# Enriching Deontic Logic

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**Abstract**

It is well-known that systems of action deontic logic emerging from a standard analysis of permission in terms of possibility of doing an action without incurring in a violation of the law are subject to paradoxes. In general, paradoxes are acknowledged as such if we have intuitions telling us that things should be different. The aim of this paper is to introduce a paradox-free deontic action system by $(i)$ identifying the basic intuitions leading to the emergence of the paradoxes and $(ii)$ exploiting these intuitions in order to develop a consistent deontic framework, where it can be shown why some phenomena seem to be paradoxical and why they are not so if interpreted in a correct way.

**Keywords:** Action deontic logic; Ought-to-be logic; Ought-to-do logic; Choice permission; Sequential permission; Contrary-to-duty obligation.

## 1 Introduction

We use deontic logics for analysing deontic systems, that is, systems of norms that allow for the preservation, as far as it is possible, of a certain ideal order in the interactions of groups of agents. If we assume that groups of agents interacting in a common environment constitute an action system, then we can simply say that deontic systems are introduced in order to regulate action systems. These in turn can be modelled as dynamical systems where a set of possible stages and a set of possible actions are initially defined in such a way that, at each stage, the set of executable actions and the set of accessible stages are determined. Accordingly, deontic systems can be modelled as dynamical systems where both possible stages and possible actions can be deontically characterized as permitted, prohibited, or obligatory, either in themselves or relative to certain constraints. In this view, a basic framework for developing deontic logics is a framework in which at least the basic theoretical concepts concerning actions and courses of actions in an action system are interpretable. Our aim here is precisely to put forward sufficient resources to introduce a framework of this kind, so as to be able not only to describe one-step actions and sequences of actions but also to distinguish between absolute prescriptions, characterizing stages and actions in themselves, and conditional prescriptions, characterizing actions at specific situations or stages. Working within this framework will enable us to introduce a first system of action deontic logic where three significant distinctions can be characterized:

1. a distinction between prescriptions on states and on actions;
2. a distinction between prescriptions on actions and on courses of actions;
3. a distinction between abstract absolute norms and actual conditional norms.

As we will see, the first distinction will allow us to integrate an *ought-to-be* logic and an *ought-to-do* logic, while the final distinction will allow us to cope with both *ideal global* prescriptions and *optimal local* prescriptions on actions and states. Given these distinctions, we will be able to develop an insightful analysis of some important paradoxes and to provide intuitive solutions for them. The plan of the paper is as follows. In the next section, we discuss the basic intuitions that our system aims at capturing as they emerge from the analysis of a simple case. In section 3, we introduce our system of deontic logic of states and actions. In the last section, we define four groups of deontic concepts and provide solutions to some of the classical deontic paradoxes.

## 2 Framing the system

Our proposal is based on the idea that, in order to account for the intuitions which generate the paradoxes, more distinctions than those which can be drawn within a standard dynamic deontic system are to be made both on the ontic and on the deontic level. In order to motivate these distinctions, let us consider the following simple case.

## 2.1 A case study

In most Italian schools, kids are required to be at school before 8:00 in the morning and they are strongly recommended to be in their classrooms 10 minutes before 8:00 in order to review their homework. Suppose that two kids, Sara and Luca, have three ways to get to the school: $\alpha_1$, by car, with their dad; $\alpha_2$, by bike; $\alpha_3$, on foot. If they go by car, it takes them 15 minutes to arrive at the school, provided their dad respects the speed limit, and 10 minutes, if their dad decides not to respect the speed limit. If they go by bike, it takes them 20 minutes to arrive at the school. Finally, if they go on foot, it takes them 30 minutes to arrive at the school. In addition, the kids need 2 minutes to go from the gate of the school to their classroom, $\alpha_4$.

| Step 1: | Arriving | Step 2: | Being at the desk |
|---|---|---|---|
| Going by car | 7:40 | Going to the classroom | 7:42 |
| Going by car | 7:45 | Going to the classroom | 7:47 |
| Going by bike | 7:50 | Going to the classroom | 7:52 |
| Going on foot | 8:00 | Going to the classroom | 8:02 |

Table 1: Luca and Sara leave their home at 7:30 am.

Table 1 illustrates what happens in a situation in which Sara and Luca leave their home at 7:30. In the case under consideration, four action types are taken explicitly into account and each of them can be instantiated in several ways. In particular, two ways of instantiating the first type, $\alpha_1$, are highlighted, i.e. going by car respecting the speed limit and going by car not respecting the speed limit.

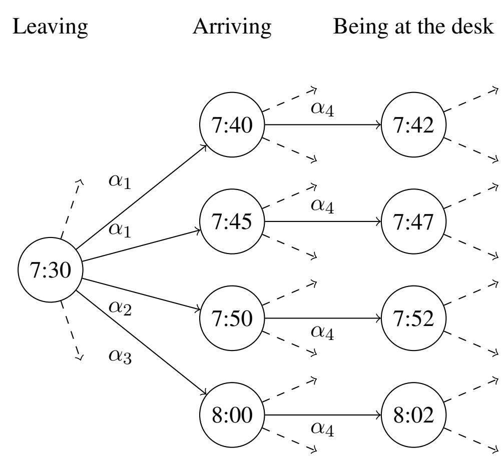



Figure 1: Ontic concepts.

Let us assume that a *story* is any, finite or infinite, sequence of stages such that every successive pair in the sequence is connected by a transition. In figure 1, four *stories* are displayed, each consisting of three *stages* connected by two *transitions*. Transitions are instances of *actions*, intended as action *types*, and go from a certain stage of a possible story to a different stage of the same story. In addition, different *states* are realized at different stages.

Given a certain story, both actions and stages are classifiable according to the deontic status of what is successively realized in that story. In our specific case, an analysis of the deontic value of the relevant actions and stages leads to the following result. As for the actions, all action types are permitted, since there are ways of performing these actions in accordance with the given norms (even if some of them, namely $\alpha_1$ and $\alpha_4$, have also prohibited instances). As for the stages, *ideal* stages, where all norms are respected, can be distinguished from stages in which some norm is violated. The latter include the stage where Luca and Sara arrive at the school by car at 7:40 and the stage where they arrive at the school at 8:02, since at these stages the norm on the speed limit and the norm of the school are respectively violated. As shown in figure 2, all other stages are ideal from the point of view of the norms we are focusing on (in the picture, the ellipses indicate the stages where no violation occurs, that is, the stages that are ideal according to the norms).

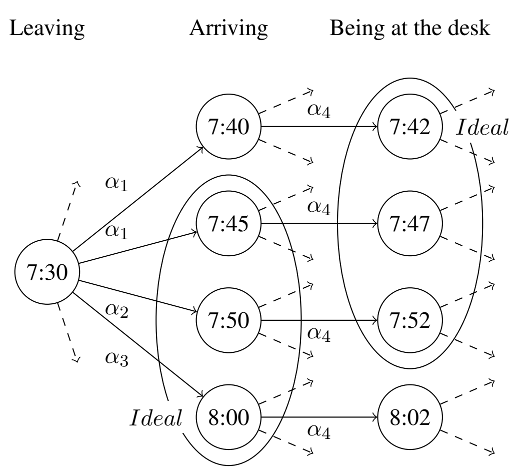



Figure 2: Global deontic concepts.

Importantly, note that the stage where Luca and Sara arrive at the school by car at 7:45 is not only ideal but also *optimal*, since in this case the two kids have time to properly review their homework, and this is the best they can achieve under circumstances. Furthermore, the deontic status of stage 7:40 is different from the deontic status of stage 8:02. To be sure, at 7:30 Luca and Sara have a way to avoid the prohibited transition leading to stage 7:40 and still reach an ideal stage, while at 8:00 there is no way for them to get to a stage where the norm of the school is respected. At this stage, the best the kids can do is to go as quickly as possible to their classroom and be there at 8:02. Hence, relative to stage 8:00, it is indeed recommended to transit in 8:02, even if this transition leads to a stage where the norm of the school is violated. Finally, observe that the norm of the school is an *ought-to-be* prescription, since kids are required *to be* at school before 8:00, while the norm on the speed limit is an *ought-to-do* prescription, since people are required *to drive* at a certain speed.

