
qed

Likewise, if proper parthood is adopted as primitive, then the asymmetry of proper parthood is redundant in the context of weak supplementation, leading to the third alternative axiomatization.[17]

locale *MM3* =
  assumes *part-eq*: $P\ x\ y \longleftrightarrow PP\ x\ y \vee x = y$
  assumes *overlap-eq*: $O\ x\ y \longleftrightarrow (\exists\ z.\ P\ z\ x \wedge P\ z\ y)$
  assumes *proper-part-transitivity*: $PP\ x\ y \Longrightarrow PP\ y\ z \Longrightarrow PP\ x\ z$
  assumes *weak-supplementation*: $PP\ y\ x \Longrightarrow (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
begin

lemma *part-reflexivity*: $P\ x\ x$

[17] See [Donnelly, 2011] p. 232 and [Cotnoir, 2018].

```
proof −
  have x = x..
  hence PP x x ∨ x = x..
  with part-eq show P x x..
qed

lemma proper-part-irreflexivity: ¬ PP x x
proof
  assume PP x x
  hence ∃ z. P z x ∧ ¬ O z x by (rule weak-supplementation)
  then obtain z where z: P z x ∧ ¬ O z x..
  hence ¬ O z x..
  from z have P z x..
  with part-reflexivity have P z z ∧ P z x..
  hence ∃ v. P v z ∧ P v x..
  with overlap-eq have O z x..
  with ‹¬ O z x› show False..
qed

end

sublocale MM3 ⊆ M4
proof
  fix x y z
  show P x y ⟷ PP x y ∨ x = y using part-eq.
  show O x y ⟷ (∃ z. P z x ∧ P z y) using overlap-eq.
  show proper-part-irreflexivity: PP x y ⟹ ¬ PP y x
  proof −
    assume PP x y
    show ¬ PP y x
    proof
      assume PP y x
      hence PP y y using ‹PP x y› by (rule proper-part-transitivity)
      with proper-part-irreflexivity show False..
    qed
  qed
  show PP x y ⟹ PP y z ⟹ PP x z using proper-part-transitivity.
qed

sublocale MM3 ⊆ MM
proof
  fix x y
  show PP y x ⟹ (∃ z. P z x ∧ ¬ O z y) using weak-supplementation.
qed

sublocale MM ⊆ MM3
proof
  fix x y z
  show P x y ⟷ (PP x y ∨ x = y) using part-eq.
```

show $O\ x\ y \longleftrightarrow (\exists z.\ P\ z\ x \wedge P\ z\ y)$ **using** *overlap-eq*.
show $PP\ x\ y \Longrightarrow PP\ y\ z \Longrightarrow PP\ x\ z$ **using** *proper-part-transitivity*.
show $PP\ y\ x \Longrightarrow \exists z.\ P\ z\ x \wedge \neg\ O\ z\ y$ **using** *weak-supplementation*.
qed

5 Extensional Mereology

Extensional mereology adds to ground mereology the axiom of strong supplementation.[18]

locale $EM = M +$
assumes *strong-supplementation*:
$\neg\ P\ x\ y \Longrightarrow (\exists z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
begin

Strong supplementation entails weak supplementation.[19]

lemma *weak-supplementation*: $PP\ x\ y \Longrightarrow (\exists z.\ P\ z\ y \wedge \neg\ O\ z\ x)$
proof –
assume $PP\ x\ y$
hence $\neg\ P\ y\ x$ **by** (*rule proper-implies-not-part*)
thus $\exists z.\ P\ z\ y \wedge \neg\ O\ z\ x$ **by** (*rule strong-supplementation*)
qed

end

So minimal mereology is a subtheory of extensional mereology.[20]

sublocale $EM \subseteq MM$
proof
fix $y\ x$
show $PP\ y\ x \Longrightarrow \exists z.\ P\ z\ x \wedge \neg\ O\ z\ y$ **using** *weak-supplementation*.
qed

Strong supplementation also entails the proper parts principle.[21]

context EM
begin

lemma *proper-parts-principle*:
$(\exists z.\ PP\ z\ x) \Longrightarrow (\forall z.\ PP\ z\ x \longrightarrow P\ z\ y) \Longrightarrow P\ x\ y$
proof –
assume $\exists z.\ PP\ z\ x$
then obtain v **where** v: $PP\ v\ x$..
hence $P\ v\ x$ **by** (*rule proper-implies-part*)
assume *antecedent*: $\forall z.\ PP\ z\ x \longrightarrow P\ z\ y$

[18] See [Simons, 1987] p. 29, [Varzi, 1996] p. 262 and [Casati and Varzi, 1999] p. 39-40.
[19] See [Simons, 1987] p. 29 and [Casati and Varzi, 1999] p. 40.
[20] [Casati and Varzi, 1999] p. 40.
[21] See [Simons, 1987] pp. 28-9 and [Varzi, 1996] p. 263.

hence $PP\ v\ x \longrightarrow P\ v\ y$**..**
hence $P\ v\ y$ **using** $\langle PP\ v\ x\rangle$**..**
with $\langle P\ v\ x\rangle$ **have** $P\ v\ x \wedge P\ v\ y$**..**
hence $\exists\, v.\ P\ v\ x \wedge P\ v\ y$**..**
with *overlap-eq* **have** $O\ x\ y$**..**
show $P\ x\ y$
proof (*rule ccontr*)
  assume $\neg\ P\ x\ y$
  hence $\exists\, z.\ P\ z\ x \wedge \neg\ O\ z\ y$
    by (*rule strong-supplementation*)
  then obtain z **where** z: $P\ z\ x \wedge \neg\ O\ z\ y$**..**
  hence $P\ z\ x$**..**
  moreover have $z \neq x$
  proof
    assume $z = x$
    moreover from z **have** $\neg\ O\ z\ y$**..**
    ultimately have $\neg\ O\ x\ y$ **by** (*rule subst*)
    thus *False* **using** $\langle O\ x\ y\rangle$**..**
  qed
  ultimately have $P\ z\ x \wedge z \neq x$**..**
  with *nip-eq* **have** $PP\ z\ x$**..**
  from *antecedent* **have** $PP\ z\ x \longrightarrow P\ z\ y$**..**
  hence $P\ z\ y$ **using** $\langle PP\ z\ x\rangle$**..**
  hence $O\ z\ y$ **by** (*rule part-implies-overlap*)
  from z **have** $\neg\ O\ z\ y$**..**
  thus *False* **using** $\langle O\ z\ y\rangle$**..**
qed
qed

Which with antisymmetry entails the extensionality of proper parthood.[22]

theorem *proper-part-extensionality*:
$(\exists\, z.\ PP\ z\ x \vee PP\ z\ y) \Longrightarrow x = y \longleftrightarrow (\forall\, z.\ PP\ z\ x \longleftrightarrow PP\ z\ y)$
proof –
  assume *antecedent*: $\exists\, z.\ PP\ z\ x \vee PP\ z\ y$
  show $x = y \longleftrightarrow (\forall\, z.\ PP\ z\ x \longleftrightarrow PP\ z\ y)$
  proof
    assume $x = y$
    moreover have $\forall\, z.\ PP\ z\ x \longleftrightarrow PP\ z\ x$ **by** *simp*
    ultimately show $\forall\, z.\ PP\ z\ x \longleftrightarrow PP\ z\ y$ **by** (*rule subst*)
  next
    assume *right*: $\forall\, z.\ PP\ z\ x \longleftrightarrow PP\ z\ y$
    have $\forall\, z.\ PP\ z\ x \longrightarrow P\ z\ y$
    proof
      fix z
      show $PP\ z\ x \longrightarrow P\ z\ y$
      proof

[22] See [Simons, 1987] p. 28, [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 40.

```
      assume PP z x
      from right have PP z x ⟷ PP z y..
      hence PP z y using ‹PP z x›..
      thus P z y by (rule proper-implies-part)
    qed
  qed
  have ∀ z. PP z y ⟶ P z x
  proof
    fix z
    show PP z y ⟶ P z x
    proof
      assume PP z y
      from right have PP z x ⟷ PP z y..
      hence PP z x using ‹PP z y›..
      thus P z x by (rule proper-implies-part)
    qed
  qed
  from antecedent obtain z where z: PP z x ∨ PP z y..
  thus x = y
  proof (rule disjE)
    assume PP z x
    hence ∃ z. PP z x..
    hence P x y using ‹∀ z. PP z x ⟶ P z y›
      by (rule proper-parts-principle)
    from right have PP z x ⟷ PP z y..
    hence PP z y using ‹PP z x›..
    hence ∃ z. PP z y..
    hence P y x using ‹∀ z. PP z y ⟶ P z x›
      by (rule proper-parts-principle)
    with ‹P x y› show x = y
      by (rule part-antisymmetry)
  next
    assume PP z y
    hence ∃ z. PP z y..
    hence P y x using ‹∀ z. PP z y ⟶ P z x›
      by (rule proper-parts-principle)
    from right have PP z x ⟷ PP z y..
    hence PP z x using ‹PP z y›..
    hence ∃ z. PP z x..
    hence P x y using ‹∀ z. PP z x ⟶ P z y›
        by (rule proper-parts-principle)
    thus x = y
      using ‹P y x› by (rule part-antisymmetry)
  qed
 qed
qed
```

It also follows from strong supplementation that parthood is definable in terms of overlap.[23]

```
lemma part-overlap-eq: P x y ⟷ (∀ z. O z x ⟶ O z y)
proof
  assume P x y
  show (∀ z. O z x ⟶ O z y)
  proof
    fix z
    show O z x ⟶ O z y
    proof
      assume O z x
      with ‹P x y› show O z y
        by (rule overlap-monotonicity)
    qed
  qed
next
  assume right: ∀ z. O z x ⟶ O z y
  show P x y
  proof (rule ccontr)
    assume ¬ P x y
    hence ∃ z. P z x ∧ ¬ O z y
      by (rule strong-supplementation)
    then obtain z where z: P z x ∧ ¬ O z y..
    hence ¬ O z y..
    from right have O z x ⟶ O z y..
    moreover from z have P z x..
    hence O z x by (rule part-implies-overlap)
    ultimately have O z y..
    with ‹¬ O z y› show False..
  qed
qed
```

Which entails the extensionality of overlap.

```
theorem overlap-extensionality: x = y ⟷ (∀ z. O z x ⟷ O z y)
proof
  assume x = y
  moreover have ∀ z. O z x ⟷ O z x
  proof
    fix z
    show O z x ⟷ O z x..
  qed
  ultimately show ∀ z. O z x ⟷ O z y
    by (rule subst)
next
  assume right: ∀ z. O z x ⟷ O z y
  have ∀ z. O z y ⟶ O z x
  proof
    fix z
```

[23] See [Parsons, 2014] p. 4.

from *right* **have** $O\ z\ x \longleftrightarrow O\ z\ y$**..**
thus $O\ z\ y \longrightarrow O\ z\ x$**..**
qed
with *part-overlap-eq* **have** $P\ y\ x$**..**
have $\forall z.\ O\ z\ x \longrightarrow O\ z\ y$
proof
fix z
from *right* **have** $O\ z\ x \longleftrightarrow O\ z\ y$**..**
thus $O\ z\ x \longrightarrow O\ z\ y$**..**
qed
with *part-overlap-eq* **have** $P\ x\ y$**..**
thus $x = y$
using ‹$P\ y\ x$› **by** (*rule part-antisymmetry*)
qed

end

6 Closed Mereology

The theory of *closed mereology* adds to ground mereology conditions guaranteeing the existence of sums and products.[24]

locale $CM = M +$
assumes *sum-eq*: $x \oplus y = (THE\ z.\ \forall v.\ O\ v\ z \longleftrightarrow O\ v\ x \lor O\ v\ y)$
assumes *sum-closure*: $\exists z.\ \forall v.\ O\ v\ z \longleftrightarrow O\ v\ x \lor O\ v\ y$
assumes *product-eq*:
$x \otimes y = (THE\ z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \land P\ v\ y)$
assumes *product-closure*:
$O\ x\ y \Longrightarrow \exists z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \land P\ v\ y$
begin

6.1 Products

lemma *product-intro*:
$(\forall w.\ P\ w\ z \longleftrightarrow (P\ w\ x \land P\ w\ y)) \Longrightarrow x \otimes y = z$
proof –
assume z: $\forall w.\ P\ w\ z \longleftrightarrow (P\ w\ x \land P\ w\ y)$
hence $(THE\ v.\ \forall w.\ P\ w\ v \longleftrightarrow P\ w\ x \land P\ w\ y) = z$
proof (*rule the-equality*)
fix v
assume v: $\forall w.\ P\ w\ v \longleftrightarrow (P\ w\ x \land P\ w\ y)$
have $\forall w.\ P\ w\ v \longleftrightarrow P\ w\ z$
proof
fix w

[24] See [Masolo and Vieu, 1999] p. 238. [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 43 give a slightly weaker version of the sum closure axiom, which is equivalent given axioms considered later.

from z **have** $P\ w\ z \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$**..**
moreover from v **have** $P\ w\ v \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$**..**
ultimately show $P\ w\ v \longleftrightarrow P\ w\ z$ **by** (*rule ssubst*)
qed
with *part-extensionality* **show** $v = z$**..**
qed
thus $x \otimes y = z$
using *product-eq* **by** (*rule subst*)
qed

lemma *product-idempotence*: $x \otimes x = x$
proof –
have $\forall w.\ P\ w\ x \longleftrightarrow P\ w\ x \wedge P\ w\ x$
proof
fix w
show $P\ w\ x \longleftrightarrow P\ w\ x \wedge P\ w\ x$
proof
assume $P\ w\ x$
thus $P\ w\ x \wedge P\ w\ x$ **using** ‹$P\ w\ x$›**..**
next
assume $P\ w\ x \wedge P\ w\ x$
thus $P\ w\ x$**..**
qed
qed
thus $x \otimes x = x$ **by** (*rule product-intro*)
qed

lemma *product-character*:
$O\ x\ y \Longrightarrow (\forall w.\ P\ w\ (x \otimes y) \longleftrightarrow (P\ w\ x \wedge P\ w\ y))$
proof –
assume $O\ x\ y$
hence $\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$ **by** (*rule product-closure*)
then obtain z **where** z: $\forall w.\ P\ w\ z \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$**..**
hence $x \otimes y = z$ **by** (*rule product-intro*)
thus $\forall w.\ P\ w\ (x \otimes y) \longleftrightarrow P\ w\ x \wedge P\ w\ y$
using z **by** (*rule ssubst*)
qed

lemma *product-commutativity*: $O\ x\ y \Longrightarrow x \otimes y = y \otimes x$
proof –
assume $O\ x\ y$
hence $O\ y\ x$ **by** (*rule overlap-symmetry*)
hence $\forall w.\ P\ w\ (y \otimes x) \longleftrightarrow (P\ w\ y \wedge P\ w\ x)$ **by** (*rule product-character*)
hence $\forall w.\ P\ w\ (y \otimes x) \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$ **by** *auto*
thus $x \otimes y = y \otimes x$ **by** (*rule product-intro*)
qed

lemma *product-in-factors*: $O\ x\ y \Longrightarrow P\ (x \otimes y)\ x \wedge P\ (x \otimes y)\ y$

proof –
assume $O\ x\ y$
hence $\forall w.\ P\ w\ (x \otimes y) \longleftrightarrow P\ w\ x \wedge P\ w\ y$ **by** (*rule product-character*)
hence $P\ (x \otimes y)\ (x \otimes y) \longleftrightarrow P\ (x \otimes y)\ x \wedge P\ (x \otimes y)\ y$..
moreover have $P\ (x \otimes y)\ (x \otimes y)$ **by** (*rule part-reflexivity*)
ultimately show $P\ (x \otimes y)\ x \wedge P\ (x \otimes y)\ y$..
qed

lemma *product-in-first-factor*: $O\ x\ y \Longrightarrow P\ (x \otimes y)\ x$
proof –
assume $O\ x\ y$
hence $P\ (x \otimes y)\ x \wedge P\ (x \otimes y)\ y$ **by** (*rule product-in-factors*)
thus $P\ (x \otimes y)\ x$..
qed

lemma *product-in-second-factor*: $O\ x\ y \Longrightarrow P\ (x \otimes y)\ y$
proof –
assume $O\ x\ y$
hence $P\ (x \otimes y)\ x \wedge P\ (x \otimes y)\ y$ **by** (*rule product-in-factors*)
thus $P\ (x \otimes y)\ y$..
qed

lemma *nonpart-implies-proper-product*:
$\neg\ P\ x\ y \wedge O\ x\ y \Longrightarrow PP\ (x \otimes y)\ x$
proof –
assume *antecedent*: $\neg\ P\ x\ y \wedge O\ x\ y$
hence $\neg\ P\ x\ y$..
from *antecedent* **have** $O\ x\ y$..
hence $P\ (x \otimes y)\ x$ **by** (*rule product-in-first-factor*)
moreover have $(x \otimes y) \neq x$
proof
assume $(x \otimes y) = x$
hence $\neg\ P\ (x \otimes y)\ y$
using $\langle \neg\ P\ x\ y \rangle$ **by** (*rule ssubst*)
moreover have $P\ (x \otimes y)\ y$
using $\langle O\ x\ y \rangle$ **by** (*rule product-in-second-factor*)
ultimately show *False*..
qed
ultimately have $P\ (x \otimes y)\ x \wedge x \otimes y \neq x$..
with *nip-eq* **show** $PP\ (x \otimes y)\ x$..
qed

lemma *common-part-in-product*: $P\ z\ x \wedge P\ z\ y \Longrightarrow P\ z\ (x \otimes y)$
proof –
assume *antecedent*: $P\ z\ x \wedge P\ z\ y$
hence $\exists z.\ P\ z\ x \wedge P\ z\ y$..
with *overlap-eq* **have** $O\ x\ y$..
hence $\forall w.\ P\ w\ (x \otimes y) \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$

by ($rule$ $product\text{-}character$)
hence $P\ z\ (x \otimes y) \longleftrightarrow (P\ z\ x \wedge P\ z\ y)$..
thus $P\ z\ (x \otimes y)$
using $\langle P\ z\ x \wedge P\ z\ y\rangle$..
qed

lemma *product-part-in-factors*:
$O\ x\ y \Longrightarrow P\ z\ (x \otimes y) \Longrightarrow P\ z\ x \wedge P\ z\ y$
proof –
assume $O\ x\ y$
hence $\forall\, w.\ P\ w\ (x \otimes y) \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$
by ($rule$ $product\text{-}character$)
hence $P\ z\ (x \otimes y) \longleftrightarrow (P\ z\ x \wedge P\ z\ y)$..
moreover assume $P\ z\ (x \otimes y)$
ultimately show $P\ z\ x \wedge P\ z\ y$..
qed

corollary *product-part-in-first-factor*:
$O\ x\ y \Longrightarrow P\ z\ (x \otimes y) \Longrightarrow P\ z\ x$
proof –
assume $O\ x\ y$
moreover assume $P\ z\ (x \otimes y)$
ultimately have $P\ z\ x \wedge P\ z\ y$
by ($rule$ $product\text{-}part\text{-}in\text{-}factors$)
thus $P\ z\ x$..
qed

corollary *product-part-in-second-factor*:
$O\ x\ y \Longrightarrow P\ z\ (x \otimes y) \Longrightarrow P\ z\ y$
proof –
assume $O\ x\ y$
moreover assume $P\ z\ (x \otimes y)$
ultimately have $P\ z\ x \wedge P\ z\ y$
by ($rule$ $product\text{-}part\text{-}in\text{-}factors$)
thus $P\ z\ y$..
qed

lemma *part-product-identity*: $P\ x\ y \Longrightarrow x \otimes y = x$
proof –
assume $P\ x\ y$
with *part-reflexivity* **have** $P\ x\ x \wedge P\ x\ y$..
hence $P\ x\ (x \otimes y)$ **by** ($rule$ $common\text{-}part\text{-}in\text{-}product$)
have $O\ x\ y$ **using** $\langle P\ x\ y\rangle$ **by** ($rule$ $part\text{-}implies\text{-}overlap$)
hence $P\ (x \otimes y)\ x$ **by** ($rule$ $product\text{-}in\text{-}first\text{-}factor$)
thus $x \otimes y = x$ **using** $\langle P\ x\ (x \otimes y)\rangle$ **by** ($rule$ $part\text{-}antisymmetry$)
qed

lemma *product-overlap*: $P\ z\ x \Longrightarrow O\ z\ y \Longrightarrow O\ z\ (x \otimes y)$
proof –

assume $P\ z\ x$
assume $O\ z\ y$
with *overlap-eq* **have** $\exists\, v.\ P\ v\ z \wedge P\ v\ y$**..**
then obtain v **where** v: $P\ v\ z \wedge P\ v\ y$**..**
hence $P\ v\ y$**..**
from v **have** $P\ v\ z$**..**
hence $P\ v\ x$ **using** $\langle P\ z\ x\rangle$ **by** (*rule part-transitivity*)
hence $P\ v\ x \wedge P\ v\ y$ **using** $\langle P\ v\ y\rangle$**..**
hence $P\ v\ (x \otimes y)$ **by** (*rule common-part-in-product*)
with $\langle P\ v\ z\rangle$ **have** $P\ v\ z \wedge P\ v\ (x \otimes y)$**..**
hence $\exists\, v.\ P\ v\ z \wedge P\ v\ (x \otimes y)$**..**
with *overlap-eq* **show** $O\ z\ (x \otimes y)$**..**
qed

lemma *disjoint-from-second-factor*:
$P\ x\ y \wedge \neg\ O\ x\ (y \otimes z) \Longrightarrow \neg\ O\ x\ z$
proof −
assume *antecedent*: $P\ x\ y \wedge \neg\ O\ x\ (y \otimes z)$
hence $\neg\ O\ x\ (y \otimes z)$**..**
show $\neg\ O\ x\ z$
proof
from *antecedent* **have** $P\ x\ y$**..**
moreover assume $O\ x\ z$
ultimately have $O\ x\ (y \otimes z)$
by (*rule product-overlap*)
with $\langle\neg\ O\ x\ (y \otimes z)\rangle$ **show** *False***..**
qed
qed

lemma *converse-product-overlap*:
$O\ x\ y \Longrightarrow O\ z\ (x \otimes y) \Longrightarrow O\ z\ y$
proof −
assume $O\ x\ y$
hence $P\ (x \otimes y)\ y$ **by** (*rule product-in-second-factor*)
moreover assume $O\ z\ (x \otimes y)$
ultimately show $O\ z\ y$
by (*rule overlap-monotonicity*)
qed

