Grassmann's epistemology: multiplication and constructivism[1]

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Introduction

Grassmann's epistemological insights have attracted less attention than his mathematical results. Mathematicians have considered him as a precursor of modern mathematical theories – vector algebra, exterior algebra, Clifford algebra, and geometric algebra –, paying attention to the development of mathematical results rather than to the philosophical differences between Grassmann's own project and modern mathematical reconstructions [cf. Cartan 1908; Hestenes 1986]. Historians of mathematics have analyzed the reception and influence of Grassmann (see for example the papers collected in [Schubring 1996]), and philosophers' attention has been driven especially to the Introduction to the 1844 edition of *Ausdehnungslehre*, or to a comparison with other philosophers.[2]

This paper will follow a different path, namely an inquiry on the reciprocal influences of Grassmann's mathematical and philosophical insights, approaching Grassmann's epistemology from the perspective of his mathematical work. The starting point will be an analysis of the notion of product between extensive magnitudes, to which the philosophical literature has paid insufficient attention. Several works of Grassmann will be examined, including the two editions of *Ausdehnungslehre* [A1, A2], *Theorie der Ebbe und Flut* [EBBE], *Kurze Uebersicht über das Wesen der Ausdehnungslehre* [Grassmann 1845], *Geometrische Analyse* [PREIS], *Sur les différents genres de multiplication* (SD) [Grassmann 1855 seq.], and *Der Ort der Hamilton'schen Quaternionen in der Ausdehnungslehre* (HQ) [Grassmann 1877].[3]

The second section of the paper will be devoted to a comparison with three vector-based systems: vector analysis, exterior algebra, and geometric algebra. Considering similarities and differences with respect to these mathematical theories, the attention will be directed to some philosophical issues, such as the question of the homogeneity of the elements, the consideration of the domain as having a fixed or a variable number of dimensions, and the general properties of multiplicative operations. I will claim that there is a certain continuity in Grassmann's epistemological program – contrary to the common tendency in the literature to emphasize the elements of discontinuity between A1 and A2.

In the last section of the paper some aspects of Grassmann's epistemology – the difference between numbers and magnitudes, the relation between geometry and extension

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[2] See for example the articles by A. C. Lewis on Schleiermacher [Lewis 1996] and M. L. Heuser on Schelling [Heuser 1996], or recent (yet unpublished) papers by S. Russ on Bolzano, A. C. Lewis on Cassirer, J. Riche on Whitehead, and M. Hartimo on Husserl.

[3] The 1844 and 1862 editions of the *Ausdehungslehre* and the latter two articles have been translated into English by L. C. Kannenberg [Grassmann 1995; 2000]. References are to the Grassmann's *Gesammelte Werke* [HGW11-HGW32]; English quotations are taken (with some modifications) from the English translation.

theory, and the development of a constructivist approach to mathematics – will be related to the role played by the operation of multiplication in Grassmann's mathematical theory.

1 The product between extensive magnitudes

To understand Grassmann's notion of product between extensive magnitudes, one has first to introduce the concept of an extensive magnitude itself, which is not an easy task, especially if one wants to take into account the various definitions and names introduced by Grassmann in different works. Since I am interested here in the philosophical implications of the choice of characterizing extensional magnitudes by means of their product, I will only briefly summarize some relevant features of the definition of extensive magnitudes.[4]

1.1 Extensive magnitudes

In A1 Grassmann distinguished between a form (or thought form), which constitutes the general object of mathematics, and a concept of extensive magnitude which constitutes the object of extension theory, i.e. the mathematical theory that studies continuous forms generated by different elements. A form, or thought form, is defined as a "particular existant that has come to be by an act of thought" [Grassmann 1995, 24]. Each form is determined by its generating elements, which might be equal or different, and by its generating act, either continuous or discrete. Forms are thus classified according to opposite concepts: equal/different, discrete/continuous. On the basis of this partition of forms in four kinds, which is dependent on their laws of generation, Grassmann classified mathematics in four branches – Number Theory, Theory of Intensive Magnitudes, Combinatorial Theory, and Theory of Extensive Magnitudes –, and claimed that each one should be grounded independently.