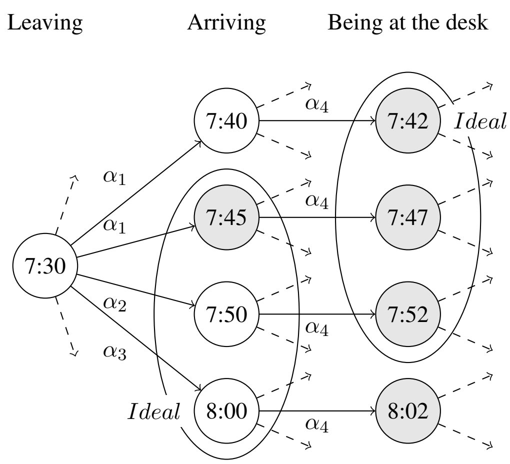



Figure 3: Local deontic concepts.

The overall picture we have obtained is schematically represented in figure 3. Here, the gray stages are the optimal ones. Note that a non-ideal stage can be optimal relative to a previous stage only if no ideal stage is accessible from that stage in one step.

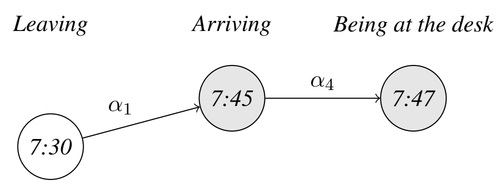
## 2.2 Introducing the basic concepts: ontic level

Let us now make explicit the conceptual framework we have illustrated so far. On the ontic level, we assume the following elements:

- stages; states;
- moves; actions; courses of actions.

The ontic part of our system will include both a logic of states and a logic of actions. We conceive of stages as elements constituting possible stories, and, as usual, we model states as sets of stages, i.e. the sets of stages in which the states are realized. In addition, we assume that actions are action types, modelled as relations defined on the stages, and that courses of action are sequences of actions. Thus, each instance of an action is a transition linking two stages and each instance of a course of action is a path through stages. Furthermore, actions can be basic or composite: moves are basic actions, while composite actions are obtained by conjoining, disjoining, and negating moves. Finally, stories are sequences of stages and actions.

**Example 1.** *Consider figure 3: we can identify the story*




*consisting of three stages and two moves constituting the course of action* $(\alpha_1, \alpha_4)$.

In this context, actions are to be intended as achievements and accomplishments, in Vendler's classification [17], or as acts and achievements in von Wright's conceptualization [19, 21]. On this conception, each action has a result, which is a state that in principle can obtain without the action having been performed. Thus, while the realization of the result of an action is essential to its performance, in the sense that an action has not been performed unless its result has been accomplished, there is no perfect correspondence between stages to which an action leads and stages where the result of that action obtains. Still, there is a perfect correspondence between stages to which an action leads and stages where *the state that the action has been done* is realized. Subsequently, we will use this correspondence in order to define actions in terms of states.[1] Schematically:

[1]In a previous paper [9], we have worked out a system of action deontic logic based on the distinction between the possible end-stages of an action and the stages in which its result possibly obtains.

$$
ontic\,part \begin{cases} static & \begin{cases} stages \\ states: \quad sets\,of\,stages \end{cases} \\ \\ dynamic & \begin{cases} short\,range: \quad actions \\ long\,range: \quad courses\,of\,action \end{cases} \end{cases}
$$

It is worth noting that, in this framework, for each stage, we can single out the set of moves which are executable at that stage. In addition, given the presence of executable moves, we can introduce the following distinctions among stages that are directly accessible by doing a specific action, stages that are directly accessible, and stages that are accessible:

1. a stage $v$ is *directly* $\alpha$*-accessible* from $w$ just in case there is an $\alpha$-transition starting at $w$ and leading to $v$;

2. a stage $v$ is *directly accessible* from $w$ just in case there is a transition starting at $w$ and leading to $v$;

3. a stage $v$ is *accessible* from $w$ just in case there is a sequence of transitions starting at $w$ and leading to $v$.

Thus, direct $\alpha$-accessibility is a particular case of direct accessibility, which in turn is a particular case of accessibility. Similarly, we can introduce the following distinctions between states that are directly realizable and states that are realizable:

1. the state that $\phi$ is *directly* $\alpha$*-realizable* at $w$ just in case there is an $\alpha$-transition starting at $w$ which has $\phi$ as a consequence;

2. the state that $\phi$ is *directly realizable* at $w$ just in case there is a transition starting at $w$ which has $\phi$ as a consequence;

3. the state that $\phi$ is *realizable* at $w$ just in case there is a sequence of transitions starting at $w$ and leading to a stage where $\phi$ is realized.

As before, direct $\alpha$-realizability is a particular case of direct realizability, which in turn is a particular case of realizability. In conclusion, at the ontic level, what we need is $(i)$ a set of stages, states being subsets of stages, $(ii)$ a function fixing what is accessible at a given stage, and $(iii)$ a set of functions fixing, for each action $\alpha$, what is $\alpha$-accessible at a given stage.

### 2.3 Introducing the basic concepts: deontic level

Turning to the deontic level, the fundamental distinction we assume is the distinction between

- what is *ideal*;
- what is *optimal*.

This distinction, which is primarily applied to stages, gives rise to a rich analysis of deontic concepts not only on stages but also on actions and courses of action. The basic general idea is as follows. An ideal stage is a stage in which the law is fulfilled, while a non-ideal stage is a stage in which some norms are not fulfilled. What ideally we should do is to ensure that our actions always lead us to an ideal stage, so as to produce a course of action which is completely safe from the point of view of the law. In those cases in which this is possible, the best we can do is to perform an action leading us not merely to an ideal stage but to one ideal stage which is among the optimal directly accessible stages. In those cases in which this is not possible, we still have the opportunity to act in an optimal way given the circumstances, that is, in a way allowing us to reach one of the optimal stages among the directly accessible ones. Therefore, the first fundamental distinction is the one between:

1. stages where the ideal is directly realizable;
2. stages where the ideal is not directly realizable.

In this regard, a fundamental deontic principle can be introduced in our system to the effect that *at any stage the ideal is realizable, either directly or indirectly*. That is, the deontic tragedy of being in a persistent condition of non-ideality is always avoidable. This is a basic and intuitive deontic principle, since we are not usually interested in past infractions: when a norm is violated, we land in a non-ideal stage, but the future stages can still be ideal, provided no further norm is violated at them.

In order to account for the primitive distinction between ideal and optimal stages, we develop both a logic of an abstract deontic ideal, represented by a set of worlds satisfying the prescriptions of a set of norms, and a logic of an actual deontic ideal, represented by a function picking out, for each action $\alpha$ and stage $w$, the set of stages accessible from $w$ by doing $\alpha$ which are optimal with respect to what can be directly accessed from $w$. This distinction constitutes the most important novelty introduced in the present framework. Crucially, while it is possible for an agent to be in a situation where no ideal stage is directly accessible, in every situation the agent can select, among the directly accessible stages, those which are optimal with respect to that situation, so that it is always possible to perform an action leading to an optimal stage. In particular, if there are actions leading from a given world $w$ to ideal stages, the optimal stages at $w$ are the best ideal stages directly accessible from $w$, that is, the best stages an agent who is complying with the law can reach; if there are no actions leading from a given world $w$ to ideal stages, there are still optimal stages at $w$ which are the best stages the agent can access from $w$. In light of this, we may assume the following procedure as our fundamental rule of conduct.

**Rule.** Ask yourself: are there executable actions leading to an ideal stage? If so, perform one of these actions, possibly an action leading to an optimal stage. If not, perform one of the actions leading to a stage which is optimal relative to the current situation.

Our rule of conduct is thus based on the maxim: always act so as to land on a *safe* stage, i.e., either on an ideal or on an optimal stage. Within this basic framework, a fine-grained analysis of the deontic status of stages, actions and courses of action can be carried out.

**Deontic concepts on stages.** Starting with stages, our rule of conduct allows us to introduce a fourfold deontic classification of accessible stages.

1. Green! stages: ideal and optimal.
2. Green stages: ideal, either optimal or non-optimal.
3. Orange stages: optimal, but not ideal.
4. Red stages: neither ideal nor optimal.

Note that, among ideal stages, it is thus possible to distinguish between the optimal and the non-optimal ones. Given this classification, our maxim becomes: *always act so as to never land on a red stage and, if you can, try to land on an optimal green stage*.

**Deontic concepts on actions.** The deontic status of an action is determined on the basis of the deontic status of the stages to which it possibly leads. More specifically, an action which possibly leads to an ideal stage is a *permitted action*, while an action which possibly leads to an optimal stage is a *recommended action*. Since it is possible that in certain situations no ideal stage is directly accessible and, yet, in every situation there are directly accessible stages which are optimal with respect to that situation, it follows that it is possible that in certain situations no action is permitted, although in all situations some actions are recommended. Crucially, when an ideal stage is directly accessible, the recommended actions are the best permitted actions; that is, those permitted actions which possibly lead to optimal directly accessible stages. We can then introduce the following deontic classification of actions.

