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Philosophical Accounts of First-Order Logical Truths

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Abstract
Starting from certain metalogical results (the completeness theorem, the soundness theorem, and Lindenbaum-Scott theorem), I argue that first-order logical truths of classical logic are *a priori* and necessary. Afterwards, I formulate two arguments for the idea that first-order logical truths are also analytic, namely, I first argue that there is a conceptual connection between aprioricity, necessity, and analyticity, such that aprioricity together with necessity entails analyticity; then, I argue that the structure of natural deduction systems for FOL displays the analyticity of its truths. Consequently, each philosophical approach to these truths should account for this *evidence*, i.e., that first-order logical truths are a priori, necessary, and analytic, and it is my contention that the *semantic* account is a better candidate.

Keywords First-order logical truths · Completeness theorem · Lindenbaum-Scott theorem

1 Introduction

The problem of explaining the nature of logical truth is, even today,[1] "one of the most important problems in the foundations of logic and perhaps in the whole theoretical philosophy" [(Carnap 1942): 56]. The main reason which makes the decision difficult among different philosophical accounts that logicians and philosophers have proposed is, I think, that there is no consensus regarding the *philosophical properties* that classical first-order logical truths (FOLT) do have, and, consequently, each philosophical approach should account for.

[1]The current debate on logical anti-exceptionalism, i.e., the idea that logic does not have a special status in our overall system of knowledge and, thus, logical truths are on the same par with the scientific truths, brings into focus the importance of the problem of explaining the nature of logical truths and could be also seen as a confirmation of *Carnap's dictum* quoted above. See (Hjortland 2017) for a discussion of logical anti-exceptionalism.

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My strategy in this paper is to establish first, from what I take to be a philosophically neutral point of view, that FOLTs have three essential properties, namely, aprioricity, necessity, and analyticity. Following (Shapiro 2000) and (Rumfitt 2015), I argue that certain metalogical results, namely, the completeness theorem, the soundness theorem, and Lindenbaum-Scott theorem, can be used to show that FOLTs are *a priori* and necessary. Although I know no metalogical theorem which could be directly used to point out that FOLTs are also analytic, i.e., true in virtue of meanings, I will argue that there is a conceptual connections between aprioricity, necessity, and analyticity, such that aprioricity together with necessity entails analyticity. Thus, once we establish that FOLTs are a priori and necessary, then, by definition, they have the third property, namely, analyticity. In addition, I will explain why the analyticity of FOLTs could be seen as emerging from the structure of Gentzen's natural deduction system.

The completeness and soundness theorems of first-order logic indicate that since we have an *a priori* warrant for each FOLT, these statements could be known independently of experience and, thus, they are a priori. In the third section, I argue that FOLTs are necessary. Lindenbaum-Scott theorem states that each implicative relation ($\Rightarrow$) is associated with a certain space ($\Pi_{\Rightarrow}$), such that the conclusion of that implication is true at every member of that space at which all the premises are true. In other terms, the theorem says that an implicative relation obtains between some premises and a conclusion if and only if truth is preserved at each member of the associated space $\Pi_{\Rightarrow}$. These members of $\Pi_{\Rightarrow}$ can be taken as 'possibilities' and, thus, the modal notions "necessary" ($\Box_{\Rightarrow}\psi$) and "possible" ($\Diamond_{\Rightarrow}\psi$), associated with a certain implicative relation, can be easily defined. Moreover, any necessity operator which is generated by an implicative relation in this way is *normal* with respect to that implicative relation, i.e., whenever $\varphi_1,\ldots,\varphi_n \Rightarrow \psi$, we have $\Box_{\Rightarrow}\varphi_1,\ldots, \Box_{\Rightarrow}\varphi_n \Rightarrow \Box_{\Rightarrow}\psi$. That is to say that $\Box_{\Rightarrow}\psi$ is true at each member of $\Pi_{\Rightarrow}$ at which all the $\Box_{\Rightarrow}\varphi_i$ are true. In the particular case when the statement ψ is a thesis ($\Rightarrow \psi$), i.e., when it follows from the empty set of premises, as the logical truths are, we have $\Box_{\Rightarrow}\psi$. Now, if we consider the space of possibilities associated with the logical implicative relation ($\Rightarrow_L$), and take $\Box_L$ to stand for logical necessity, we have that every logical truth is logically[2] necessary, i.e., $\Box_L\psi$ if and only if $\Rightarrow_L \psi$.

In the fourth section, I discuss the distinctions among metaphysical, epistemological, and semantic categories and I argue that once we abandon the speculative idea that we can have a priori knowledge about facts, then aprioricity together with necessity entails analyticity. In addition, I argue that the analytic character of FOLTs may be seen as being displayed by the

[2] The term "logic" and its derivatives will occur often in my article, so the reader may expect some remarks on the notion of logicality. However, since I am explicitly concerned in this article with classical FOL and we know from practice which are the logical terms of this system (we can simply enumerate them), for the present purposes the problem of logicality is assumed as being settled and, thus, the term "logic" is taken rather as primitive. I do not think that this treatment raises difficulties since, as [(Tarski 1986): 145] emphasizes, even a criterion of logicality has to be in agreement in the end, for adequacy to the intuitive concepts, with "at least with one usage which is actually encountered in practice".

structure of Gentzen's natural deduction system for FOL. Once these three properties of FOLTs are established, the question that concerns me is: *which available philosophical approach of FOLTs accounts better for them?* I will argue that the best available account of FOLTs seems to be the *semantic* one. In particular, I will show that (what I take to be) the main philosophical approaches to logical truths do not account for at least one of these three features that FOLTs do have. Finally, I will discuss the idea that the semantic philosophical approach of logical truths is a better candidate.