Grassmann introduced an extensive formation as the totality of elements obtained by variation of the generating element [HGW11, 28]. He then defined an extensive magnitude as the class of extensive forms that are generated according to the same law by means of equal variations [Grassmann 1995, 47], in other words, the class of all extensive formations that have the same direction, the same orientation and the same size. In modern parlance, an extensive magnitude of first grade is a vector, while an extensive formation of first grade is a bound vector. Continuous numbers are numerical magnitudes, obtained as the quotient of magnitudes of the same grade [HGW11, 130].

In A2 Grassmann seems to drop his former plan to build extension theory independently from all other mathematical branches, as he assumes from the beginning a continuous system of numbers, and he defines extensive magnitudes by means of them.[5] The

[4] I developed a more detailed analysis in the second part of my PhD thesis [Cantù 2003, 153–345], where the accent was put on the discontinuities between the two editions, and on Grassmann's criticism of the 'traditional' definition of mathematics as a theory of magnitudes.

[5] "I define as an *extensive magnitude* any expression that is derived from a system of units (none of which need to be the absolute unit) by numbers, and I call the numbers that belong to the units the *derivation numbers* of that magnitude; for example the polynomial $a_1e_1 + a_2e_2 + \ldots$ or $\Sigma a\mathbf{e}$ or $\Sigma a_r\mathbf{e}_r$, where a_1, a_2, … are real

use of numbers to define magnitudes seems to contradict A1, where numbers were introduced as quotients of magnitudes, but it is to be noted that Grassmann added that "the presentation chosen here very closely follows arithmetic, but in the sense that it assumes the numerical magnitude as a continuous magnitude"[Grassmann 2000, XIV], i.e. as one of the extensive magnitudes introduced in A1. Besides, the following passage – omitted without explanation in the English translation – shows that Grassmann still considered numerical magnitudes as derivable from the definition of extensive magnitudes: "If the system [of units, i.e. an extensive magnitude] consists only of the absolute unity (1), then the derived magnitude is not an extensive but a numerical magnitude" [HGW12, 12].

1.2 The product between extensive magnitudes

Magnitudes of second, third, …and *n*-th grade were obtained by means of the introduction of a multiplication of the generating elements or units. In A2 for example $v=a_1e_1+a_2e_2$ is a quantity of first grade, while $w=a_1e_1+a_2e_1e_2$ is a quantity of second grade.

In EBBE, the inner (linear) product of two segments was defined as the algebraic product of a segment by the orthogonal projection of the second on the first [HGW31, 40, 212]. A similar definition was maintained in A1, although the product was only briefly mentioned in the Introduction [HGW11, 11]. In A2 the inner product of two arbitrary magnitudes was defined by means of the outer product: it is equal to the outer product of the first by the supplement of the second, that is, $[A|B]$ is the inner product of the magnitudes A and B. If the grade of A is a and the number of dimensions of the system is n, the grade of the supplement of A is $n-a$ [HGW12, 93–94]. This definition thus includes in some sense the previously defined notion of inner product.

In EBBE the 'geometric' product of two segments was the oriented surface of the parallelogram thereby delimited [HGW31, 30]. The definition of the 'outer' product in A1 was analogous [HGW11, 80–81]. In SD [HGW21, 214–15] and in A2 [HGW12, 37–38] the definition was based on two axiomatic conditions: the outer product is anti-commutative: $e_re_s=e_se_r$, and it is equal to 0 when the magnitudes are linearly dependent: $e_re_r=0$.

In A1 Grassmann introduced another notion of product, which he called regressive product or 'eingewandt' (a sort of counter-product) [HGW11, 206]. The regressive product is relative to the system that the two extensive magnitudes have in common (e.g. the intersection of the systems they belong to), and the grade of the resulting magnitude depends on the number of dimensions of such a system. This system might vary, and the grade of the result varies according to the system. But the product of two magnitudes can also be considered as relative to a fixed system, e.g. to a system of dimension 3, as in the case of geometry [HGW11, 243], where the product can be considered as applied.[6] If the grade of the result of

numbers and e_1, e_2 …form a system of units, is an extensive magnitude, specifically the one derived from the units e_1, e_2 … by the numbers a_1, a_2, …belonging to them" [Grassmann 2000, 4].