lemma *part-product-in-whole-product*:
$O\ x\ y \Longrightarrow P\ x\ v \wedge P\ y\ z \Longrightarrow P\ (x \otimes y)\ (v \otimes z)$
proof −
assume $O\ x\ y$
assume $P\ x\ v \wedge P\ y\ z$
have $\forall\, w.\ P\ w\ (x \otimes y) \longrightarrow P\ w\ (v \otimes z)$
proof
fix w
show $P\ w\ (x \otimes y) \longrightarrow P\ w\ (v \otimes z)$
proof

assume $P\ w\ (x \otimes y)$
with $\langle O\ x\ y\rangle$ **have** $P\ w\ x \wedge P\ w\ y$
  by (*rule product-part-in-factors*)
have $P\ w\ v \wedge P\ w\ z$
proof
  from $\langle P\ w\ x \wedge P\ w\ y\rangle$ **have** $P\ w\ x$..
  moreover from $\langle P\ x\ v \wedge P\ y\ z\rangle$ **have** $P\ x\ v$..
  ultimately show $P\ w\ v$ **by** (*rule part-transitivity*)
next
  from $\langle P\ w\ x \wedge P\ w\ y\rangle$ **have** $P\ w\ y$..
  moreover from $\langle P\ x\ v \wedge P\ y\ z\rangle$ **have** $P\ y\ z$..
  ultimately show $P\ w\ z$ **by** (*rule part-transitivity*)
qed
thus $P\ w\ (v \otimes z)$ **by** (*rule common-part-in-product*)
qed
qed
hence $P\ (x \otimes y)\ (x \otimes y) \longrightarrow P\ (x \otimes y)\ (v \otimes z)$..
moreover have $P\ (x \otimes y)\ (x \otimes y)$ **by** (*rule part-reflexivity*)
ultimately show $P\ (x \otimes y)\ (v \otimes z)$..
qed

lemma *right-associated-product*: $(\exists\, w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z) \Longrightarrow$
$(\forall\, w.\ P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge (P\ w\ y \wedge P\ w\ z))$
proof −
assume *antecedent*: $(\exists\, w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z)$
then obtain w **where** w: $P\ w\ x \wedge P\ w\ y \wedge P\ w\ z$..
hence $P\ w\ x$..
from w **have** $P\ w\ y \wedge P\ w\ z$..
hence $\exists\, w.\ P\ w\ y \wedge P\ w\ z$..
with *overlap-eq* **have** $O\ y\ z$..
hence *yz*: $\forall\, w.\ P\ w\ (y \otimes z) \longleftrightarrow (P\ w\ y \wedge P\ w\ z)$
  by (*rule product-character*)
hence $P\ w\ (y \otimes z) \longleftrightarrow (P\ w\ y \wedge P\ w\ z)$..
hence $P\ w\ (y \otimes z)$
  using $\langle P\ w\ y \wedge P\ w\ z\rangle$..
with $\langle P\ w\ x\rangle$ **have** $P\ w\ x \wedge P\ w\ (y \otimes z)$..
hence $\exists\, w.\ P\ w\ x \wedge P\ w\ (y \otimes z)$..
with *overlap-eq* **have** $O\ x\ (y \otimes z)$..
hence *xyz*: $\forall\, w.\ P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge P\ w\ (y \otimes z)$
  by (*rule product-character*)
show $\forall\, w.\ P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge (P\ w\ y \wedge P\ w\ z)$
proof
  fix w
  from *yz* **have** *wyz*: $P\ w\ (y \otimes z) \longleftrightarrow (P\ w\ y \wedge P\ w\ z)$..
  moreover from *xyz* **have**
    $P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge P\ w\ (y \otimes z)$..
  ultimately show
    $P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge (P\ w\ y \wedge P\ w\ z)$
    by (*rule subst*)

qed
qed

lemma *left-associated-product*: $(\exists\, w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z) \Longrightarrow$
$(\forall\, w.\ P\ w\ ((x \otimes y) \otimes z) \longleftrightarrow (P\ w\ x \wedge P\ w\ y) \wedge P\ w\ z)$
proof $-$
assume *antecedent*: $(\exists\, w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z)$
then obtain w **where** w: $P\ w\ x \wedge P\ w\ y \wedge P\ w\ z$..
hence $P\ w\ y \wedge P\ w\ z$..
hence $P\ w\ y$..
have $P\ w\ z$
using $\langle P\ w\ y \wedge P\ w\ z\rangle$..
from w **have** $P\ w\ x$..
hence $P\ w\ x \wedge P\ w\ y$
using $\langle P\ w\ y\rangle$..
hence $\exists\, z.\ P\ z\ x \wedge P\ z\ y$..
with *overlap-eq* **have** $O\ x\ y$..
hence *xy*: $\forall\, w.\ P\ w\ (x \otimes y) \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$
by (*rule product-character*)
hence $P\ w\ (x \otimes y) \longleftrightarrow (P\ w\ x \wedge P\ w\ y)$..
hence $P\ w\ (x \otimes y)$
using $\langle P\ w\ x \wedge P\ w\ y\rangle$..
hence $P\ w\ (x \otimes y) \wedge P\ w\ z$
using $\langle P\ w\ z\rangle$..
hence $\exists\, w.\ P\ w\ (x \otimes y) \wedge P\ w\ z$..
with *overlap-eq* **have** $O\ (x \otimes y)\ z$..
hence *xyz*: $\forall\, w.\ P\ w\ ((x \otimes y) \otimes z) \longleftrightarrow P\ w\ (x \otimes y) \wedge P\ w\ z$
by (*rule product-character*)
show $\forall\, w.\ P\ w\ ((x \otimes y) \otimes z) \longleftrightarrow (P\ w\ x \wedge P\ w\ y) \wedge P\ w\ z$
proof
fix v
from *xy* **have** *vxy*: $P\ v\ (x \otimes y) \longleftrightarrow (P\ v\ x \wedge P\ v\ y)$..
moreover from *xyz* **have**
$P\ v\ ((x \otimes y) \otimes z) \longleftrightarrow P\ v\ (x \otimes y) \wedge P\ v\ z$..
ultimately show $P\ v\ ((x \otimes y) \otimes z) \longleftrightarrow (P\ v\ x \wedge P\ v\ y) \wedge P\ v\ z$
by (*rule subst*)
qed
qed

theorem *product-associativity*:
$(\exists\, w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z) \Longrightarrow x \otimes (y \otimes z) = (x \otimes y) \otimes z$
proof $-$
assume *ante*:$(\exists\, w.\ P\ w\ x \wedge P\ w\ y \wedge P\ w\ z)$
hence $(\forall\, w.\ P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ x \wedge (P\ w\ y \wedge P\ w\ z))$
by (*rule right-associated-product*)
moreover from *ante* **have**
$(\forall\, w.\ P\ w\ ((x \otimes y) \otimes z) \longleftrightarrow (P\ w\ x \wedge P\ w\ y) \wedge P\ w\ z)$
by (*rule left-associated-product*)
ultimately have $\forall\, w.\ P\ w\ (x \otimes (y \otimes z)) \longleftrightarrow P\ w\ ((x \otimes y) \otimes z)$

by *simp*
with *part-extensionality* **show** $x \otimes (y \otimes z) = (x \otimes y) \otimes z$**..**
qed

end

6.2 Differences

Some writers also add to closed mereology the axiom of difference closure.[25]

locale $CMD = CM +$
assumes *difference-eq*:
$x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y)$
assumes *difference-closure*:
$(\exists w.\ P\ w\ x \land \neg\ O\ w\ y) \Longrightarrow (\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y)$
begin

lemma *difference-intro*:
$(\forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y) \Longrightarrow x \ominus y = z$
proof –
assume *antecedent*: $(\forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y)$
hence $(THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y) = z$
proof (*rule the-equality*)
fix v
assume v: $(\forall w.\ P\ w\ v \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y)$
have $\forall w.\ P\ w\ v \longleftrightarrow P\ w\ z$
proof
fix w
from *antecedent* **have** $P\ w\ z \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y$**..**
moreover from v **have** $P\ w\ v \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y$**..**
ultimately show $P\ w\ v \longleftrightarrow P\ w\ z$ **by** (*rule ssubst*)
qed
with *part-extensionality* **show** $v = z$**..**
qed
with *difference-eq* **show** $x \ominus y = z$ **by** (*rule ssubst*)
qed

lemma *difference-idempotence*: $\neg\ O\ x\ y \Longrightarrow (x \ominus y) = x$
proof –
assume $\neg\ O\ x\ y$
hence $\neg\ O\ y\ x$ **by** (*rule disjoint-symmetry*)
have $\forall w.\ P\ w\ x \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y$
proof
fix w
show $P\ w\ x \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y$
proof

[25] See, for example, [Varzi, 1996] p. 263 and [Masolo and Vieu, 1999] p. 238.

assume $P\ w\ x$
hence $\neg\ O\ y\ w$ **using** $\langle\neg\ O\ y\ x\rangle$
by (*rule disjoint-demonotonicity*)
hence $\neg\ O\ w\ y$ **by** (*rule disjoint-symmetry*)
with $\langle P\ w\ x\rangle$ **show** $P\ w\ x \wedge \neg\ O\ w\ y$**..**
next
assume $P\ w\ x \wedge \neg\ O\ w\ y$
thus $P\ w\ x$**..**
qed
qed
thus $(x \ominus y) = x$ **by** (*rule difference-intro*)
qed

lemma *difference-character*: $(\exists\, w.\ P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow$
$(\forall\, w.\ P\ w\ (x \ominus y) \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$
proof −
assume $\exists\, w.\ P\ w\ x \wedge \neg\ O\ w\ y$
hence $\exists\, z.\ \forall\, w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$ **by** (*rule difference-closure*)
then obtain z **where** z: $\forall\, w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$**..**
hence $(x \ominus y) = z$ **by** (*rule difference-intro*)
thus $\forall\, w.\ P\ w\ (x \ominus y) \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$ **using** z **by** (*rule ssubst*)
qed

lemma *difference-disjointness*:
$(\exists\, z.\ P\ z\ x \wedge \neg\ O\ z\ y) \Longrightarrow \neg\ O\ y\ (x \ominus y)$
proof −
assume $\exists\, z.\ P\ z\ x \wedge \neg\ O\ z\ y$
hence *xmy*: $\forall\, w.\ P\ w\ (x \ominus y) \longleftrightarrow (P\ w\ x \wedge \neg\ O\ w\ y)$
by (*rule difference-character*)
show $\neg\ O\ y\ (x \ominus y)$
proof
assume $O\ y\ (x \ominus y)$
with *overlap-eq* **have** $\exists\, v.\ P\ v\ y \wedge P\ v\ (x \ominus y)$**..**
then obtain v **where** v: $P\ v\ y \wedge P\ v\ (x \ominus y)$**..**
from *xmy* **have** $P\ v\ (x \ominus y) \longleftrightarrow (P\ v\ x \wedge \neg\ O\ v\ y)$**..**
moreover from v **have** $P\ v\ (x \ominus y)$**..**
ultimately have $P\ v\ x \wedge \neg\ O\ v\ y$**..**
hence $\neg\ O\ v\ y$**..**
moreover from v **have** $P\ v\ y$**..**
hence $O\ v\ y$ **by** (*rule part-implies-overlap*)
ultimately show *False***..**
qed
qed

end

6.3 The Universe

Another closure condition sometimes considered is the existence of the universe.[26]

```
locale CMU = CM +
  assumes universe-eq: u = (THE z. ∀ w. P w z)
  assumes universe-closure: ∃ y. ∀ x. P x y
begin

lemma universe-intro: (∀ w. P w z) ⟹ u = z
proof –
  assume z: ∀ w. P w z
  hence (THE z. ∀ w. P w z) = z
  proof (rule the-equality)
    fix v
    assume v: ∀ w. P w v
    have ∀ w. P w v ⟷ P w z
    proof
      fix w
      show P w v ⟷ P w z
      proof
        assume P w v
        from z show P w z..
      next
        assume P w z
        from v show P w v..
      qed
    qed
    with part-extensionality show v = z..
  qed
  thus u = z using universe-eq by (rule subst)
qed

lemma universe-character: P x u
proof –
  from universe-closure obtain y where y: ∀ x. P x y..
  hence u = y by (rule universe-intro)
  hence ∀ x. P x u using y by (rule ssubst)
  thus P x u..
qed

lemma ¬ PP u x
proof
  assume PP u x
  hence ¬ P x u by (rule proper-implies-not-part)
  thus False using universe-character..
qed
```

[26] See, for example, [Varzi, 1996] p. 264 and [Casati and Varzi, 1999] p. 45.

lemma *product-universe-implies-factor-universe*:
  $O\ x\ y \Longrightarrow x \otimes y = u \Longrightarrow x = u$
proof –
  assume $x \otimes y = u$
  moreover assume $O\ x\ y$
  hence $P\ (x \otimes y)\ x$
    by (*rule product-in-first-factor*)
  ultimately have $P\ u\ x$
    by (*rule subst*)
  with *universe-character* **show** $x = u$
    by (*rule part-antisymmetry*)
qed

end

6.4 Complements

As is a condition ensuring the existence of complements.[27]

locale *CMC* = *CM* +
  assumes *complement-eq*: $-x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
  assumes *complement-closure*:
    $(\exists w.\ \neg\ O\ w\ x) \Longrightarrow (\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
  assumes *difference-eq*:
    $x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \land \neg\ O\ w\ y)$
begin

lemma *complement-intro*:
  $(\forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x) \Longrightarrow -x = z$
proof –
  assume *antecedent*: $\forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x$
  hence $(THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x) = z$
  proof (*rule the-equality*)
    fix v
    assume v: $\forall w.\ P\ w\ v \longleftrightarrow \neg\ O\ w\ x$
    have $\forall w.\ P\ w\ v \longleftrightarrow P\ w\ z$
    proof
      fix w
      from *antecedent* **have** $P\ w\ z \longleftrightarrow \neg\ O\ w\ x$..
      moreover from v **have** $P\ w\ v \longleftrightarrow \neg\ O\ w\ x$..
      ultimately show $P\ w\ v \longleftrightarrow P\ w\ z$ **by** (*rule ssubst*)
    qed
    with *part-extensionality* **show** $v = z$..
  qed
  with *complement-eq* **show** $-x = z$ **by** (*rule ssubst*)
qed

[27] See, for example, [Varzi, 1996] p. 264 and [Casati and Varzi, 1999] p. 45.

lemma *complement-character*:
  $(\exists\, w.\ \neg\ O\ w\ x) \Longrightarrow (\forall\, w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x)$
proof –
  assume $\exists\, w.\ \neg\ O\ w\ x$
  hence $(\exists\, z.\ \forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$ **by** (*rule complement-closure*)
  then obtain z **where** z: $\forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x$**..**
  hence $-x = z$ **by** (*rule complement-intro*)
  thus $\forall\, w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
    using z **by** (*rule ssubst*)
qed

lemma *not-complement-part*: $\exists\, w.\ \neg\ O\ w\ x \Longrightarrow \neg\ P\ x\ (-x)$
proof –
  assume $\exists\, w.\ \neg\ O\ w\ x$
  hence $\forall\, w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
    by (*rule complement-character*)
  hence $P\ x\ (-x) \longleftrightarrow \neg\ O\ x\ x$**..**
  show $\neg\ P\ x\ (-x)$
  proof
    assume $P\ x\ (-x)$
    with $\langle P\ x\ (-x) \longleftrightarrow \neg\ O\ x\ x\rangle$ **have** $\neg\ O\ x\ x$**..**
    thus *False* **using** *overlap-reflexivity***..**
  qed
qed

lemma *complement-part*: $\neg\ O\ x\ y \Longrightarrow P\ x\ (-y)$
proof –
  assume $\neg\ O\ x\ y$
  hence $\exists\, z.\ \neg\ O\ z\ y$**..**
  hence $\forall\, w.\ P\ w\ (-y) \longleftrightarrow \neg\ O\ w\ y$
    by (*rule complement-character*)
  hence $P\ x\ (-y) \longleftrightarrow \neg\ O\ x\ y$**..**
  thus $P\ x\ (-y)$ **using** $\langle\neg\ O\ x\ y\rangle$**..**
qed

lemma *complement-overlap*: $\neg\ O\ x\ y \Longrightarrow O\ x\ (-y)$
proof –
  assume $\neg\ O\ x\ y$
  hence $P\ x\ (-y)$
    by (*rule complement-part*)
  thus $O\ x\ (-y)$
    by (*rule part-implies-overlap*)
qed

lemma *or-complement-overlap*: $\forall\, y.\ O\ y\ x \lor O\ y\ (-x)$
proof
  fix y
  show $O\ y\ x \lor O\ y\ (-x)$
  proof *cases*

assume $O\ y\ x$
thus $O\ y\ x \lor O\ y\ (-x)$..
next
assume $\neg\ O\ y\ x$
hence $O\ y\ (-x)$
by (*rule complement-overlap*)
thus $O\ y\ x \lor O\ y\ (-x)$..
qed
qed

lemma *complement-disjointness*: $\exists\, v.\ \neg\ O\ v\ x \Longrightarrow \neg\ O\ x\ (-x)$
proof −
assume $\exists\, v.\ \neg\ O\ v\ x$
hence w: $\forall\, w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
by (*rule complement-character*)
show $\neg\ O\ x\ (-x)$
proof
assume $O\ x\ (-x)$
with *overlap-eq* **have** $\exists\, v.\ P\ v\ x \land P\ v\ (-x)$..
then obtain v **where** v: $P\ v\ x \land P\ v\ (-x)$..
from w **have** $P\ v\ (-x) \longleftrightarrow \neg\ O\ v\ x$..
moreover from v **have** $P\ v\ (-x)$..
ultimately have $\neg\ O\ v\ x$..
moreover from v **have** $P\ v\ x$..
hence $O\ v\ x$ **by** (*rule part-implies-overlap*)
ultimately show *False*..
qed
qed

lemma *part-disjoint-from-complement*:
$\exists\, v.\ \neg\ O\ v\ x \Longrightarrow P\ y\ x \Longrightarrow \neg\ O\ y\ (-x)$
proof
assume $\exists\, v.\ \neg\ O\ v\ x$
hence $\neg\ O\ x\ (-x)$ **by** (*rule complement-disjointness*)
assume $P\ y\ x$
assume $O\ y\ (-x)$
with *overlap-eq* **have** $\exists\, v.\ P\ v\ y \land P\ v\ (-x)$..
then obtain v **where** v: $P\ v\ y \land P\ v\ (-x)$..
hence $P\ v\ y$..
hence $P\ v\ x$ **using** ‹$P\ y\ x$› **by** (*rule part-transitivity*)
moreover from v **have** $P\ v\ (-x)$..
ultimately have $P\ v\ x \land P\ v\ (-x)$..
hence $\exists\, v.\ P\ v\ x \land P\ v\ (-x)$..
with *overlap-eq* **have** $O\ x\ (-x)$..
with ‹$\neg\ O\ x\ (-x)$› **show** *False*..
qed

lemma *product-complement-character*: $(\exists\, w.\ P\ w\ x \land \neg\ O\ w\ y) \Longrightarrow$
$(\forall\, w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow (P\ w\ x \land (\neg\ O\ w\ y)))$

proof –
assume *antecedent*: $\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y$
then obtain w **where** w: $P\ w\ x \wedge \neg\ O\ w\ y$**..**
hence $P\ w\ x$**..**
moreover from w **have** $\neg\ O\ w\ y$**..**
hence $P\ w\ (-y)$ **by** (*rule complement-part*)
ultimately have $P\ w\ x \wedge P\ w\ (-y)$**..**
hence $\exists w.\ P\ w\ x \wedge P\ w\ (-y)$**..**
with *overlap-eq* **have** $O\ x\ (-y)$**..**
hence *prod*: $(\forall w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow (P\ w\ x \wedge P\ w\ (-y)))$
by (*rule product-character*)
show $\forall w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow (P\ w\ x \wedge (\neg\ O\ w\ y))$
proof
fix v
from w **have** $\neg\ O\ w\ y$**..**
hence $\exists w.\ \neg\ O\ w\ y$**..**
hence $\forall w.\ P\ w\ (-y) \longleftrightarrow \neg\ O\ w\ y$
by (*rule complement-character*)
hence $P\ v\ (-y) \longleftrightarrow \neg\ O\ v\ y$**..**
moreover have $P\ v\ (x \otimes (-y)) \longleftrightarrow (P\ v\ x \wedge P\ v\ (-y))$
using *prod***..**
ultimately show $P\ v\ (x \otimes (-y)) \longleftrightarrow (P\ v\ x \wedge (\neg\ O\ v\ y))$
by (*rule subst*)
qed
qed

theorem *difference-closure*: $(\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow$
$(\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$
proof –
assume $\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y$
hence $\forall w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$
by (*rule product-complement-character*)
thus $(\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y)$ **by** (*rule exI*)
qed

end

sublocale *CMC* $\subseteq$ *CMD*
proof
fix $x\ y$
show $x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
using *difference-eq***.**
show $(\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow$
$(\exists z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
using *difference-closure***.**
qed

corollary (**in** *CMC*) *difference-is-product-of-complement*:
$(\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y) \Longrightarrow (x \ominus y) = x \otimes (-y)$

proof −
  assume *antecedent*: $\exists w.\ P\ w\ x \wedge \neg\ O\ w\ y$
  hence $\forall w.\ P\ w\ (x \otimes (-y)) \longleftrightarrow P\ w\ x \wedge \neg\ O\ w\ y$
    by (*rule product-complement-character*)
  thus $(x \ominus y) = x \otimes (-y)$ **by** (*rule difference-intro*)
qed

Universe and difference closure entail complement closure, since the difference of an individual and the universe is the individual's complement.

locale *CMUD* = *CMU* + *CMD* +
  assumes *complement-eq*: $-x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
begin

lemma *universe-difference*:
  $(\exists w.\ \neg\ O\ w\ x) \Longrightarrow (\forall w.\ P\ w\ (u \ominus x) \longleftrightarrow \neg\ O\ w\ x)$
proof −
  assume $\exists w.\ \neg\ O\ w\ x$
  then obtain w **where** w: $\neg\ O\ w\ x$..
  from *universe-character* **have** $P\ w\ u$.
  hence $P\ w\ u \wedge \neg\ O\ w\ x$ **using** ‹$\neg\ O\ w\ x$›..
  hence $\exists z.\ P\ z\ u \wedge \neg\ O\ z\ x$..
  hence *ux*: $\forall w.\ P\ w\ (u \ominus x) \longleftrightarrow (P\ w\ u \wedge \neg\ O\ w\ x)$
    by (*rule difference-character*)
  show $\forall w.\ P\ w\ (u \ominus x) \longleftrightarrow \neg\ O\ w\ x$
  proof
    fix w
    from *ux* **have** *wux*: $P\ w\ (u \ominus x) \longleftrightarrow (P\ w\ u \wedge \neg\ O\ w\ x)$..
    show $P\ w\ (u \ominus x) \longleftrightarrow \neg\ O\ w\ x$
    proof
      assume $P\ w\ (u \ominus x)$
      with *wux* **have** $P\ w\ u \wedge \neg\ O\ w\ x$..
      thus $\neg\ O\ w\ x$..
    next
      assume $\neg\ O\ w\ x$
      from *universe-character* **have** $P\ w\ u$.
      hence $P\ w\ u \wedge \neg\ O\ w\ x$ **using** ‹$\neg\ O\ w\ x$›..
      with *wux* **show** $P\ w\ (u \ominus x)$..
    qed
  qed
qed

theorem *complement-closure*:
  $(\exists w.\ \neg\ O\ w\ x) \Longrightarrow (\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
proof −
  assume $\exists w.\ \neg\ O\ w\ x$
  hence $\forall w.\ P\ w\ (u \ominus x) \longleftrightarrow \neg\ O\ w\ x$
    by (*rule universe-difference*)
  thus $\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x$..