2 Completeness, Soundness, and Aprioricity

The main aim of this section is to show that a completeness theorem for a sound system of logic can be used to indicate that each logical truth of that system is knowable a priori. To fully understand the relation between completeness and aprioricity, we must have a clear-cut understanding of the term "a priori". Thus, some brief remarks about this term, and about its correlate, "a posteriori", are necessary.

A statement is defined as being *a priori* if and only if it can be known independently of any experience.[3] Nevertheless, it is important to emphasize that if a statement is a priori, this does not entail that it would not be possible for us, human beings, to know it by experience, i.e., a posteriori. As [(Kripke 1980/2001): 35] convincingly argued, "'can be known *a priori*' doesn't mean 'must be known *a priori*'". Hence, although experience is not necessary for a priori knowledge, sometimes it is sufficient for knowing some a priori truths (for instance, we may find out the solution of a mathematical problem by simply using a computing machine). The modal ingredient expressed by 'can' is, thus, essential for the concept of aprioricity.

In contrast with *a priori*, if a statement is *a posteriori*, then it could be known *only* through experience, otherwise, if it could be known independently of experience, then—needless to say—it would be *a priori*. Thus, we have an exhaustive classification of the ways in which statements could be known: only through experience and independently of experience. Therefore, to show that FOLTs are *a priori*, we must show why we *could* know them, in principle, independently of any experience.

No doubt, the main opponent of the idea that there are *a priori* truths is Quine's naturalistic conception. For Quine, since "logic, as any science, has as its business the pursuit of truth" [(Quine 1950/1966): xi], and "there is no higher access to truth than empirically testable hypotheses" [(Quine 1995): 251], it follows that logic, as the entire human knowledge, has the same status, namely, it is a posteriori. The strategy that I want to pursue below, stressing on (Shapiro 2000)'s analysis, is to show that certain metalogical results offer a

[3] This definition goes back to [(Kant 1781/1998): 137]'s *Critique of Pure Reason*: "we will understand by a priori cognitions not those that occur independently of this or that experience, but rather those that occur *absolutely* independently of all experience. Opposed to them are empirical cognitions, or those that are possible only a posteriori, i.e., through experience.", and closely resembles [(Kripke 1980/2001): 34]'s formulation of Kant's idea: "the traditional characterization from Kant goes something like: a priori truths are those which can be known independently of any experience".

justification for the idea that FOLTs are a priori and then to formulate the argument independently of Quine's naturalistic conception.

Quine's web of beliefs confronts the experience tribunal as a whole and new observations may bring changes within the web. Due to the links among the statements of the web, logical statements are also exposed to revision and they will be changed if this conducts to a less drastic revision of the web (in accordance with the principles of conservatism and simplicity). To measure which change is less drastic, however, logic is directly used because there are logical interconnections among statements. As [(Quine 1951): 39] puts it: "re-evaluation of some statements entails re-evaluation of others, because of their logical interconnections –the logical laws being in turn further statements in the system, certain further elements of the field". For instance, if we have formed the belief that $(\varphi \vee \psi) \rightarrow \theta$, and a relevant observation shows us that "θ" is not the case, then, according to *modus tollens*, we will deny $\varphi \vee \psi$, and, consequently, according to DeMorgan rules, we will further deny both "φ" and "ψ".

Since Quine talks about both logical truths, that are statements of the web, and logical interconnections among statements, we may wonder which is the relation between these two constituents of the web. [(Quine 1950/1966): xv] thinks[4] that the statements of logical interconnections, i.e., statements about the relation of logical consequence, are not *really* new constituents of the web because whenever a statement (ψ) is a logical consequence of another statement (φ), there is a third conditional statement in the web ($\varphi \rightarrow \psi$) that is logically true. Moreover, he emphasizes in the same place that to say that "ψ logically follows from φ" is the same as saying that "the statement 'if φ, then ψ' is logically true".

Due to Quine's extensionalist commitments, I think that the best way to interpret that the metastatements "ψ logically follows from φ" and "the statement 'if φ, then ψ' is logically true" have the same meaning is to say that they are equivalent. Thus, the equivalence between logical deduction and logical truth is assumed to hold in the web, i.e., a deduction from the statements $\varphi_1,\ldots,\varphi_n$ to the statement ψ is valid if and only if the corresponding conditional statement "if $\varphi_1,\ldots,\varphi_n$, then ψ" is logically true. If we let '⊢' to stand for the relation of syntactic logical consequence (i.e., logical deducibility)[5] and '⊨' for the relation of semantic logical consequence, then the equivalence—or the transfer principle, as [(Shapiro 2000): 337] labels it—states that $\varphi_1,\ldots,\varphi_n \vdash \psi$ if and only if $\vDash (\varphi_1 \& \ldots \& \varphi_n) \rightarrow \psi$. We should note, however, that his equivalence holds only if some other things also hold. In particular, if we are allowed to deduce a conclusion from an infinite number of premises and that conclusion does not follow from a finite subset of the initial set of premises, then the mentioned equivalence does not hold.[6] Thus, the logic of the web must have the property of finite logical

[4] "Now of these relations of statements to statements, one of conspicuous importance is the relation of logical implication: the relation of any statement to any that follows logically from it. […] Yet implication is not really an added factor; for, to say that one statement logically implies a second is the same as saying that a third statement of the system, an 'if-then' compound formed from the other two is logically true, or 'valid'."

