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DLEAC AND THE REJECTION PARADOX

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Abstract

In this paper we first develop a Dialethic Logic with Exclusive Assumptions and Conclusions, DLEAC. We adopt the semantics of the *logic of paradox* (LP) extended with a notion of *model* suitable for DLEAC, and we modify its *proof theory* by refining the notions of *assumption* and *conclusion*, which are understood as speech acts. We introduce a new paradox – the *rejectability paradox* – first informally, then formally. We then provide its derivation in an extension of DLEAC containing the *rejectability* predicate.

1 Introduction

Rejection is standardly considered a speech act that expresses an attitude of dissent. In the last years, some calculi – whose aim is to formalize such a notion, such as the *refutation* or *rejection calculi* – have been proposed.

For a general introduction to these calculi, specifically in propositional logic, see [19]. Let me review two examples: Skura's refutation calculi (developed in [16]) and Wansing's [21] natural deduction calculus. Skura's refutation calculi (see [16] but also [17], and [18]) is based on a Łukasiewicz-style refutation calculi for propositional logics (see on this [20]). Skura proposed a system for the modal logic of *S4* in [17]. With the same purpose, H. Wansing, in his [21] (and in other papers), introduced a natural deduction calculus whose central idea was to begin with pairs comprising a set of assertions and a set of rejections, obtaining a similar pair by inference. Wansing's idea was to dualize the introduction and elimination rules for intuitionistic propositional logic with a primitive notion of dual proof to obtain a

kind of *bi-intuitionistic* propositional logic that combines *verification* and its dual, i.e. *falsification*.

Here I propose a refutation calculus based on a dialethic conception of *negation* and *refutation* or *denial*.

Dialethic *negation* is not exclusive, whereas *denial* is: in a dialethic framework A and $\neg A$ may both be true, but you cannot correctly assert and deny A . This is how Priest in [10] tries to recover the exclusivity of negation by introducing the notion of *rejection* or *denial*¹ as a *speech act*. He claims that while it is possible to *accept* both a sentence and its negation², one *cannot* accept and reject the same sentence. Assertion and rejection or denial are *incompatible* speech acts.

In this paper, I take the impossibility of accepting and rejecting the same sentence as primitive. In this way, I conceive the rejection of sentence A as a speech act that – in virtue of its very meaning – expresses the fact that A is *only* false. Similarly, the act of rejecting $\neg A$ expresses the fact that A is *only* true.

This dialethic use of rejection suggests a theory of natural deduction, where the acts of *assuming* and *concluding* may be understood in an *ordinary* or *exclusive mode*. To assume a sentence in an ordinary mode amounts to supposing that it is *at least* true; to assume it in an exclusive mode amounts to supposing that it is *only* true.

To assume A in the ordinary mode then corresponds to the assertion of A , whereas to assume A in the exclusive mode corresponds to the rejection of $\neg A$.

Any sentence can be rightfully assumed in an *ordinary* or *exclusive* mode at will. Similarly, to prove a sentence in an ordinary mode proves that it is (at least) true (under certain assumptions); to prove it in an exclusive mode proves that it is only true.

Accordingly, concluding in the ordinary mode is to be understood as the assertion of the conclusion, and concluding in the exclusive mode as the rejection of the negation of the conclusion.

The acts of proving A and $\neg A$ in an exclusive mode are incompatible because they both indefeasibly lead to the rejection of some assumptions they depend on. Specifically, concluding A and $\neg A$ in an exclusive mode – independent of any hypothesis – cannot in principle be performed by any rational human being. In this way, I realize the dialethic aim of taking exclusivity as extraneous to the meaning of logical negation and embedded in the speech acts of assuming and concluding. I am going to formalize such speech acts within a modified natural deduction, where they will be governed by *indefeasible rules*.

¹ For a general background on denial in non-classical theories, see [13, §3].

² On the thesis see, also, [8]. For a recent discussion of the topic see also [7].

The goal of this paper is to formulate the above-mentioned modified natural deduction, via a dialethic logic with exclusive assumptions and conclusions DLEAC, where *exclusivity* is expressed *via* certain speech acts. Specifically, in DLEAC, exclusivity is expressed using the speech acts of assuming and concluding. In this paper, I adopt the semantics of the *logic of paradox* (LP)³ extended with a notion of *model* suitable for DLEAC and I modify its *proof theory* by refining the notions of assumption and conclusion, which are understood as speech acts (I follow, in this part of the paper, [5]). In the second part of the paper, I introduce a new paradox – the *rejectability paradox* – first informally, then formally; I give its derivation in an extension of DLEAC.