[6] My use of the term 'applied' should not be confused with Grassmann's use of the term 'eingewandt' to denote the regressive product as opposed to the outer or progressive product. An applied product, as I intend it, is a regressive product whose result is univocally determined, since it is relative to a fixed system.

the product is $> n$, $< n$, or $= n$, the product increases, decreases, or either increases or decreases the grade of the magnitudes. Geometry is an application of extension theory, exactly because of the 3-dimensionality of space: the product is always relative to it. In A2 the regressive product was not abandoned, as a general notion of a product relative to a system of given dimension was developed: the grade of the result of the product of two magnitudes A and B of grade q and r respectively in a domain of dimension n is defined as $q+r$ ($mod\ n$) [HGW12, 66]. Progressive and regressive products correspond to the outer and to the inner product. The latter is generalized by means of the notion of the supplement of a given magnitude in a domain of fixed dimension, so as to decrease the grade of magnitudes, but still expresses orthogonality. It is precisely in A2 that the idea of introducing products as a means to increase but also decrease the dimension of the magnitudes themselves becomes fully explicit. It is exactly because this fact went generally unnoticed that a huge discontinuity between the two editions has been defended in the philosophical and mathematical literature. On the contrary, the general notion of product considered in A1 as relative to a variable domain is restricted in A2 to the case of what we have called an applied product, i.e. a product that is relative to a system of fixed dimension. This seems to be a consequence of Grassmann's aim at unifying the two products under the name of a 'product relative to a principal domain'.

Finally, in HQ, a late writing that Grassmann published in order to prove the originality of his own theory with respect to Hamilton's quaternions, a third kind of operation is considered: the median product, which is defined as the sum of the inner and the outer product [HGW21, 268]: $ab = \Lambda[a|b]+\mu[ab]$. This notion of product has been later developed by Clifford, and by the defenders of geometric algebra [Hestenes 1986].

2 A comparative philosophical analysis

In order to appreciate the philosophical interest of Grassmann's ideas, I will briefly ccompare it to three other vector systems based on the notion of product: vector analysis, multivector algebra and geometric algebra.

2.1 The product between vectors and multivectors

The first approach has been developed by Gibbs and Heaviside in the case of a three-dimensional space. They considered the case $V=R^3$ and defined two products. The dot product of two vectors $V \times V \rightarrow R$ is the product of the moduli of the vectors by the cosine of the angle between them: since the resultant is a scalar and not itself a vector, it is also called a scalar product. The cross product of two vectors $V \times V \rightarrow V$ is defined as the vector that is perpendicular to both vectors, so directed that the triple of vectors might be a right triple, and whose modulus is equal to the product of the moduli by the sine of the angle between them. The cross product is thus itself a vector. The algebra generated by the dot product is different from the algebra generated by the cross product: the former is commutative, and not associative; the latter is anti-commutative and not associative.

The second approach, exterior algebra, is based on an operation of multiplication – the wedge product – defined as $\wedge : V^{n} \rightarrow \Lambda(V)$, , which generates multivectors of different grade. Scalars are multivectors of grade 0: $K = \Lambda^{0}(V)$. The domain of this algebra, which is associative and anti-commutative, is the direct sum of the subspaces containing the entities of a given grade k. V is thus itself a subspace of $\Lambda(V)$.

Geometric algebra, first presented by D. Hestenes as a foundation for classical mechanics [cf. Hestenes and Sobczyk], was inspired by Clifford's and Grassmann's works. The main difference with respect to exterior algebra consists in the definition of a single notion of multiplication that allows the construction of a unique algebra for inner and outer product. The basic idea is to introduce a general product to explain – like Grassmann's median product – both the increase and the decrease of a magnitude's grade. The inner product $a{\cdot}b$ of two vectors a and b is considered as a product that decreases the grade, whereas the outer product $a\wedge b$ is conceived as a product that increases the grade. As in multivector algebra, there are magnitudes of various grade, which are called scalars if $K = 0$, and k-blades if $k > 0$, to emphasise the fact that unlike scalars they have directional features.