[5] It is natural to treat here the relation of logical consequence syntactically, since the logic of the web will have an associated formal system in which the deductions will be carried out –in the process of revision, in particular.

[6] The equivalence would fail in this case because there would be no conditional sentence to correspond to that infinitary deduction –the implicit assumption being here that any statement from the web has a finite length. In particular, the commitment to this equivalence forbids us to accept deductions governed by the ω-rule, i.e., the rule which allows to infer $(\forall n)\varphi(n)$ from the infinite number of premises $\varphi(0)$, $\varphi(1)$, $\varphi(2)$,….

consequence. In addition, the transition from the syntactic logical consequence $\varphi_1,\ldots,\varphi_n \vdash \psi$ to the logical truth $\vDash (\varphi_1\&\ldots\&\varphi_n) \rightarrow \psi$ presupposes the validity of the deduction theorem, i.e., $\varphi_1,\ldots,\varphi_n \vdash \psi$ if and only if $\vdash (\varphi_1\&\ldots\&\varphi_n) \rightarrow \psi$. Thus, the deduction theorem must also hold in the web's logic.[7]

Finally, if we consider the particular case of in which we have a deduction of a statement from the empty set of premises on the left side of the equivalence (i.e., a logical theorem), then the transfer principle assures us that we will have a logical truth on the right side (if $\vdash\psi$, then $\vDash\psi$). Thus, the commitment to the transfer principle requires the soundness of the web's logic. Conversely, if on the right side of the equivalence we have a logical truth, then the equivalence guarantees that there is a corresponding sound deduction on the left side. Thus, the logic of the web has to be complete (at least in the weak sense: if $\vDash\psi$, then $\vdash\psi$). Therefore, the acceptance of the transfer principle commits Quine to the acceptance of a sound and complete system of logic.

Due to the completeness theorem,[8] however, each logical truth has a proof in the standard deduction systems. A proof for a logical theorem ψ is a sequence of formulas in the formal language ending with ψ, such that ψ depends on no premises or assumptions. More precisely, ψ is obtained from assumptions that are afterwards discharged by the natural deduction rules. However, since ψ is independent of *any* assumptions,[9] it will *a fortiori* also be independent of those assumptions that require empirical justification. It follows, thus, that we can determine the truth of each first-order logically true statement independently of experience.

In addition, since each FOLT has a proof in the standard deduction systems, then we may take those particular proofs as warrants or justifications for the corresponding truths.[10] Nevertheless, since the proofs for the FOLTs use assumptions that are discharged in the end by the natural deduction rules, the mentioned warrants do not depend on experience and

[7] Certainly, as one reviewer kindly pointed out, if we take the transfer principle as primitive, then the deduction theorem is deducible from it in conjunction with the soundness and weak completeness theorems: assuming $\varphi_1,\ldots,\varphi_n \vdash \psi$, it follows by the transfer principle that $\vDash (\varphi_1\&\ldots\&\varphi_n) \rightarrow \psi$, and from that by weak completeness that $\vdash (\varphi_1\&\ldots\&\varphi_n) \rightarrow \psi$. Conversely, if $\vdash (\varphi_1\&\ldots\&\varphi_n) \rightarrow \psi$, then it follows by soundness that $\vDash (\varphi_1\&\ldots\&\varphi_n) \rightarrow \psi$ and from there by the transfer principle that $\varphi_1,\ldots,\varphi_n \vdash \psi$. However, I think that we should not take the transfer principle as primitive. If we explicitly distinguish between (L1) *the logic of the web* and (L2) *the formal logical system meant to formalize this logic of the web*, then we can easily recognize that the transfer principle belongs prima facie to L1. The identification of the metatheoretical presuppositions of the transfer principle is the problem of finding out which are the correspondent metatheoretical properties of that formal logical system, meant to be an adequate formalization of L1, in which the transfer principle would be valid. In this sense, the transfer principle presupposes the deduction theorem, i.e., if the deduction theorem does not hold in L2, then the transfer principle also does not hold in L2 and, thus, that formal system cannot be an adequate formalization of the logic of the web (i.e., of L1).

[8] The completeness theorem is used in fact by Quine as one of the arguments for the idea that first-order logic is the only *logic*. Of course, there are different notions of completeness that logicians work with (see for instance [(Manzano and Alonso 2013): 51–53]), but the relevant one for the present discussion is the notion of weak completeness of a calculus, i.e., each logical truth has a proof in that calculus (if $\vDash \Phi$ then $\vdash \Phi$). In addition, since the deduction theorem holds in FOL (i.e., $\varphi_1,\ldots,\varphi_n \vdash \psi$ is equivalent to $\vdash (\varphi_1\&\ldots\&\varphi_n) \rightarrow \psi$) the notion of strong completeness can be accounted *via* weak completeness.

[9] The conclusion is free of any assumption in the sense that it does not rely on any other formula of the object language. The conclusion is still dependent, for instance, on the structural rules that define the concept of logical consequence in classical logic. These structural assumptions, however, seem to be unproblematic since they hold even in the formal logical system that was taken by some logicians to display the empirical character of logic, i.e., quantum logic (see for instance (Bacciagaluppi 2009) for a recent discussion of (Putnam 1975)).