2 The Basics of DLEAC

Let me first introduce the basic elements of DLEAC, specifically its syntax and semantics.

Let L be a language of first-order logic with identity (FOL =) with individual constants and predicates of any *ariety*. For the sake of simplicity, I omit function symbols in L . I adopt the semantics for LP extended with a new, generalized notion of the model.

Let me briefly review the semantics for LP.⁴

A dialethic interpretation of the propositional logic consists of an evaluation v that assigns to each atomic formula a member of the set $\{\{1\}, \{0\}, \{0, 1\}\}$. The v is extended to the complex formulas using the following clauses:

- (\vee) $v(A \vee B) = \{1\}$ if either $0 \notin v(A)$ or $0 \notin v(B)$;
 $v(A \vee B) = \{0\}$ if $1 \notin v(A)$ and $1 \notin v(B)$;
 $v(A \vee B) = \{0, 1\}$ otherwise.
- (\wedge) $v(A \wedge B) = \{1\}$ if $0 \notin v(A)$ and $0 \notin v(B)$;
 $v(A \wedge B) = \{0\}$ if either $1 \notin v(A)$ or $1 \notin v(B)$;
 $v(A \wedge B) = \{0, 1\}$ otherwise.
- (\neg) $v(\neg A) = \{1\}$ if $v(A) = \{0\}$;
 $v(\neg A) = \{0\}$ if $v(A) = \{1\}$;
 $v(\neg A) = \{0, 1\}$ otherwise.

A sentence A is true if $1 \in v(A)$, is false if $0 \in v(A)$;
 A is exclusively true if $0 \notin v(A)$, is exclusively false if $1 \notin v(A)$.

³For a general background on LP, see [1], [2], [9], [15], [3].

⁴For details see [11, sez. 5.2, 5.3].

This semantics is extended in a similar way to first order logic with identity. I simplify, making the assumption that there is a name in the language L for every object of the domain D of quantification.

An evaluation v assigns to every individual constant a member of the domain D , and assigns to every unary predicate P two subsets of D : the extension P^+ and the counter-extension P^- , possibly overlapping, with the only constraint that $P^+ \cup P^- = D$. Then:

$$\begin{aligned} v(Pa) &= \{ 1 \} \text{ if } a \in P^+ - P^- \\ v(Pa) &= \{ 0 \} \text{ if } a \in P^- - P^+ \\ v(Pa) &= \{ 0, 1 \} \text{ if } a \in P^+ \cap P^- \end{aligned}$$

Similarly for predicates of degree > 1 .

The constraints for the identity sign ($=$) are the following:

$$(=)^+ = \{(a, a) : a \in D\}, \text{ while } (=)^- \text{ is arbitrary with the only constraint that } (=)^+ \cup (=)^- = D.$$

The clauses for the universal and existential quantifiers are analogous to those of conjunction and disjunction, respectively.

I extend the semantics of LP by introducing a notion of model suitable for DLEAC.

Let S be any set of sentences of a first order language L , some of which may be starred (i.e. marked by a star $*$). Observe that stars $*$ do not belong to the object language L .

A model M of S is an LP-interpretation in which all sentences of S are true and the starred ones are exclusively true.

A sentence A (a starred sentence A^*) is a *semantic consequence* of a set S of possibly starred sentences, in symbols $S \models A^*$, if it is true (exclusively true) in every model of S .

3 DLEAC: Deductive rules

Let $A, B, C \dots$ be formulas of a first order language L , and let Γ be a finite set of possibly starred formulas.

A sequent is an expression of the form:

$$\Gamma: C^*$$

to be read: “From the assumptions in Γ , one can infer the conclusion C (in an ordinary or exclusive mode).”

The non-starred formulas in Γ are assumed to be in an ordinary mode, and the starred ones in an exclusive mode. Similarly, the conclusion C can be understood in an ordinary or in an exclusive mode.

3.1 Basic deductive rules for DLEAC

In this section I list the primitive inference rules (I follow [5]). When stars occur in parentheses () the deductive rule holds in the double form:

- with all stars in parentheses at work
- with all stars in parentheses deleted.