1. Green! actions: recommended and permitted.

2. Green actions: permitted, either recommended or not recommended.

3. Orange actions: recommended, but not permitted.

4. Red actions: neither permitted nor recommended.

These four cases can then be illustrated as shown in figure 4, where the fan originating at stage $w_i$ represents the set of possible transitions starting at $w_i$.

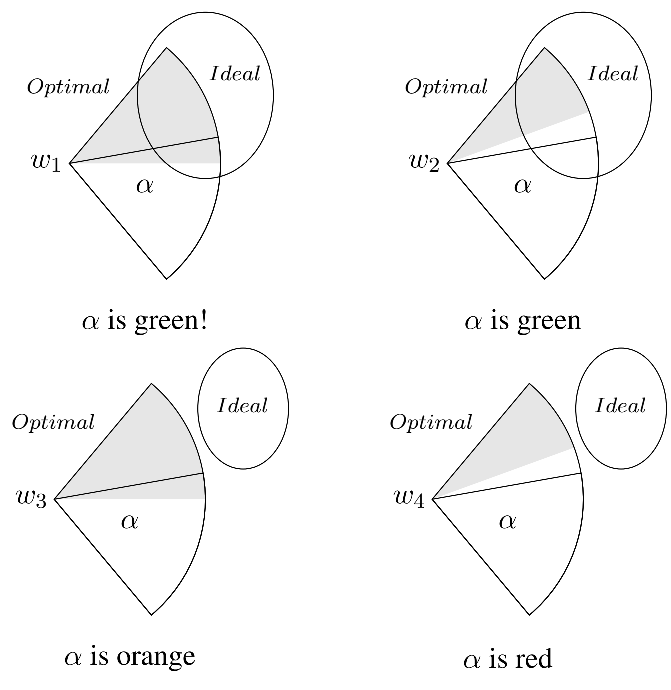



Figure 4: Deontic concepts on actions.

In this framework our maxim becomes: *never perform a red action, and, if you can, perform an optimal green action*. Hence, for an action $\alpha$ to be obligatory it is not necessary that its performance ensures that an ideal or optimal stage is reached: the fact that $\alpha$ is obligatory does not exclude the possibility that certain ways of doing $\alpha$ result in red stages. In this sense, our maxim encodes a notion of obligation which allows for the possibility that obligatory actions are risky.

**Deontic concepts on courses of action.** Deontic concepts on stages and actions allow us to introduce deontic concepts on courses of action. Unlike an action, a course of action can be assessed under two perspectives, depending on whether $(i)$ it possibly leads to a safe stage, and whether $(ii)$ it possibly corresponds to a safe path, that is, to a sequence of transitions consisting only of safe steps. The assumption of the former perspective gives rise to what in the literature has been called *goal norms*, while the assumption of the latter to what has been called *process norms* [3, 4].[2] In light of the distinctions between ideal and optimal stages and between ideal and optimal actions, the two perspectives can be further refined. In particular, if we assess courses of action on the basis of their ending stages, we can distinguish:

1. courses of action instantiated by some paths ending in an ideal stage
   (possibly ending in green stages);

2. courses of action instantiated by some paths ending in an optimal stage
   (possibly ending in green or orange stages).

If we assess courses of action on the basis of the transitions corresponding to them, we can distinguish

1. courses of action instantiated by some paths entirely consisting of permitted actions
   (possibly passing through green stages only);

2. courses of action instantiated by paths entirely consisting of recommended actions
   (possibly passing through green or orange stages only).

In conclusion, at the deontic level, what we need is $(i)$ a set that fixes what is globally ideal; $(ii)$ a set of functions fixing, for each action $\alpha$, what is optimal given what is $\alpha$-accessible at a given stage.

We are now ready to introduce our basic system $\mathbf{ADL}$ of action deontic logic.

[2] The distinction between goal and process norms is already present in Van Der Meyden [16].

## 3 The basic system ADL

The language $\mathcal{L}(\mathbf{ADL})$ of the system $\mathbf{ADL}$ of action deontic logic contains a set $Tm(\mathcal{L}(\mathbf{ADL}))$ of terms and a set $Fm(\mathcal{L}(\mathbf{ADL}))$ of formulas. Assuming a standard distinction between action types and individual actions, let $A$ be a countable set of action types variables. Then $Tm(\mathcal{L}(\mathbf{ADL}))$ is defined according to the following grammar:

$$\alpha ::= a_i \mid 1 \mid \overline{\alpha} \mid \alpha \sqcup \beta \mid \alpha \sqcap \beta$$

where $a_i \in A$. Intuitively, the set $Tm(\mathcal{L}(\mathbf{ADL}))$ is the set of moves or one-step actions. More specifically, 1 is the action type instantiated by any action whatsoever; $\overline{\alpha}$ is the action type instantiated by any action which does not instantiate the type $\alpha$; $\alpha \sqcup \beta$ is the action type instantiated by any action which instantiates either the type $\alpha$ or the type $\beta$ or both; $\alpha \sqcap \beta$ is the action type instantiated by any action which instantiates the types $\alpha$ and $\beta$ in parallel. We assume that an individual action can instantiate different action types. Accordingly, when we say that an action is a token of $a_i$ we do not exclude the possibility of its being also a token of a different type $a_j$.

Turning to the set of formulas of $\mathcal{L}(\mathbf{ADL})$, let $P$ be a countable set of propositional variables. Then $Fm(\mathcal{L}(\mathbf{ADL}))$ is defined according to the following grammar:

$$\phi ::= p_i \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi \mid [F]\phi \mid done(\alpha) \mid [1]\phi \mid [!]\phi \mid I$$

where $p_i \in P$ and $\alpha \in Tm(\mathcal{L}(\mathbf{ADL}))$. The dual modalities $\Diamond, \langle F \rangle, \langle 1 \rangle, \langle ! \rangle$, are defined in the standard way.

The intended interpretation of the modal formulas is as follows. $\Box$ is a standard universal modality characterizing propositions that hold at all possible stages, while $[F]$ is a standard modality characterizing propositions that hold at all the stages that are accessible from the current stage. The introduction of these modalities allows us to distinguish between the stages which are possible and the stages which are both possible and accessible. The distinction is crucial because what is no longer accessible from a given world cannot be optimal with respect to that world. $done$ is a modal property of actions such that $done(\alpha)$ holds at all the stages where $\alpha$ has *just* been performed. [1] is a dynamic ontic modality such that $[1]\phi$ is true when $\phi$ holds in all stages accessible by doing an action, while [!] is a dynamic deontic modality such that $[!]\phi$ is true when $\phi$ holds in all optimal stages accessible by doing an action. Finally, $I$ is the deontic ideal, characterizing stages where no norm is violated.

## 3.1 Semantics

The conceptual framework we adopt is based on the following notion of frame.

**Definition 1.** *(Frame) A frame for $\mathcal{L}(\mathbf{ADL})$ is a tuple $F = \langle W, R, D, R_1, S_1, Ideal \rangle$ such that*

***1.*** $W \neq \varnothing$
***2.*** $R : W \to \wp(W)$
***3.*** $D : Tm(\mathcal{L}(\mathbf{ADL})) \to \wp(W)$
***4.*** $R_1 : W \to \wp(W)$
***5.*** $S_1 : W \to \wp(W)$
***6.*** $Ideal \subseteq W$

Let us comment on each item in turn.

**1.** $W$ is our set of possible stages.

**2.** $R$ is the function that determines the set of stages which are accessible at a given stage and is characterized by the following conditions.

**Constraints on $R$:**
(a) for each $w \in W$, $w \in R(w)$
(b) for each $w, v \in W$, $v \in R(w) \Rightarrow R(v) \subseteq R(w)$

Thus, for each stage $w \in W$, $R(w)$ is the cone containing the stages that are accessible from $w$ by performing any course of action.

**3.** $D$ is the function that determines the set of stages at which a given action has just been performed and has to satisfy the following constraints:

**Constraints on $D$:**
(a) $D(1) = W$
(b) $D(\overline{\alpha}) = W - D(\alpha)$
(c) $D(\alpha \sqcup \beta) = D(\alpha) \cup D(\beta)$
(d) $D(\alpha \sqcap \beta) = D(\alpha) \cap D(\beta)$

Thus, for each action $\alpha$, $D(\alpha)$ is the set containing the stages in which $\alpha$ has just been instantiated and the constraints provide a straightforward connection between the algebra of actions and the algebra of states corresponding to the instantiations of the action types. Hence: (a) some action is instantiated at any stage; (b) instantiating the negation of an action coincides with not instantiating that action; (c) instantiating the disjunction of two actions coincides with instantiating either the first or the second action; (d) instantiating the conjunction of two actions coincides with instantiating both the first and the second action.