[10] In general, in scientific practice, and it is hard to see how it can be otherwise, the evidence for a logico-mathematical statement is taken to be (given by) a proof for that statement.

are *a priori*, in which case the logical truths themselves are *a priori*. Moreover, we said that a proof is a sequence of *valid* deductions. The validity of deductions assures us that the warrants are legitimate. The validity of deductions is justified by the soundness theorem for the logic in question. It is useful to have in mind the natural deduction system when we speak about proofs because it does not require any axiom to start with, whose truth-value should be also justified. Each logical truth can be derived from the empty set of premises only by using the rules of the system.[11] If we deduce a conclusion from some premises or assumptions, then, by discharging those premises or assumptions, we obtain a logical theorem and, due to the soundness theorem, a logical truth.

We may now formulate the argument sketched above, starting from Quine's naturalism, in a philosophically neutral manner, as follows:

> **CSA_P1**: If a system of logic is complete, then each logical truth from that system has a proof in that system. (Completeness)
> **CSA_P2**: A proof for a statement constitutes a warrant for that statement.
> **CSA_P3**: A proof is a sequence of valid deductions. (Soundness)
> **CSA_P4**: A proof for a logical truth makes it independent of *any* assumptions and, thus, of assumptions that require empirical justification.
> **CSA_C**: Logical truths are knowable independently of experience, i.e., they are a priori.

In conjunction with certain other philosophical assumptions (for instance, that proofs are warrants), that seem quite reasonable to me, the soundness and completeness theorems play an essential role in this argument. Of course, we may wonder whether these theorems are *a priori* or a posteriori. One may think that if these theorems have an a posteriori status, i.e., the metatheory of logic is itself a posteriori, being part of the web, then something is not right. How could some statements be a priori if their justification also depends on a posteriori warrants?[12]

It is hard to see how the completeness theorem may be a posteriori when its proof contains only *logical steps*, but let us assume, for the sake of the argument, that the metatheory is a posteriori. Would this fact change the conclusion of the above argument? I think that the conclusion could still be maintained, and that the tension between the *a priori* status of the FOLTs and the potential a posteriori status of the metatheory appears because we tend to believe that an a priori statement should not depend on an a posteriori warrant. Nevertheless, as [(Shapiro, 2000): 341] emphasizes, we should distinguish between something being locally *a priori* and being fundamentally a priori. A statement is fundamentally a priori if it has an a priori warrant and that warrant has an a priori one and so on. If it has an a priori warrant which is based, somewhere, on an *a posteriori* warrant, then it is locally a priori. In this sense, even if the meta-theory of first-order logic is a posteriori, then logical truths still are *locally* a priori and the local level is relevant in this context because it is the level at which the justification takes place in scientific practice.

[11] Some remarks about the status of these rules are formulated in section IV.

[12] Although each logical truth is based on its proof, its justification also depends on the metatheorems because the proof is basically for the logical theorem, and in order to make the correspondence between the logical theorems and the logical truths we need to involve the soundness and completeness theorems.

Thus far, I have argued that first-order truths are *a priori*. In fact, if a logical truth belongs to a sound and complete system of logic, then we have a strong warrant to assert that it is a priori. Hence, the above argument may be generalized for every sound and complete system of logic, when completeness is understood in the sense mentioned above.[13] The received view is that if something is a priori then it is also necessary,[14] but since there are arguments for the existence of a priori contingent statements, then aprioricity by itself does not, or may not, entail necessity. In the next section, I argue, stressing on (Koslow 1992)'s and (Rumfitt 2015)'s analyses, that Lindenbaum-Scott theorem[15] offers a good warrant for the idea that FOLTs are also necessary.

3 Lindenbaum-Scott Theorem and the Necessity of Logical Truths

There is a long-standing debate whether logical consequence has a modal ingredient, namely, whether it is only truth-preserving[16] or it is necessarily truth-preserving, i.e., whether it is *logically necessary* that if the premises are true, then so is the conclusion. The well-known Tarskian model-theoretic account of logical consequence generated many discussions with respect to the modal ingredient of logical consequence. Some philosophers believe that this modal feature is accounted by Tarski's approach, some others believe otherwise.[17] (Tarski 1936/1956) considered that the model-theoretic concept of logical consequence should capture two important features that the intuitive concept of logical consequence has, namely, it is formally truth-preserving and it is known independently of experience, i.e., it is a priori. However, if we accept that aprioricity does not directly entail necessity, then we may agree that the Tarskian approach does not directly account for the necessary character of logical consequence.

The idea that logical consequence is necessarily truth-preserving is present in Aristotle's definition of syllogism and we may label it, following [(Rumfitt 2015): 69], as *Aristotle's Thesis*. Now, the question is: could we justify the idea that logical consequence is necessarily truth preserving and that logical truths are necessary? I will argue below that Lindenbaum-Scott theorem offers a strong warrant for the idea that FOLTs are necessary.

To state Lindenbaum-Scott theorem, we have to introduce first some notions. Let $I = \langle S, \Rightarrow \rangle$ be an *implication structure* on S, where S is a non-empty set of statements and $\Rightarrow$ is an implication relation, i.e., a relation satisfying Gentzen's structural rules (reflexivity, monotonicity, and cut). We define a *bisection* of S as a pair $T = \langle K, L \rangle$ of non-empty subsets of S such that K and L are disjoint and their union is the whole of S. With the notion of bisection, we can define a *bisection implication* ($\Rightarrow^T$) as follows: for

[13] Certainly, I do not want to suggest that completeness is a necessary condition for aprioricity. For instance, Peano Arithmetic may be a priori even though it is incomplete.

[14] For a critical discussion and analysis of this received view, which is in fact one of the tenets of logical empiricism, see for instance (Sloman 1965).