qed

end

sublocale $CMUD \subseteq CMC$
proof
  fix $x\ y$
  show $-x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow (\neg\ O\ w\ x))$
    using *complement-eq*.
  show $\exists w.\ \neg\ O\ w\ x \Longrightarrow \exists z.\ \forall w.\ P\ w\ z \longleftrightarrow (\neg\ O\ w\ x)$
    using *complement-closure*.
  show $x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
    using *difference-eq*.
qed

corollary (**in** *CMUD*) *complement-universe-difference*:
  $(\exists y.\ \neg\ O\ y\ x) \Longrightarrow -x = (u \ominus x)$
proof –
  assume $\exists w.\ \neg\ O\ w\ x$
  hence $\forall w.\ P\ w\ (u \ominus x) \longleftrightarrow \neg\ O\ w\ x$
    by (*rule universe-difference*)
  thus $-x = (u \ominus x)$
    by (*rule complement-intro*)
qed

7 Closed Extensional Mereology

Closed extensional mereology combines closed mereology with extensional mereology.[28]

locale $CEM = CM + EM$

Likewise, closed minimal mereology combines closed mereology with minimal mereology.[29]

locale $CMM = CM + MM$

But famously closed minimal mereology and closed extensional mereology are the same theory, because in closed minimal mereology product closure and weak supplementation entail strong supplementation.[30]

sublocale $CMM \subseteq CEM$
proof
  fix $x\ y$

[28] See [Varzi, 1996] p. 263 and [Casati and Varzi, 1999] p. 43.
[29] See [Casati and Varzi, 1999] p. 43.
[30] See [Simons, 1987] p. 31 and [Casati and Varzi, 1999] p. 44.

```
  show strong-supplementation: ¬ P x y ⟹ (∃ z. P z x ∧ ¬ O z y)
  proof –
    assume ¬ P x y
    show ∃ z. P z x ∧ ¬ O z y
    proof cases
      assume O x y
      with ‹¬ P x y› have ¬ P x y ∧ O x y..
      hence PP (x ⊗ y) x by (rule nonpart-implies-proper-product)
     hence ∃ z. P z x ∧ ¬ O z (x ⊗ y) by (rule weak-supplementation)
      then obtain z where z: P z x ∧ ¬ O z (x ⊗ y)..
      hence ¬ O z y by (rule disjoint-from-second-factor)
      moreover from z have P z x..
      hence  P z x ∧ ¬ O z y
        using ‹¬ O z y›..
      thus ∃ z. P z x ∧ ¬ O z y..
    next
      assume ¬ O x y
      with part-reflexivity have P x x ∧ ¬ O x y..
      thus (∃ z. P z x ∧ ¬ O z y)..
    qed
  qed
qed

sublocale CEM ⊆ CMM..
```

7.1 Sums

```
context CEM
begin

lemma sum-intro:
  (∀ w. O w z ⟷ (O w x ∨ O w y)) ⟹ x ⊕ y = z
proof –
  assume sum: ∀ w. O w z ⟷ (O w x ∨ O w y)
  hence (THE v. ∀ w. O w v ⟷ (O w x ∨ O w y)) = z
  proof (rule the-equality)
    fix a
    assume a: ∀ w. O w a ⟷ (O w x ∨ O w y)
    have ∀ w. O w a ⟷ O w z
    proof
      fix w
      from sum have O w z ⟷ (O w x ∨ O w y)..
      moreover from a have O w a ⟷ (O w x ∨ O w y)..
      ultimately show O w a ⟷ O w z by (rule ssubst)
      qed
      with overlap-extensionality show a = z..
  qed
  thus x ⊕ y = z
    using sum-eq by (rule subst)
```

```
qed

lemma sum-idempotence: x ⊕ x = x
proof −
  have ∀ w. O w x ⟷ (O w x ∨ O w x)
  proof
    fix w
    show O w x ⟷ (O w x ∨ O w x)
    proof (rule iffI)
      assume O w x
      thus O w x ∨ O w x..
    next
      assume O w x ∨ O w x
      thus O w x by (rule disjE)
    qed
  qed
  thus x ⊕ x = x by (rule sum-intro)
qed

lemma part-sum-identity: P y x ⟹ x ⊕ y = x
proof −
  assume P y x
  have ∀ w. O w x ⟷ (O w x ∨ O w y)
  proof
    fix w
    show O w x ⟷ (O w x ∨ O w y)
    proof
      assume O w x
      thus O w x ∨ O w y..
    next
      assume O w x ∨ O w y
      thus O w x
      proof
        assume O w x
        thus O w x.
      next
        assume O w y
        with ‹P y x› show O w x
          by (rule overlap-monotonicity)
      qed
    qed
  qed
  thus x ⊕ y = x by (rule sum-intro)
qed

lemma sum-character: ∀ w. O w (x ⊕ y) ⟷ (O w x ∨ O w y)
proof −
  from sum-closure have (∃ z. ∀ w. O w z ⟷ (O w x ∨ O w y)).
  then obtain a where a: ∀ w. O w a ⟷ (O w x ∨ O w y)..
```

hence $x \oplus y = a$ **by** (*rule sum-intro*)
thus $\forall\ w.\ O\ w\ (x \oplus y) \longleftrightarrow (O\ w\ x \lor O\ w\ y)$
  using a **by** (*rule ssubst*)
qed

lemma *sum-overlap*: $O\ w\ (x \oplus y) \longleftrightarrow (O\ w\ x \lor O\ w\ y)$
  using *sum-character*..

lemma *sum-part-character*:
  $P\ w\ (x \oplus y) \longleftrightarrow (\forall\ v.\ O\ v\ w \longrightarrow O\ v\ x \lor O\ v\ y)$
proof
  assume $P\ w\ (x \oplus y)$
  show $\forall\ v.\ O\ v\ w \longrightarrow O\ v\ x \lor O\ v\ y$
  proof
    fix v
    show $O\ v\ w \longrightarrow O\ v\ x \lor O\ v\ y$
    proof
      assume $O\ v\ w$
      with $\langle P\ w\ (x \oplus y)\rangle$ **have** $O\ v\ (x \oplus y)$
        by (*rule overlap-monotonicity*)
      with *sum-overlap* **show** $O\ v\ x \lor O\ v\ y$..
    qed
  qed
next
  assume *right*: $\forall\ v.\ O\ v\ w \longrightarrow O\ v\ x \lor O\ v\ y$
  have $\forall\ v.\ O\ v\ w \longrightarrow O\ v\ (x \oplus y)$
  proof
    fix v
    from *right* **have** $O\ v\ w \longrightarrow O\ v\ x \lor O\ v\ y$..
    with *sum-overlap* **show** $O\ v\ w \longrightarrow O\ v\ (x \oplus y)$
      by (*rule ssubst*)
  qed
  with *part-overlap-eq* **show** $P\ w\ (x \oplus y)$..
qed

lemma *sum-commutativity*: $x \oplus y = y \oplus x$
proof −
  from *sum-character* **have** $\forall\ w.\ O\ w\ (y \oplus x) \longleftrightarrow O\ w\ y \lor O\ w\ x$.
  hence $\forall\ w.\ O\ w\ (y \oplus x) \longleftrightarrow O\ w\ x \lor O\ w\ y$ **by** *metis*
  thus $x \oplus y = y \oplus x$ **by** (*rule sum-intro*)
qed

lemma *first-summand-overlap*: $O\ z\ x \Longrightarrow O\ z\ (x \oplus y)$
proof −
  assume $O\ z\ x$
  hence $O\ z\ x \lor O\ z\ y$..
  with *sum-overlap* **show** $O\ z\ (x \oplus y)$..
qed

lemma *first-summand-disjointness*: $\neg$ O z $(x \oplus y) \Longrightarrow \neg$ O z x
proof −
  assume $\neg$ O z $(x \oplus y)$
  show $\neg$ O z x
  proof
    assume O z x
    hence O z $(x \oplus y)$ **by** (*rule first-summand-overlap*)
    with ‹$\neg$ O z $(x \oplus y)$› **show** *False*..
  qed
qed

lemma *first-summand-in-sum*: P x $(x \oplus y)$
proof −
  have $\forall$ w. O w $x \longrightarrow O$ w $(x \oplus y)$
  proof
    fix w
    show O w $x \longrightarrow O$ w $(x \oplus y)$
    proof
      assume O w x
      thus O w $(x \oplus y)$
        by (*rule first-summand-overlap*)
    qed
  qed
  with *part-overlap-eq* **show** P x $(x \oplus y)$..
qed

lemma *common-first-summand*: P x $(x \oplus y) \wedge P$ x $(x \oplus z)$
proof
  from *first-summand-in-sum* **show** P x $(x \oplus y)$.
  from *first-summand-in-sum* **show** P x $(x \oplus z)$.
qed

lemma *common-first-summand-overlap*: O $(x \oplus y)$ $(x \oplus z)$
proof −
  from *first-summand-in-sum* **have** P x $(x \oplus y)$.
  moreover from *first-summand-in-sum* **have** P x $(x \oplus z)$.
  ultimately have P x $(x \oplus y) \wedge P$ x $(x \oplus z)$..
  hence $\exists$ v. P v $(x \oplus y) \wedge P$ v $(x \oplus z)$..
  with *overlap-eq* **show** *?thesis*..
qed

lemma *second-summand-overlap*: O z $y \Longrightarrow O$ z $(x \oplus y)$
proof −
  assume O z y
  from *sum-character* **have** O z $(x \oplus y) \longleftrightarrow (O$ z $x \vee O$ z $y)$..
  moreover from ‹O z y› **have** O z $x \vee O$ z y..
  ultimately show O z $(x \oplus y)$..
qed

lemma *second-summand-disjointness*: $\neg\ O\ z\ (x \oplus y) \Longrightarrow \neg\ O\ z\ y$
proof –
  assume $\neg\ O\ z\ (x \oplus y)$
  show $\neg\ O\ z\ y$
  proof
    assume $O\ z\ y$
    hence $O\ z\ (x \oplus y)$
      by (*rule second-summand-overlap*)
    with ‹$\neg\ O\ z\ (x \oplus y)$› **show** *False*..
  qed
qed

lemma *second-summand-in-sum*: $P\ y\ (x \oplus y)$
proof –
  have $\forall\ w.\ O\ w\ y \longrightarrow O\ w\ (x \oplus y)$
  proof
    fix w
    show $O\ w\ y \longrightarrow O\ w\ (x \oplus y)$
    proof
      assume $O\ w\ y$
      thus $O\ w\ (x \oplus y)$
        by (*rule second-summand-overlap*)
    qed
  qed
  with *part-overlap-eq* **show** $P\ y\ (x \oplus y)$..
qed

lemma *second-summands-in-sums*: $P\ y\ (x \oplus y) \wedge P\ v\ (z \oplus v)$
proof
  show $P\ y\ (x \oplus y)$ **using** *second-summand-in-sum*.
  show $P\ v\ (z \oplus v)$ **using** *second-summand-in-sum*.
qed

lemma *disjoint-from-sum*: $\neg\ O\ z\ (x \oplus y) \longleftrightarrow \neg\ O\ z\ x \wedge \neg\ O\ z\ y$
proof –
  from *sum-character* **have** $O\ z\ (x \oplus y) \longleftrightarrow (O\ z\ x \vee O\ z\ y)$..
  thus *?thesis* **by** *simp*
qed

lemma *summands-part-implies-sum-part*:
  $P\ x\ z \wedge P\ y\ z \Longrightarrow P\ (x \oplus y)\ z$
proof –
  assume *antecedent*: $P\ x\ z \wedge P\ y\ z$
  have $\forall\ w.\ O\ w\ (x \oplus y) \longrightarrow O\ w\ z$
  proof
    fix w
    have w: $O\ w\ (x \oplus y) \longleftrightarrow (O\ w\ x \vee O\ w\ y)$
      using *sum-character*..
    show $O\ w\ (x \oplus y) \longrightarrow O\ w\ z$

```
    proof
      assume O w (x ⊕ y)
      with w have O w x ∨ O w y..
      thus O w z
      proof
        from antecedent have P x z..
        moreover assume O w x
        ultimately show O w z
          by (rule overlap-monotonicity)
      next
        from antecedent have P y z..
        moreover assume O w y
        ultimately show O w z
          by (rule overlap-monotonicity)
      qed
    qed
  qed
  with part-overlap-eq show P (x ⊕ y) z..
qed

lemma sum-part-implies-summands-part:
  P (x ⊕ y) z ⟹ P x z ∧ P y z
proof −
  assume antecedent: P (x ⊕ y) z
  show P x z ∧ P y z
  proof
    from first-summand-in-sum show P x z
      using antecedent by (rule part-transitivity)
  next
    from second-summand-in-sum show P y z
      using antecedent by (rule part-transitivity)
  qed
qed

lemma in-second-summand: P z (x ⊕ y) ∧ ¬ O z x ⟹ P z y
proof −
  assume antecedent: P z (x ⊕ y) ∧ ¬ O z x
  hence P z (x ⊕ y)..
  show P z y
  proof (rule ccontr)
    assume ¬ P z y
    hence ∃ v. P v z ∧ ¬ O v y
      by (rule strong-supplementation)
    then obtain v where v: P v z ∧ ¬ O v y..
    hence ¬ O v y..
    from v have P v z..
    hence P v (x ⊕ y)
      using ‹P z (x ⊕ y)› by (rule part-transitivity)
    hence O v (x ⊕ y) by (rule part-implies-overlap)
```

from *sum-character* **have** $O\ v\ (x \oplus y) \longleftrightarrow O\ v\ x \lor O\ v\ y$**..**
hence $O\ v\ x \lor O\ v\ y$ **using** ‹$O\ v\ (x \oplus y)$›**..**
thus *False*
proof (*rule disjE*)
  from *antecedent* **have** $\neg\ O\ z\ x$**..**
  moreover assume $O\ v\ x$
  hence $O\ x\ v$ **by** (*rule overlap-symmetry*)
  with ‹$P\ v\ z$› **have** $O\ x\ z$
    by (*rule overlap-monotonicity*)
  hence $O\ z\ x$ **by** (*rule overlap-symmetry*)
  ultimately show *False***..**
next
  assume $O\ v\ y$
  with ‹$\neg\ O\ v\ y$› **show** *False***..**
qed
qed
qed

lemma *disjoint-second-summands*:
$P\ v\ (x \oplus y) \land P\ v\ (x \oplus z) \Longrightarrow \neg\ O\ y\ z \Longrightarrow P\ v\ x$
proof −
assume *antecedent*: $P\ v\ (x \oplus y) \land P\ v\ (x \oplus z)$
hence $P\ v\ (x \oplus z)$**..**
assume $\neg\ O\ y\ z$
show $P\ v\ x$
proof (*rule ccontr*)
  assume $\neg\ P\ v\ x$
  hence $\exists\ w.\ P\ w\ v \land \neg\ O\ w\ x$ **by** (*rule strong-supplementation*)
  then obtain w **where** w: $P\ w\ v \land \neg\ O\ w\ x$**..**
  hence $\neg\ O\ w\ x$**..**
  from w **have** $P\ w\ v$**..**
  moreover from *antecedent* **have** $P\ v\ (x \oplus z)$**..**
  ultimately have $P\ w\ (x \oplus z)$ **by** (*rule part-transitivity*)
  hence $P\ w\ (x \oplus z) \land \neg\ O\ w\ x$ **using** ‹$\neg\ O\ w\ x$›**..**
  hence $P\ w\ z$ **by** (*rule in-second-summand*)
  from *antecedent* **have** $P\ v\ (x \oplus y)$**..**
  with ‹$P\ w\ v$› **have** $P\ w\ (x \oplus y)$ **by** (*rule part-transitivity*)
  hence $P\ w\ (x \oplus y) \land \neg\ O\ w\ x$ **using** ‹$\neg\ O\ w\ x$›**..**
  hence $P\ w\ y$ **by** (*rule in-second-summand*)
  hence $P\ w\ y \land P\ w\ z$ **using** ‹$P\ w\ z$›**..**
  hence $\exists\ w.\ P\ w\ y \land P\ w\ z$**..**
  with *overlap-eq* **have** $O\ y\ z$**..**
  with ‹$\neg\ O\ y\ z$› **show** *False***..**
qed
qed

lemma *right-associated-sum*:
$O\ w\ (x \oplus (y \oplus z)) \longleftrightarrow O\ w\ x \lor (O\ w\ y \lor O\ w\ z)$
proof −

from *sum-character* **have** $O\ w\ (y \oplus z) \longleftrightarrow O\ w\ y \lor O\ w\ z$..
moreover from *sum-character* **have**
$O\ w\ (x \oplus (y \oplus z)) \longleftrightarrow (O\ w\ x \lor O\ w\ (y \oplus z))$..
ultimately show *?thesis*
by (*rule subst*)
qed

lemma *left-associated-sum*:
$O\ w\ ((x \oplus y) \oplus z) \longleftrightarrow (O\ w\ x \lor O\ w\ y) \lor O\ w\ z$
proof –
from *sum-character* **have** $O\ w\ (x \oplus y) \longleftrightarrow (O\ w\ x \lor O\ w\ y)$..
moreover from *sum-character* **have**
$O\ w\ ((x \oplus y) \oplus z) \longleftrightarrow O\ w\ (x \oplus y) \lor O\ w\ z$..
ultimately show *?thesis*
by (*rule subst*)
qed

theorem *sum-associativity*: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
proof –
have $\forall\ w.\ O\ w\ (x \oplus (y \oplus z)) \longleftrightarrow O\ w\ ((x \oplus y) \oplus z)$
proof
fix w
have $O\ w\ (x \oplus (y \oplus z)) \longleftrightarrow (O\ w\ x \lor O\ w\ y) \lor O\ w\ z$
using *right-associated-sum* **by** *simp*
with *left-associated-sum* **show**
$O\ w\ (x \oplus (y \oplus z)) \longleftrightarrow O\ w\ ((x \oplus y) \oplus z)$ **by** (*rule ssubst*)
qed
with *overlap-extensionality* **show** $x \oplus (y \oplus z) = (x \oplus y) \oplus z$..
qed

7.2 Distributivity

The proofs in this section are adapted from [Pietruszczak, 2018] pp. 102-4.

lemma *common-summand-in-product*: $P\ x\ ((x \oplus y) \otimes (x \oplus z))$
using *common-first-summand* **by** (*rule common-part-in-product*)

lemma *product-in-first-summand*:
$\neg\ O\ y\ z \Longrightarrow P\ ((x \oplus y) \otimes (x \oplus z))\ x$
proof –
assume $\neg\ O\ y\ z$
have $\forall\ v.\ P\ v\ ((x \oplus y) \otimes (x \oplus z)) \longrightarrow P\ v\ x$
proof
fix v
show $P\ v\ ((x \oplus y) \otimes (x \oplus z)) \longrightarrow P\ v\ x$
proof
assume $P\ v\ ((x \oplus y) \otimes (x \oplus z))$
with *common-first-summand-overlap* **have**
$P\ v\ (x \oplus y) \land P\ v\ (x \oplus z)$ **by** (*rule product-part-in-factors*)

thus $P\ v\ x$ **using** ‹¬ $O\ y\ z$› **by** (*rule disjoint-second-summands*)
qed
qed
hence $P\ ((x \oplus y) \otimes (x \oplus z))\ ((x \oplus y) \otimes (x \oplus z)) \longrightarrow$
$P\ ((x \oplus y) \otimes (x \oplus z))\ x$..
thus $P\ ((x \oplus y) \otimes (x \oplus z))\ x$ **using** *part-reflexivity*..
qed

lemma *product-is-first-summand*:
$\neg\ O\ y\ z \Longrightarrow (x \oplus y) \otimes (x \oplus z) = x$
proof –
assume $\neg\ O\ y\ z$
hence $P\ ((x \oplus y) \otimes (x \oplus z))\ x$
by (*rule product-in-first-summand*)
thus $(x \oplus y) \otimes (x \oplus z) = x$
using *common-summand-in-product*
by (*rule part-antisymmetry*)
qed

lemma *sum-over-product-left*: $O\ y\ z \Longrightarrow P\ (x \oplus (y \otimes z))\ ((x \oplus y) \otimes (x \oplus z))$
proof –
assume $O\ y\ z$
hence $P\ (y \otimes z)\ ((x \oplus y) \otimes (x \oplus z))$ **using** *second-summands-in-sums*
by (*rule part-product-in-whole-product*)
with *common-summand-in-product* **have**
$P\ x\ ((x \oplus y) \otimes (x \oplus z)) \wedge P\ (y \otimes z)\ ((x \oplus y) \otimes (x \oplus z))$..
thus $P\ (x \oplus (y \otimes z))\ ((x \oplus y) \otimes (x \oplus z))$
by (*rule summands-part-implies-sum-part*)
qed

lemma *sum-over-product-right*:
$O\ y\ z \Longrightarrow P\ ((x \oplus y) \otimes (x \oplus z))\ (x \oplus (y \otimes z))$
proof –
assume $O\ y\ z$
show $P\ ((x \oplus y) \otimes (x \oplus z))\ (x \oplus (y \otimes z))$
proof (*rule ccontr*)
assume $\neg\ P\ ((x \oplus y) \otimes (x \oplus z))\ (x \oplus (y \otimes z))$
hence $\exists\ v.\ P\ v\ ((x \oplus y) \otimes (x \oplus z)) \wedge \neg\ O\ v\ (x \oplus (y \otimes z))$
by (*rule strong-supplementation*)
then obtain v **where** v:
$P\ v\ ((x \oplus y) \otimes (x \oplus z)) \wedge \neg\ O\ v\ (x \oplus (y \otimes z))$..
hence $\neg\ O\ v\ (x \oplus (y \otimes z))$..
with *disjoint-from-sum* **have** *vd*: $\neg\ O\ v\ x \wedge \neg\ O\ v\ (y \otimes z)$..
hence $\neg\ O\ v\ (y \otimes z)$..
from *vd* **have** $\neg\ O\ v\ x$..
from v **have** $P\ v\ ((x \oplus y) \otimes (x \oplus z))$..
with *common-first-summand-overlap* **have**
vs: $P\ v\ (x \oplus y) \wedge P\ v\ (x \oplus z)$ **by** (*rule product-part-in-factors*)

hence $P\ v\ (x \oplus y)$..
hence $P\ v\ (x \oplus y) \land \neg\ O\ v\ x$ **using** ‹$\neg\ O\ v\ x$›..
hence $P\ v\ y$ **by** (*rule in-second-summand*)
moreover from *vs* **have** $P\ v\ (x \oplus z)$..
hence $P\ v\ (x \oplus z) \land \neg\ O\ v\ x$ **using** ‹$\neg\ O\ v\ x$›..
hence $P\ v\ z$ **by** (*rule in-second-summand*)
ultimately have $P\ v\ y \land P\ v\ z$..
hence $P\ v\ (y \otimes z)$ **by** (*rule common-part-in-product*)
hence $O\ v\ (y \otimes z)$ **by** (*rule part-implies-overlap*)
with ‹$\neg\ O\ v\ (y \otimes z)$› **show** *False*..
qed
qed