**4.** $R_1$ is the function that determines the set of stages which are accessible in one step at a given stage and is characterized by the following conditions.

**Constraints on $R_1$:**
(a) for each $w \in W$, $R_1(w) \neq \varnothing$
(b) for each $w \in W$, $R_1(w) \subseteq R(w)$

Accordingly, it is assumed that some stage is always directly accessible to the agent and that what is accessible in one step is accessible in general.

**5.** $S_1$ is the function that determines the set of optimal stages which are accessible in one step at a given stage and is characterized by the following conditions.

**Constraints on $S_1$:**
(a) for each $w \in W$, $S_1(w) \neq \varnothing$
(b) for each $w \in W$, $S_1(w) \subseteq R_1(w)$

Accordingly, it is assumed that some optimal stage is always accessible to the agent and, of course, that what is optimally accessible in one step is accessible in one step. The conditions on $S_1$ ensure that condition (a) on $R_1$ is always satisfied.

**6.** $Ideal$ is the subset of $W$ containing the best possible stages from the point of view of the law, which are the stages where the law is wholly fulfilled.

**Constraints on $I$:**
(a) for each $w \in W$, $R(w) \cap Ideal \neq \varnothing$
(b) for each $w \in W$, $R_1(w) \cap Ideal \neq \varnothing \Rightarrow S_1(w) \subseteq Ideal$

According to the conditions on $Ideal$, (a) the set of accessible stages always contains some ideal stages and (b) only ideal stages are optimal, if some ideal stages is directly accessible. Importantly, note that we allow for the possibility that the set of ideal worlds directly accessible from a given stage is larger than the set of optimal stages with respect to that stage. In this way, we can account for the idea that some ideal stages may be better than others from a deontic perspective which takes into account not only the current law but also other possible criteria.

*Remark* 1. We do not assume that $R(w)$ coincides with $R_1(w)$, since, as clarified in section 2.2, we allow for a difference between the stages which are accessible and the stages which are directly accessible. This is crucial to account for the fact that the ideal, although always accessible through some course of action, in certain cases, for example at stage 8:00 in our case study, might not be realizable by performing any action.

We are now able to introduce the definitions of the crucial functions of our action deontic logic.

**Definition 2.** *($\alpha$-transitions) $R_\alpha$ is a map $R_\alpha : W \to \wp(W)$ such that, for each $w \in W$, $R_\alpha(w) = R_1(w) \cap D(\alpha)$.*

$R_\alpha$ is a function that, for each possible stage $w$, returns the outcomes of the transitions associated with $\alpha$, when $\alpha$ is performed at $w$, so that $R_\alpha(w)$ is the set of *all stages* that are accessible by doing $\alpha$ at $w$. What we assume is that at any stage an agent is endowed with a set of actions and think of these actions as ways of getting to further stages within a cone of accessible stages. Since actions are conceived of as action types that can be instantiated in different ways, every action corresponds to a set $R_\alpha(w)$ of transitions between stages. What is new in our approach is that $R_\alpha(w)$ is identified with the set of transitions that lead from $w$ to stages where $\alpha$ has been instantiated.[3]

**Definition 3.** *(Optimal $\alpha$-transitions) $S_\alpha$ is a map $S_\alpha : W \to \wp(W)$ such that, for each $w \in W$, $S_\alpha(w) = S_1(w) \cap D(\alpha)$.*

$S_\alpha$ is a function that, for each possible stage $w$, returns the outcomes of the optimal transitions associated with $\alpha$, when $\alpha$ is performed at $w$, so that $S_\alpha(w)$ is the set of *optimal stages* that are accessible by doing $\alpha$ at $w$.

As it is not difficult to see, our notion of frame allows us to capture all the distinctions concerning the deontic status of stages and actions we made in the previous section.

**Deontic status of stages.** With respect to $w$, $v$ is

| | | | |
|---|---|---|---|
| *Green!*: | $v \in S_1(w) \cap Ideal$ | *Orange*: | $v \in S_1(w) - Ideal$ |
| *Green*: | $v \in R_1(w) \cap Ideal$ | *Red*: | $v \notin S_1(w) \cup Ideal$ |

**Deontic status of actions.** With respect to $w$, $\alpha$ is

| | | | |
|---|---|---|---|
| *Green!*: | $S_\alpha(w) \cap Ideal \neq \varnothing$ | *Orange*: | $S_\alpha(w) - Ideal \neq \varnothing$ |
| *Green*: | $R_\alpha(w) \cap Ideal \neq \varnothing$ | *Red*: | $S_\alpha(w) \cup (R_\alpha(w) \cap Ideal) = \varnothing$ |

**Definition 4.** *(Model) A model for $\mathcal{L}(\mathbf{ADL})$ is a pair $M = \langle F, V \rangle$, where*

*(i) $F$ is a frame for $\mathcal{L}(\mathbf{ADL})$*
*(ii) $V$ is a function that maps propositional variables in $\wp(W)$.*

**Definition 5.** *(Truth in a model) The notion of truth in a model for $\mathcal{L}(\mathbf{ADL})$ is inductively defined as follows:*

$$M, w \models p_i \;\Leftrightarrow\; w \in V(p_i)$$

[3]We wish to thank Frederik Van De Putte for insightful discussions on this point.

$M, w \models \neg\phi \;\Leftrightarrow\; M, w \not\models \phi$
$M, w \models \phi \wedge \psi \;\Leftrightarrow\; M, w \models \phi$ and $M, w \models \psi$
$M, w \models \Box\phi \;\Leftrightarrow\; \forall v \in W(M, v \models \phi)$
$M, w \models [F]\phi \;\Leftrightarrow\; \forall v \in W(v \in R(w) \Rightarrow M, v \models \phi)$
$M, w \models done(\alpha) \Leftrightarrow\; w \in D(\alpha)$
$M, w \models [1]\phi \;\Leftrightarrow\; \forall v \in W(v \in R_1(w) \Rightarrow M, v \models \phi)$
$M, w \models [!]\phi \;\Leftrightarrow\; \forall v \in W(v \in S_1(w) \Rightarrow M, v \models \phi)$
$M, w \models I \Leftrightarrow\; w \in Ideal$

The main operators of standard action logic can now be explicitly defined.

**Definition 6.** *Ontic and deontic dynamic operators.*

*(i) $\phi$ is a consequence of doing $\alpha$: $[\alpha]\phi := [1](done(\alpha) \rightarrow \phi)$;*
*(ii) $\phi$ is a consequence of optimally doing $\alpha$: $[!_\alpha]\phi := [!](done(\alpha) \rightarrow \phi)$;*

The dual notions $\langle\alpha\rangle$ and $\langle!_\alpha\rangle$ are defined accordingly. Interestingly, given these definitions, the following equivalences turn out to be valid:

- $\alpha$ is executable: $\langle\alpha\rangle \top \leftrightarrow \langle 1\rangle\, done(\alpha)$.
- $\alpha$ is executable and optimal: $\langle!_\alpha\rangle \top \leftrightarrow \langle !\rangle\, done(\alpha)$.

In particular, it is worth noting that there are two ways in which $\langle!_\alpha\rangle \top$ can fail. First, because $\alpha$ is not executable. Second, because, despite being executable, $\alpha$ does not result in optimal stages. Furthermore, the basic distinctions on actions and courses of actions highlighted in section 2.3 are now expressible in our language.

**Deontic status of actions.** We can see actions as being coloured.

| | | | |
|---|---|---|---|
| $\alpha$ is *green!*: | $\langle!_\alpha\rangle\, I$ | $\alpha$ is *orange*: | $\neg\langle 1\rangle\, I \wedge \langle!_\alpha\rangle \top$ |
| $\alpha$ is *green*: | $\langle\alpha\rangle\, I$ | $\alpha$ is *red*: | $\neg\langle 1\rangle\, I \wedge \neg\langle!_\alpha\rangle \top$ |

**Deontic status of courses of actions.** We can distinguish between process and goal permissions.

| | Process permissions | | Goal permissions |
|---|---|---|---|
| Ideal | $\langle\alpha\rangle\,(I \wedge \langle\beta\rangle\, I)$ | $\rightarrow$ | $\langle\alpha\rangle\,\langle\beta\rangle\, I$ |
| Optimal | $\langle!_\alpha\rangle\,\langle!_\beta\rangle \top$ | $\rightarrow$ | $\langle\alpha\rangle\,\langle!_\beta\rangle \top$ |

The distinction between process and goal norms has been presented only relative to courses of action consisting of two actions, but the definition can be extended to longer courses of action in a straightforward manner.