[15] This theorem is a result proved by (Scott 1974), who saw it as generalization of Lindenbaum's Theorem (see (Scott 1974: 416). This is why some authors mention it as Lindenbaum-Scott Theorem (see (Koslow 1992) and (Rumfitt 2015)) and some others as Scott-Lindenbaum Theorem (see Payette and Schotch 2014).

[16] [(Russell 1919): 153–154], for instance, argued that logical consequence has no modal ingredient and that its essential feature is only truth-preservingness.

[17] See (Etchemendy 1990 and Prawitz 2005) for a criticism of Tarski's account on this aspect and (García-Carpintero 2003) for an analysis of the debates generated by Etchemendy's book.

every $\varphi_1,\ldots,\varphi_n$ and ψ in S, $\varphi_1,\ldots,\varphi_n \Rightarrow^T \psi$ if and only if either some $\varphi_i \in K$ or $\psi \in L$. A bisection T = <K, L> on an implication structure is a *strong bisection* on I if and only if L is strongly closed under the implication relation $\Rightarrow$ of the structure, i.e., for every $\varphi_1,\ldots,\varphi_n$ and ψ in S, if $\varphi_1,\ldots,\varphi_n \Rightarrow \psi$, and each $\varphi_i \in L$, then $\psi \in L$.

Having introduced these notions, we may now formulate the *Lindenbaum-Scott theorem*: let I be any non-trivial implication structure (there are at least two elements of it neither of which implies the other). Then, $\varphi_1,\ldots,\varphi_n \Rightarrow \psi$ if and only if $\varphi_1,\ldots,\varphi_n \Rightarrow^T \psi$, for all strong bisection relation $\Rightarrow^T$ on the structure.[18]

In [(Scott 1974): 416]'s original terms, the theorem says that an implication relation is the intersection of all the consistent and complete extensions of it, where an implicative relation is *consistent* if and only if there is no member of S such that it is both a thesis (i.e., it is implied by everything from S) and an antithesis (i.e., it implies everything from S) and it is *complete* if and only if for each member of S it is either a thesis or an antithesis.[19]

How does the content of this theorem relate with necessity? Lindenbaum-Scott theorem guarantees that whenever we have an implicative relation ($\Rightarrow$), there will be a related topological space ($\Pi_\Rightarrow$). Essentially, the theorem states that there is a relation between an implicative relation and the preservation of truth at any member of the related space $\Pi_\Rightarrow$, namely, an implicative relation obtains between some premises and a conclusion if and only if truth is preserved at every member of the related space $\Pi_\Rightarrow$. In other words, Lindenbaum-Scott theorem says that some premises stand in the implicative relation $\Rightarrow$ with a conclusion if and only if the conclusion is true in each situation in which the premises are true, i.e., it guarantees the existence of a space $\Pi_\Rightarrow$ of such situations. For example, if the following implicative relation holds, $\varphi_1, \varphi_2 \Rightarrow \psi$, then the bisection T = <K, L>, such that both φ_1 and φ_2 belong to L, but ψ belongs to K, will not be a bisective extension of $\Rightarrow$. The associated space $\Pi_\Rightarrow$ excludes a situation of this kind, i.e., it is not a possibility relative to the implicative relation in question and, thus, not a member of $\Pi_\Rightarrow$.

Thus, taking the members of $\Pi_\Rightarrow$ as *possibilities*,[20] any implicative relation obtains if truth is preserved at any possibility associated with that implicative relation, i.e., at every member of $\Pi_\Rightarrow$. In the next step, we may define the modal operators associated with a certain implicative relation as follows:

> $\Diamond_\Rightarrow\psi$ is true if and only if ψ is true at some member of the space of possibilities $\Pi_\Rightarrow$.
>
> $\Box_\Rightarrow\psi$ is true if and only if ψ is true at every member of the space of possibilities $\Pi_\Rightarrow$.

[18] For this formulation of the theorem, and for a proof of it, see [(Koslow 1992): 50–51; Theorem 8.11 and (Koslow 2010): 120–121].

[19] For this version of the theorem and its relation with the version mentioned above see also the theorems 8.13 and 8.15 from (Koslow 1992). (Scott 1974) formulated the theorem in a multiple-conclusion framework, but its restriction to single conclusion does not raise difficulties. For a discussion of the relation between implication and valuations see also [(Scott 1971): 795–798].

[20] The relation between the possibilities generated by an implicative relation and the models of the system of logic that defines that implicative relation requires further investigations. It seems natural to take every model of FOL as representing a possibility from $\Pi_\Rightarrow$ and to take every possibility from $\Pi_\Rightarrow$ as being represented by a model of FOL, but, as [(Shapiro 2018): 5] points out, if we move to second order calculus (where we have both standard and Henkin models), this correspondence is no longer so straightforward.

In addition, any necessity operator which is generated by an implicative relation is, according to the theorem, *normal* with respect to that implicative relation, i.e., if $\varphi_1,\ldots,\varphi_n \Rightarrow \psi$, then $\Box_{\Rightarrow}\varphi_1,\ldots, \Box_{\Rightarrow}\varphi_n \Rightarrow \Box_{\Rightarrow}\psi$. In the particular case in which the statement ψ is a thesis, i.e., it follows from the empty set of premises, we have $\Box_{\Rightarrow}\psi$. Now, if we consider the space of possibilities associated with the logical implicative relation($\Rightarrow_L$), and take $\Box_L$ to stand for logical necessity, we obtain that every logical truth is logically necessary, i.e., $\Box_L\psi$ if and only if $\Rightarrow_L\psi$. In other words, each first-order logical truth is necessary in the space ($\Pi_{\Rightarrow L}$) associated with the logical implicative relation in FOL.[21]