Reflexivity:

$$A(*): A(*)$$

$$A*: A$$

The informal reading of the first rule is the following: From the assumption that A is only true (at least true), it follows that A is only true (at least true). The informal reading of the second rule is: From the assumption that A is only true it follows that A is (at least) true.

Weakening:

$$\frac{\Gamma : A(*)}{\Gamma \Delta : A(*)}$$

Cut:

$$\frac{\Gamma : A(*), \Delta A(*): B}{\Gamma \Delta : B}$$

$$\frac{\Gamma : A(*), \Delta A(*) : B^*}{\Gamma \Delta : B^*}$$

Conjunction:

$$I_{\wedge} \frac{\Gamma : A(*), \Delta : B(*)}{\Gamma \Delta : A \wedge B(*)}$$

$$E_{\wedge} \frac{\Gamma : A \wedge B(*)}{\Gamma : A(*)}$$

$$E_{\wedge} \frac{\Gamma : A \wedge B(*)}{\Gamma : B(*)}$$

Disjunction:

$$I_{\vee} \frac{\Gamma : A (*)}{\Gamma : A \vee B (*)}$$

$$E_{\vee} \frac{\Gamma A : C(*), \Delta B : C(*), \Lambda : A \vee B}{\Gamma \Delta \Lambda : C(*)}$$

$$E_{\vee} \frac{\Gamma A^* : C(*), \Delta B^* : C(*), \Lambda : A \vee B^*}{\Gamma \Delta \Lambda : C(*)}$$

Double negation:

$$A (*) : \neg\neg A(*)$$

$$\neg\neg A (*) : A(*)$$

Introduction of absurd (IA):

$$\frac{\Gamma : A^*, \Delta : \neg A}{\Gamma \Delta : A \wedge \neg A^*}$$

The informal justification of IA is the following: From A and $\neg A$ follows $A \wedge \neg A$. Furthermore, since A is only true, it cannot be a *dialetheia*; therefore $\neg A$ also cannot be a *dialetheia*. As a result, neither of the conjuncts of $A \wedge \neg A$ can also be false, and therefore, $A \wedge \neg A$ is *only true*.

Since $\neg(A \wedge \neg A)$ is a *dialethic logical law*, the conclusion $A \wedge \neg A^*$ is an *authentic absurd*, i.e. a conclusion unacceptable even by a dialetheist. Since, dialethically, $A \wedge \neg A$ might be true, it does not count as an absurd. For this reason, by an *absurd*, I mean a formula $A \wedge \neg A$ that is *only true*.

Reductio ad absurdum (RAA):

$$\frac{\Gamma A^*: B \wedge \neg B^*}{\Gamma : \neg A}$$

$$\frac{\Gamma A: B \wedge \neg B^*}{\Gamma : \neg A^*}$$

Informally, RAA works in this way: If the assumption that A is true (only true) leads to the *authentic absurd*, it cannot be true (only true), hence it is *only false* (at least false).

The rules for the quantifiers are analogous to those of conjunction (\wedge) and disjunction (\vee). The rules for identity are as follows:

Introduction of identity (I =):

$$: x = x$$

Elimination of identity (E =):

$$x = y, Px : Py$$

$$E = \frac{\Gamma A^*: \neg(t = t)^*}{\Gamma : \neg A}$$

$$E = \frac{\Gamma A: \neg(t = t)^*}{\Gamma : \neg A^*}$$

Observe that, according to the semantics of identity (=), a sentence having the form $(t = t)$ ⁵ cannot be *exclusively false*.

3.2 Derived deductive rules for DLEAC

In this section, I introduce some derived rules of DLEAC:

Material conditional:

$$\frac{\Gamma A (*): B(*)}{\Gamma : \neg A \vee B}$$

$$\frac{\Gamma A^*: B}{\Gamma : \neg A \vee B}$$

$$\frac{\Gamma A: B(*)}{\Gamma : \neg A \vee B(*)}$$

Elimination of absurd (Ex absurdo quodlibet) (EA):

$$EA \frac{\Gamma : A \wedge \neg A^*}{\Gamma : B^*}$$

Notice that EA is a derived rule.