### 3.2 Axiomatization

The axiom system $\mathbf{ADL}$ consists of the following groups of axioms and rules. The first two groups take into account the static part of the system, while the other groups take into account the dynamic part, both from an ontic and from a deontic point of view.

**Group 1:** axioms and rules for $\Box$

**A1.1** $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

**A1.2** $\Box\phi \to \phi$

**A1.3** $\neg\Box\phi \to \Box\neg\Box\phi$

**R1.1** $\phi/\Box\phi$

**Group 2:** axioms for $[F]$

**A2.1** $[F](\phi \to \psi) \to ([F]\phi \to [F]\psi)$

**A2.2** $[F]\phi \to \phi$

**A2.3** $[F]\phi \to [F][F]\phi$

**A2.4** $\Box\phi \to [F]\phi$

**Group 3:** axioms for *done*

**A3.1** $done(1)$

**A3.2** $done(\alpha) \leftrightarrow \neg done(\bar{\alpha})$

**A3.3** $done(\alpha \sqcup \beta) \leftrightarrow done(\alpha) \vee done(\beta)$

**A3.4** $done(\alpha \sqcap \beta) \leftrightarrow done(\alpha) \wedge done(\beta)$

**Group 4:** axioms for $[1]$ and $[!]$

**A4.1** $[1](\phi \to \psi) \to ([1]\phi \to [1]\psi)$

**A4.2** $[F]\phi \to [1]\phi$

**A4.3** $[!](\phi \to \psi) \to ([!]\phi \to [!]\psi)$

**A4.4** $[1]\phi \to [!]\phi \wedge \langle ! \rangle\, \phi$

**Group 5:** axioms for $I$

**A5.1** $\langle F \rangle\, I$

**A5.2** $\langle 1 \rangle\, I \to [!]I$

The crucial deontic axioms are **A4.4**, **A5.1** and **A5.2**. From **A4.4** it follows that, for every situation, it is possible to single out a set of directly accessible stages which are optimal with respect to that situation; that is, $\langle ! \rangle\, \top$ is derivable in $\mathbf{ADL}$. The first axiom on $I$ ensures that we can always reach an ideal stage through an appropriate course of action, while the second axiom on $I$ ensures that, whenever an ideal stage is directly accessible, the optimal stages are among the ideal stages. The presence of $I$ and $\Box$ in the language allows us to introduce the following deontic concepts on *states*.

**Definition 7.** *Andersonian deontic operators on states based on* $\Box$ *and* $I$.
$[I]\phi := \Box(I \to \phi)$ *and* $\langle I \rangle\, \phi := \Diamond(I \wedge \phi)$.

$[I]\phi$ is a standard concept of obligation for states, as proposed in [1]. It is not difficult to see that $[I]$ is a $KD45$ modality, since we can derive:[4]

[4]The choice of an $S5$-based logic gives us theorems like $O\phi \to \Box O\phi$ and $P\phi \to \Box P\phi$. These

(i) $[I](\phi \to \psi) \to ([I]\phi \to [I]\psi)$
(ii) $[I]\phi \to \langle I \rangle \phi$
(iii)$[I]\phi \to [I][I]\phi$
(iv) $\langle I \rangle \phi \to [I] \langle I \rangle \phi$
(v) $\phi/[I]\phi$

The fundamental distinction we want to highlight here concerns $\langle I \rangle \phi$ and $\langle ! \rangle \phi$. While $\langle I \rangle \phi$ states that $\phi$ holds in some ideal worlds, $\langle ! \rangle \phi$ states that $\phi$ holds in some stages which are optimal with respect to the current situation. As we will see, this distinction gives rise to two different operators of permission.

**Corollary 1.** *Given the definitions of* $[\alpha]$ *and* $[!_\alpha]$ *the following propositions are derivable.*

| | | | |
|---|---|---|---|
| **C1.1** | $[\overline{\overline{\alpha}}]\phi \leftrightarrow [\alpha]\phi$ | **C1.7** | $[!_{\overline{\overline{\alpha}}}]\phi \leftrightarrow [!_\alpha]\phi$ |
| **C1.2** | $[\alpha]\phi \vee [\beta]\phi \to [\alpha \sqcap \beta]\phi$ | **C1.8** | $[!_{\alpha_1}]\phi \vee [!_{\alpha_2}]\phi \to [!_{\alpha_1 \sqcap \alpha_2}]\phi$ |
| **C1.3** | $[\alpha \sqcup \beta]\phi \leftrightarrow [\alpha]\phi \wedge [\beta]\phi$ | **C1.9** | $[!_{\alpha_1 \sqcup \alpha_2}]\phi \leftrightarrow [!_{\alpha_1}]\phi \wedge [!_{\alpha_2}]\phi$ |
| **C1.4** | $[\overline{\alpha \sqcap \beta}]\phi \leftrightarrow [\overline{\alpha}]\phi \wedge [\overline{\beta}]\phi$ | **C1.10** | $[!_{\overline{\alpha_1 \sqcap \alpha_2}}]\phi \leftrightarrow [!_{\overline{\alpha_1}}]\phi \wedge [!_{\overline{\alpha_2}}]\phi$ |
| **C1.5** | $[\overline{\alpha}]\phi \vee [\overline{\beta}]\phi \to [\overline{\alpha \sqcup \beta}]\phi$ | **C1.11** | $[!_{\overline{\alpha_1}}]\phi \vee [!_{\overline{\alpha_2}}]\phi \to [!_{\overline{\alpha_1 \sqcup \alpha_2}}]\phi$ |
| **C1.6** | $\langle \alpha \rangle \top \leftrightarrow \langle 1 \rangle\, done(\alpha)$ | **C1.12** | $[!]\phi \to [!_\alpha]\phi$ |

Since **C1.1** to **C1.5** are derivable, our system is powerful enough to interpret the system proposed by Meyer in [11], except for the axiom on the negation of sequential actions. In addition, due to **C1.6**, the executability of an action, expressed by $\langle \alpha \rangle \top$, is to be distinguished from the abstract possibility of an action, expressed by $\Diamond done(\alpha)$. In fact, while $\langle \alpha \rangle \top \to \Diamond done(\alpha)$, it is possible for $\Diamond done(\alpha)$ to hold even if $\alpha$ is currently not executable.

**Corollary 2.** *The following propositions on* $I$ *are derivable.*

**C2.1** $\langle \alpha \rangle I \to [!_\alpha]I$

**C2.2** $\langle 1 \rangle I \wedge \langle !_\alpha \rangle \top \to \langle \alpha \rangle I$

**C2.1** is a formalization of the basic principle that, if an ideal stage is directly $\alpha$-accessible, then $\alpha$ is recommended only if its performance leads to an ideal

principles are justified by the intended interpretation of a formula like $[I]\phi$. To be sure, $I$ is an ideal state determined by a specific legal code, and we assume that the distinction between what is prescribed and what is not prescribed is also fixed by that same code. In light of this, given that $O\phi$ is interpreted as $\phi$ is prescribed by the code that fixes $I$, the previous principles turn out to be intuitive, since it is impossible to change what is prescribed according to the code without changing that code as well.

stage. **C2.2** is a formalization of the basic principle that, if an ideal stage is directly accessible, then $\alpha$ is recommended only if it is permitted. Since $\langle\alpha\rangle I \rightarrow \langle !_\alpha\rangle\top$ is not derivable (in fact, recall that we allow for the possibility that not all directly accessible ideal stages are also optimal), **C2.2** also tells us that, even when the ideal is accessible, Meyer's concepts of permitted action, which is $\langle\alpha\rangle I$, is weaker than our concept of a green action. In fact, when the ideal is directly accessible, $\langle !_\alpha\rangle \top$ says that $\alpha$ is not only permitted but also recommended.

*Remark* 2. Although we have focused on the deontic part of our axiom system, a brief note on the operator $done$ is here in order. In [11], Meyer suggests the possibility of introducing the modality $done$, which is interpreted as stating that an action has been performed and which is characterized by the axioms in group 3. Meyer also suggests that it would be of interest to go through the logic of $done$. What we have shown is not only how to develop the logic of $done$, but also that, once $done$ is at our disposal, the entire system of dynamic modalities proposed by Meyer can be introduced in terms of [1] and $done$. With respect to other recent proposals of developing a logic of $done$ [6, 14], our proposal has also the advantage of being based on a simple semantics, which does not presuppose the apparatus of temporal logic.