The argument sketched above may be stated more systematically as follows:

> **LSN_P1**: Any implicative relation is associated with a related space $\Pi_{\Rightarrow}$, such that this relation obtains if and only if truth is preserved at any member of $\Pi_{\Rightarrow}$.
> **LSN_P2**: The members of $\Pi_{\Rightarrow}$ can be taken as possibilities.
> **LSN_P3**: Any necessity operator generated by an implicative relation is normal with respect to that implicative relation, i.e., if $\varphi_1,\ldots,\varphi_n \Rightarrow \psi$, then $\Box_{\Rightarrow}\varphi_1,\ldots, \Box_{\Rightarrow}\varphi_n \Rightarrow \Box_{\Rightarrow}\psi$, and in particular if $\Rightarrow\psi$, then $\Box_{\Rightarrow}\psi$.
> **LSN_P4**: FOLTs follow from the empty set of premises.
> **LSN_C**: FOLTs are logically necessary.

More generally, Lindenbaum-Scott theorem states that each implicative relation preserves more than *bare truth*. Due to the fact that each implicative relation has associated a certain space of possibilities, that relation will preserve a genuine kind of necessity. Of course, the relation of logical consequence will primarily preserve the *logical necessity* that it generates and it is a very important implicative relation because it could be used to extend any other implicative relation (geometrical, physical et al.).[22] Moreover, since FOLTs are logical consequences of the empty set of premises, the theorem assures us that they are modalized with the necessity operator relatively defined to the relation of logical consequence in FOL. Thus, FOLTs are logically necessary—in the sense of necessity specified above.

So far, I have argued that FOLTs are *a priori* and necessary. In the next section, I advance two arguments for the idea that they are also analytic. First, on the assumption that there is no *a priori* knowledge of facts, I will point out a conceptual connection between aprioricity, necessity, and analyticity—such that the first two notions are necessary and sufficient conditions for the latter-, and then, I will argue that the structure of natural deduction systems for FOL displays the analytic character of its truths.

4 Analyticity, Necessity, and Aprioricity

Traditionally, it was a widespread conviction among philosophers that there is a strong connection among analyticity, aprioricity, and necessity. For instance, G.W. Leibniz

[21] As (Quine 1953) argued, the combination between modal operators and quantifiers raises problems when we are allowed to attach the necessity sign to open sentences and then to quantify into these modal contexts. However, this problem does not affect the present discussion of the necessity of FOLTs because they are closed statements and, thus, we can treat their modalized form as having, in Quine's terms, only two grades of modal involvement (as statement predicate or statement operator). We do not need to consider formulas of this sort $(\exists x)\Box Fx$ since they are part of first order modal logic, but not of FOL. See also [(Rumfitt 2015): 76] for a brief discussion of this aspect.

[22] See [(Rumfitt 2015): 52–55] for a discussion of this idea.

believed that necessary truths are truths of reason and, thus, they could be known only independently of experience, i.e., they are a priori. Although on different reasons, D. Hume also thought that necessary truths are referring only to relations among ideas that, of course, are not subject to empirical experience. If we introduce the term "analyticity", as I. Kant explicitly did, then, for Leibniz and Hume, the analytic truths are both a priori and necessary.[23] For Kant, aprioricity is coextensive with necessity, but neither of them is coextensive with analyticity, since Kant believed that there are also synthetic a priori truths. However, especially due to the development of non-Euclidean geometries, the idea of synthetic a priori truths was abandoned by most of, if not all, philosophers, and this was one of the fundamental ideas of logical empiricism.

Continuing the Humean empiricist tradition and strongly influenced by Wittgenstein's *Tractatus*, the Vienna Circle's members thought that every statement for which we need experience to justify its truth value is contingent, synthetic, and a priori. In order to explain the necessity usually attributed to the statements of logic and mathematics, they took the statements of these disciplines to be true in virtue of meanings[24] and, thus, they considered those statements to be analytic, a priori, and necessary. On this view, the terms 'analytic', '*a priori*', and 'necessary' are coextensive with each other and disjoint with the corresponding terms 'synthetic', '*a posteriori*', and 'contingent'. Quine strongly criticized the distinction between analytic and synthetic statements, the first dogma of empiricism, and together with it the idea that there are statements which could be known independently of experience. Nevertheless, as we saw in the argument based on completeness, the idea that every statement is known only a posteriori could not be reasonably maintained until the end.

In 'Naming and Necessity', Saul Kripke strongly criticized the coextensiveness of necessity and aprioricity and admitted the existence of both necessary a posteriori statements and contingent a priori statements, but he did not reject the distinction between analytic and synthetic statements. In fact, he accepted that, by definition, an analytic statement is both *a priori* and necessary, although he argued that analyticity can do the job of neither aprioricity, nor of necessity. Thus, for Kripke, if a statement is both a priori and necessary, as I have argued that FOLTs are, then it is also analytic.[25]

[23] More precisely, according to this line of thought, logico-mathematical statements are a priori because they are necessary.See (McEvoy 2013) for a discussion of this idea.

[24] The logical empiricist view on logic is usually described by saying that logical truths are true by linguistic convention. However, Rudolf Carnap disapproved the use of "linguistic conventions" as applying to his account to logical truths. The choice of the meanings of the logical terms may be a matter of convention, but once these meanings are fixed, there is not conventional at all which statements are logically true: "once the meanings of the individual words in a sentence … are given (which may be regarded as a matter of convention), then it is no longer a matter of convention or of arbitrary choice whether or not to regard the sentence as true; the truth of such sentence is determined by the logical relations holding between given meanings" [(Carnap 1963): 915–916]. The distinction is important and, as [(Hacking 1979): 317] emphasized, "to reject truth by convention is not to reject truth in virtue of meaning".