The geometric product is defined as the sum of the inner and outer product: $ab{=}a{\cdot}b{+}a\wedge b$, where a and b are magnitudes of any grade. If a and b are 1-blades, i.e. vectors, the definition implies for example the addition of a scalar and a 2-blade, i.e. a bivector: geometrically interpreted, this means that one should add a number and an oriented plane segment.

Other geometrical properties such as coplanarity or perpendicularity are expressed by means of the commutative or anti-commutative property of the product: vectors are collinear if and only if their geometrical product is commutative, and they are orthogonal if and only if the geometrical product is anti-commutative. Apart from the non-commutativity of multiplication, which of course has some consequences on the definition of division, the algebra thus obtained is very similar to the algebra of scalar quantities (numbers), and "facilitates the transfer of skills in manipulations with scalar algebra to manipulations with geometric algebra" [Hestenes 1986, 36].

2.2 Domain and homogeneity

A comparison of the different 'vector algebras' raises some questions concerning the nature of the domain of extensive magnitudes. Is it closed under multiplication? Is it possible to define addition between non homogeneous elements?

In the Gibbs-Heaviside approach, the domain is closed under the additive and the multiplicative operation: the cross product of two vectors is again a vector, i.e. a magnitude of the same dimension. Besides, only homogeneous magnitudes are added or multiplied.

Grassmann's product on the contrary might produce magnitudes of higher or smaller grade; besides, it can be defined between magnitudes of different grade. Addition on the contrary is generally defined between homogeneous elements in such a way that the result is again an element of the same dimension [HGW12, 49]. Yet homogeneity properly applies only to ‘real’ addition, and not to ‘formal’ addition.[7] A kind of addition can be defined even between non homogeneous magnitudes, as it is the case in A1, where the addition of a segment and a point gives a point. But here the addition is just ‘formal’, and the symbol of addition has to be interpreted as a movement from one point to another point rather than as a concatenation of magnitudes [HGW11, 166]. The ‘formal’ addition is not a sum of extensive magnitudes, but an operation that shares the same algebraic properties.

Exterior algebra similarly admits that the product might generate elements of different grade, but does not introduce a geometrical interpretation of addition of entities of different grade. Besides, the multivector space is generated by the wedge product from an *n*-tuple of one-dimensional vectors, so the multiplication is primarily introduced between homogeneous elements.

The geometric product defined by Hestenes as the sum of an inner and an outer product is closed under the domain of *k*-blades. In other words the product is defined between any two graded magnitudes and is again a graded magnitude. Grassmann’s notion of a unique, applied product (i.e. relative to a principal domain) can be nicely described, yet the idea of a product of magnitudes relative to a variable domain can hardly be explained from this perspective, given the fact that the domain is closed under the product. Besides, geometric algebra explicitly refuses the law of homogeneity, arguing that it is not only possible but also useful to abandon this mathematical taboo [Hestenes 1986, 30].

This brief comparison between vector analysis, Grassmann’s extension theory, exterior algebra and geometric algebra has already shown that there are some differences, both from a technical and from a philosophical point of view with respect to Grassman’s own theory. On the one hand I have shown that modern algebras are not a complete description of Grassmann’s notion of extensive magnitude but rather capture only some aspects of a more general philosophical project. On the other hand I have directed attention to some philosophical questions that rise from the texts of Grassmann, as he distinguishes between the generation of a magnitude of grade *n* by means of a *n*-tuple of one-dimensional generators and an applied product, or between a non homogeneous formal addition and a homogeneous real addition.

3 Conclusion

As a conclusion, I will now claim that the product between extensive magnitudes is related to Grassmann’s non reductionist interpretation of the relation between numbers and magnitudes, to a new way of introducing the distinction between abstract and applied mathematics, and to the development of a constructivist approach in mathematics.