Modus ponens (MPP):

$$MPP \frac{\Gamma : A^*, \Delta : \neg A \vee B}{\Gamma \Delta : B}$$

⁵where 't' is an individual constant or a variable.

$$\text{MPP1} \frac{\Gamma : A, \Delta : \neg A \vee B^*}{\Gamma \Delta : B^*}$$

For an example of how DLEAC works, here is the derivation of MPP1:

1	1.	A	Assumption
2	2.	$\neg A \vee B^*$	Assumption
3	3.	$\neg A^*$	Assumption
1, 3	4.	$A \wedge \neg A^*$	IA
1, 3	5.	B^*	EA
6	6.	B^*	Assumption
6	7.	B^*	<i>Reflexivity</i>
1, 2	8.	B^*	E \vee

Following LP, the material conditional is not a genuine conditional because, in general, it does not permit the validity of MPP.⁶ In this approach, the validity of MPP is appropriate under a starred assumption. This way, we obtain the following reading of the quasi-validity of MPP for a dialetheist: MPP is appropriate when at least one of the two premises is starred.

The following are other derived rules of DLEAC.

De Morgan rules:

$$\frac{\Gamma : \neg(A \wedge B)(*)}{\Gamma : \neg A \vee \neg B (*)}$$

$$\frac{\Gamma : \neg A \vee \neg B (*)}{\Gamma : \neg(A \wedge B)(*)}$$

$$\frac{\Gamma : \neg(A \vee B)(*)}{\Gamma : \neg A \wedge \neg B(*)}$$

$$\frac{\Gamma : \neg A \wedge \neg B(*)}{\Gamma : \neg(A \vee B)(*)}$$

The Law of non-contradiction:

⁶For an extended discussion of this topic see [4].

$$\Gamma : \neg(A \wedge \neg A)$$

The Law of the excluded middle:

$$\Gamma : (A \vee \neg A)$$

4 The Completeness of DLEAC

Let S be any set of possibly starred sentences.

I suggest that S is *dialetheically consistent* (*d – consistent*) if no conclusion of form $(A \wedge \neg A)^*$ is derivable from S .

Theorem 1. If S is *d – consistent*, then it has a *model* \mathcal{M} .

Proof. Let S be *d – consistent*. Extend the language L to a language L' with an infinite sequence of new individual constants $c_1, c_2, \dots, c_n, \dots$. Let

$$A_1, A_2, \dots, A_n, \dots$$

be a sequence of all L' -sentences. I inductively define the sequence:

$$S_0, S_1, \dots, S_n, \dots$$

of sets of (possibly starred) L' -sentences as follows:

1. $S_0 = S$;
2. $S_{n+1} = S_n$ if A_{n+1} is derivable from S_n and is not an existential sentence;
3. $S_{n+1} = S_n \cup \{B(c)^*\}$ if $A_{n+1} = \exists xB(x)$ and $S_n \vdash \exists xB(x)^*$, where c is the first constant not occurring in S_n nor in A_{n+1} ;
4. $S_{n+1} = S_n \cup \{\neg A_{n+1}^*\}$ if A_{n+1} is not derivable from S_n .

Let us consider the following definition:

$$S_\omega = \cup_{n \in \mathbb{N}} S_n$$

One can prove by induction that each S_n is *d-consistent*, so that S_ω is *d-consistent*.

Consider, for example, 3. Suppose, by reduction, that S_{n+1} is inconsistent. If $S_{n+1} = S_n \cup \{B(c)\}$, then $S_n \vdash \neg B(c)^*$ and hence $S_n \vdash \forall x \neg B(x)^*$, against the *d-consistency* of S_n . If $S_{n+1} = S_n \cup \{B(c)^*\}$, then $A_{n+1} = \exists x B(x)^*$. Then $S_n \vdash \neg B(c)$ and hence $S_n \vdash \forall x \neg B(x)$, against the *d-consistency* of S_n . □

S_ω is *deductively complete*: for any L' -sentence, if not $S_\omega \vdash A$, then $S_\omega \vdash \neg A^*$.

An interpretation I of L' can be defined as follows. Take the set D of all individual constants as domain. Evaluation v can be defined as follows:

$1 \in v(A)$ iff $S_\omega \vdash A$, $0 \in v(A)$ iff $S_\omega \vdash \neg A$, for every atomic L' -sentence.

One can prove, by induction on the complexity of a sentence A , that $v(A) = \{1\}$ iff $S_\omega \vdash A^*$, $v(A) = \{0\}$ iff $S_\omega \vdash \neg A^*$, $v(A) = \{0, 1\}$ iff $S_\omega \vdash A$ and $S_\omega \vdash \neg A$.