[25] [(Boghossian 1996): 363] distinguishes between two notions of analyticity: an epistemological and a metaphysical one. In the latter sense, a statement is analytic if it owes its truth value completely to its meaning, and not at all to 'the facts'. A statement S is analytic in the former sense if mere grasp of S's meaning by a person P sufficed for P's being justified in holding S true. Boghossian's concern is to show that the analytic theory of *a priori* could be maintained if analyticity is understood in the epistemological sense, and that the metaphysical notion of analyticity, which is supposed to ground the linguistic theory of necessity, could not be defended. However, since I do not want to ground neither aprioricity, nor necessity in analyticity, I do not find this distinction helpful. It would be so if we want to pass directly from analyticity to aprioricity, or to necessity.

Which is the moral of this brief historical detour? The moral is that, historically, only certain combinations among the epistemological, metaphysical, and semantic categories mentioned above have been taken as possible. For instance, Kant admitted three possible combinations, namely, <a priori, necessary, analytic>, <a posteriori, contingent, synthetic>, and <a priori, necessary, synthetic>. Due to the rejection of synthetic a priori statements, logical empiricism admitted only two combinations, namely, <a priori, necessary, analytic>, <a posteriori, contingent, synthetic>. Kripke, by disentangling the epistemological categories from the metaphysical ones, admitted the two possibilities asserted by the logical empiricists, but he also recognized other two possibilities, namely, <a priori, contingent, synthetic> and <a posteriori, necessary, synthetic>. However, once we abandon the idea that we can know empirical facts independently of experience, as I think we should do if we consider the development of modern science,[26] we have no reason to accept that there could be necessary a priori statements which are also synthetic, and, therefore, a statement which is both a priori and necessary will be, *ipso facto*, analytic.

The argument for the conceptual connection among necessity, aprioricity, and analyticity may be briefly formulated as follows: if a statement is both a priori and necessary, then it is either analytic or synthetic. However, if it is synthetic, then synthetic a priori knowledge is presupposed. Since synthetic a priori knowledge is quite problematic for being endorsed, it follows that a statement which is both a priori and necessary is *ipso facto* analytic.

Another argument that may be advanced for the idea that FOLTs are analytic, a version of which was also suggested by [(Shapiro 2000): 340], goes in the following direction: if we take the natural deduction rules of a deductive system for FOL as constituting the meanings of the logical terms (as the inferentialists do[27]), then, since logical deduction proceeds only by introduction and elimination rules (that can be reduced to the introduction ones), it follows that logical truths are analytic, i.e., true merely in virtue of the meanings of the logical terms.

If we focus our attention on Gentzen's natural deduction system for FOL, the analyticity of FOLTs could be easily understood. In the natural deduction systems,

[26] The philosophers who endorse the view that we can know empirical facts independently of experience usually postulate a special faculty of our mind, called intuition (or rational insight). Prominent scientists have recognized the importance of intuition (*anschaulicher Einsicht*) for the construction of mathematical and physical theories (see [(Hilbert 1930/2005): 1161–63; (Einstein 1934): 163–165]), but the principles gained by this intuitive insight, as [(Einstein 1934): 165] explicitly states it, have no a priori *justification*. Their justification is obtained by confronting the whole theory with the observational data. Kant's assertion of the existence of synthetic a priori knowledge rests in the end on the assumption that we impose on experience, in sense perception, the relational structure of space and time. However, the development of modern geometry and physics undermined Kant's assumption in the specific form that he gave it (see also [(Hilbert 1930/2005): 1162–63]). More generally, as [(Hintikka 1999): 145] emphasizes, the assertion of synthetic a priori knowledge rests on the failure to disentangle between heuristics and justification. (The reader may find in Hintikka's article an excellent analysis and criticism of the concept of 'intuitive knowledge'.)

[27] Due to Carnap's proofs from his 1943 book, *Formalization of Logic,* a problem with this approach is that if we start only from the formal system, the formal rules and axioms are not categorical in their standard formulation, i.e., they do not uniquely determine the meanings of the logical terms. Nevertheless, this problem could be easily solved if we adopt a multiple conclusion calculus and transfinite rules for the quantifiers, as (Carnap 1943) did, or we simply impose some semantic restrictions on interpretations, as (Bonnay and Westerståhl 2016) proposed. In the present context of the analysis of FOLTs, the use of both model and proof-theoretic instruments is allowed since they do not engage empirical considerations.

there are no axioms, but only introduction and elimination rules of deduction for the logical terms. The introduction rules, however, according to [(Gentzen 1969): 80], "represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions". This 'final analysis' that Gentzen mentions was fruitfully analyzed and explained by (Prawitz 2005, 2006), by showing how the elimination rules could be reduced to the introduction ones; the introduction rules are taken by Prawitz to be justified by the meanings of the logical that terms they introduce and the elimination ones are taken to be justified, via reduction, by the introduction ones.[28]

What this reduction shows is that the elimination rules do not go beyond the meanings of the logical terms defined by the introduction ones. We know, however, that FOLTs are obtained from the empty set of premises only by applying these two types of rules. Thus, since these truths are assertable in virtue of the sound introduction and elimination rules of deduction and these deductions are finally justified by the meanings of the logical terms, it follows that each FOLT is justified by the meanings of the logical terms and, thus, analytic.