### 3.3 Characterization

The previous axiomatic system is sound and complete with respect to the class of all models for $\mathcal{L}(\mathbf{ADL})$. The proof of soundness is straightforward. The proof of completeness follows from the proof of the fact that any $\mathbf{ADL}$-consistent set of formulas $X$ is satisfiable in a model for $\mathcal{L}(\mathbf{ADL})$, which in turn derives from the construction of a canonical model based on an $\mathbf{ADL}$-complete set $x$ extending $X$.

**Definition 8.** *(Canonical model) Let $x$ be an $\mathbf{ADL}$-complete set of formulas of $\mathcal{L}(\mathbf{ADL})$. The canonical model for $\mathcal{L}(\mathbf{ADL})$ is the tuple $\langle W, R, D, R_1, S_1, Ideal, V\rangle$ such that*

- $W = \{w \mid w$ *is* $\mathbf{ADL}$*-complete and* $x/\Box \subseteq w\}$, *where* $x/\Box = \{\phi \mid \Box\phi \in x\}$
- $R$ *is such that* $v \in R(w) \Leftrightarrow w/F \subseteq v$, *where* $w/F = \{\phi \mid [F]\phi \in w\}$
- $D$ *is such that* $D(\alpha) = \{v \mid done(\alpha) \in v\}$
- $R_1$ *is such that* $v \in R_1(w) \Leftrightarrow w/1 \subseteq v$, *where* $w/1 = \{\phi \mid [1]\phi \in w\}$
- $S_1$ *is such that* $v \in S_1(w) \Leftrightarrow w/! \subseteq v$, *where* $w/! = \{\phi \mid [!]\phi \in w\}$

– $Ideal = \{v \mid I \in v\}$

The proof of the fact that this is indeed a model for $\mathcal{L}(\mathbf{ADL})$ that verifies every formula in $x$ is not difficult to carry out and is left to the reader.

# 4 Deontic Concepts and Paradoxes

Having introduced a basic system of action deontic logic which encodes the conceptual framework introduced in section 2, we can finally provide an explicit characterization of the fundamental deontic concepts on states and actions we will use in order to clarify the intuitions that will help us to solve some of the main paradoxes in deontic logic. Before going into the details, we present here the general schema of the relevant concepts and the basic intuitions underlying them. Importantly, in what follows, we will provide definitions only of permission and obligation; the notion of prohibition will always be intended as the negation of permission.

1. Deontic concepts on states.

   (a) *Permission*
   (Intuition: we are allowed to be in a state)

   (b) *Obligation*
   (Intuition: we are not allowed to exit a state)

2. Deontic concepts on actions.

   (a) *Plain permission*
   (Intuition: we are allowed to perform an action)

   (b) *Plain obligation*
   (Intuition: we are required to perform an action)

3. Deontic concepts on actions.

   (a) *Choice permission*
   (Intuition: we are given a choice and we are allowed to choose)

   (b) *Choice obligation*
   (Intuition: we are given a constraint and we are allowed to choose)

4. Deontic concepts on courses of actions.

(a) *Sequential permission*
(Intuition: we are allowed to tread a path)

(b) *Sequential obligation*
(Intuition: we are not allowed to exit a path)

5. Deontic concepts on actions after a violation.

   (a) *Compensatory obligation*
   (Intuition: after a violation, we are required to compensate)

As we will see, within our framework compensatory obligations will allow us to model contrary to duty obligations in an intuitive sense.

## 4.1 Deontic Concepts on States

Starting with the logic of the *ought-to-be*, within our system it is possible to introduce three main concepts of obligation and permission on states, which further articulate the intuition beyond the Andersonian notions of obligation and permission introduced in section 3.2. The idea is that, under circumstances, it is possible for an agent both to be obliged to do something that, with respect to the set of ideal stages, is merely permitted and to be recommended to do something that, with respect to the set of ideal stages, is not obligatory.

**Definition 9.** *Basic prescriptions on states.*

| | *Basic permission* | *Basic obligation* |
|---|---|---|
| *Global* | $\langle I \rangle \phi := \Diamond(I \wedge \phi)$ | $[I]\phi := \Box(I \to \phi)$ |
| *Local* | $P(\phi) := \langle 1 \rangle (I \wedge \phi)$ | $O(\phi) := \langle 1 \rangle I \wedge [1](I \to \phi)$ |
| *Optimal* | $\langle ! \rangle \phi$ | $[!]\phi$ |

The relations among these concepts are displayed in figure 5.

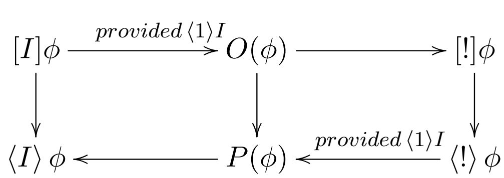
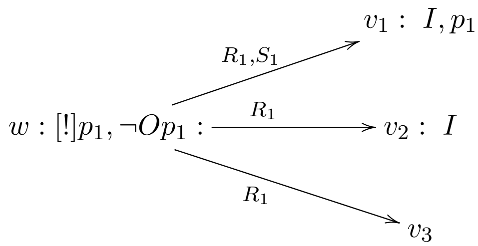



Figure 5: Global, local and optimal deontic concepts on states.

Note that, even when the ideal is accessible, $[!]\phi$ does not entail $O(\phi)$. This depends on the fact that, under our interpretation, it is possible that the stages which are optimal with respect to $w$ are fewer than the ideal stages which are directly accessible from $w$, as shown in the model below (where arrows corresponding to $R$ and reflexive arrows are omitted). This is interesting because it suggests that local and optimal notions of obligations are in line with those consequentialist approaches to obligation which allow for the thesis that what we ought to do is to obtain an acceptable amount of positive consequences instead of a maximum amount thereof.[5]




As final, but extremely significant operators on states, we introduce the following effective operators.

**Definition 10.** *Effective prescriptions on states.*

*(i) Effective permission:* $\mathbf{P}(\phi) := P(\phi) \vee \langle ! \rangle \phi$;
*(ii) Effective obligation:* $\mathbf{O}(\phi) := O(\phi) \vee (\neg \langle 1 \rangle I \wedge [!]\phi)$.

These operators have two pleasant properties: $(i)$ if something is either permitted or obligatory at a given stage, then it is so regardless of whether or not the ideal is directly realizable at that stage (this is not true in many systems of deontic logic, where everything becomes obligatory whenever ideal stages are not (directly) accessible); $(ii)$ it is always true that $\mathbf{O}(\phi) \rightarrow \mathbf{P}(\phi)$, since $O(\phi) \rightarrow P(\phi)$, by the definition of $O$, and $[!]\phi \rightarrow \langle ! \rangle \phi$, by the axioms on $[!]$. Using these operators, we can now introduce our main deontic concepts on actions.

## 4.2 Deontic concepts on actions

Moving from the *ought-to-be* to the *ought-to-do* perspective, let us start by spelling out the distinction between global, local and optimal deontic notions on actions. Global deontic notions are defined in terms of the global modality and give rise to formulas that hold in all possible stages, while local deontic notions give rise to

[5] See [12] for an introduction to these approaches.

formulas that might hold at a stage without being globally true. Finally, optimal deontic notions express what the agent is recommended to do at a given stage.

**Definition 11.** *Basic prescriptions on actions.*

| | *Basic permission* | *Basic obligation* |
|---|---|---|
| *Global* | $\langle I\rangle\, done(\alpha)$ | $[I]done(\alpha)$ |
| *Local* | $P(done(\alpha))$ | $O(done(\alpha))$ |
| *Optimal* | $\langle !\rangle\, done(\alpha)$ | $[!]done(\alpha)$ |

Basic prescriptions on actions are thus specific instances of basic prescriptions on states, the instance being $done(\alpha)$. It is worth noting that our definition of local permission agrees with Meyer's definition of permission, while our local obligation is stronger than Meyer's notion of obligation (cf. [11]). To be sure, it is not difficult to check that the following equivalences hold:

- $P(done(\alpha)) \leftrightarrow \langle 1\rangle\,(I \wedge done(\alpha)) \leftrightarrow \langle\alpha\rangle\, I$
- $O(done(\alpha)) \leftrightarrow \langle 1\rangle\, I \wedge [1](I \rightarrow done(\alpha)) \leftrightarrow \langle\alpha\rangle\, I \wedge [\overline{\alpha}]\neg I$

As in the case of deontic notions on states, local and optimal deontic notions on actions can be used to define operators for effective prescriptions. These can be subdivided into absolute and conditional prescriptions, where conditional prescriptions are to be intended as giving rise to permissions or obligations that only hold once a certain action has been performed.