It follows that I is a model of S_ω and hence of S .

Completeness. If $S \models A^*$ then $S \vdash A^*$

Proof. Let $S \models A$. Suppose, by reduction, that it is not the case that $S \vdash A$. Then $S \cup \{\neg A^*\}$ is *d-consistent* and hence has a model where $\neg A$ is only true, against the hypothesis. Similarly if $S \models A^*$. □

5 Extending a theory with the truth predicate

In this section, I show (I refer to what is done in [5]) that any dialethic interpretation of a first order language L can be extended to an interpretation of a language L' capable of expressing its own truth predicate.

Let L be a first order language with predicates and individual constants (for simplicity, I ignore functions). Let I be any interpretation of L and D its domain of quantification. Extend L with a new predicate symbol T and infinitely many individual constants. Extend D to D' by adding all L' -sentences to D . Let I' map the new constants 1-1 onto D' so that any member of D' has an L' -name. If A is an L' -sentence, we indicate by $[A]$ its name.

I' puts all sentences in the counter-extension of the L -predicates and the members of D in the counter-extension of T . As shown (in [5]), it is possible to fix the

interpretation of \mathbb{T} in such a way that it turns out to be the truth predicate of I' , so that, for all L' -sentences A , A and $\mathbb{T}(\lceil A \rceil)$ have the same truth values.

Theorem 2. There is an extension of I to an interpretation I' of L' such that, for every L' -sentence A , A and $\mathbb{T}(\lceil A \rceil)$ have the same truth values, while the values of the L -sentences, relativized to D are unchanged.

An evaluation v' is a *sub-evaluation* of v , in symbols $v' \subset v$, if v' is obtained from v by suppressing a truth value of some atomic dialetheias.

Lemma. If a sentence has a unique v -value, this is also the unique v' -value, for any $v' \subset v$.

Proof. The proof is obtained by an induction on the complexity of the sentence. \square

Proof of the theorem 2. The following sequence can be defined by transfinite induction:

$$v_0 \supset v_1 \supset \dots \supset v_\alpha, \dots \text{ (for all ordinals).}$$

They are evaluations of sentences of form $\mathbb{T}(\lceil A \rceil)$ for all L' -sentences A :

$$v_0(\mathbb{T}(\lceil A \rceil)) = \{0, 1\} \text{ for all } A;$$

$v_{\alpha+1}(\mathbb{T}(\lceil A \rceil))$ is defined by cases:

- (i) $v_{\alpha+1}(\mathbb{T}(\lceil A \rceil)) = v_\alpha(A)$ if $v_\alpha(A)$ is a singleton, while $v_\alpha(\mathbb{T}(\lceil A \rceil))$ is not;
- (ii) $v_{\alpha+1}(\mathbb{T}(\lceil A \rceil)) = v_\alpha(\mathbb{T}(\lceil A \rceil))$ otherwise.

$$v_\beta(\mathbb{T}(\lceil A \rceil)) = \bigcap_{\alpha < \beta} v_\alpha(\mathbb{T}(\lceil A \rceil)) \text{ for } \beta \text{ limit.}$$

One can prove, by transfinite induction, that:

for all α , if $v_\alpha(\mathbb{T}(\lceil A \rceil))$ is a singleton, then $v_\alpha(\mathbb{T}(\lceil A \rceil)) = v_\alpha(A)$.

1. $\alpha = 0$. Trivial

2. $\alpha = \beta + 1$. Let $v_\alpha (\mathbb{T}[A])$ be a singleton. We distinguish the two cases above (i) and (ii):

(i) $v_\alpha (\mathbb{T}[A]) = v_\beta (A)$ and, since $v_\beta (A)$ is a singleton and v_α is a sub-evaluation of v_β , by the *lemma* $v_\alpha (A) = v_\beta(A)$.

(ii) $v_\alpha (\mathbb{T}[A]) = v_\beta (\mathbb{T}[A])$. Therefore, $v_\beta (\mathbb{T}[A])$ is a singleton. By the induction hypothesis, $v_\beta (\mathbb{T}[A]) = v_\beta(A)$ and, by the lemma, $v_\alpha(A) = v_\beta(A)$.