Sums distribute over products.

theorem *sum-over-product*:
$O\ y\ z \Longrightarrow x \oplus (y \otimes z) = (x \oplus y) \otimes (x \oplus z)$
proof −
assume $O\ y\ z$
hence $P\ (x \oplus (y \otimes z))\ ((x \oplus y) \otimes (x \oplus z))$
by (*rule sum-over-product-left*)
moreover have $P\ ((x \oplus y) \otimes (x \oplus z))\ (x \oplus (y \otimes z))$
using ‹$O\ y\ z$› **by** (*rule sum-over-product-right*)
ultimately show $x \oplus (y \otimes z) = (x \oplus y) \otimes (x \oplus z)$
by (*rule part-antisymmetry*)
qed

lemma *product-in-factor-by-sum*:
$O\ x\ y \Longrightarrow P\ (x \otimes y)\ (x \otimes (y \oplus z))$
proof −
assume $O\ x\ y$
hence $P\ (x \otimes y)\ x$
by (*rule product-in-first-factor*)
moreover have $P\ (x \otimes y)\ y$
using ‹$O\ x\ y$› **by** (*rule product-in-second-factor*)
hence $P\ (x \otimes y)\ (y \oplus z)$
using *first-summand-in-sum* **by** (*rule part-transitivity*)
with ‹$P\ (x \otimes y)\ x$› **have** $P\ (x \otimes y)\ x \land P\ (x \otimes y)\ (y \oplus z)$..
thus $P\ (x \otimes y)\ (x \otimes (y \oplus z))$
by (*rule common-part-in-product*)
qed

lemma *product-of-first-summand*:
$O\ x\ y \Longrightarrow \neg\ O\ x\ z \Longrightarrow P\ (x \otimes (y \oplus z))\ (x \otimes y)$
proof −
assume $O\ x\ y$
hence $O\ x\ (y \oplus z)$
by (*rule first-summand-overlap*)
assume $\neg\ O\ x\ z$
show $P\ (x \otimes (y \oplus z))\ (x \otimes y)$

proof (*rule ccontr*)
assume $\neg\ P\ (x \otimes (y \oplus z))\ (x \otimes y)$
hence $\exists\ v.\ P\ v\ (x \otimes (y \oplus z)) \wedge \neg\ O\ v\ (x \otimes y)$
by (*rule strong-supplementation*)
then obtain v **where** v: $P\ v\ (x \otimes (y \oplus z)) \wedge \neg\ O\ v\ (x \otimes y)$..
hence $P\ v\ (x \otimes (y \oplus z))$..
with ‹$O\ x\ (y \oplus z)$› **have** $P\ v\ x \wedge P\ v\ (y \oplus z)$
by (*rule product-part-in-factors*)
hence $P\ v\ x$..
moreover from v **have** $\neg\ O\ v\ (x \otimes y)$..
ultimately have $P\ v\ x \wedge \neg\ O\ v\ (x \otimes y)$..
hence $\neg\ O\ v\ y$ **by** (*rule disjoint-from-second-factor*)
from ‹$P\ v\ x \wedge P\ v\ (y \oplus z)$› **have** $P\ v\ (y \oplus z)$..
hence $P\ v\ (y \oplus z) \wedge \neg\ O\ v\ y$ **using** ‹$\neg\ O\ v\ y$›..
hence $P\ v\ z$ **by** (*rule in-second-summand*)
with ‹$P\ v\ x$› **have** $P\ v\ x \wedge P\ v\ z$..
hence $\exists\ v.\ P\ v\ x \wedge P\ v\ z$..
with *overlap-eq* **have** $O\ x\ z$..
with ‹$\neg\ O\ x\ z$› **show** *False*..
qed
qed

theorem *disjoint-product-over-sum*:
$O\ x\ y \Longrightarrow \neg\ O\ x\ z \Longrightarrow x \otimes (y \oplus z) = x \otimes y$
proof −
assume $O\ x\ y$
moreover assume $\neg\ O\ x\ z$
ultimately have $P\ (x \otimes (y \oplus z))\ (x \otimes y)$
by (*rule product-of-first-summand*)
moreover have $P\ (x \otimes y)(x \otimes (y \oplus z))$
using ‹$O\ x\ y$› **by** (*rule product-in-factor-by-sum*)
ultimately show $x \otimes (y \oplus z) = x \otimes y$
by (*rule part-antisymmetry*)
qed

lemma *product-over-sum-left*:
$O\ x\ y \wedge O\ x\ z \Longrightarrow P\ (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
proof −
assume $O\ x\ y \wedge O\ x\ z$
hence $O\ x\ y$..
hence $O\ x\ (y \oplus z)$ **by** (*rule first-summand-overlap*)
show $P\ (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
proof (*rule ccontr*)
assume $\neg\ P\ (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
hence $\exists\ v.\ P\ v\ (x \otimes (y \oplus z)) \wedge \neg\ O\ v\ ((x \otimes y) \oplus (x \otimes z))$
by (*rule strong-supplementation*)
then obtain v **where** v:
$P\ v\ (x \otimes (y \oplus z)) \wedge \neg\ O\ v\ ((x \otimes y) \oplus (x \otimes z))$..
hence $\neg\ O\ v\ ((x \otimes y) \oplus (x \otimes z))$..

with *disjoint-from-sum* **have** *oxyz*:
$\neg\ O\ v\ (x \otimes y) \wedge \neg\ O\ v\ (x \otimes z)$..
from v **have** $P\ v\ (x \otimes (y \oplus z))$..
with $\langle O\ x\ (y \oplus z)\rangle$ **have** *pxyz*: $P\ v\ x \wedge P\ v\ (y \oplus z)$
by (*rule product-part-in-factors*)
hence $P\ v\ x$..
moreover from *oxyz* **have** $\neg\ O\ v\ (x \otimes y)$..
ultimately have $P\ v\ x \wedge \neg\ O\ v\ (x \otimes y)$..
hence $\neg\ O\ v\ y$ **by** (*rule disjoint-from-second-factor*)
from *oxyz* **have** $\neg\ O\ v\ (x \otimes z)$..
with $\langle P\ v\ x\rangle$ **have** $P\ v\ x \wedge \neg\ O\ v\ (x \otimes z)$..
hence $\neg\ O\ v\ z$ **by** (*rule disjoint-from-second-factor*)
with $\langle \neg\ O\ v\ y\rangle$ **have** $\neg\ O\ v\ y \wedge \neg\ O\ v\ z$..
with *disjoint-from-sum* **have** $\neg\ O\ v\ (y \oplus z)$..
from *pxyz* **have** $P\ v\ (y \oplus z)$..
hence $O\ v\ (y \oplus z)$ **by** (*rule part-implies-overlap*)
with $\langle \neg\ O\ v\ (y \oplus z)\rangle$ **show** *False*..
qed
qed

lemma *product-over-sum-right*:
$O\ x\ y \wedge O\ x\ z \Longrightarrow P((x \otimes y) \oplus (x \otimes z))(x \otimes (y \oplus z))$
proof –
assume *antecedent*: $O\ x\ y \wedge O\ x\ z$
have $P\ (x \otimes y)\ (x \otimes (y \oplus z)) \wedge P\ (x \otimes z)\ (x \otimes (y \oplus z))$
proof
from *antecedent* **have** $O\ x\ y$..
thus $P\ (x \otimes y)\ (x \otimes (y \oplus z))$
by (*rule product-in-factor-by-sum*)
next
from *antecedent* **have** $O\ x\ z$..
hence $P\ (x \otimes z)\ (x \otimes (z \oplus y))$
by (*rule product-in-factor-by-sum*)
with *sum-commutativity* **show** $P\ (x \otimes z)\ (x \otimes (y \oplus z))$
by (*rule subst*)
qed
thus $P((x \otimes y) \oplus (x \otimes z))(x \otimes (y \oplus z))$
by (*rule summands-part-implies-sum-part*)
qed

theorem *product-over-sum*:
$O\ x\ y \wedge O\ x\ z \Longrightarrow x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$
proof –
assume *antecedent*: $O\ x\ y \wedge O\ x\ z$
hence $P\ (x \otimes (y \oplus z))((x \otimes y) \oplus (x \otimes z))$
by (*rule product-over-sum-left*)
moreover have $P((x \otimes y) \oplus (x \otimes z))(x \otimes (y \oplus z))$
using *antecedent* **by** (*rule product-over-sum-right*)
ultimately show $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$

by (*rule part-antisymmetry*)
qed

lemma *joint-identical-sums*:
$v \oplus w = x \oplus y \Longrightarrow O\ x\ v \wedge O\ x\ w \Longrightarrow ((x \otimes v) \oplus (x \otimes w)) = x$
proof −
assume $v \oplus w = x \oplus y$
moreover assume $O\ x\ v \wedge O\ x\ w$
hence $x \otimes (v \oplus w) = x \otimes v \oplus x \otimes w$
by (*rule product-over-sum*)
ultimately have $x \otimes (x \oplus y) = x \otimes v \oplus x \otimes w$ **by** (*rule subst*)
moreover have $(x \otimes (x \oplus y)) = x$ **using** *first-summand-in-sum*
by (*rule part-product-identity*)
ultimately show $((x \otimes v) \oplus (x \otimes w)) = x$ **by** (*rule subst*)
qed

lemma *disjoint-identical-sums*:
$v \oplus w = x \oplus y \Longrightarrow \neg\ O\ y\ v \wedge \neg\ O\ w\ x \Longrightarrow x = v \wedge y = w$
proof −
assume *identical*: $v \oplus w = x \oplus y$
assume *disjoint*: $\neg\ O\ y\ v \wedge \neg\ O\ w\ x$
show $x = v \wedge y = w$
proof
from *disjoint* **have** $\neg\ O\ y\ v$**..**
hence $(x \oplus y) \otimes (x \oplus v) = x$
by (*rule product-is-first-summand*)
with *identical* **have** $(v \oplus w) \otimes (x \oplus v) = x$
by (*rule ssubst*)
moreover from *disjoint* **have** $\neg\ O\ w\ x$**..**
hence $(v \oplus w) \otimes (v \oplus x) = v$
by (*rule product-is-first-summand*)
with *sum-commutativity* **have** $(v \oplus w) \otimes (x \oplus v) = v$
by (*rule subst*)
ultimately show $x = v$ **by** (*rule subst*)
next
from *disjoint* **have** $\neg\ O\ w\ x$**..**
hence $(y \oplus w) \otimes (y \oplus x) = y$
by (*rule product-is-first-summand*)
moreover from *disjoint* **have** $\neg\ O\ y\ v$**..**
hence $(w \oplus y) \otimes (w \oplus v) = w$
by (*rule product-is-first-summand*)
with *sum-commutativity* **have** $(w \oplus y) \otimes (v \oplus w) = w$
by (*rule subst*)
with *identical* **have** $(w \oplus y) \otimes (x \oplus y) = w$
by (*rule subst*)
with *sum-commutativity* **have** $(w \oplus y) \otimes (y \oplus x) = w$
by (*rule subst*)
with *sum-commutativity* **have** $(y \oplus w) \otimes (y \oplus x) = w$
by (*rule subst*)

```
    ultimately show y = w
      by (rule subst)
  qed
qed

end
```

7.3 Differences

```
locale CEMD = CEM + CMD
begin

lemma plus-minus: PP y x ⟹ y ⊕ (x ⊖ y) = x
proof –
  assume PP y x
  hence ∃ z. P z x ∧ ¬ O z y by (rule weak-supplementation)
  hence xmy:∀ w. P w (x ⊖ y) ⟷ (P w x ∧ ¬ O w y)
    by (rule difference-character)
  have ∀ w. O w x ⟷ (O w y ∨ O w (x ⊖ y))
  proof
    fix w
    from xmy have w: P w (x ⊖ y) ⟷ (P w x ∧ ¬ O w y)..
    show O w x ⟷ (O w y ∨ O w (x ⊖ y))
    proof
      assume O w x
      with overlap-eq have ∃ v. P v w ∧ P v x..
      then obtain v where v: P v w ∧ P v x..
      hence P v w..
      from v have P v x..
      show O w y ∨ O w (x ⊖ y)
      proof cases
        assume O v y
        hence O y v by (rule overlap-symmetry)
        with ‹P v w› have O y w by (rule overlap-monotonicity)
        hence O w y by (rule overlap-symmetry)
        thus O w y ∨ O w (x ⊖ y)..
      next
        from xmy have P v (x ⊖ y) ⟷ (P v x ∧ ¬ O v y)..
        moreover assume ¬ O v y
        with ‹P v x› have  P v x ∧ ¬ O v y..
        ultimately have P v (x ⊖ y)..
        with ‹P v w› have P v w ∧ P v (x ⊖ y)..
        hence ∃ v. P v w ∧ P v (x ⊖ y)..
        with overlap-eq have O w (x ⊖ y)..
        thus O w y ∨ O w (x ⊖ y)..
      qed
    next
      assume O w y ∨ O w (x ⊖ y)
      thus O w x
```

    proof
      from ‹$PP\ y\ x$› **have** $P\ y\ x$
        by (*rule proper-implies-part*)
      moreover assume $O\ w\ y$
      ultimately show $O\ w\ x$
        by (*rule overlap-monotonicity*)
    next
      assume $O\ w\ (x \ominus y)$
      with *overlap-eq* **have** $\exists\ v.\ P\ v\ w \land P\ v\ (x \ominus y)$..
      then obtain v **where** v: $P\ v\ w \land P\ v\ (x \ominus y)$..
      hence $P\ v\ w$..
      from *xmy* **have** $P\ v\ (x \ominus y) \longleftrightarrow (P\ v\ x \land \neg\ O\ v\ y)$..
      moreover from v **have** $P\ v\ (x \ominus y)$..
      ultimately have $P\ v\ x \land \neg\ O\ v\ y$..
      hence $P\ v\ x$..
      with ‹$P\ v\ w$› **have** $P\ v\ w \land P\ v\ x$..
      hence $\exists\ v.\ P\ v\ w \land P\ v\ x$..
      with *overlap-eq* **show** $O\ w\ x$..
    qed
  qed
 qed
 thus $y \oplus (x \ominus y) = x$
  by (*rule sum-intro*)
qed

end

7.4 The Universe

locale $CEMU = CEM + CMU$
begin

lemma *something-disjoint*: $x \neq u \Longrightarrow (\exists\ v.\ \neg\ O\ v\ x)$
proof −
 assume $x \neq u$
 with *universe-character* **have** $P\ x\ u \land x \neq u$..
 with *nip-eq* **have** $PP\ x\ u$..
 hence $\exists\ v.\ P\ v\ u \land \neg\ O\ v\ x$
  by (*rule weak-supplementation*)
 then obtain v **where** $P\ v\ u \land \neg\ O\ v\ x$..
 hence $\neg\ O\ v\ x$..
 thus $\exists\ v.\ \neg\ O\ v\ x$..
qed

lemma *overlaps-universe*: $O\ x\ u$
proof −
 from *universe-character* **have** $P\ x\ u$.
 thus $O\ x\ u$ **by** (*rule part-implies-overlap*)
qed

lemma *universe-absorbing*: $x \oplus u = u$
proof –
  from *universe-character* **have** $P\ (x \oplus u)\ u$**.**
  thus $x \oplus u = u$ **using** *second-summand-in-sum*
    by (*rule part-antisymmetry*)
qed

lemma *second-summand-not-universe*: $x \oplus y \neq u \Longrightarrow y \neq u$
proof –
  assume *antecedent*: $x \oplus y \neq u$
  show $y \neq u$
  proof
    assume $y = u$
    hence $x \oplus u \neq u$ **using** *antecedent* **by** (*rule subst*)
    thus *False* **using** *universe-absorbing***..**
  qed
qed

lemma *first-summand-not-universe*: $x \oplus y \neq u \Longrightarrow x \neq u$
proof –
  assume $x \oplus y \neq u$
  with *sum-commutativity* **have** $y \oplus x \neq u$ **by** (*rule subst*)
  thus $x \neq u$ **by** (*rule second-summand-not-universe*)
qed

end

7.5 Complements

locale *CEMC* = *CEM* + *CMC* +
  assumes *universe-eq*: $u = (THE\ x.\ \forall\ y.\ P\ y\ x)$
begin

lemma *complement-sum-character*: $\forall\ y.\ P\ y\ (x \oplus (-x))$
proof
  fix y
  have $\forall\ v.\ O\ v\ y \longrightarrow O\ v\ x \lor O\ v\ (-x)$
  proof
    fix v
    show $O\ v\ y \longrightarrow O\ v\ x \lor O\ v\ (-x)$
    proof
      assume $O\ v\ y$
      show $O\ v\ x \lor O\ v\ (-x)$
        using *or-complement-overlap***..**
    qed
  qed
  with *sum-part-character* **show** $P\ y\ (x \oplus (-x))$**..**
qed

lemma *universe-closure*: $\exists\ x.\ \forall\ y.\ P\ y\ x$
  using *complement-sum-character* **by** (*rule exI*)

end

sublocale *CEMC* $\subseteq$ *CEMU*
proof
  show $u = (THE\ z.\ \forall\ w.\ P\ w\ z)$ **using** *universe-eq***.**
  show $\exists\ x.\ \forall\ y.\ P\ y\ x$ **using** *universe-closure***.**
qed

sublocale *CEMC* $\subseteq$ *CEMD*
proof
qed

context *CEMC*
begin

corollary *universe-is-complement-sum*: $u = x \oplus (-x)$
  using *complement-sum-character* **by** (*rule universe-intro*)

lemma *strong-complement-character*:
  $x \neq u \Longrightarrow (\forall\ v.\ P\ v\ (-x) \longleftrightarrow \neg\ O\ v\ x)$
proof −
  assume $x \neq u$
  hence $\exists\ v.\ \neg\ O\ v\ x$ **by** (*rule something-disjoint*)
  thus $\forall\ v.\ P\ v\ (-x) \longleftrightarrow \neg\ O\ v\ x$ **by** (*rule complement-character*)
qed

lemma *complement-part-not-part*: $x \neq u \Longrightarrow P\ y\ (-x) \Longrightarrow \neg\ P\ y\ x$
proof −
  assume $x \neq u$
  hence $\forall\ w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
    by (*rule strong-complement-character*)
  hence *y*: $P\ y\ (-x) \longleftrightarrow \neg\ O\ y\ x$**..**
  moreover assume $P\ y\ (-x)$
  ultimately have $\neg\ O\ y\ x$**..**
  thus $\neg\ P\ y\ x$
    by (*rule disjoint-implies-not-part*)
qed

lemma *complement-involution*: $x \neq u \Longrightarrow x = -(-x)$
proof −
  assume $x \neq u$
  have $\neg\ P\ u\ x$
  proof
    assume $P\ u\ x$
    with *universe-character* **have** $x = u$

by (*rule part-antisymmetry*)
with $\langle x \neq u \rangle$ **show** *False*..
qed
hence $\exists\ v.\ P\ v\ u \wedge \neg\ O\ v\ x$
by (*rule strong-supplementation*)
then obtain v **where** v: $P\ v\ u \wedge \neg\ O\ v\ x$..
hence $\neg\ O\ v\ x$..
hence $\exists\ v.\ \neg\ O\ v\ x$..
hence *notx*: $\forall\ w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
by (*rule complement-character*)
have $-x \neq u$
proof
assume $-x = u$
hence $\forall\ w.\ P\ w\ u \longleftrightarrow \neg\ O\ w\ x$ **using** *notx* **by** (*rule subst*)
hence $P\ x\ u \longleftrightarrow \neg\ O\ x\ x$..
hence $\neg\ O\ x\ x$ **using** *universe-character*..
thus *False* **using** *overlap-reflexivity*..
qed
have $\neg\ P\ u\ (-x)$
proof
assume $P\ u\ (-x)$
with *universe-character* **have** $-x = u$
by (*rule part-antisymmetry*)
with $\langle -x \neq u \rangle$ **show** *False*..
qed
hence $\exists\ v.\ P\ v\ u \wedge \neg\ O\ v\ (-x)$
by (*rule strong-supplementation*)
then obtain w **where** w: $P\ w\ u \wedge \neg\ O\ w\ (-x)$..
hence $\neg\ O\ w\ (-x)$..
hence $\exists\ v.\ \neg\ O\ v\ (-x)$..
hence *notnotx*: $\forall\ w.\ P\ w\ (-(-x)) \longleftrightarrow \neg\ O\ w\ (-x)$
by (*rule complement-character*)
hence $P\ x\ (-(-x)) \longleftrightarrow \neg\ O\ x\ (-x)$..
moreover have $\neg\ O\ x\ (-x)$
proof
assume $O\ x\ (-x)$
with *overlap-eq* **have** $\exists\ s.\ P\ s\ x \wedge P\ s\ (-x)$..
then obtain s **where** s: $P\ s\ x \wedge P\ s\ (-x)$..
hence $P\ s\ x$..
hence $O\ s\ x$ **by** (*rule part-implies-overlap*)
from *notx* **have** $P\ s\ (-x) \longleftrightarrow \neg\ O\ s\ x$..
moreover from s **have** $P\ s\ (-x)$..
ultimately have $\neg\ O\ s\ x$..
thus *False* **using** $\langle O\ s\ x \rangle$..
qed
ultimately have $P\ x\ (-(-x))$..
moreover have $P\ (-(-x))\ x$
proof (*rule ccontr*)
assume $\neg\ P\ (-(-x))\ x$