In A1 continuous numbers are defined in the Newtonian’s way as the quotient of extensive magnitudes; in A2 they are used to define the notion of extensive magnitude, and they are introduced themselves as numerical magnitudes. In SD, the article written after Cauchy’s presentation at the French Academy of Sciences of his work on the ‘clefs

[7] I thank Dominique Flament for directing my attention to this point.

algébriques', the analysis of multiplication allows to better distinguish the main algebraic difference between numbers and magnitudes, i.e. the commutativity or anticommutativity of the product [HGW11, 214–15]. So, continuous numbers are defined by means of magnitudes, and not viceversa. Natural numbers are independently defined in the *Lehrbuch der Arithmetik*, as the result of a discrete generation from equal elements [LA, 2–3]. The independent foundation of arithmetic and extension theory has the aim of bringing attention to the structural similarities based on the analogy in the construction process. The product between extensive magnitudes is analogous to the operation of addition in arithmetic, because they are both generating laws. The product is used in the axiomatic definition of numbers and magnitudes: both are characterized by linearity, but the product of the former commutes, thus making the operations simpler, while the product of the latter does not, which makes it easier to "grasp the gradually emerging magnitudes in their simplest concepts" [Grassmann 2000, 27]. Geometric algebra best explains Grassmann's idea that numerical magnitudes can be indifferently defined as the quotient of two magnitudes of the same grade, or as the only magnitudes whose product commute, as it defines scalars as vectors of grade 0 that satisfy commutativity. The notion of product is thus essential in all works of Grassmann to grasp the difference between numbers and magnitudes, and also the reasons why he refuses a reductionist foundation of the former on the latter, or viceversa.

Such a reductionism was quite widespread in analysis and in the development of algebra as a universal mathesis, but Grassmann developed a new notion of extensive magnitudes, based on abstract constraints concerning their multiplicative generation rather than on a definition of magnitude based on addition, order, and homogeneity conditions (as in the generalized theory of proportions). Besides, he classified the branches of mathematics according to their different operational rules, rather than on empirical criteria of abstraction or on the distinction of different species belonging to a common genus [cf. Cantù 2008]. In the case of extensive magnitudes the fundamental operational rule is exactly the product. Finally, considering geometry as an application of extension theory, and geometrical figures as multivectors generated by vectors rather than sets of points generated by a point, he suggested a radical alternative to analytical geometry. Grassmann's epistemological shift in 19th century geometry is due to the previously mentioned factors, which are all connected to the introduction of a new notion of product between extended magnitudes: that's the reason why the originality of Grassmann's philosophical project cannot be appreciated without driving attention to the notion of product.

In particular, the product is related to Grassmann's constructivism, which is based on a different notion of generalization, and on the fact that the knowledge of mathematical forms relies on the understanding of the rules of generation of the forms themselves. Parting from the traditional definition of mathematics as a science of magnitudes, Grassmann considers mathematical forms as particulars rather than universals. Generalization is not conceived as an enlargement of a given domain by means of the addition of new elements; it is rather obtained by modifiying the defining conditions of the fundamental operations. In particular, the ambivalent role played by geometry in A1 and often reproached to Grassmann as an incapacity of achieving a truly abstract perspective, should be considered in the light of the distinction between the 'general' notion of product defined in A1 for two magnitudes with respect to the domain generated by the magnitudes themselves (regressive product in its general formulation), and the 'particular', regressive, applied product defined both in A1 and in A2 with respect to a system of *n* dimensions. Geometry is an application of extension theory, essentially because multiplication in space is relative to the fixed number of dimensions of the space itself. This is the primary sense in which geometrical magnitudes are 'embodied' extensive forms: the operations on geometrical figures are relative to a fixed system, the 3-dimensional space.