3. α limit. If $v_\alpha (\mathbb{T}[A])$ is a singleton, then $v_\alpha (\mathbb{T}[A]) = v_\beta (\mathbb{T}[A]) = v_\beta(A)$, for some $\beta < \alpha$. By the lemma, $v_\alpha(A) = v_\beta(A)$.

□

Observe that if $v_\beta \neq v_\alpha$ with $\beta < \alpha$, there is some sentence A such that $v_\beta(A) = \{0, 1\}$, while $v_\gamma(A)$ is a singleton for all $\gamma > \beta$. And since only countably many sentences can satisfy—for some ordinal—this condition, it follows that at least from the first uncountable ordinal on, the sequence becomes stationary. If δ is such an ordinal, v_δ is clearly the required evaluation.

Consider then the following familiar rules for a truth predicate.

Primitive Tarki's rules:

$$\frac{\Gamma : A (*)}{\Gamma : \mathbb{T}(\mathbb{T}[A])(*)}$$

$$\frac{\Gamma : \mathbb{T}(\mathbb{T}[A])(*)}{\Gamma : A (*)}$$

From the above-mentioned rules, it follows that, semantically, $\mathbb{T}(\mathbb{T}[A])$ and A possess the same truth-values.

Derived Tarki's rules:

$$\frac{\Gamma : \neg \mathbb{T}(\mathbb{T}[A])(*)}{\Gamma : \mathbb{T}(\mathbb{T}[\neg A])(*)}$$

$$\frac{\Gamma : \mathsf{T}([\neg A])(*)}{\Gamma : \neg \mathsf{T}([A])(*)}$$

The guiding idea of the evaluation constructed above suggests that, when Tarski's rules fail to determine a unique value for a sentence of form $\mathsf{T}([A])$, there is no reason to arbitrarily choose one of the two truth values for $\mathsf{T}([A])$, which will, therefore, be evaluated as a dialetheia.

Conservativity. The extension of any theory by means of the predicate T with Tarski's rules is *conservative*.

Proof. It is derivable from *Completeness* and *Theorem 1*. □

6 The refutation paradox in DLEAC

Let us go back to the speech acts of *assertion* and *denial*. Classically, to deny A is equivalent to asserting $\neg A$: A is correctly denied iff $\neg A$ is correctly asserted,⁷ but the dialethic denial of A is *stronger* than the assertion of $\neg A$. As Littman and Simmons observed in [6], because the dialetheist appeals to “non-standard relations between assertability and deniability”, a full account of these notions is required. Specifically, any such account would need to deal with apparent paradoxes that turn on the notion of *assertability* and/or *deniability*.

In their paper Littman and Simmons proposed a paradox concerning assertion, *the assertability paradox* ([6], 320). Take a sentence α having the form:

$$(\alpha) \alpha \text{ is not assertable.}$$

They argue that (α) is a dialetheia. Here is the proof they give:

Proof.

Suppose (α) is *true*. Then what it says is the case. So (α) (i.e. $(\alpha) \alpha$ is not assertable) is not assertable. But we have just asserted (α) . So (α) is assertable—and we have a contradiction.

Suppose, on the other hand, that (α) is *false*. Then what (α) says is not the case, and (α) is assertable. So we may assert: (α) is not assertable. Again, we have a contradiction. □

⁷[14] calls this the *denial equivalence*.

If α is a dialetheia, then it is both assertable and not assertable. But how is it possible to both assert and not assert a sentence? This seems to be impossible also for a dialetheist. While acknowledging that certain sentences can be both true and false, a dialetheist cannot admit that a sentence is assertable and not assertable: “there seems to be no room for manoeuvre. So, the dialetheist will need to say more” ([6], 320).

Note that there is a problem in the proof of the assertability paradox: The mere supposition that α is *true does not imply its assertability*. Indeed, assertability implies the recognition, not just the mere supposition, of the truth of (α) .

In the following text, I propose a revised version of the assertability paradox called *the amended assertability paradox*. To amend the argument, a reasoning by which the truth of (α) can be recognized is provided. The reasoning is given by the following proof.

Let us prove dialetheically that α is true by distinguishing the following two cases:

- (1) Assume that (α) is false. Then, its negation is true, so (α) is assertable and then it is true.
- (2) Assume that (α) is true. Then it is true.