hence $\exists\ s.\ P\ s\ (-(-x)) \wedge \neg\ O\ s\ x$
by (*rule strong-supplementation*)
then obtain s **where** s: $P\ s\ (-(-x)) \wedge \neg\ O\ s\ x$**..**
hence $\neg\ O\ s\ x$**..**
from *notnotx* **have** $P\ s\ (-(-x)) \longleftrightarrow (\neg\ O\ s\ (-x))$**..**
moreover from s **have** $P\ s\ (-(-x))$**..**
ultimately have $\neg\ O\ s\ (-x)$**..**
from *or-complement-overlap* **have** $O\ s\ x \vee O\ s\ (-x)$**..**
thus *False*
proof
assume $O\ s\ x$
with $\langle\neg\ O\ s\ x\rangle$ **show** *False***..**
next
assume $O\ s\ (-x)$
with $\langle\neg\ O\ s\ (-x\ )\rangle$ **show** *False***..**
qed
qed
ultimately show $x = -(-x)$
by (*rule part-antisymmetry*)
qed

lemma *part-complement-reversal*: $y \neq u \Longrightarrow P\ x\ y \Longrightarrow P\ (-y)\ (-x)$
proof $-$
assume $y \neq u$
hence *ny*: $\forall\ w.\ P\ w\ (-y) \longleftrightarrow \neg\ O\ w\ y$
by (*rule strong-complement-character*)
assume $P\ x\ y$
have $x \neq u$
proof
assume $x = u$
hence $P\ u\ y$ **using** $\langle P\ x\ y\rangle$ **by** (*rule subst*)
with *universe-character* **have** $y = u$
by (*rule part-antisymmetry*)
with $\langle y \neq u\rangle$ **show** *False***..**
qed
hence $\forall\ w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
by (*rule strong-complement-character*)
hence $P\ (-y)\ (-x) \longleftrightarrow \neg\ O\ (-y)\ x$**..**
moreover have $\neg\ O\ (-y)\ x$
proof
assume $O\ (-y)\ x$
with *overlap-eq* **have** $\exists\ v.\ P\ v\ (-y) \wedge P\ v\ x$**..**
then obtain v **where** v: $P\ v\ (-y) \wedge P\ v\ x$**..**
hence $P\ v\ (-y)$**..**
from *ny* **have** $P\ v\ (-y) \longleftrightarrow \neg\ O\ v\ y$**..**
hence $\neg\ O\ v\ y$ **using** $\langle P\ v\ (-y)\rangle$**..**
moreover from v **have** $P\ v\ x$**..**
hence $P\ v\ y$ **using** $\langle P\ x\ y\rangle$
by (*rule part-transitivity*)

hence $O\ v\ y$
by (*rule part-implies-overlap*)
ultimately show *False*..
qed
ultimately show $P\ (-y)\ (-x)$..
qed

lemma *complements-overlap*: $x \oplus y \neq u \Longrightarrow O(-x)(-y)$
proof –
assume $x \oplus y \neq u$
hence $\exists\ z.\ \neg\ O\ z\ (x \oplus y)$
by (*rule something-disjoint*)
then obtain z **where** z:$\neg\ O\ z\ (x \oplus y)$..
hence $\neg\ O\ z\ x$ **by** (*rule first-summand-disjointness*)
hence $P\ z\ (-x)$ **by** (*rule complement-part*)
moreover from z **have** $\neg\ O\ z\ y$
by (*rule second-summand-disjointness*)
hence $P\ z\ (-y)$ **by** (*rule complement-part*)
ultimately show $O(-x)(-y)$
by (*rule overlap-intro*)
qed

lemma *sum-complement-in-complement-product*:
$x \oplus y \neq u \Longrightarrow P(-(x \oplus y))(-x \otimes -y)$
proof –
assume $x \oplus y \neq u$
hence $O\ (-x)\ (-y)$
by (*rule complements-overlap*)
hence $\forall\ w.\ P\ w\ (-x \otimes -y) \longleftrightarrow (P\ w\ (-x) \wedge P\ w\ (-y))$
by (*rule product-character*)
hence $P(-(x \oplus y))(-x \otimes -y) \longleftrightarrow (P(-(x \oplus y))(-x) \wedge P(-(x \oplus y))(-y))$..
moreover have $P\ (-(x \oplus y))(-x) \wedge P\ (-(x \oplus y))(-y)$
proof
show $P\ (-(x \oplus y))(-x)$ **using** ‹$x \oplus y \neq u$› *first-summand-in-sum*
by (*rule part-complement-reversal*)
next
show $P\ (-(x \oplus y))(-y)$ **using** ‹$x \oplus y \neq u$› *second-summand-in-sum*
by (*rule part-complement-reversal*)
qed
ultimately show $P\ (-(x \oplus y))(-x \otimes -y)$..
qed

lemma *complement-product-in-sum-complement*:
$x \oplus y \neq u \Longrightarrow P(-x \otimes -y)(-(x \oplus y))$
proof –
assume $x \oplus y \neq u$
hence $\forall w.\ P\ w\ (-(x \oplus y)) \longleftrightarrow \neg\ O\ w\ (x \oplus y)$
by (*rule strong-complement-character*)

hence $P\ (-x \otimes -y)\ (-(x \oplus y)) \longleftrightarrow (\neg\ O\ (-x \otimes -y)\ (x \oplus y))$..
moreover have $\neg\ O\ (-x \otimes -y)\ (x \oplus y)$
proof
 have $O(-x)(-y)$ **using** ‹$x \oplus y \neq u$› **by** (*rule complements-overlap*)
  hence p: $\forall\ v.\ P\ v\ ((-x) \otimes (-y)) \longleftrightarrow (P\ v\ (-x) \wedge P\ v\ (-y))$
   by (*rule product-character*)
  have $O(-x \otimes -y)(x \oplus y) \longleftrightarrow (O(-x \otimes -y)\ x \vee O(-x \otimes -y)y)$
   using *sum-character*..
  moreover assume $O\ (-x \otimes -y)(x \oplus y)$
  ultimately have $O\ (-x \otimes -y)\ x \vee O\ (-x \otimes -y)\ y$..
  thus *False*
  proof
   assume $O\ (-x \otimes -y)\ x$
   with *overlap-eq* **have** $\exists\ v.\ P\ v\ (-x \otimes -y) \wedge P\ v\ x$..
   then obtain v **where** v: $P\ v\ (-x \otimes -y) \wedge P\ v\ x$..
   hence $P\ v\ (-x \otimes -y)$..
   from p **have** $P\ v\ ((-x) \otimes (-y)) \longleftrightarrow (P\ v\ (-x) \wedge P\ v\ (-y))$..
   hence $P\ v\ (-x) \wedge P\ v\ (-y)$ **using** ‹$P\ v\ (-x \otimes -y)$›..
   hence $P\ v\ (-x)$..
   have $x \neq u$ **using** ‹$x \oplus y \neq u$›
    by (*rule first-summand-not-universe*)
   hence $\forall w.\ P\ w\ (-x) \longleftrightarrow \neg\ O\ w\ x$
    by (*rule strong-complement-character*)
   hence $P\ v\ (-x) \longleftrightarrow \neg\ O\ v\ x$..
   hence $\neg\ O\ v\ x$ **using** ‹$P\ v\ (-x)$›..
   moreover from v **have** $P\ v\ x$..
   hence $O\ v\ x$ **by** (*rule part-implies-overlap*)
   ultimately show *False*..
  next
   assume $O\ (-x \otimes -y)\ y$
   with *overlap-eq* **have** $\exists\ v.\ P\ v\ (-x \otimes -y) \wedge P\ v\ y$..
   then obtain v **where** v: $P\ v\ (-x \otimes -y) \wedge P\ v\ y$..
   hence $P\ v\ (-x \otimes -y)$..
   from p **have** $P\ v\ ((-x) \otimes (-y)) \longleftrightarrow (P\ v\ (-x) \wedge P\ v\ (-y))$..
   hence $P\ v\ (-x) \wedge P\ v\ (-y)$ **using** ‹$P\ v\ (-x \otimes -y)$›..
   hence $P\ v\ (-y)$..
   have $y \neq u$ **using** ‹$x \oplus y \neq u$›
    by (*rule second-summand-not-universe*)
   hence $\forall w.\ P\ w\ (-y) \longleftrightarrow \neg\ O\ w\ y$
    by (*rule strong-complement-character*)
   hence $P\ v\ (-y) \longleftrightarrow \neg\ O\ v\ y$..
   hence $\neg\ O\ v\ y$ **using** ‹$P\ v\ (-y)$›..
   moreover from v **have** $P\ v\ y$..
   hence $O\ v\ y$ **by** (*rule part-implies-overlap*)
   ultimately show *False*..
  qed
 qed
 ultimately show $P\ (-x \otimes -y)\ (-(x \oplus y))$..
qed

theorem *sum-complement-is-complements-product*:
$x \oplus y \neq u \Longrightarrow -(x \oplus y) = (-x \otimes -y)$
proof –
assume $x \oplus y \neq u$
show $-(x \oplus y) = (-x \otimes -y)$
proof (*rule part-antisymmetry*)
show $P\ (-\ (x \oplus y))\ (-\ x \otimes -\ y)$ **using** $\langle x \oplus y \neq u\rangle$
by (*rule sum-complement-in-complement-product*)
show $P\ (-\ x \otimes -\ y)\ (-\ (x \oplus y))$ **using** $\langle x \oplus y \neq u\rangle$
by (*rule complement-product-in-sum-complement*)
qed
qed

lemma *complement-sum-in-product-complement*:
$O\ x\ y \Longrightarrow x \neq u \Longrightarrow y \neq u \Longrightarrow P\ ((-x) \oplus (-y))(-(x \otimes y))$
proof –
assume $O\ x\ y$
assume $x \neq u$
assume $y \neq u$
have $x \otimes y \neq u$
proof
assume $x \otimes y = u$
with $\langle O\ x\ y\rangle$ **have** $x = u$
by (*rule product-universe-implies-factor-universe*)
with $\langle x \neq u\rangle$ **show** *False*..
qed
hence *notxty*: $\forall\ w.\ P\ w\ (-(x \otimes y)) \longleftrightarrow \neg\ O\ w\ (x \otimes y)$
by (*rule strong-complement-character*)
hence $P((-x)\oplus(-y))(-(x \otimes y)) \longleftrightarrow \neg O((-x)\oplus(-y))(x \otimes y)$..
moreover have $\neg\ O\ ((-x) \oplus (-y))\ (x \otimes y)$
proof
from *sum-character* **have**
$\forall\ w.\ O\ w\ ((-x) \oplus (-y)) \longleftrightarrow (O\ w\ (-x) \vee O\ w\ (-y))$.
hence $O(x \otimes y)((-x)\oplus(-y)) \longleftrightarrow (O(x \otimes y)(-x) \vee O(x \otimes y)(-y))$..
moreover assume $O\ ((-x) \oplus (-y))\ (x \otimes y)$
hence $O\ (x \otimes y)\ ((-x) \oplus (-y))$ **by** (*rule overlap-symmetry*)
ultimately have $O\ (x \otimes y)\ (-x) \vee O\ (x \otimes y)\ (-y)$..
thus *False*
proof
assume $O\ (x \otimes y)(-x)$
with *overlap-eq* **have** $\exists\ v.\ P\ v\ (x \otimes y) \wedge P\ v\ (-x)$..
then obtain v **where** v: $P\ v\ (x \otimes y) \wedge P\ v\ (-x)$..
hence $P\ v\ (-x)$..
with $\langle x \neq u\rangle$ **have** $\neg\ P\ v\ x$
by (*rule complement-part-not-part*)
moreover from v **have** $P\ v\ (x \otimes y)$..
with $\langle O\ x\ y\rangle$ **have** $P\ v\ x$ **by** (*rule product-part-in-first-factor*)

ultimately show *False*..
next
assume $O\ (x \otimes y)\ (-y)$
with *overlap-eq* **have** $\exists\ v.\ P\ v\ (x \otimes y) \wedge P\ v\ (-y)$..
then obtain v **where** v: $P\ v\ (x \otimes y) \wedge P\ v\ (-y)$..
hence $P\ v\ (-y)$..
with $\langle y \neq u\rangle$ **have** $\neg\ P\ v\ y$
by (*rule complement-part-not-part*)
moreover from v **have** $P\ v\ (x \otimes y)$..
with $\langle O\ x\ y\rangle$ **have** $P\ v\ y$ **by** (*rule product-part-in-second-factor*)
ultimately show *False*..
qed
qed
ultimately show $P\ ((-x) \oplus (-y))(-(x \otimes y))$..
qed

lemma *product-complement-in-complements-sum*:
$x \neq u \Longrightarrow y \neq u \Longrightarrow P(-(x \otimes y))((-x) \oplus (-y))$
proof $-$
assume $x \neq u$
hence $x = -(-x)$
by (*rule complement-involution*)
assume $y \neq u$
hence $y = -(-y)$
by (*rule complement-involution*)
show $P\ (-(x \otimes y))((-x) \oplus (-y))$
proof *cases*
assume $-x \oplus -y = u$
thus $P\ (-(x \otimes y))((-x) \oplus (-y))$
using *universe-character* **by** (*rule ssubst*)
next
assume $-x \oplus -y \neq u$
hence $-x \oplus -y = -(-(-x \oplus -\ y))$
by (*rule complement-involution*)
moreover have $-(-x \oplus -y) = -(-x) \otimes -(-y)$
using $\langle -x \oplus -y \neq u\rangle$
by (*rule sum-complement-is-complements-product*)
with $\langle x = -(-x)\rangle$ **have** $-(-x \oplus -y) = x \otimes -(-y)$
by (*rule ssubst*)
with $\langle y = -(-y)\rangle$ **have** $-(-x \oplus -y) = x \otimes y$
by (*rule ssubst*)
hence $P\ (-(x \otimes y))(-(-(-x \oplus -y)))$
using *part-reflexivity* **by** (*rule subst*)
ultimately show $P\ (-(x \otimes y))(-x \oplus -y)$
by (*rule ssubst*)
qed
qed

theorem *complement-of-product-is-sum-of-complements*:

$O\ x\ y \Longrightarrow x \oplus y \neq u \Longrightarrow -(x \otimes y) = (-x) \oplus (-y)$
proof –
  assume $O\ x\ y$
  assume $x \oplus y \neq u$
  show $-(x \otimes y) = (-x) \oplus (-y)$
  proof (*rule part-antisymmetry*)
    have $x \neq u$ **using** ‹$x \oplus y \neq u$›
      by (*rule first-summand-not-universe*)
    have $y \neq u$ **using** ‹$x \oplus y \neq u$›
      by (*rule second-summand-not-universe*)
    show $P\ (-\ (x \otimes y))\ (-\ x \oplus -\ y)$
    using ‹$x \neq u$› ‹$y \neq u$› **by** (*rule product-complement-in-complements-sum*)
    show $P\ (-\ x \oplus -\ y)\ (-\ (x \otimes y))$
    using ‹$O\ x\ y$› ‹$x \neq u$› ‹$y \neq u$› **by** (*rule complement-sum-in-product-complement*)
  qed
qed

end

8 General Mereology

The theory of *general mereology* adds the axiom of fusion to ground mereology.[31]

locale $GM = M +$
  assumes *fusion*:
    $\exists\ x.\ \varphi\ x \Longrightarrow \exists\ z.\ \forall\ y.\ O\ y\ z \longleftrightarrow (\exists\ x.\ \varphi\ x \wedge O\ y\ x)$
begin

Fusion entails sum closure.

theorem *sum-closure*: $\exists\ z.\ \forall\ w.\ O\ w\ z \longleftrightarrow (O\ w\ a \vee O\ w\ b)$
proof –
  have $a = a$..
  hence $a = a \vee a = b$..
  hence $\exists\ x.\ x = a \vee x = b$..
  hence $(\exists\ z.\ \forall\ y.\ O\ y\ z \longleftrightarrow (\exists\ x.\ (x = a \vee x = b) \wedge O\ y\ x))$
    by (*rule fusion*)
  then obtain z **where** z:
    $\forall\ y.\ O\ y\ z \longleftrightarrow (\exists\ x.\ (x = a \vee x = b) \wedge O\ y\ x)$..
  have $\forall\ w.\ O\ w\ z \longleftrightarrow (O\ w\ a \vee O\ w\ b)$
  proof
    fix w
    from z **have** w: $O\ w\ z \longleftrightarrow (\exists\ x.\ (x = a \vee x = b) \wedge O\ w\ x)$..
    show $O\ w\ z \longleftrightarrow (O\ w\ a \vee O\ w\ b)$
    proof
      assume $O\ w\ z$

[31] See [Simons, 1987] p. 36, [Varzi, 1996] p. 265 and [Casati and Varzi, 1999] p. 46.

```
        with w have ∃ x. (x = a ∨ x = b) ∧ O w x..
        then obtain x where x: (x = a ∨ x = b) ∧ O w x..
        hence O w x..
        from x have x = a ∨ x = b..
        thus O w a ∨ O w b
        proof (rule disjE)
          assume x = a
          hence O w a using ‹O w x› by (rule subst)
          thus O w a ∨ O w b..
        next
          assume x = b
          hence O w b using ‹O w x› by (rule subst)
          thus O w a ∨ O w b..
        qed
      next
        assume O w a ∨ O w b
        hence ∃ x. (x = a ∨ x = b) ∧ O w x
        proof (rule disjE)
          assume O w a
          with ‹a = a ∨ a = b› have (a = a ∨ a = b) ∧ O w a..
          thus ∃ x. (x = a ∨ x = b) ∧ O w x..
        next
          have b = b..
          hence b = a ∨ b = b..
          moreover assume O w b
          ultimately have (b = a ∨ b = b) ∧ O w b..
          thus ∃ x. (x = a ∨ x = b) ∧ O w x..
        qed
        with w show O w z..
      qed
    qed
    thus ∃ z. ∀ w. O w z ⟷ (O w a ∨ O w b)..
  qed

end
```

9 General Minimal Mereology

The theory of *general minimal mereology* adds general mereology to minimal mereology.[32]

locale *GMM* = *GM* + *MM*
begin

It is natural to assume that just as closed minimal mereology and closed extensional mereology are the same theory, so are general

[32]See [Casati and Varzi, 1999] p. 46.

minimal mereology and general extensional mereology.[33] But this is not the case, since the proof of strong supplementation in closed minimal mereology required the product closure axiom. However, in general minimal mereology, the fusion axiom does not entail the product closure axiom. So neither product closure nor strong supplementation are theorems.

lemma *product-closure*:
  $O\ x\ y \Longrightarrow (\exists\ z.\ \forall\ v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
  nitpick [*expect* = *genuine*] **oops**

lemma *strong-supplementation*: $\neg\ P\ x\ y \Longrightarrow (\exists\ z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
  nitpick [*expect* = *genuine*] **oops**

end

10 General Extensional Mereology

The theory of *general extensional mereology*, also known as *classical extensional mereology* adds general mereology to extensional mereology.[34]

locale *GEM* = *GM* + *EM* +
  assumes *sum-eq*: $x \oplus y = (THE\ z.\ \forall v.\ O\ v\ z \longleftrightarrow O\ v\ x \vee O\ v\ y)$
  assumes *product-eq*:
    $x \otimes y = (THE\ z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
  assumes *difference-eq*:
    $x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
  assumes *complement-eq*: $-\ x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
  assumes *universe-eq*: $u = (THE\ x.\ \forall y.\ P\ y\ x)$
  assumes *fusion-eq*: $\exists x.\ F\ x \Longrightarrow$
    $(\sigma\ x.\ F\ x) = (THE\ x.\ \forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
  assumes *general-product-eq*: $(\pi\ x.\ F\ x) = (\sigma\ x.\ \forall y.\ F\ y \longrightarrow P\ x\ y)$

sublocale $GEM \subseteq GMM$
proof
qed

10.1 General Sums

context *GEM*
begin

[33] For this mistake see [Simons, 1987] p. 37 and [Casati and Varzi, 1999] p. 46. The mistake is corrected in [Pontow, 2004] and [Hovda, 2009]. For discussion of the significance of this issue see, for example, [Varzi, 2009] and [Cotnoir, 2016].