**Definition 12.** *Effective prescriptions on actions.*

ABSOLUTE*:*
*(i)* $\alpha$ *is permitted:* $\mathbf{P}(\alpha) := \mathbf{P}(done(\alpha))$*;*
*(ii)* $\alpha$ *is obligatory:* $\mathbf{O}(\alpha) := \mathbf{O}(done(\alpha))$.

CONDITIONAL*:*
*(i)* $\alpha$ *is permitted given* $\phi$ *:* $\mathbf{P}(\alpha \mid \phi) := \langle 1\rangle\, \phi \wedge [1](\phi \rightarrow \mathbf{P}(\alpha))$*;*
*(ii)* $\alpha$ *is obligatory given* $\phi$*:* $\mathbf{O}(\alpha \mid \phi) := \langle 1\rangle\, \phi \wedge [1](\phi \rightarrow \mathbf{O}(\alpha))$.

Hence, in accordance with our maxim, i.e. *never perform a red action and, if you can, perform a green action*, we are always allowed to perform non-red actions and required to perform either green or orange actions. It is easy to see that, since in every situation there are actions leading to optimal stages, it is always permitted to do something and it is never obligatory to do everything, not even when no ideal stage is directly accessible. That is, our concept of permission is never empty and our concept of obligation never undergoes trivialization. Furthermore, and importantly, the notions of effective permission and obligation are such that performing a prescribed action is always possible and may be risky: for an action to be either permitted or obligatory, it is in fact not necessary that *all* ways of doing it are either green or orange. Lastly, the distinction between the absolute and the conditional notion of obligation is crucial for the subsequent interpretation of compensatory and contrary-to-duty obligations (but more on this later).

Looking at the properties of $\mathbf{O}$ and $\mathbf{P}$ more closely, we can see that our concepts of effective permission and obligation satisfy the so-called Paradox of Choice Permission and the Ross's Paradox:

- $\mathbf{P}(\alpha \sqcup \beta) \leftrightarrow \mathbf{P}(\alpha) \vee \mathbf{P}(\beta)$
- $\mathbf{O}(\alpha) \rightarrow \mathbf{O}(\alpha \sqcup \beta)$

These principles are not problematic once we focus on their intended interpretation. In fact, recall that $\alpha \sqcup \beta$ is a type that is instantiated just in case either $\alpha$ is instantiated or $\beta$ is instantiated. Then, the first principle just states that it is permitted to instantiate an action type like $\alpha \sqcup \beta$ precisely when it is permitted to instantiate $\alpha$ or to instantiate $\beta$. Similarly, the second principle just states that every required instance of $\alpha$ is a required instance of $\alpha \sqcup \beta$, since every instance of $\alpha$ is an instance of $\alpha \sqcup \beta$. Surely, it may be observed that what we have proposed is not the standard interpretation of statements of permission and obligation. We agree, noting that, within our system, we can still capture the standard interpretation in two different ways. Firstly, we can opt for using notions of strong prescriptions, along the lines of Meyer [11]. Secondly, and more interestingly, we can define specific notions of choice prescriptions.

**Definition 13.** *Choice prescriptions on actions.*

*(i) The choice between $\alpha$ and $\beta$ is permitted:* $\mathbf{P}(\alpha + \beta) := \mathbf{P}(\alpha) \wedge \mathbf{P}(\beta)$.
*(ii) The choice between $\alpha$ and $\beta$ is obliged:* $\mathbf{O}(\alpha + \beta) := \mathbf{O}(\alpha \sqcup \beta) \wedge \mathbf{P}(\alpha + \beta)$.

Definition 13 tells us that a choice between $\alpha$ and $\beta$ is permitted if and only if either one of the following cases obtains:

1. $P(done(\alpha)) \wedge P(done(\beta))$, i.e. $\langle \alpha \rangle I \wedge \langle \beta \rangle I$
2. $P(done(\alpha)) \wedge \langle ! \rangle done(\beta)$, i.e. $\langle \alpha \rangle I \wedge \langle !_\beta \rangle \top$
3. $\langle ! \rangle done(\alpha) \wedge P(done(\beta))$, i.e. $\langle !_\alpha \rangle \top \wedge \langle \beta \rangle I$
4. $\langle ! \rangle done(\alpha) \wedge \langle ! \rangle done(\beta)$, i.e. $\langle !_\alpha \rangle \top \wedge \langle !_\beta \rangle \top$

Thus, in full accordance with our maxim, a choice is permitted if and only if, whatever alternative we choose, we will do at least an orange action. In addition, our definitions agree with the intuition that being permitted to make a specific choice coincides with $(i)$ being presented with a number of alternatives and $(ii)$ being allowed to select one of them. Similarly, being obliged to make a specific choice coincides with $(i)$ being presented with a number of permitted alternatives and $(ii)$ being required to select one of them. More specifically, a choice is required if and only if any action not instantiating at least one of the alternatives is red and, whatever alternative we choose, we will do at least an orange action. As before, what is interesting is that choosing may be risky, even when the choice is in fact required. Yet, unlike before, once definition 13 is adopted, the Paradox of Choice Permission and Ross's Paradox are no longer derivable, as it is not difficult to check.

**Definition 14.** *Prescriptions on courses of action.*

*(i) A path is permitted:* $\mathbf{P}(\alpha;\beta) := \mathbf{P}(done(\alpha) \wedge \mathbf{P}(\beta))$*;*
*(ii) A path is obligatory:* $\mathbf{O}(\alpha;\beta) := \mathbf{O}(\alpha) \wedge \mathbf{O}(done(\alpha) \rightarrow \mathbf{O}(\beta))$.

**Corollary 3.** $\mathbf{P}(\alpha;\beta) \rightarrow \mathbf{P}(\alpha)$ *and* $\mathbf{O}(\alpha;\beta) \rightarrow \mathbf{O}(\alpha)$.

Hence, a path is permitted just in case either one of the following cases obtains:

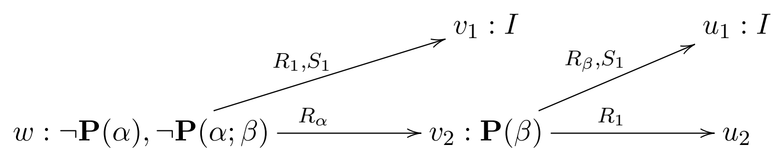
1. $\langle\alpha\rangle (I \wedge \langle\beta\rangle I)$: the path is wholly green;
2. $\langle\alpha\rangle (I \wedge \langle !_\beta\rangle \top)$: the path is green and then green/orange;
3. $\langle !_\alpha\rangle \langle\beta\rangle I$: the path is green/orange and then green;
4. $\langle !_\alpha\rangle \langle !_\beta\rangle \top$: the path is wholly green/orange.

Our notion of permission of a course of action is thus a notion of process permission, in the sense of sections 2.3 and 3.1. In a similar way, our notion of obligation of a course of action is a notion of process obligation. In fact, a course of action is obliged if and only if either one of the following cases obtains:

1. $\langle 1\rangle I \wedge [1](I \rightarrow done(\alpha)) \wedge [1](I \wedge done(\alpha) \rightarrow \mathbf{O}(\beta))$;
2. $\neg \langle 1\rangle I \wedge [!]done(\alpha) \wedge [!](done(\alpha) \rightarrow \mathbf{O}(\beta))$.

That is, if ideal stages are directly accessible, then doing $\alpha$-and-then-$\beta$ is obligatory just in case ideal stages are accessible from the current situation only by doing $\alpha$ and, in all such stages, $\beta$ is effectively obligatory; if no ideal stage is directly accessible, then doing $\alpha$-and-then-$\beta$ is obligatory just in case optimal stages are accessible from the current situations only by doing $\alpha$ and, in all such stages, $\beta$ is effectively obligatory.