According to the Law of the excluded middle – as formulated in classical first order logic – (α) is true. In this way, we have a proof – not just a supposition – of the truth of (α) , and we can assert it. So (α) is assertable, in opposition to what (α) claims and is false. Therefore it is a dialetheia. That is the *revised assertability paradox*.

Now, again, let us repeat our question: If α is a dialetheia, it is both assertable and not assertable, but how is it possible to both assert and not assert a sentence?

Is this a *real problem* for a dialetheist? The quick answer is, ‘No’. Why should it be a problem for a dialetheist to admit that a sentence is *both* assertable and *not* assertable? Once the exclusivity of logical negation has been rejected, the non assertability of a sentence does not exclude its assertability, even if it is far from clear what it means that a certain sentence *is and is not assertable*.

Here I am not interested in giving a philosophical answer to the above questions. They concern the philosophical *status* of these speech acts, and this is not the place to discuss them.

Priest, in ([12]) proposed something similar to the *assertability paradox*: the *irrationalist paradox*. Let I be a sentence having the form:

(I) : it is not rational to accept I .

You can both accept and and reject (I).
 Priest's derivation is as follows ([12] 121).

Proof.

Let Rat be an operator expressing *rational acceptance* and $R = \neg\text{Rat}(R)$, Priest derives R from the schema

$$(P) \neg\text{Rat}(A \wedge \neg\text{Rat}(A)),$$

as follows:

$$\frac{\frac{\frac{\neg\text{Rat}(R \wedge \neg\text{Rat}(R))}{\neg\text{Rat}(R \wedge R)}}{\neg\text{Rat}(R)}}{R}.$$

That is, R , and hence $\neg\text{Rat}(R)$, is deducible from P : $P \vdash R$.

Assuming that rational acceptance (Rat) is closed under single-premise deducibility, and that P is rationally acceptable (*and it seems to be: if someone believes A , and, at the same time, believes that it is not rationally permissible to believe A , that would seem to be pretty irrational – not something that is itself rationally permissible*), we have $\text{Rat}(P) \vdash \text{Rat}(R)$, and we have a contradiction. □

Priest calls this kind of paradox a *rational dilemma*. He observes that a dialetheist cannot rule out a priori the occurrence of rational dilemmas:

Arguably, the existence of dilemmas is simply a fact of life ([10], 111).

Moreover, he maintains that the irrationalist's paradox is much more problematic for a classicist than for a dialetheist. For the latter it is not irrational to believe both a sentence α and that it is irrational to believe α , if such a belief is also rational, an option clearly closed to a fan of classical logic. This argument is in line with the observation to the *assertability paradox* done before: if negation is non-exclusive, a dialetheist can rightly assert a non-assertable sentence, if she has recognized that it is also true.

Again, it is not the aim of this paper to debate Priest's argument on *the irrationalist paradox*.

I would like just to expand the *revised assertability paradox* in a *rejection direction*. In what follow I first informally introduce *the rejectability (or deniability)*

paradox, then I give its derivation in an extension of DLEAC with the *rejectability* predicate.

Let R be a sentence having the form:

(R) the sentence R is rejectable.

You can both accept and reject (R).

Proof.

Assuming that (R) is true, then it is rejectable. So, there is a state of knowledge in which one can *reject* it. In such a state, one recognizes that what (R) says is true, so that one is in a position to *assert* (R). So you *both reject and accept* R . Thus, the assumption of (R) leads to a state (of knowledge) in which *one can both assert and reject* (R), and that is *dialetheically unacceptable*.

It then follows that (R) *cannot be true*. But, then, we can *reject* it, recognize its truth and *assert* it, which, again, is in opposition of Priest's thesis of *the impossibility of accepting and rejecting the same sentence*. Again: *dialetheically unacceptable*. □

Notice – *en passant* – that, in contrast with the *assertability paradox*, the *deniability paradox* goes against the dialetheist thesis that *assertion* and rejection are *incompatible* speech acts.

Since, as in the amended assertability paradox, evidence of truth or falsity must be available, the agent is assumed to be able to recognize them by the reasoning displayed in the informal proof. The agent is then assumed to have some minimal logical, semantical, and pragmatical skills.⁸ Such requirements are fairly minimal and are ordinarily satisfied by normal agents in normal circumstances. If a dialetheist tried to avoid the paradox by rejecting such an assumption, she would make dialetheism a viable view only for limited cognitive agents. Such an extreme move would be like invoking the ghost of Tarski at the cognitive level, by limiting the cognitive resources of an agent instead of the expressive capacity of the language.