[34] For this axiomatization see [Varzi, 1996] p. 265 and [Casati and Varzi, 1999] p. 46.

lemma *fusion-intro*:
$(\forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)) \Longrightarrow (\sigma\ x.\ F\ x) = z$
proof −
  assume *antecedent*: $(\forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x))$
  hence $(\mathit{THE}\ x.\ \forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z)) = z$
  proof (*rule the-equality*)
    fix a
    assume a: $(\forall y.\ O\ y\ a \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x))$
    have $\forall x.\ O\ x\ a \longleftrightarrow O\ x\ z$
    proof
      fix b
      from *antecedent* **have** $O\ b\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ b\ x)$**..**
      moreover from a **have** $O\ b\ a \longleftrightarrow (\exists x.\ F\ x \wedge O\ b\ x)$**..**
      ultimately show $O\ b\ a \longleftrightarrow O\ b\ z$ **by** (*rule ssubst*)
    qed
    with *overlap-extensionality* **show** $a = z$**..**
  qed
  moreover from *antecedent* **have** $O\ z\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ z\ x)$**..**
  hence $\exists x.\ F\ x \wedge O\ z\ x$ **using** *overlap-reflexivity***..**
  hence $\exists x.\ F\ x$ **by** *auto*
  hence $(\sigma\ x.\ F\ x) = (\mathit{THE}\ x.\ \forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
    by (*rule fusion-eq*)
  ultimately show $(\sigma\ v.\ F\ v) = z$ **by** (*rule subst*)
qed

lemma *fusion-idempotence*: $(\sigma\ x.\ z = x) = z$
proof −
  have $\forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ z = x \wedge O\ y\ x)$
  proof
    fix y
    show $O\ y\ z \longleftrightarrow (\exists x.\ z = x \wedge O\ y\ x)$
    proof
      assume $O\ y\ z$
      with *refl* **have** $z = z \wedge O\ y\ z$**..**
      thus $\exists x.\ z = x \wedge O\ y\ x$**..**
    next
      assume $\exists x.\ z = x \wedge O\ y\ x$
      then obtain x **where** x: $z = x \wedge O\ y\ x$**..**
      hence $z = x$**..**
      moreover from x **have** $O\ y\ x$**..**
      ultimately show $O\ y\ z$ **by** (*rule ssubst*)
    qed
  qed
  thus $(\sigma\ x.\ z = x) = z$
    by (*rule fusion-intro*)
qed

The whole is the sum of its parts.

lemma *fusion-absorption*: $(\sigma\ x.\ P\ x\ z) = z$

proof –
have $(\forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ P\ x\ z \wedge O\ y\ x))$
proof
fix y
show $O\ y\ z \longleftrightarrow (\exists x.\ P\ x\ z \wedge O\ y\ x)$
proof
assume $O\ y\ z$
with *part-reflexivity* **have** $P\ z\ z \wedge O\ y\ z$**..**
thus $\exists x.\ P\ x\ z \wedge O\ y\ x$**..**
next
assume $\exists x.\ P\ x\ z \wedge O\ y\ x$
then obtain x **where** x: $P\ x\ z \wedge O\ y\ x$**..**
hence $P\ x\ z$**..**
moreover from x **have** $O\ y\ x$**..**
ultimately show $O\ y\ z$ **by** (*rule overlap-monotonicity*)
qed
qed
thus $(\sigma\ x.\ P\ x\ z) = z$
by (*rule fusion-intro*)
qed

lemma *part-fusion*: $P\ w\ (\sigma\ v.\ P\ v\ x) \Longrightarrow P\ w\ x$
proof –
assume $P\ w\ (\sigma\ v.\ P\ v\ x)$
with *fusion-absorption* **show** $P\ w\ x$ **by** (*rule subst*)
qed

lemma *fusion-character*:
$\exists x.\ F\ x \Longrightarrow (\forall y.\ O\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x))$
proof –
assume $\exists x.\ F\ x$
hence $\exists z.\ \forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)$
by (*rule fusion*)
then obtain z **where** z: $\forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)$**..**
hence $(\sigma\ v.\ F\ v) = z$ **by** (*rule fusion-intro*)
thus $\forall y.\ O\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)$ **using** z **by** (*rule ssubst*)
qed

The next lemma characterises fusions in terms of parthood.[35]

lemma *fusion-part-character*: $\exists x.\ F\ x \Longrightarrow$
$(\forall y.\ P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v)))$
proof –
assume $(\exists x.\ F\ x)$
hence F: $\forall y.\ O\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\exists x.\ F\ x \wedge O\ y\ x)$
by (*rule fusion-character*)
show $\forall y.\ P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge O\ w\ v))$
proof

[35] See [Pontow, 2004] pp. 202-9.

```
    fix y
    show P y (σ v. F v) ⟷ (∀ w. P w y ⟶ (∃ v. F v ∧ O w v))
    proof
      assume P y (σ v. F v)
      show ∀ w. P w y ⟶ (∃ v. F v ∧ O w v)
      proof
        fix w
        from F have w: O w (σ v. F v) ⟷ (∃ x. F x ∧ O w x)..
        show P w y ⟶ (∃ v. F v ∧ O w v)
        proof
          assume P w y
          hence P w (σ v. F v) using ‹P y (σ v. F v)›
            by (rule part-transitivity)
          hence O w (σ v. F v) by (rule part-implies-overlap)
          with w show ∃ x. F x ∧ O w x..
        qed
      qed
    next
      assume right: ∀ w. P w y ⟶ (∃ v. F v ∧ O w v)
      show P y (σ v. F v)
      proof (rule ccontr)
        assume ¬ P y (σ v. F v)
        hence ∃ v. P v y ∧ ¬ O v (σ v. F v)
          by (rule strong-supplementation)
        then obtain v where v: P v y ∧ ¬ O v (σ v. F v)..
        hence ¬ O v (σ v. F v)..
        from right have P v y ⟶ (∃ w. F w ∧ O v w)..
        moreover from v have P v y..
        ultimately have ∃ w. F w ∧ O v w..
        from F have O v (σ v. F v) ⟷ (∃ x. F x ∧ O v x)..
        hence O v (σ v. F v) using ‹∃ w. F w ∧ O v w›..
        with ‹¬ O v (σ v. F v)› show False..
      qed
    qed
  qed
qed

lemma fusion-part: F x ⟹ P x (σ x. F x)
proof –
  assume F x
  hence ∃ x. F x..
  hence ∀ y. P y (σ v. F v) ⟷ (∀ w. P w y ⟶ (∃ v. F v ∧ O w v))
    by (rule fusion-part-character)
  hence P x (σ v. F v) ⟷ (∀ w. P w x ⟶ (∃ v. F v ∧ O w v))..
  moreover have ∀ w. P w x ⟶ (∃ v. F v ∧ O w v)
  proof
    fix w
    show P w x ⟶ (∃ v. F v ∧ O w v)
    proof
```

```
      assume P w x
      hence O w x by (rule part-implies-overlap)
      with ‹F x› have F x ∧ O w x..
      thus ∃ v. F v ∧ O w v..
    qed
  qed
  ultimately show P x (σ v. F v)..
qed

lemma common-part-fusion:
  O x y ⟹ (∀ w. P w (σ v. (P v x ∧ P v y)) ⟷ (P w x ∧ P w y))
proof −
  assume O x y
  with overlap-eq have ∃ z. (P z x ∧ P z y)..
  hence sum: (∀ w. P w (σ v. (P v x ∧ P v y)) ⟷
    (∀ z. P z w ⟶ (∃ v. (P v x ∧ P v y) ∧ O z v)))
    by (rule fusion-part-character)
  show ∀ w. P w (σ v. (P v x ∧ P v y)) ⟷ (P w x ∧ P w y)
  proof
    fix w
    from sum have w: P w (σ v. (P v x ∧ P v y))
      ⟷ (∀ z. P z w ⟶ (∃ v. (P v x ∧ P v y) ∧ O z v))..
    show P w (σ v. (P v x ∧ P v y)) ⟷ (P w x ∧ P w y)
    proof
      assume P w (σ v. (P v x ∧ P v y))
      with w have bla:
        (∀ z. P z w ⟶ (∃ v. (P v x ∧ P v y) ∧ O z v))..
      show P w x ∧ P w y
      proof
        show P w x
        proof (rule ccontr)
          assume ¬ P w x
          hence ∃ z. P z w ∧ ¬ O z x
            by (rule strong-supplementation)
          then obtain z where z: P z w ∧ ¬ O z x..
          hence ¬ O z x..
          from bla have P z w ⟶ (∃ v. (P v x ∧ P v y) ∧ O z v)..
          moreover from z have P z w..
          ultimately have ∃ v. (P v x ∧ P v y) ∧ O z v..
          then obtain v where v: (P v x ∧ P v y) ∧ O z v..
          hence P v x ∧ P v y..
          hence P v x..
          moreover from v have O z v..
          ultimately have O z x
            by (rule overlap-monotonicity)
          with ‹¬ O z x› show False..
        qed
        show P w y
        proof (rule ccontr)
```

```
          assume ¬ P w y
          hence ∃ z. P z w ∧ ¬ O z y
            by (rule strong-supplementation)
          then obtain z where z: P z w ∧ ¬ O z y..
          hence ¬ O z y..
          from bla have P z w ⟶ (∃ v. (P v x ∧ P v y) ∧ O z v)..
          moreover from z have P z w..
          ultimately have ∃ v. (P v x ∧ P v y) ∧ O z v..
          then obtain v where v: (P v x ∧ P v y) ∧ O z v..
          hence P v x ∧ P v y..
          hence P v y..
          moreover from v have O z v..
          ultimately have O z y
            by (rule overlap-monotonicity)
          with ‹¬ O z y› show False..
        qed
      qed
    next
      assume P w x ∧ P w y
      thus P w (σ v. (P v x ∧ P v y))
        by (rule fusion-part)
    qed
  qed
qed

theorem product-closure:
  O x y ⟹ (∃ z. ∀ w. P w z ⟷ (P w x ∧ P w y))
proof −
  assume O x y
  hence (∀ w. P w (σ v. (P v x ∧ P v y)) ⟷ (P w x ∧ P w y))
    by (rule common-part-fusion)
  thus ∃ z. ∀ w. P w z ⟷ (P w x ∧ P w y)..
qed

end

sublocale GEM ⊆ CEM
proof
  fix x y
  show ∃ z. ∀ w. O w z = (O w x ∨ O w y)
    using sum-closure.
  show x ⊕ y = (THE z. ∀ v. O v z ⟷ O v x ∨ O v y)
    using sum-eq.
  show x ⊗ y = (THE z. ∀ v. P v z ⟷ P v x ∧ P v y)
    using product-eq.
  show O x y ⟹ (∃ z. ∀ w. P w z = (P w x ∧ P w y))
    using product-closure.
qed
```

```
context GEM
begin

corollary O x y ⟹ x ⊗ y = (σ v. P v x ∧ P v y)
proof −
  assume O x y
  hence (∀ w. P w (σ v. (P v x ∧ P v y)) ⟷ (P w x ∧ P w y))
    by (rule common-part-fusion)
  thus x ⊗ y = (σ v. P v x ∧ P v y) by (rule product-intro)
qed

lemma disjoint-fusion:
  ∃ w. ¬ O w x ⟹ (∀ w. P w (σ z. ¬ O z x) ⟷ ¬ O w x)
proof −
  assume antecedent: ∃ w. ¬ O w x
  hence ∀ y. O y (σ v. ¬ O v x) ⟷ (∃ v. ¬ O v x ∧ O y v)
    by (rule fusion-character)
  hence x: O x (σ v. ¬ O v x) ⟷ (∃ v. ¬ O v x ∧ O x v)..
  show ∀ w. P w (σ z. ¬ O z x) ⟷ ¬ O w x
  proof
    fix y
    show P y (σ z. ¬ O z x) ⟷ ¬ O y x
    proof
      assume P y (σ z. ¬ O z x)
      moreover have ¬ O x (σ z. ¬ O z x)
      proof
        assume O x (σ z. ¬ O z x)
        with x have (∃ v. ¬ O v x ∧ O x v)..
        then obtain v where v: ¬ O v x ∧ O x v..
        hence ¬ O v x..
        from v have O x v..
        hence O v x by (rule overlap-symmetry)
        with ‹¬ O v x› show False..
      qed
      ultimately have ¬ O x y
        by (rule disjoint-demonotonicity)
      thus ¬ O y x by (rule disjoint-symmetry)
    next
      assume ¬ O y x
      thus P y (σ v. ¬ O v x)
        by (rule fusion-part)
    qed
  qed
qed

theorem complement-closure:
  ∃ w. ¬ O w x ⟹ (∃ z. ∀ w. P w z ⟷ ¬ O w x)
proof −
  assume (∃ w. ¬ O w x)
```

  hence $\forall w.\ P\ w\ (\sigma\ z.\ \neg\ O\ z\ x) \longleftrightarrow \neg\ O\ w\ x$
    by (*rule disjoint-fusion*)
  thus $\exists z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x$..
qed

end

sublocale *GEM* $\subseteq$ *CEMC*
proof
  fix $x\ y$
  show $-\ x = (\mathit{THE}\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
    using *complement-eq*.
  show $(\exists w.\ \neg\ O\ w\ x) \Longrightarrow (\exists z.\ \forall w.\ P\ w\ z = (\neg\ O\ w\ x))$
    using *complement-closure*.
  show $x \ominus y = (\mathit{THE}\ z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
    using *difference-eq*.
  show $u = (\mathit{THE}\ x.\ \forall y.\ P\ y\ x)$
    using *universe-eq*.
qed

context *GEM*
begin

corollary *complement-is-disjoint-fusion*:
  $\exists w.\ \neg\ O\ w\ x \Longrightarrow -\ x = (\sigma\ z.\ \neg\ O\ z\ x)$
proof –
  assume $\exists w.\ \neg\ O\ w\ x$
  hence $\forall w.\ P\ w\ (\sigma\ z.\ \neg\ O\ z\ x) \longleftrightarrow \neg\ O\ w\ x$
    by (*rule disjoint-fusion*)
  thus $-\ x = (\sigma\ z.\ \neg\ O\ z\ x)$
    by (*rule complement-intro*)
qed

theorem *strong-fusion*: $\exists x.\ F\ x \Longrightarrow$
  $\exists x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
proof –
  assume $\exists x.\ F\ x$
  have $(\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ v.\ F\ v)) \wedge$
     $(\forall y.\ P\ y\ (\sigma\ v.\ F\ v) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
  proof
    show $\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ v.\ F\ v)$
    proof
      fix y
      show $F\ y \longrightarrow P\ y\ (\sigma\ v.\ F\ v)$
      proof
        assume $F\ y$
        thus $P\ y\ (\sigma\ v.\ F\ v)$
          by (*rule fusion-part*)
      qed

qed
next
have $(\forall y.\ P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow$
$(\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge\ O\ w\ v)))$
using $\langle\exists x.\ F\ x\rangle$ **by** (*rule fusion-part-character*)
hence $P\ (\sigma\ v.\ F\ v)\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall w.\ P\ w\ (\sigma\ v.\ F\ v) \longrightarrow$
$(\exists v.\ F\ v \wedge\ O\ w\ v))$**..**
thus $\forall w.\ P\ w\ (\sigma\ v.\ F\ v) \longrightarrow (\exists v.\ F\ v \wedge\ O\ w\ v)$ **using** *part-reflexivity***..**
qed
thus *?thesis***..**
qed

theorem *strong-fusion-eq*: $\exists x.\ F\ x \Longrightarrow (\sigma\ x.\ F\ x) =$
$(\mathit{THE}\ x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge\ O\ y\ z)))$
proof −
assume $\exists x.\ F\ x$
have $(\mathit{THE}\ x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge\ O\ y\ z))) = (\sigma\ x.\ F\ x)$
proof (*rule the-equality*)
show $(\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)) \wedge (\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge\ O\ y\ z))$
proof
show $\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)$
proof
fix y
show $F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)$
proof
assume $F\ y$
thus $P\ y\ (\sigma\ x.\ F\ x)$
by (*rule fusion-part*)
qed
qed
next
show $(\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge\ O\ y\ z))$
proof
fix y
show $P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge\ O\ y\ z)$
proof
have $\forall y.\ P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge\ O\ w\ v))$
using $\langle\exists x.\ F\ x\rangle$ **by** (*rule fusion-part-character*)
hence $P\ y\ (\sigma\ v.\ F\ v) \longleftrightarrow (\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge\ O\ w\ v))$**..**
moreover assume $P\ y\ (\sigma\ x.\ F\ x)$
ultimately have $\forall w.\ P\ w\ y \longrightarrow (\exists v.\ F\ v \wedge\ O\ w\ v)$**..**
hence $P\ y\ y \longrightarrow (\exists v.\ F\ v \wedge\ O\ y\ v)$**..**
thus $\exists v.\ F\ v \wedge\ O\ y\ v$ **using** *part-reflexivity***..**
qed

qed
qed
next
fix x
assume x: $(\forall\, y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall\, y.\ P\ y\ x \longrightarrow (\exists\, z.\ F\ z \wedge O\ y$
$z))$
have $\forall\, y.\ O\ y\ x \longleftrightarrow (\exists\, z.\ F\ z \wedge O\ y\ z)$
proof
fix y
show $O\ y\ x \longleftrightarrow (\exists\, z.\ F\ z \wedge O\ y\ z)$
proof
assume $O\ y\ x$
with *overlap-eq* **have** $\exists\, v.\ P\ v\ y \wedge P\ v\ x$..
then obtain v **where** v: $P\ v\ y \wedge P\ v\ x$..
from x **have** $\forall\, y.\ P\ y\ x \longrightarrow (\exists\, z.\ F\ z \wedge O\ y\ z)$..
hence $P\ v\ x \longrightarrow (\exists\, z.\ F\ z \wedge O\ v\ z)$..
moreover from v **have** $P\ v\ x$..
ultimately have $\exists\, z.\ F\ z \wedge O\ v\ z$..
then obtain z **where** z: $F\ z \wedge O\ v\ z$..
hence $F\ z$..
from v **have** $P\ v\ y$..
moreover from z **have** $O\ v\ z$..
hence $O\ z\ v$ **by** (*rule overlap-symmetry*)
ultimately have $O\ z\ y$ **by** (*rule overlap-monotonicity*)
hence $O\ y\ z$ **by** (*rule overlap-symmetry*)
with $\langle F\ z\rangle$ **have** $F\ z \wedge O\ y\ z$..
thus $\exists\, z.\ F\ z \wedge O\ y\ z$..
next
assume $\exists\, z.\ F\ z \wedge O\ y\ z$
then obtain z **where** z: $F\ z \wedge O\ y\ z$..
from x **have** $\forall\, y.\ F\ y \longrightarrow P\ y\ x$..
hence $F\ z \longrightarrow P\ z\ x$..
moreover from z **have** $F\ z$..
ultimately have $P\ z\ x$..
moreover from z **have** $O\ y\ z$..
ultimately show $O\ y\ x$
by (*rule overlap-monotonicity*)
qed
qed
hence $(\sigma\ x.\ F\ x) = x$
by (*rule fusion-intro*)
thus $x = (\sigma\ x.\ F\ x)$..
qed
thus *?thesis*..
qed

lemma *strong-sum-eq*: $x \oplus y = (THE\ z.\ (P\ x\ z \wedge P\ y\ z) \wedge (\forall\, w.\ P\ w$
$z \longrightarrow O\ w\ x \vee O\ w\ y))$
proof −

have $(THE\ z.\ (P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y))$
$= x \oplus y$
proof (*rule the-equality*)
show $(P\ x\ (x \oplus y) \wedge P\ y\ (x \oplus y)) \wedge (\forall w.\ P\ w\ (x \oplus y) \longrightarrow O\ w$
$x \vee O\ w\ y)$
proof
show $P\ x\ (x \oplus y) \wedge P\ y\ (x \oplus y)$
proof
show $P\ x\ (x \oplus y)$ **using** *first-summand-in-sum*.
show $P\ y\ (x \oplus y)$ **using** *second-summand-in-sum*.
qed
show $\forall w.\ P\ w\ (x \oplus y) \longrightarrow O\ w\ x \vee O\ w\ y$
proof
fix w
show $P\ w\ (x \oplus y) \longrightarrow O\ w\ x \vee O\ w\ y$
proof
assume $P\ w\ (x \oplus y)$
hence $O\ w\ (x \oplus y)$ **by** (*rule part-implies-overlap*)
with *sum-overlap* **show** $O\ w\ x \vee O\ w\ y$..
qed
qed
qed
fix z
assume z: $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)$
hence $P\ x\ z \wedge P\ y\ z$..
have $\forall w.\ O\ w\ z \longleftrightarrow (O\ w\ x \vee O\ w\ y)$
proof
fix w
show $O\ w\ z \longleftrightarrow (O\ w\ x \vee O\ w\ y)$
proof
assume $O\ w\ z$
with *overlap-eq* **have** $\exists v.\ P\ v\ w \wedge P\ v\ z$..
then obtain v **where** v: $P\ v\ w \wedge P\ v\ z$..
hence $P\ v\ w$..
from z **have** $\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y$..
hence $P\ v\ z \longrightarrow O\ v\ x \vee O\ v\ y$..
moreover from v **have** $P\ v\ z$..
ultimately have $O\ v\ x \vee O\ v\ y$..
thus $O\ w\ x \vee O\ w\ y$
proof
assume $O\ v\ x$
hence $O\ x\ v$ **by** (*rule overlap-symmetry*)
with $\langle P\ v\ w \rangle$ **have** $O\ x\ w$ **by** (*rule overlap-monotonicity*)
hence $O\ w\ x$ **by** (*rule overlap-symmetry*)
thus $O\ w\ x \vee O\ w\ y$..
next
assume $O\ v\ y$
hence $O\ y\ v$ **by** (*rule overlap-symmetry*)
with $\langle P\ v\ w \rangle$ **have** $O\ y\ w$ **by** (*rule overlap-monotonicity*)

hence $O\ w\ y$ **by** (*rule overlap-symmetry*)
thus $O\ w\ x \lor O\ w\ y$..
qed
next
assume $O\ w\ x \lor O\ w\ y$
thus $O\ w\ z$
proof
from ‹$P\ x\ z \land P\ y\ z$› **have** $P\ x\ z$..
moreover assume $O\ w\ x$
ultimately show $O\ w\ z$
by (*rule overlap-monotonicity*)
next
from ‹$P\ x\ z \land P\ y\ z$› **have** $P\ y\ z$..
moreover assume $O\ w\ y$
ultimately show $O\ w\ z$
by (*rule overlap-monotonicity*)
qed
qed
qed
hence $x \oplus y = z$ **by** (*rule sum-intro*)
thus $z = x \oplus y$..
qed
thus *?thesis*..
qed