The idea of considering as more 'general' the product relative to a variable domain – a domain that is not closed under the operation but rather a result of our carrying out the operation itself –, is one of Grassmann's most interesting philosophical ideas that lacks an adequate representation in the mentioned mathematical theories. One could claim that the general notion of product (relative to a variable domain) was substituted in A2 by a general notion of product (relative to a principal domain) because of technical mathematical difficulties. The problem with this answer is that it does not take into account the fact that Grassmann never really abandoned the idea of considering operations as determined independently from the domain they are applied to. Firstly, both in SD and A2 Grassmann developed an axiomatic definition of the multiplication that is not relative to a principal domain. Besides, the refusal to admit a domain of elements given prior to, or independently from the generation of the elements themselves, is an idea that Grassmann never abandons, and a basic assumption of his epistemological "constructivism". The latter is grounded on the distinction between formal sciences – where no constraint on the domain is taken as granted, and the forms are one and the same with their construction –, and real sciences, where some constraints are accepted from the onset, and forms are thus 'embodied' in a fixed domain. Grassmann's constructivism is based on the idea that there are some fixed fundamental operations rather than a fixed domain; besides, a general notion is particularized when further conditions are fixed, as in the case of the regressive product which is less general, if considered as relative to a unique domain.

References

Barnabei, Marilena, Andrea Brini and Gian-Carlo Rota. 1985. On the Exterior Calculus of Invariant Theory. *Journal of Algebra* 96: 120–160.

Cantù, Paola. 2003. *The Definition of Mathematics: From a Theory of Quantities to a Theory of Forms. Grassmann's Extension Theory* (Italian). PhD thesis, Università di Genova.

Cantù, Paola. 2008. Aristotle's prohibition rule on kind-crossing and the definition of mathematics as a science of quantities. *Synthese* 2008.

Cartan, Élie Joseph. 1908. Nombres complexes. Exposé, d'après l'article allemand de E. Study. *Encyclopédie des sciences mathématiques pures et appliqués*, volume I.1, Paris: Gauthier-Villars, 329–468.

Gibbs, Josiah Willard. 1961. *The Scientific Papers of J. W. Gibbs*, volume 1–2, New York: Dover.

Graßmann, Hermann. 1845. Kurze Uebersicht über das Wesen der Ausdehnungslehre. *Archiv der Mathematik und Physik* 6 (1845): 337–350. Reprint in: HGW11, 297–312.

Graßmann, Hermann. 1855 seq. Sur les différents genres de multiplication. *Journal für die reine und angewandte Mathematik* 49 (1855): 123–141. Reprint in: HGW21, 199–217.

Graßmann, Hermann. 1877. Der Ort der Hamiltonschen Quaternionen in der Ausdehnungslehre. *Mathematische Annalen* 12 (1877): 375–386. Reprint in: HGW31, 268–282.

Grassmann, Hermann Günther. 1994. *La science de la grandeur extensive*. French transl. by Dominique Flament and B. Bekemeier, Blanchard, Paris.

Grassmann, Hermann Günther. 1995. *A New Branch of Mathematics: The "Ausdehnungslehre" of 1844 and Other Works*. Engl. transl. by Lloyd C. Kannenberg. Chicago, Ill. [u.a.]: Open Court.

Grassmann, Hermann Günther. 2000. *Extension Theory*. Engl. transl. by Lloyd C. Kannenberg. Providence, R.I. : American Mathematical Society.

Hestenes. David. 1986. *New Foundations for Classical Mechanics*. Dordrecht: Kluwer.

Hestenes, David and Garret Sobczyk. 1984. *Clifford Algebra to Geometric Calculus*. Dordrecht [u.a.]: Reidel.

Heuser, Marie-Luise. 1996. Geometrical Product - Exponentiation - Evolution. Justus Günther Graßmann and Dynamist Naturphilosophie. In Schubring, ed. 1996, 47-58.

Lewis, Albert C. 1996. The influence of Grassmann's theory of tides on the 'Ausdehnungslehre'. In Schubring, ed. 1996, 29-35.

Schubring, Gert, ed. 1996. *Hermann Günther Grassmann (1809-1877): Visionary Mathematician, Scientist and Neohumanist Scholar: Papers from a Sesquicentennial Conference*. Dordrecht: Kluwer.