In the next part of this section I logically argue that, in a dialethic logic, such as DLEAC (which is compatible with dialetheism), expanded with a rejectability predicate (which is used in natural languages with some intuitive derivation rules for the introduction and elimination of rejection) a strong absurd can be derived.

⁸In particular, she must be able to understand (R) and truth and falsehood predicates, be capable of rejecting and asserting, having elementary logical skills, and some ability to reflect on her own reasoning processes.

Let ‘R’ be the rejectability predicate $R(\lceil A \rceil)$ is to be read: *A is rejectable*; or more explicitly: *an ideal rational human can reach indefeasible reasons for excluding the truth of A*.

While assuming an ideal agent with such epistemic capacities might be problematic in general, in the context of (*R*) it is justified. As the informal argument of the paradox shows, evidence for truth and falsity of (*R*) is indeed attainable. Generalizing the inference rules of (*R*) to other contexts would be possible at the price of introducing complications unnecessary for present purposes.⁹

The inference rules for R are:

$$\text{ER} \frac{\Gamma : R(\lceil A \rceil)}{\Gamma : \neg A^*}$$

$$\text{IR} \frac{}{\lceil \neg A^* \rceil : R(\lceil A \rceil)}$$

Let *k* be a sentence of form $R(\lceil k \rceil)$:

1	1.	$R(\lceil k \rceil)$	Assumption
1	2.	$\neg R(\lceil k \rceil)^*$	ER
1	3.	$R(\lceil k \rceil) \wedge \neg R(\lceil k \rceil)^*$	IA
	4.	$\neg R(\lceil k \rceil)^*$	RAA
	5.	$R(\lceil k \rceil)$	IR
	6.	$R(\lceil k \rceil) \wedge \neg R(\lceil k \rceil)^*$	IA

Notice that (6) is a *strong absurd* in DLEAC.

On the contrary, observe that – according to *conservativity* as it was formulated in Sec.5 – it is impossible to use the liar paradox to obtain an *absurd*. Indeed, while you obtain a formula saying that the liar is false, there is no formula saying that it is *only false*, as is shown in the following proof.

Let *A* be a sentence of form $\neg T(\lceil A \rceil)$. You have that $T(\lceil A \rceil)$ is a dialetheia.

⁹For example, (IR) should also require that evidence is available for $\neg A^*$, which is the case, however, in the case of (*R*). To be fully complete, the proof should also incorporate the reasoning through which the agent recovers such evidence.

1	1.	$\top(\ulcorner A \urcorner)$	Assumption
1	2.	$\neg\top(\ulcorner A \urcorner)$	Tarski
	3.	$\neg\top(\ulcorner A \urcorner) \vee \neg\top(\ulcorner A \urcorner)$	<i>Material conditional</i>
	4.	$\neg\top(\ulcorner A \urcorner)$	Assumption
	5.	$\neg\top(\ulcorner A \urcorner)$	<i>Reflexivity</i>
	6.	$\neg\top(\ulcorner A \urcorner)$	$E\vee$
	7.	$\top(\ulcorner A \urcorner)$	Tarski
	8.	$\neg\top(\ulcorner A \urcorner) \wedge \top(\ulcorner A \urcorner)$	$I\wedge$

Observe that—even using the starred assumptions—you can not get an *absurd*, as in the following proof:

1	1.	$\top(\ulcorner A \urcorner)^*$	Assumption
1	2.	$\neg\top(\ulcorner A \urcorner)^*$	Tarski
1	3.	$\neg\top(\ulcorner A \urcorner) \wedge \top(\ulcorner A \urcorner)^*$	IA
	4.	$\neg\top(\ulcorner A \urcorner)$	RAA
	5.	$\top(\ulcorner A \urcorner)$	Tarski
	6.	$\neg\top(\ulcorner A \urcorner) \wedge \top(\ulcorner A \urcorner)$	$I\wedge$

7 Conclusion

In the first part of the paper I exposed DLEAC, a dialethic logic in which *exclusivity* is expressed *via* the speech acts of assuming and concluding. An expansion of DLEAC with a predicate for rejection and some intuitive derivation rules for its *introduction* and *elimination* led to a strong absurd, a problem for a dialetheist.

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