10.2 General Products

lemma *general-product-intro*: $(\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \land O\ y\ z)) \Longrightarrow (\pi\ x.\ F\ x) = x$
proof −
assume $\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ F\ y \longrightarrow P\ z\ y) \land O\ y\ z)$
hence $(\sigma\ x.\ \forall y.\ F\ y \longrightarrow P\ x\ y) = x$ **by** (*rule fusion-intro*)
with *general-product-eq* **show** $(\pi\ x.\ F\ x) = x$ **by** (*rule ssubst*)
qed

lemma *general-product-idempotence*: $(\pi\ z.\ z = x) = x$
proof −
have $\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ y = x \longrightarrow P\ z\ y) \land O\ y\ z)$
by (*meson overlap-eq part-reflexivity part-transitivity*)
thus $(\pi\ z.\ z = x) = x$ **by** (*rule general-product-intro*)
qed

lemma *general-product-absorption*: $(\pi\ z.\ P\ x\ z) = x$
proof −
have $\forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ (\forall y.\ P\ x\ y \longrightarrow P\ z\ y) \land O\ y\ z)$
by (*meson overlap-eq part-reflexivity part-transitivity*)
thus $(\pi\ z.\ P\ x\ z) = x$ **by** (*rule general-product-intro*)
qed

lemma *general-product-character*: $\exists z. \forall y. F\ y \longrightarrow P\ z\ y \Longrightarrow$
$\forall y. O\ y\ (\pi\ x. F\ x) \longleftrightarrow (\exists z. (\forall y. F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z)$
proof −
assume $(\exists z. \forall y. F\ y \longrightarrow P\ z\ y)$
hence $(\exists x. \forall y. O\ y\ x \longleftrightarrow (\exists z. (\forall y. F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z))$
by (*rule fusion*)
then obtain x **where** x:
$\forall y. O\ y\ x \longleftrightarrow (\exists z. (\forall y. F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z)$**..**
hence $(\pi\ x. F\ x) = x$ **by** (*rule general-product-intro*)
thus $(\forall y. O\ y\ (\pi\ x. F\ x) \longleftrightarrow (\exists z. (\forall y. F\ y \longrightarrow P\ z\ y) \wedge O\ y\ z))$
using x **by** (*rule ssubst*)
qed

corollary $\neg (\exists x. F\ x) \Longrightarrow u = (\pi\ x. F\ x)$
proof −
assume *antecedent*: $\neg (\exists x. F\ x)$
have $\forall y. P\ y\ (\pi\ x. F\ x)$
proof
fix y
show $P\ y\ (\pi\ x. F\ x)$
proof (*rule ccontr*)
assume $\neg\ P\ y\ (\pi\ x. F\ x)$
hence $\exists z. P\ z\ y \wedge \neg\ O\ z\ (\pi\ x. F\ x)$ **by** (*rule strong-supplementation*)
then obtain z **where** z: $P\ z\ y \wedge \neg\ O\ z\ (\pi\ x. F\ x)$**..**
hence $\neg\ O\ z\ (\pi\ x. F\ x)$**..**
from *antecedent* **have** *bla*: $\forall\ y. F\ y \longrightarrow P\ z\ y$ **by** *simp*
hence $\exists\ v. \forall\ y. F\ y \longrightarrow P\ v\ y$**..**
hence $(\forall y. O\ y\ (\pi\ x. F\ x) \longleftrightarrow (\exists z. (\forall y. F\ y \longrightarrow P\ z\ y) \wedge O\ y$
$z))$ **by** (*rule general-product-character*)
hence $O\ z\ (\pi\ x. F\ x) \longleftrightarrow (\exists v. (\forall y. F\ y \longrightarrow P\ v\ y) \wedge O\ z\ v)$**..**
moreover from *bla* **have** $(\forall\ y. F\ y \longrightarrow P\ z\ y) \wedge O\ z\ z$
using *overlap-reflexivity***..**
hence $\exists\ v. (\forall\ y. F\ y \longrightarrow P\ v\ y) \wedge O\ z\ v$**..**
ultimately have $O\ z\ (\pi\ x. F\ x)$**..**
with $\langle\neg\ O\ z\ (\pi\ x. F\ x)\rangle$ **show** *False***..**
qed
qed
thus $u = (\pi\ x. F\ x)$
by (*rule universe-intro*)
qed

end

10.3 Strong Fusion

An alternative axiomatization of general extensional mereology adds a stronger version of the fusion axiom to minimal mereology, with correspondingly stronger definitions of sums and general sums.[36]

locale *GEM1* = *MM* +
assumes *strong-fusion*: $\exists x.\ F\ x \Longrightarrow \exists x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
assumes *strong-sum-eq*: $x \oplus y = (THE\ z.\ (P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y))$
assumes *product-eq*:
$x \otimes y = (THE\ z.\ \forall v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge P\ v\ y)$
assumes *difference-eq*:
$x \ominus y = (THE\ z.\ \forall w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
assumes *complement-eq*: $-\ x = (THE\ z.\ \forall w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
assumes *universe-eq*: $u = (THE\ x.\ \forall y.\ P\ y\ x)$
assumes *strong-fusion-eq*: $\exists x.\ F\ x \Longrightarrow (\sigma\ x.\ F\ x) = (THE\ x.\ (\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$
assumes *general-product-eq*: $(\pi\ x.\ F\ x) = (\sigma\ x.\ \forall y.\ F\ y \longrightarrow P\ x\ y)$
begin

theorem *fusion*:
$\exists x.\ \varphi\ x \Longrightarrow (\exists z.\ \forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ \varphi\ x \wedge O\ y\ x))$
proof –
assume $\exists x.\ \varphi\ x$
hence $\exists x.\ (\forall y.\ \varphi\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ \varphi\ z \wedge O\ y\ z))$ **by** (*rule strong-fusion*)
then obtain x **where** x:
$(\forall y.\ \varphi\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ \varphi\ z \wedge O\ y\ z))$**..**
have $\forall y.\ O\ y\ x \longleftrightarrow (\exists v.\ \varphi\ v \wedge O\ y\ v)$
proof
fix y
show $O\ y\ x \longleftrightarrow (\exists v.\ \varphi\ v \wedge O\ y\ v)$
proof
assume $O\ y\ x$
with *overlap-eq* **have** $\exists z.\ P\ z\ y \wedge P\ z\ x$**..**
then obtain z **where** z: $P\ z\ y \wedge P\ z\ x$**..**
hence $P\ z\ x$**..**
from x **have** $\forall y.\ P\ y\ x \longrightarrow (\exists v.\ \varphi\ v \wedge O\ y\ v)$**..**
hence $P\ z\ x \longrightarrow (\exists v.\ \varphi\ v \wedge O\ z\ v)$**..**
hence $\exists v.\ \varphi\ v \wedge O\ z\ v$ **using** ‹$P\ z\ x$›**..**
then obtain v **where** v: $\varphi\ v \wedge O\ z\ v$**..**
hence $O\ z\ v$**..**
with *overlap-eq* **have** $\exists w.\ P\ w\ z \wedge P\ w\ v$**..**
then obtain w **where** w: $P\ w\ z \wedge P\ w\ v$**..**
hence $P\ w\ z$**..**
moreover from z **have** $P\ z\ y$**..**
ultimately have $P\ w\ y$
by (*rule part-transitivity*)
moreover from w **have** $P\ w\ v$**..**
ultimately have $P\ w\ y \wedge P\ w\ v$**..**

[36] See [Tarski, 1983] p. 25. The proofs in this section are adapted from [Hovda, 2009].

hence $\exists w.\ P\ w\ y \wedge P\ w\ v$**..**
with *overlap-eq* **have** $O\ y\ v$**..**
from v **have** $\varphi\ v$**..**
hence $\varphi\ v \wedge O\ y\ v$ **using** $\langle O\ y\ v\rangle$**..**
thus $\exists v.\ \varphi\ v \wedge O\ y\ v$**..**
next
assume $\exists v.\ \varphi\ v \wedge O\ y\ v$
then obtain v **where** v: $\varphi\ v \wedge O\ y\ v$**..**
hence $O\ y\ v$**..**
with *overlap-eq* **have** $\exists z.\ P\ z\ y \wedge P\ z\ v$**..**
then obtain z **where** z: $P\ z\ y \wedge P\ z\ v$**..**
hence $P\ z\ v$**..**
from x **have** $\forall y.\ \varphi\ y \longrightarrow P\ y\ x$**..**
hence $\varphi\ v \longrightarrow P\ v\ x$**..**
moreover from v **have** $\varphi\ v$**..**
ultimately have $P\ v\ x$**..**
with $\langle P\ z\ v\rangle$ **have** $P\ z\ x$
by (*rule part-transitivity*)
from z **have** $P\ z\ y$**..**
thus $O\ y\ x$ **using** $\langle P\ z\ x\rangle$
by (*rule overlap-intro*)
qed
qed
thus $(\exists z.\ \forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ \varphi\ x \wedge O\ y\ x))$**..**
qed

lemma *pair*: $\exists v.\ (\forall w.\ (w = x \vee w = y) \longrightarrow P\ w\ v) \wedge (\forall w.\ P\ w\ v \longrightarrow (\exists z.\ (z = x \vee z = y) \wedge O\ w\ z))$
proof −
have $x = x$**..**
hence $x = x \vee x = y$**..**
hence $\exists v.\ v = x \vee v = y$**..**
thus *?thesis*
by (*rule strong-fusion*)
qed

lemma *or-id*: $(v = x \vee v = y) \wedge O\ w\ v \Longrightarrow O\ w\ x \vee O\ w\ y$
proof −
assume v: $(v = x \vee v = y) \wedge O\ w\ v$
hence $O\ w\ v$**..**
from v **have** $v = x \vee v = y$**..**
thus $O\ w\ x \vee O\ w\ y$
proof
assume $v = x$
hence $O\ w\ x$ **using** $\langle O\ w\ v\rangle$ **by** (*rule subst*)
thus $O\ w\ x \vee O\ w\ y$**..**
next
assume $v = y$
hence $O\ w\ y$ **using** $\langle O\ w\ v\rangle$ **by** (*rule subst*)

thus $O\ w\ x \lor O\ w\ y$**..**
qed
qed

lemma *strong-sum-closure*:
$\exists z.\ (P\ x\ z \land P\ y\ z) \land (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \lor O\ w\ y)$
proof –
from *pair* **obtain** z **where** z: $(\forall w.\ (w = x \lor w = y) \longrightarrow P\ w\ z) \land (\forall w.\ P\ w\ z \longrightarrow (\exists v.\ (v = x \lor v = y) \land O\ w\ v))$**..**
have $(P\ x\ z \land P\ y\ z) \land (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \lor O\ w\ y)$
proof
from z **have** *allw*: $\forall w.\ (w = x \lor w = y) \longrightarrow P\ w\ z$**..**
hence $x = x \lor x = y \longrightarrow P\ x\ z$**..**
moreover have $x = x \lor x = y$ **using** *refl***..**
ultimately have $P\ x\ z$**..**
from *allw* **have** $y = x \lor y = y \longrightarrow P\ y\ z$**..**
moreover have $y = x \lor y = y$ **using** *refl***..**
ultimately have $P\ y\ z$**..**
with ‹$P\ x\ z$› **show** $P\ x\ z \land P\ y\ z$**..**
next
show $\forall w.\ P\ w\ z \longrightarrow O\ w\ x \lor O\ w\ y$
proof
fix w
show $P\ w\ z \longrightarrow O\ w\ x \lor O\ w\ y$
proof
assume $P\ w\ z$
from z **have** $\forall w.\ P\ w\ z \longrightarrow (\exists v.\ (v = x \lor v = y) \land O\ w\ v)$**..**
hence $P\ w\ z \longrightarrow (\exists v.\ (v = x \lor v = y) \land O\ w\ v)$**..**
hence $\exists v.\ (v = x \lor v = y) \land O\ w\ v$ **using** ‹$P\ w\ z$›**..**
then obtain v **where** v: $(v = x \lor v = y) \land O\ w\ v$**..**
thus $O\ w\ x \lor O\ w\ y$ **by** (*rule or-id*)
qed
qed
qed
thus *?thesis***..**
qed

end

sublocale *GEM1* $\subseteq$ *GMM*
proof
fix $x\ y\ \varphi$
show $(\exists x.\ \varphi\ x) \Longrightarrow (\exists z.\ \forall y.\ O\ y\ z \longleftrightarrow (\exists x.\ \varphi\ x \land O\ y\ x))$ **using** *fusion***.**
qed

context *GEM1*
begin

lemma *least-upper-bound*:
assumes *sf*:
$((\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$
shows *lub*:
$(\forall y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall z.\ (\forall y.\ F\ y \longrightarrow P\ y\ z) \longrightarrow P\ x\ z)$
proof
from *sf* **show** $\forall y.\ F\ y \longrightarrow P\ y\ x$..
next
show $(\forall z.\ (\forall y.\ F\ y \longrightarrow P\ y\ z) \longrightarrow P\ x\ z)$
proof
fix z
show $(\forall y.\ F\ y \longrightarrow P\ y\ z) \longrightarrow P\ x\ z$
proof
assume z: $\forall y.\ F\ y \longrightarrow P\ y\ z$
from *pair* **obtain** v **where** v: $(\forall w.\ (w = x \vee w = z) \longrightarrow P\ w\ v)$
$\wedge (\forall w.\ P\ w\ v \longrightarrow (\exists y.\ (y = x \vee y = z) \wedge O\ w\ y))$..
hence *left*: $(\forall w.\ (w = x \vee w = z) \longrightarrow P\ w\ v)$..
hence $(x = x \vee x = z) \longrightarrow P\ x\ v$..
moreover have $x = x \vee x = z$ **using** *refl*..
ultimately have $P\ x\ v$..
have $z = v$
proof (*rule ccontr*)
assume $z \neq v$
from *left* **have** $z = x \vee z = z \longrightarrow P\ z\ v$..
moreover have $z = x \vee z = z$ **using** *refl*..
ultimately have $P\ z\ v$..
hence $P\ z\ v \wedge z \neq v$ **using** ‹$z \neq v$›..
with *nip-eq* **have** $PP\ z\ v$..
hence $\exists w.\ P\ w\ v \wedge \neg\ O\ w\ z$ **by** (*rule weak-supplementation*)
then obtain w **where** w: $P\ w\ v \wedge \neg\ O\ w\ z$..
hence $P\ w\ v$..
from v **have** *right*:
$\forall w.\ P\ w\ v \longrightarrow (\exists y.\ (y = x \vee y = z) \wedge O\ w\ y)$..
hence $P\ w\ v \longrightarrow (\exists y.\ (y = x \vee y = z) \wedge O\ w\ y)$..
hence $\exists y.\ (y = x \vee y = z) \wedge O\ w\ y$ **using** ‹$P\ w\ v$›..
then obtain s **where** s: $(s = x \vee s = z) \wedge O\ w\ s$..
hence $s = x \vee s = z$..
thus *False*
proof
assume $s = x$
moreover from s **have** $O\ w\ s$..
ultimately have $O\ w\ x$ **by** (*rule subst*)
with *overlap-eq* **have** $\exists t.\ P\ t\ w \wedge P\ t\ x$..
then obtain t **where** t: $P\ t\ w \wedge P\ t\ x$..
hence $P\ t\ x$..
from *sf* **have** $(\forall y.\ P\ y\ x \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$..
hence $P\ t\ x \longrightarrow (\exists z.\ F\ z \wedge O\ t\ z)$..
hence $\exists z.\ F\ z \wedge O\ t\ z$ **using** ‹$P\ t\ x$›..
then obtain a **where** a: $F\ a \wedge O\ t\ a$..

hence $F\ a$**..**
from *sf* **have** *ub*: $\forall\, y.\ F\ y \longrightarrow P\ y\ x$**..**
hence $F\ a \longrightarrow P\ a\ x$**..**
hence $P\ a\ x$ **using** $\langle F\ a \rangle$**..**
moreover from a **have** $O\ t\ a$**..**
ultimately have $O\ t\ x$
by (*rule overlap-monotonicity*)
from t **have** $P\ t\ w$**..**
moreover have $O\ z\ t$
proof −
from z **have** $F\ a \longrightarrow P\ a\ z$**..**
moreover from a **have** $F\ a$**..**
ultimately have $P\ a\ z$**..**
moreover from a **have** $O\ t\ a$**..**
ultimately have $O\ t\ z$
by (*rule overlap-monotonicity*)
thus $O\ z\ t$ **by** (*rule overlap-symmetry*)
qed
ultimately have $O\ z\ w$
by (*rule overlap-monotonicity*)
hence $O\ w\ z$ **by** (*rule overlap-symmetry*)
from w **have** $\neg\ O\ w\ z$**..**
thus *False* **using** $\langle O\ w\ z \rangle$**..**
next
assume $s = z$
moreover from s **have** $O\ w\ s$**..**
ultimately have $O\ w\ z$ **by** (*rule subst*)
from w **have** $\neg\ O\ w\ z$**..**
thus *False* **using** $\langle O\ w\ z \rangle$**..**
qed
qed
thus $P\ x\ z$ **using** $\langle P\ x\ v \rangle$ **by** (*rule ssubst*)
qed
qed
qed

corollary *strong-fusion-intro*: $(\forall\, y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall\, y.\ P\ y\ x \longrightarrow (\exists\, z.\ F\ z \wedge O\ y\ z)) \Longrightarrow (\sigma\ x.\ F\ x) = x$
proof −
assume *antecedent*: $(\forall\, y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall\, y.\ P\ y\ x \longrightarrow (\exists\, z.\ F\ z \wedge O\ y\ z))$
with *least-upper-bound* **have** *lubx*:
$(\forall\, y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall\, z.\ (\forall\, y.\ F\ y \longrightarrow P\ y\ z) \longrightarrow P\ x\ z)$**.**
from *antecedent* **have** $\forall\, y.\ P\ y\ x \longrightarrow (\exists\, z.\ F\ z \wedge O\ y\ z)$**..**
hence $P\ x\ x \longrightarrow (\exists\, z.\ F\ z \wedge O\ x\ z)$**..**
hence $\exists\, z.\ F\ z \wedge O\ x\ z$ **using** *part-reflexivity***..**
then obtain z **where** z: $F\ z \wedge O\ x\ z$**..**
hence $F\ z$**..**
hence $\exists\, z.\ F\ z$**..**

hence (σ x. F x) = (*THE* x. ($\forall$ y. F y $\longrightarrow$ P y x) $\wedge$ ($\forall$ y. P y x $\longrightarrow$ ($\exists$ z. F z $\wedge$ O y z))) **by** (*rule strong-fusion-eq*)
moreover have (*THE* x. ($\forall$ y. F y $\longrightarrow$ P y x) $\wedge$ ($\forall$ y. P y x $\longrightarrow$ ($\exists$ z. F z $\wedge$ O y z))) = x
proof (*rule the-equality*)
show ($\forall$ y. F y $\longrightarrow$ P y x) $\wedge$ ($\forall$ y. P y x $\longrightarrow$ ($\exists$ z. F z $\wedge$ O y z))
using *antecedent*.
next
fix w
assume w:
($\forall$ y. F y $\longrightarrow$ P y w) $\wedge$ ($\forall$ y. P y w $\longrightarrow$ ($\exists$ z. F z $\wedge$ O y z))
with *least-upper-bound* **have** *lubw*:
($\forall$ y. F y $\longrightarrow$ P y w) $\wedge$ ($\forall$ z. ($\forall$ y. F y $\longrightarrow$ P y z) $\longrightarrow$ P w z).
hence ($\forall$ z. ($\forall$ y. F y $\longrightarrow$ P y z) $\longrightarrow$ P w z)..
hence ($\forall$ y. F y $\longrightarrow$ P y x) $\longrightarrow$ P w x..
moreover from *antecedent* **have** $\forall$ y. F y $\longrightarrow$ P y x..
ultimately have P w x..
from *lubx* **have** ($\forall$ z. ($\forall$ y. F y $\longrightarrow$ P y z) $\longrightarrow$ P x z)..
hence ($\forall$ y. F y $\longrightarrow$ P y w) $\longrightarrow$ P x w..
moreover from *lubw* **have** ($\forall$ y. F y $\longrightarrow$ P y w)..
ultimately have P x w..
with ‹P w x› **show** w = x
by (*rule part-antisymmetry*)
qed
ultimately show (σ x. F x) = x **by** (*rule ssubst*)
qed

lemma *strong-fusion-character*: $\exists$ x. F x $\Longrightarrow$ (($\forall$ y. F y $\longrightarrow$ P y (σ x. F x)) $\wedge$ ($\forall$ y. P y (σ x. F x) $\longrightarrow$ ($\exists$ z. F z $\wedge$ O y z)))
proof –
assume $\exists$ x. F x
hence ($\exists$ x. ($\forall$ y. F y $\longrightarrow$ P y x) $\wedge$ ($\forall$ y. P y x $\longrightarrow$ ($\exists$ z. F z $\wedge$ O y z))) **by** (*rule strong-fusion*)
then obtain x **where** x:
($\forall$ y. F y $\longrightarrow$ P y x) $\wedge$ ($\forall$ y. P y x $\longrightarrow$ ($\exists$ z. F z $\wedge$ O y z))..
hence (σ x. F x) = x **by** (*rule strong-fusion-intro*)
thus *?thesis* **using** x **by** (*rule ssubst*)
qed

lemma *F-in*: $\exists$ x. F x $\Longrightarrow$ ($\forall$ y. F y $\longrightarrow$ P y (σ x. F x))
proof –
assume $\exists$ x. F x
hence (($\forall$ y. F y $\longrightarrow$ P y (σ x. F x)) $\wedge$ ($\forall$ y. P y (σ x. F x) $\longrightarrow$ ($\exists$ z. F z $\wedge$ O y z))) **by** (*rule strong-fusion-character*)
thus $\forall$ y. F y $\longrightarrow$ P y (σ x. F x)..
qed

lemma *parts-overlap-Fs*:
$\exists$ x. F x $\Longrightarrow$ ($\forall$ y. P y (σ x. F x) $\longrightarrow$ ($\exists$ z. F z $\wedge$ O y z))

proof −
  assume $\exists x.\ F\ x$
  hence $((\forall y.\ F\ y \longrightarrow P\ y\ (\sigma\ x.\ F\ x)) \wedge (\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)))$ **by** (*rule strong-fusion-character*)
  thus $(\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$..
qed

lemma *in-strong-fusion*: $P\ z\ (\sigma\ x.\ z = x)$
proof −
  have $\exists y.\ z = y$ **using** *refl*..
  hence $\forall y.\ z = y \longrightarrow P\ y\ (\sigma\ x.\ z = x)$
    by (*rule F-in*)
  hence $z = z \longrightarrow P\ z\ (\sigma\ x.\ z = x)$..
  thus $P\ z\ (\sigma\ x.\ z = x)$ **using** *refl*..
qed

lemma *strong-fusion-in*: $P\ (\sigma\ x.\ z = x)\ z$
proof −
  have $\exists y.\ z = y$ **using** *refl*..
  hence *sf*:
    $(\forall y.\ z = y \longrightarrow P\ y\ (\sigma\ x.\ z = x)) \wedge (\forall y.\ P\ y\ (\sigma\ x.\ z = x) \longrightarrow (\exists v.\ z = v \wedge O\ y\ v))$
    by (*rule strong-fusion-character*)
  with *least-upper-bound* **have** *lub*: $(\forall y.\ z = y \longrightarrow P\ y\ (\sigma\ x.\ z = x)) \wedge (\forall v.\ (\forall y.\ z = y \longrightarrow P\ y\ v) \longrightarrow P\ (\sigma\ x.\ z = x)\ v)$.
  hence $(\forall v.\ (\forall y.\ z = y \longrightarrow P\ y\ v) \longrightarrow P\ (\sigma\ x.\ z = x)\ v)$..
  hence $(\forall y.\ z = y \longrightarrow P\ y\ z) \longrightarrow P\ (\sigma\ x.\ z = x)\ z$..
  moreover have $(\forall y.\ z = y \longrightarrow P\ y\ z)$
  proof
    fix y
    show $z = y \longrightarrow P\ y\ z$
    proof
      assume $z = y$
      thus $P\ y\ z$ **using** *part-reflexivity* **by** (*rule subst*)
    qed
  qed
  ultimately show $P\ (\sigma\ x.\ z = x)\ z$..
qed

lemma *strong-fusion-idempotence*: $(\sigma\ x.\ z = x) = z$
 using *strong-fusion-in in-strong-fusion* **by** (*rule part-antisymmetry*)

10.4 Strong Sums

lemma *pair-fusion*: $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y) \longrightarrow (\sigma\ z.\ z = x \vee z = y) = z$
proof
 assume *z*: $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)$
  have $(\forall v.\ v = x \vee v = y \longrightarrow P\ v\ z) \wedge (\forall v.\ P\ v\ z \longrightarrow (\exists z.\ (z = x$