It is worth noting that the proposed concept of path permission is not subject to Van Der Meyden's paradox [16], according to which, if there is a way of killing someone after which it is permitted to confess the crime, then it is permitted to kill-someone-and-then-confess-the-crime. Formally: $\langle\alpha\rangle\,\mathbf{P}(\beta) \rightarrow \mathbf{P}(\alpha;\beta)$. Van Der Meyden's paradox follows from neglecting the difference between the fact that a course of action ends in an ideal stage and the fact that the transitions leading to this stage are permitted, in the sense that no step in the path infringes the law, or fails to be what is recommended to the agent in the present circumstances. Our notion of path permission allows us to avoid the paradox precisely because this notion is defined by taking into account the way the path is: what is permitted is not merely what possibly leads to a safe stage but what possibly leads to a safe stage in a safe way. Thus, in the following model, $\langle\alpha\rangle\,\mathbf{P}(\beta) \rightarrow \mathbf{P}(\alpha)$ fails (as above, arrows representing $R$ and reflexive arrows are omitted).




Furthermore, it turns out that, if we consider the notions of prohibition corresponding to those of basic and path permission just introduced, we can avoid Anglberger's paradox [2], according to which everything becomes forbidden once a red action has been performed: $\mathbf{F}(\alpha) \rightarrow [\alpha]\mathbf{F}(\beta)$. Intuitively, this paradox follows from neglecting the difference between the absolutely ideal stages, in which no norm is violated, and the relatively ideal stages, in which the best conditions realizable by the agent in the present situation are in fact realized. Let us then consider the following notions of prohibition on actions and courses of action.

**Definition 15.** *Prohibitions on action and courses of action.*

*(i)* $\mathbf{F}(\alpha) := \neg\mathbf{P}(\alpha)$*, i.e.* $[1](I \rightarrow done(\overline{\alpha})) \wedge [!](done(\overline{\alpha}))$*;*

*(ii)* $\mathbf{F}(\alpha;\beta) := \neg\mathbf{P}(\alpha;\beta)$*, i.e.* $[1](I \wedge done(\alpha) \rightarrow \mathbf{F}(\beta)) \wedge [!](done(\alpha) \rightarrow \mathbf{F}(\beta))$.

Given (ii), it is clear that $\mathbf{F}(\alpha) \rightarrow \mathbf{F}(\alpha;\beta)$. Yet, this does not lead to the paradox, precisely because the notion of prohibition is here defined by taking into account the intuition that, even in non-ideal stages, it is possible to act in an optimal way, given the circumstances. Accordingly, although reaching a stage by performing a prohibited action can only give rise to prohibited courses of action, the fact that a course of action is prohibited is, by itself, not sufficient to exclude the possibility that the last step constituting it is the best the agent can do in the relevant situation, nor that this step actually leads, from this situation, to an ideal stage.

**Definition 16.** *Compensatory obligation.*
*(i) $\alpha$ is obligatory as compensation:* $\mathbf{O}(\beta) \wedge \mathbf{O}(\alpha \mid done(\overline{\beta}))$

Thus, an obligation is compensatory when its being effective depends on the violation of other obligations. We analyse this phenomenon by assuming that it is obligatory to do $\alpha$ after the violation of $\mathbf{O}(\beta)$ when $\beta$ is obligatory at the current stage and, in all the stages where this obligation is violated, $\alpha$ is obligatory. The emergence of a contrary-to-duty obligation can then be modelled as follows. Let us consider cases of contrary-to-duty obligations instantiating this standard schema:

You have to do $\beta$, but you don't do it.

It ought to be that if you do $\beta$ then you have to do $\alpha$.

It ought to be that if you don't do $\beta$ then you have not to do $\alpha$.

In our framework, the most intuitive analysis is based on the distinction between two stages: a stage $w_1$, where $\overline{\beta}$ has not been performed yet, and a stage $w_2$, where $\overline{\beta}$ has been performed. Then, cases of contrary to duty obligations are cases where an obligation is violated, whose fulfillment would have required to perform an action $\alpha$, and a compensatory obligation to perform the opposite action $\overline{\alpha}$ arises. As shown in figure 6, we thus get:

| At $w_1$: | At $w_2$: |
|---|---|
| $\mathbf{O}(\beta)$; $\mathbf{O}(\alpha \mid done(\beta))$; $\mathbf{O}(\overline{\alpha} \mid done(\overline{\beta}))$ | $done(\overline{\beta})$; $\mathbf{O}(\overline{\alpha})$ |

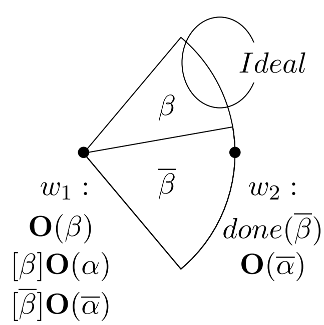



Figure 6: Contrary-to-duty obligations

According to this analysis, the principal duty is given by a local obligation, while the contrary-to-duty obligation is a conditional local obligation.[6] In this case, no contradiction follows, since any situation in which $\alpha$ is prohibited agrees with the law, while the obligation to perform $\alpha$ only arises at stages where the law is violated.[7]

As a final observation, we note that our notion of conditional obligation satisfies dynamical versions of ontic and deontic detachment. This can be seen at the semantic level, where we have

**DFD**: dynamic factual detachment

$$\frac{M, w \models \mathbf{O}(\alpha \mid \varphi) \text{ and } v \in R_1(w) \quad M, v \models \varphi}{M, v \models \mathbf{O}(\alpha)}$$

**DDD**: dynamic deontic detachment

$$\frac{M, w \models \mathbf{O}(\alpha \mid \varphi) \wedge O(\phi) \text{ and } v \in R_1(w) \quad M, v \models I}{M, v \models \mathbf{O}(\alpha)}$$

*Remark* 3. Besides being fundamental as tools to consistently express contrary to duty obligations, it is worth briefly noting that conditional prescriptions also relate in an interesting way to path prescriptions. Focusing on the concept of permission just to give an idea, consider in fact the following:

(1) $P(\alpha) \wedge \mathbf{P}(\beta \mid done(\alpha))$, which is $\langle\alpha\rangle\, I \wedge [\alpha]\mathbf{P}(\beta)$;

(2) $P(done(\alpha) \wedge \mathbf{P}(\beta))$, which is $\langle\alpha\rangle\, (I \wedge \mathbf{P}(\beta))$;

(3) $\mathbf{P}(\alpha;\beta)$, which is $\mathbf{P}(\alpha;\beta)$.

It is then evident that $(1) \rightarrow (2) \rightarrow (3)$, while the converse implications do not hold.

# 5 Conclusion

In this paper, we have proposed a basic system of action deontic logic which captures the fundamental distinctions needed to model deontic systems, intended as systems of norms introduced to regulate action systems, and in which the main deontic paradoxes arising in a dynamic framework can be overcome. The solutions to

[6]The first condition can be relaxed when modeling nested contrary-to-duty obligations.

[7]In [11], Meyer proposes a similar solution of the contrary-to-duty problem. In [2], Anglberger shows that, due to the fact that Meyer's notion of prohibition satisfies $F(\alpha) \rightarrow [\alpha]F(\beta)$, this solution contradicts a basic intuition concerning the structure of the problem. We can see that our solution is immune to this criticism.

the paradoxes we have advanced are based on the introduction of two new deontic categories, namely the categories of optimal stages and of recommended actions. As we have seen, the interaction between these new categories and the standard categories of ideal stages and ideal actions gives rise to a particularly rich analysis of deontic concepts on stages, actions and courses of actions, which allows us to identify the basic maxim: *never perform a red action and, if you can, perform a green action*. It is noteworthy that, despite having the expressive power to account for many important distinctions and concepts both at the ontic and at the deontic level, our system $\mathbf{ADL}$, supplied with the crucial notions of effective prescriptions we have defined in it, is ultimately based on this maxim exclusively. As we have shown, it is in this way that within $\mathbf{ADL}$ it is possible to account for intuitions that, if neglected, lead to paradoxical consequences. To be sure, given our maxim, it is natural to define original notions of choice permission and choice obligation that, besides not being subject to standard paradoxes, also take into account the riskiness of choices. Similarly, notions of path permission and path obligation based on the sequential application of our maxim do not incur in well-known paradoxes concerning the sequential execution of actions. Within this very same framework, an interesting notion of a compensatory obligation can also be introduced which can be used to provide a consistent analysis of cases involving contrary-to-duty obligations. Finally, we wish to emphasize once again that, under the plausible assumption that it is always possible to act in an optimal way, our maxim gives rise to concepts of permission and obligation which are indeed *effective*: if an action is either permitted or obligatory in certain circumstances, then it is so independently of whether or not, in those circumstances, some ideal stages are directly accessible; as a consequence, the actions of the agent can be qualified in a non-trivial and consistent way, even in those case in which she cannot act so as to reach the ideal.

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