Hence, lacking the knowledge of a metalogical theorem which could be directly used to justify the analyticity of FOLTs, we may use either the conceptual connection between aprioricity, necessity, and analyticity or the Gentzenian analysis of logical truths as reasonable and strong warrants for the analytic character of FOLTs. Now, having this *evidence* about FOLTs of classical logic, the question that I want to focus on is the following: *which available philosophical approach to FOLTs accounts better for this evidence and, thus, could be considered the best one?*

5 Best Available Philosophical Account of FOLTs

If we accept the arguments formulated above and, thus, we accept that FOLTs are a priori, necessary, and analytic, then this result could be used as a criterion for choosing among different philosophical accounts of FOLTs. Different philosophical accounts of FOLTs usually attribute to these truths those properties that fit better in their general philosophical framework and do not account for what *is there*, i.e., for the properties that FOLTs really have, if we consider the structure and practice of logic.

Since each philosophical approach to FOLTs should account for at least the three mentioned properties, the question which must be answered now is: *which available philosophical approach accounts better for these properties?* Of course, an answer to this question would presuppose an exhaustive enumeration of all available accounts, which is not an easy task, and, certainly, would go beyond the limits of this paper. What I want to do, rather, is to briefly state the relevant underlying ideas of what I take to be the main kinds of interpretations of FOLTs and to see which of them accounts for at least the three properties that I have argued for above.[29]

[28] The elimination rules are not *reduced* to the introduction ones in the common sense of reduction, i.e., that of being formally derived, but rather by showing how "they can be justified in some semantic way" [(Prawitz 2006):510].

[29] A detailed description and analysis of these philosophical accounts of FOLTs are found in (Brîncuş 2016). For a more comprehensive historical survey of the main philosophical approaches to logic, in analogy with those from the philosophy of mathematics, see (Maddy 2012).

There are, I think, three main kinds of philosophical accounts to FOLTs: an ontological (or metaphysical) account, an empirical account, and a linguistic (or semantic) account. The ontological account takes logical truths to be descriptive of the most general structure of reality and considers them to be made true by that very structure. Roughly, this account entails that logical truths are necessary and synthetic. They are necessary because they describe the invariant structure of the real world and they are synthetic because they are made true by at least some empirical facts, i.e., by the structure of the world. The empirical account takes logical truths to be very reliable empirical hypothesis that have a very special role in our overall system of knowledge, and, in the most instantiations of this account, logical truths are taken to be *a posteriori* and synthetic. The semantic account takes logical truths to be true in virtue of the meanings of the logical terms within a certain linguistic framework and, thus, a priori, necessary, and analytic.

The ontological account (see (Sher 2011), (Tahko 2014)) fails to account for the analytic character of logical truths and if in certain instantiations it accounts for the *a priori* character, then it assumes an a priori knowledge about at least some empirical facts, i.e., about the real structure of the world. However, judging in accordance with the scientific methodology of inquiry, it is very problematic to maintain that we can have knowledge about empirical facts by pure thought (the practice of scientific inquiry offers no warrant for this idea). The empirical approach to logical truths (see for instance: Mill, Quine, Maddy) fails to account for at least one of the three properties I argued that FOLTs do have, namely, analyticity, and some of them, as Quine's naturalistic view, fail to account for all three of them. As I will explain below, the semantic approach accounts for all these three features of FOLTs and—judging on this criterion—it seems to be the best available account.

The particular semantic account that I have in mind is the one described by Rudolf Carnap in many of his writings. (Carnap 1950) has fruitfully explained the idea that logical truths are true in virtue of meanings with the help of the concept of *linguistic framework*. As analytic statements, logical principles are the core of a linguistic framework. They are constitutive for a certain framework by providing the grammar and the rules for operating in that framework. In this sense, they are necessary precisely because they are constitutive for the framework. They delimitate the space of possibilities associated with the relation of logical implication. Once someone disobeys them, she/he simply refuses to work within that framework. Working in a framework is analogous with playing a game. If someone does not accept the rules of a game, then she/he does not play *that* game. For that game, for that linguistic framework, the rules are constitutive, and thus necessary—from this internal perspective. The framework, of course, on pragmatic reasons, may be changed; its adoption is a contingent matter. Moreover, this characterization of logical truths, as "framework principles", also reveals their regulative function. Since they indicate how one should work in a given framework, they are regulative for the activities performed in that framework.

Of course, I do not want to suggest that the semantic view on logic and mathematics is the right one, in general. The conclusion is restricted to the FOLTs of classical logic.

6 Final Remarks

The upshot of this paper can be formulated as an argument modeled by an abductive inference to the best philosophical account of first-order logical truths and could be stated as follows:

P1_(Evidence): FOLTs are a priori, necessary, and analytic.
P2_(Hypotheses): The ontological, empirical and semantic accounts are the main available philosophical accounts of the FOLTs.
P3_(Best Account): The semantic account accounts better for the evidence from P1.
Conclusion: The semantic account of FOLTs is the best candidate we have.

The evidence stated in the first premise has been discussed from a relatively neutral philosophical perspective, namely, on the basis of certain meta-theoretical results. Once this first premise is accepted, any philosophical approach to FOLTs should account for the evidence stated in it. However, the ontological and the empirical approaches fail to account for at least one of the three properties that FOLTs have. The semantic account to FOLTs seems to be the best one because it accounts for all these three *philosophical properties* of FOLTs, i.e., it fits better with the philosophical properties warranted by metalogic. One moral of this paper is that, although there may be undecidable or borderline cases, at least something is necessary, a priori, and analytic in *the whole system of science,* i.e., FOLTs of classical logic.

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