$\vee\ z = y) \wedge O\ v\ z))$
proof
show $\forall v.\ v = x \vee v = y \longrightarrow P\ v\ z$
proof
fix w
from z **have** $P\ x\ z \wedge P\ y\ z$**..**
show $w = x \vee w = y \longrightarrow P\ w\ z$
proof
assume $w = x \vee w = y$
thus $P\ w\ z$
proof
assume $w = x$
moreover from $\langle P\ x\ z \wedge P\ y\ z\rangle$ **have** $P\ x\ z$**..**
ultimately show $P\ w\ z$ **by** (*rule ssubst*)
next
assume $w = y$
moreover from $\langle P\ x\ z \wedge P\ y\ z\rangle$ **have** $P\ y\ z$**..**
ultimately show $P\ w\ z$ **by** (*rule ssubst*)
qed
qed
qed
show $\forall v.\ P\ v\ z \longrightarrow (\exists z.\ (z = x \vee z = y) \wedge O\ v\ z)$
proof
fix v
show $P\ v\ z \longrightarrow (\exists z.\ (z = x \vee z = y) \wedge O\ v\ z)$
proof
assume $P\ v\ z$
from z **have** $\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y$**..**
hence $P\ v\ z \longrightarrow O\ v\ x \vee O\ v\ y$**..**
hence $O\ v\ x \vee O\ v\ y$ **using** $\langle P\ v\ z\rangle$**..**
thus $\exists z.\ (z = x \vee z = y) \wedge O\ v\ z$
proof
assume $O\ v\ x$
have $x = x \vee x = y$ **using** *refl***..**
hence $(x = x \vee x = y) \wedge O\ v\ x$ **using** $\langle O\ v\ x\rangle$**..**
thus $\exists z.\ (z = x \vee z = y) \wedge O\ v\ z$**..**
next
assume $O\ v\ y$
have $y = x \vee y = y$ **using** *refl***..**
hence $(y = x \vee y = y) \wedge O\ v\ y$ **using** $\langle O\ v\ y\rangle$**..**
thus $\exists z.\ (z = x \vee z = y) \wedge O\ v\ z$**..**
qed
qed
qed
qed
thus $(\sigma\ z.\ z = x \vee z = y) = z$
by (*rule strong-fusion-intro*)
qed

corollary *strong-sum-fusion*: $x \oplus y = (\sigma\ z.\ z = x \vee z = y)$
proof –
have $(THE\ z.\ (P\ x\ z \wedge P\ y\ z) \wedge$
$(\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)) = (\sigma\ z.\ z = x \vee z = y)$
proof (*rule the-equality*)
have $x = x \vee x = y$ **using** *refl*..
hence *exz*: $\exists z.\ z = x \vee z = y$..
hence *allw*: $(\forall w.\ w = x \vee w = y \longrightarrow P\ w\ (\sigma\ z.\ z = x \vee z = y))$
by (*rule F-in*)
show $(P\ x\ (\sigma\ z.\ z = x \vee z = y) \wedge P\ y\ (\sigma\ z.\ z = x \vee z = y)) \wedge$
$(\forall w.\ P\ w\ (\sigma\ z.\ z = x \vee z = y) \longrightarrow O\ w\ x \vee O\ w\ y)$
proof
show $(P\ x\ (\sigma\ z.\ z = x \vee z = y) \wedge P\ y\ (\sigma\ z.\ z = x \vee z = y))$
proof
from *allw* **have** $x = x \vee x = y \longrightarrow P\ x\ (\sigma\ z.\ z = x \vee z = y)$..
thus $P\ x\ (\sigma\ z.\ z = x \vee z = y)$
using $\langle x = x \vee x = y \rangle$..
next
from *allw* **have** $y = x \vee y = y \longrightarrow P\ y\ (\sigma\ z.\ z = x \vee z = y)$..
moreover have $y = x \vee y = y$
using *refl*..
ultimately show $P\ y\ (\sigma\ z.\ z = x \vee z = y)$..
qed
next
show $\forall w.\ P\ w\ (\sigma\ z.\ z = x \vee z = y) \longrightarrow O\ w\ x \vee O\ w\ y$
proof
fix w
show $P\ w\ (\sigma\ z.\ z = x \vee z = y) \longrightarrow O\ w\ x \vee O\ w\ y$
proof
have $\forall v.\ P\ v\ (\sigma\ z.\ z = x \vee z = y) \longrightarrow (\exists z.\ (z = x \vee z = y)$
$\wedge\ O\ v\ z)$ **using** *exz* **by** (*rule parts-overlap-Fs*)
hence $P\ w\ (\sigma\ z.\ z = x \vee z = y) \longrightarrow (\exists z.\ (z = x \vee z = y) \wedge$
$O\ w\ z)$..
moreover assume $P\ w\ (\sigma\ z.\ z = x \vee z = y)$
ultimately have $(\exists z.\ (z = x \vee z = y) \wedge O\ w\ z)$..
then obtain z **where** z: $(z = x \vee z = y) \wedge O\ w\ z$..
thus $O\ w\ x \vee O\ w\ y$ **by** (*rule or-id*)
qed
qed
qed
next
fix z
assume z: $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)$
with *pair-fusion* **have** $(\sigma\ z.\ z = x \vee z = y) = z$..
thus $z = (\sigma\ z.\ z = x \vee z = y)$..
qed
with *strong-sum-eq* **show** $x \oplus y = (\sigma\ z.\ z = x \vee z = y)$
by (*rule ssubst*)
qed

corollary *strong-sum-intro*:
  $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y) \longrightarrow x \oplus y = z$
proof
  assume z: $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)$
  with *pair-fusion* **have** $(\sigma\ z.\ z = x \vee z = y) = z$**..**
  with *strong-sum-fusion* **show** $(x \oplus y) = z$
    by (*rule ssubst*)
qed

corollary *strong-sum-character*: $(P\ x\ (x \oplus y) \wedge P\ y\ (x \oplus y)) \wedge (\forall w.$
$P\ w\ (x \oplus y) \longrightarrow O\ w\ x \vee O\ w\ y)$
proof −
  from *strong-sum-closure* **obtain** z **where** z:
    $(P\ x\ z \wedge P\ y\ z) \wedge (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \vee O\ w\ y)$**..**
 with *strong-sum-intro* **have** $x \oplus y = z$**..**
 thus *?thesis* **using** z **by** (*rule ssubst*)
qed

corollary *summands-in*: $(P\ x\ (x \oplus y) \wedge P\ y\ (x \oplus y))$
  using *strong-sum-character***..**

corollary *first-summand-in*: $P\ x\ (x \oplus y)$ **using** *summands-in***..**

corollary *second-summand-in*: $P\ y\ (x \oplus y)$ **using** *summands-in***..**

corollary *sum-part-overlap*: $(\forall w.\ P\ w\ (x \oplus y) \longrightarrow O\ w\ x \vee O\ w\ y)$
using *strong-sum-character***..**

lemma *strong-sum-absorption*: $y = (x \oplus y) \Longrightarrow P\ x\ y$
proof −
 assume $y = (x \oplus y)$
 thus $P\ x\ y$ **using** *first-summand-in* **by** (*rule ssubst*)
qed

theorem *strong-supplementation*: $\neg\ P\ x\ y \Longrightarrow (\exists z.\ P\ z\ x \wedge \neg\ O\ z\ y)$
proof −
 assume $\neg\ P\ x\ y$
 have $\neg\ (\forall z.\ P\ z\ x \longrightarrow O\ z\ y)$
 proof
  assume z: $\forall z.\ P\ z\ x \longrightarrow O\ z\ y$
  have $(\forall v.\ y = v \longrightarrow P\ v\ (x \oplus y)) \wedge$
    $(\forall v.\ P\ v\ (x \oplus y) \longrightarrow (\exists z.\ y = z \wedge O\ v\ z))$
  proof
   show $\forall v.\ y = v \longrightarrow P\ v\ (x \oplus y)$
   proof
    fix v
    show $y = v \longrightarrow P\ v\ (x \oplus y)$
    proof

assume $y = v$
thus $P\ v\ (x \oplus y)$
using *second-summand-in* **by** (*rule subst*)
qed
qed
show $\forall v.\ P\ v\ (x \oplus y) \longrightarrow (\exists z.\ y = z \land O\ v\ z)$
proof
fix v
show $P\ v\ (x \oplus y) \longrightarrow (\exists z.\ y = z \land O\ v\ z)$
proof
assume $P\ v\ (x \oplus y)$
moreover from *sum-part-overlap* **have**
$P\ v\ (x \oplus y) \longrightarrow O\ v\ x \lor O\ v\ y$**..**
ultimately have $O\ v\ x \lor O\ v\ y$ **by** (*rule rev-mp*)
hence $O\ v\ y$
proof
assume $O\ v\ x$
with *overlap-eq* **have** $\exists w.\ P\ w\ v \land P\ w\ x$**..**
then obtain w **where** w: $P\ w\ v \land P\ w\ x$**..**
from z **have** $P\ w\ x \longrightarrow O\ w\ y$**..**
moreover from w **have** $P\ w\ x$**..**
ultimately have $O\ w\ y$**..**
with *overlap-eq* **have** $\exists t.\ P\ t\ w \land P\ t\ y$**..**
then obtain t **where** t: $P\ t\ w \land P\ t\ y$**..**
hence $P\ t\ w$**..**
moreover from w **have** $P\ w\ v$**..**
ultimately have $P\ t\ v$
by (*rule part-transitivity*)
moreover from t **have** $P\ t\ y$**..**
ultimately show $O\ v\ y$
by (*rule overlap-intro*)
next
assume $O\ v\ y$
thus $O\ v\ y$**.**
qed
with *refl* **have** $y = y \land O\ v\ y$**..**
thus $\exists z.\ y = z \land O\ v\ z$**..**
qed
qed
qed
hence $(\sigma\ z.\ y = z) = (x \oplus y)$ **by** (*rule strong-fusion-intro*)
with *strong-fusion-idempotence* **have** $y = x \oplus y$ **by** (*rule subst*)
hence $P\ x\ y$ **by** (*rule strong-sum-absorption*)
with $\langle \neg\ P\ x\ y \rangle$ **show** *False***..**
qed
thus $\exists z.\ P\ z\ x \land \neg\ O\ z\ y$ **by** *simp*
qed

lemma *sum-character*: $\forall v.\ O\ v\ (x \oplus y) \longleftrightarrow (O\ v\ x \lor O\ v\ y)$

proof
  fix v
  show $O\ v\ (x \oplus y) \longleftrightarrow (O\ v\ x \vee O\ v\ y)$
  proof
    assume $O\ v\ (x \oplus y)$
    with *overlap-eq* **have** $\exists w.\ P\ w\ v \wedge P\ w\ (x \oplus y)$..
    then obtain w **where** w: $P\ w\ v \wedge P\ w\ (x \oplus y)$..
    hence $P\ w\ v$..
    have $P\ w\ (x \oplus y) \longrightarrow O\ w\ x \vee O\ w\ y$ **using** *sum-part-overlap*..
    moreover from w **have** $P\ w\ (x \oplus y)$..
    ultimately have $O\ w\ x \vee O\ w\ y$..
    thus $O\ v\ x \vee O\ v\ y$
    proof
      assume $O\ w\ x$
      hence $O\ x\ w$
        by (*rule overlap-symmetry*)
      with ‹$P\ w\ v$› **have** $O\ x\ v$
        by (*rule overlap-monotonicity*)
      hence $O\ v\ x$
        by (*rule overlap-symmetry*)
      thus $O\ v\ x \vee O\ v\ y$..
    next
      assume $O\ w\ y$
      hence $O\ y\ w$
        by (*rule overlap-symmetry*)
      with ‹$P\ w\ v$› **have** $O\ y\ v$
        by (*rule overlap-monotonicity*)
      hence $O\ v\ y$ **by** (*rule overlap-symmetry*)
      thus $O\ v\ x \vee O\ v\ y$..
    qed
  next
    assume $O\ v\ x \vee O\ v\ y$
    thus $O\ v\ (x \oplus y)$
    proof
      assume $O\ v\ x$
      with *overlap-eq* **have** $\exists w.\ P\ w\ v \wedge P\ w\ x$..
      then obtain w **where** w: $P\ w\ v \wedge P\ w\ x$..
      hence $P\ w\ v$..
      moreover from w **have** $P\ w\ x$..
      hence $P\ w\ (x \oplus y)$ **using** *first-summand-in*
        by (*rule part-transitivity*)
      ultimately show $O\ v\ (x \oplus y)$
        by (*rule overlap-intro*)
    next
      assume $O\ v\ y$
      with *overlap-eq* **have** $\exists w.\ P\ w\ v \wedge P\ w\ y$..
      then obtain w **where** w: $P\ w\ v \wedge P\ w\ y$..
      hence $P\ w\ v$..
      moreover from w **have** $P\ w\ y$..

hence $P\ w\ (x \oplus y)$ **using** *second-summand-in*
by (*rule part-transitivity*)
ultimately show $O\ v\ (x \oplus y)$
by (*rule overlap-intro*)
qed
qed
qed

lemma *sum-eq*: $x \oplus y = (THE\ z.\ \forall v.\ O\ v\ z = (O\ v\ x \lor O\ v\ y))$
proof –
have $(THE\ z.\ \forall v.\ O\ v\ z \longleftrightarrow (O\ v\ x \lor O\ v\ y)) = x \oplus y$
proof (*rule the-equality*)
show $\forall v.\ O\ v\ (x \oplus y) \longleftrightarrow (O\ v\ x \lor O\ v\ y)$ **using** *sum-character*.
next
fix z
assume z: $\forall v.\ O\ v\ z \longleftrightarrow (O\ v\ x \lor O\ v\ y)$
have $(P\ x\ z \land P\ y\ z) \land (\forall w.\ P\ w\ z \longrightarrow O\ w\ x \lor O\ w\ y)$
proof
show $P\ x\ z \land P\ y\ z$
proof
show $P\ x\ z$
proof (*rule ccontr*)
assume $\neg\ P\ x\ z$
hence $\exists v.\ P\ v\ x \land \neg\ O\ v\ z$
by (*rule strong-supplementation*)
then obtain v **where** v: $P\ v\ x \land \neg\ O\ v\ z$..
hence $\neg\ O\ v\ z$..
from z **have** $O\ v\ z \longleftrightarrow (O\ v\ x \lor O\ v\ y)$..
moreover from v **have** $P\ v\ x$..
hence $O\ v\ x$ **by** (*rule part-implies-overlap*)
hence $O\ v\ x \lor O\ v\ y$..
ultimately have $O\ v\ z$..
with ‹$\neg\ O\ v\ z$› **show** *False*..
qed
next
show $P\ y\ z$
proof (*rule ccontr*)
assume $\neg\ P\ y\ z$
hence $\exists v.\ P\ v\ y \land \neg\ O\ v\ z$
by (*rule strong-supplementation*)
then obtain v **where** v: $P\ v\ y \land \neg\ O\ v\ z$..
hence $\neg\ O\ v\ z$..
from z **have** $O\ v\ z \longleftrightarrow (O\ v\ x \lor O\ v\ y)$..
moreover from v **have** $P\ v\ y$..
hence $O\ v\ y$ **by** (*rule part-implies-overlap*)
hence $O\ v\ x \lor O\ v\ y$..
ultimately have $O\ v\ z$..
with ‹$\neg\ O\ v\ z$› **show** *False*..
qed

qed
show $\forall w.\ P\ w\ z \longrightarrow (O\ w\ x \vee O\ w\ y)$
proof
fix w
show $P\ w\ z \longrightarrow (O\ w\ x \vee O\ w\ y)$
proof
from z **have** $O\ w\ z \longleftrightarrow O\ w\ x \vee O\ w\ y$**..**
moreover assume $P\ w\ z$
hence $O\ w\ z$ **by** (*rule part-implies-overlap*)
ultimately show $O\ w\ x \vee O\ w\ y$**..**
qed
qed
qed
with *strong-sum-intro* **have** $x \oplus y = z$**..**
thus $z = x \oplus y$**..**
qed
thus *?thesis***..**
qed

theorem *fusion-eq*: $\exists x.\ F\ x \Longrightarrow$
$(\sigma\ x.\ F\ x) = (THE\ x.\ \forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z))$
proof –
assume $\exists x.\ F\ x$
hence *bla*: $\forall y.\ P\ y\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ y\ z)$
by (*rule parts-overlap-Fs*)
have $(THE\ x.\ \forall y.\ O\ y\ x \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z)) = (\sigma\ x.\ F\ x)$
proof (*rule the-equality*)
show $\forall y.\ O\ y\ (\sigma\ x.\ F\ x) \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z)$
proof
fix y
show $O\ y\ (\sigma\ x.\ F\ x) \longleftrightarrow (\exists z.\ F\ z \wedge O\ y\ z)$
proof
assume $O\ y\ (\sigma\ x.\ F\ x)$
with *overlap-eq* **have** $\exists v.\ P\ v\ y \wedge P\ v\ (\sigma\ x.\ F\ x)$**..**
then obtain v **where** v: $P\ v\ y \wedge P\ v\ (\sigma\ x.\ F\ x)$**..**
hence $P\ v\ y$**..**
from *bla* **have** $P\ v\ (\sigma\ x.\ F\ x) \longrightarrow (\exists z.\ F\ z \wedge O\ v\ z)$**..**
moreover from v **have** $P\ v\ (\sigma\ x.\ F\ x)$**..**
ultimately have $(\exists z.\ F\ z \wedge O\ v\ z)$**..**
then obtain z **where** z: $F\ z \wedge O\ v\ z$**..**
hence $F\ z$**..**
moreover from z **have** $O\ v\ z$**..**
hence $O\ z\ v$ **by** (*rule overlap-symmetry*)
with $\langle P\ v\ y \rangle$ **have** $O\ z\ y$ **by** (*rule overlap-monotonicity*)
hence $O\ y\ z$ **by** (*rule overlap-symmetry*)
ultimately have $F\ z \wedge O\ y\ z$**..**
thus $(\exists z.\ F\ z \wedge O\ y\ z)$**..**
next
assume $\exists z.\ F\ z \wedge O\ y\ z$

```
      then obtain z where z: F z ∧ O y z..
      from‹∃ x. F x› have (∀ y. F y ⟶ P y (σ x. F x))
        by (rule F-in)
      hence F z ⟶  P z (σ x. F x)..
      moreover from z have F z..
      ultimately have P z (σ x. F x)..
      moreover from z have O y z..
      ultimately show O y (σ x. F x)
        by (rule overlap-monotonicity)
    qed
  qed
next
  fix x
  assume x: ∀ y. O y x ⟷ (∃ v. F v ∧ O y v)
  have (∀ y. F y ⟶ P y x) ∧ (∀ y. P y x ⟶ (∃ z. F z ∧ O y z))
  proof
    show ∀ y. F y ⟶ P y x
    proof
      fix y
      show F y ⟶ P y x
      proof
        assume F y
        show P y x
        proof (rule ccontr)
          assume ¬ P y x
          hence ∃ z. P z y ∧ ¬ O z x
            by (rule strong-supplementation)
          then obtain z where z: P z y ∧ ¬ O z x..
          hence ¬ O z x..
          from x have O z x ⟷ (∃ v. F v ∧ O z v)..
          moreover from z have P z y..
          hence O z y by (rule part-implies-overlap)
          with ‹F y› have F y ∧ O z y..
          hence ∃ y. F y ∧ O z y..
          ultimately have O z x..
          with ‹¬ O z x› show False..
        qed
      qed
    qed
    show ∀ y. P y x ⟶ (∃ z. F z ∧ O y z)
    proof
      fix y
      show P y x ⟶ (∃ z. F z ∧ O y z)
      proof
        from x have O y x ⟷ (∃ z. F z ∧ O y z)..
        moreover assume P y x
        hence O y x by (rule part-implies-overlap)
        ultimately show ∃ z. F z ∧ O y z..
      qed
```

qed
qed
hence $(\sigma\ x.\ F\ x) = x$
by (*rule strong-fusion-intro*)
thus $x = (\sigma\ x.\ F\ x)$**..**
qed
thus $(\sigma\ x.\ F\ x) = (\mathit{THE}\ x.\ \forall\, y.\ O\ y\ x \longleftrightarrow (\exists\, z.\ F\ z \wedge\ O\ y\ z))$**..**
qed

end

sublocale $\mathit{GEM1} \subseteq \mathit{GEM}$
proof
fix $x\ y\ F$
show $\neg\ P\ x\ y \Longrightarrow \exists\, z.\ P\ z\ x \wedge \neg\ O\ z\ y$
using *strong-supplementation*.
show $x \oplus y = (\mathit{THE}\ z.\ \forall\, v.\ O\ v\ z \longleftrightarrow (O\ v\ x \vee\ O\ v\ y))$
using *sum-eq*.
show $x \otimes y = (\mathit{THE}\ z.\ \forall\, v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge\ P\ v\ y)$
using *product-eq*.
show $x \ominus y = (\mathit{THE}\ z.\ \forall\, w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
using *difference-eq*.
show $-\ x = (\mathit{THE}\ z.\ \forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$
using *complement-eq*.
show $u = (\mathit{THE}\ x.\ \forall\, y.\ P\ y\ x)$
using *universe-eq*.
show $\exists\, x.\ F\ x \Longrightarrow (\sigma\ x.\ F\ x) = (\mathit{THE}\ x.\ \forall\, y.\ O\ y\ x \longleftrightarrow (\exists\, z.\ F\ z \wedge\ O\ y\ z))$ **using** *fusion-eq*.
show $(\pi\ x.\ F\ x) = (\sigma\ x.\ \forall\, y.\ F\ y \longrightarrow P\ x\ y)$
using *general-product-eq*.
qed

sublocale $\mathit{GEM} \subseteq \mathit{GEM1}$
proof
fix $x\ y\ F$
show $\exists\, x.\ F\ x \Longrightarrow (\exists\, x.\ (\forall\, y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall\, y.\ P\ y\ x \longrightarrow (\exists\, z.\ F\ z \wedge\ O\ y\ z)))$ **using** *strong-fusion*.
show $\exists\, x.\ F\ x \Longrightarrow (\sigma\ x.\ F\ x) = (\mathit{THE}\ x.\ (\forall\, y.\ F\ y \longrightarrow P\ y\ x) \wedge (\forall\, y.\ P\ y\ x \longrightarrow (\exists\, z.\ F\ z \wedge\ O\ y\ z)))$ **using** *strong-fusion-eq*.
show $(\pi\ x.\ F\ x) = (\sigma\ x.\ \forall\, y.\ F\ y \longrightarrow P\ x\ y)$ **using** *general-product-eq*.
show $x \oplus y = (\mathit{THE}\ z.\ (P\ x\ z \wedge\ P\ y\ z) \wedge (\forall\, w.\ P\ w\ z \longrightarrow O\ w\ x \vee\ O\ w\ y))$ **using** *strong-sum-eq*.
show $x \otimes y = (\mathit{THE}\ z.\ \forall\, v.\ P\ v\ z \longleftrightarrow P\ v\ x \wedge\ P\ v\ y)$
using *product-eq*.
show $x \ominus y = (\mathit{THE}\ z.\ \forall\, w.\ P\ w\ z = (P\ w\ x \wedge \neg\ O\ w\ y))$
using *difference-eq*.
show $-\ x = (\mathit{THE}\ z.\ \forall\, w.\ P\ w\ z \longleftrightarrow \neg\ O\ w\ x)$ **using** *complement-eq*.
show $u = (\mathit{THE}\ x.\ \forall\, y.\ P\ y\ x)$ **using** *universe-eq*.
qed

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