On philosophical motivations for paraconsistency: an ontology-free interpretation of the logics of formal inconsistency

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Abstract

In this paper we present a philosophical motivation for the *logics of formal inconsistency*, a family of paraconsistent logics whose distinctive feature is that of having resources for expressing the notion of consistency within the object language in such a way that consistency may be logically independent of non-contradiction. We defend the view according to which logics of formal inconsistency may be interpreted as theories of logical consequence of an epistemological character. We also argue that in order to philosophically justify paraconsistency there is no need to endorse dialetheism, the thesis that there are true contradictions. Furthermore, we argue that an intuitive reading of the bivalued semantics for the logic *mbC*, a logic of formal inconsistency based on classical logic, fits in well with the basic ideas of an intuitive interpretation of contradictions. On this interpretation, the acceptance of a pair of propositions A and $\neg A$ does not mean that A is simultaneously true and false, but rather that there is conflicting *evidence* about the truth value of A.

The aim of this paper is twofold. Firstly, we want to present a philosophical motivation for the logics of formal inconsistency (LFIs), a family of paraconsistent logics whose distinctive feature is that of having resources for expressing the notion of consistency within the object language in such a way that consistency may be logically independent of non-contradiction.

We shall defend the view according to which logics of formal inconsistency may be interpreted as theories of logical consequence of a normative and epistemological character that tell us how to make inferences in the presence of contradictions. We will see that in order to justify paraconsistency from the philosophical viewpoint there is no need to endorse dialetheism, the thesis that there are true contradictions.

Secondly, we want to show that an intuitive reading of the bivalued semantics presented in Carnielli *et al.* (2007) for the logic *mbC*[1], a LFI based on classical logic, can be maintained. The idea is to intuitively interpret paraconsistent negation in the following way. The acceptance of $\neg A$ means that there is some evidence that A is not the case. If such evidence is non-conclusive, it may be that there is simultaneously some evidence that A is the case. Conclusive evidence is tantamount to truth, and if there is conclusive evidence for A, it cancels any evidence for $\neg A$ (*mutatis mutandis* for $\neg A$ and A). Therefore, the acceptance of a pair of contradictory propositions A and $\neg A$ need not to be taken in the strong sense that both are true.[2]

In section 1, in order to give a general view of the problem dealt with by paraconsistent logics, namely, that of dealing with contexts of reasoning in which contradictions occur, we shall present the distinction between explosiveness and contradictoriness. This distinction is essential for grasping the distinction between classical and paraconsistent logics. In section 2, we present an axiomatic system for *mbC* with a correct and complete semantics. We intend to show that *mbC* is a minimal logic with some basic features suitable to the intuitive interpretation of contradictions as conflicting evidence. In section 3, we shall examine the problem of the nature of logic, namely, whether logic as a theory of logical consequence has primarily an ontological or epistemological character. The point is not to give a definite answer to this problem but, rather, to clarify and understand important aspects of paraconsistent logic. We defend the

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[1] The name 'mbC' stands for 'a minimal logic with the axiom bc1', and 'bc' stands intuitively for 'basic property of consistency'.

[2] The symbol '$\neg$' will be used here as a paraconsistent negation, in contrast to classical negation '$\sim$'.

view that logics of formal inconsistency are well suited to the epistemological side of logic and fit an intuitive justification of paraconsistency that is not committed to dialetheism.[3]

1. Contradictions and explosions

Classical logic does not accept contradictions. This is not only because it endorses the validity of the principle of non-contradiction, $\sim(A \wedge \sim A)$, but more importantly because, classically, everything follows from a contradiction. This is the inference rule called *ex falso quodlibet*, or law of explosion,

(1) $A, \sim A \vdash B$.

From a pair of propositions A and $\sim A$, we can prove any proposition through logic, from '2+2=5' to 'snow is black'.

Before the rise of modern logic (i.e., that which emerged in the late nineteenth century with the works of Boole and Frege), the validity of the principle of explosion was still a somewhat contentious issue. However, in Frege's *Begriffsschrift*, where we find for the first time a complete system of what later would be called first-order logic, the principle of explosion holds (proposition 36 of part II). It is worth noting in passing that Frege's logicist project failed precisely because his system was inconsistent, and therefore trivial. Frege strongly attacked the influence of psychologism in logic (we will return to this point in section 3), and in his writings we find a realist conception of mathematics and logic that contributed to emphasizing the ontological and realist vein of classical logic. It is true that one may endorse classical logic without being a realist. However, the unrestricted validity of excluded middle and non-contradiction, an essential feature of classical logic, is perfectly suited to the realist view according to which reality ultimately decides the truth value of every meaningful proposition independently of our ability to know it.

The account of logical consequence that was established as standard in the first half of twentieth century through the works of Russell, Tarski, and Quine, among others, is

[3] One of the intentions of this paper is to present the logics of formal inconsistency to the non-technical minded reader. We presuppose only a basic knowledge of classical logic. The paper, we hope, is as intuitive as it can be within the space allowed, while at the same time dealing with aspects of logics of formal inconsistency that are yet unexplored.

classical. Classical logic is invariably the logic we first study in introductory logic books, and the law of explosion holds in this system, where consistency is tantamount to freedom from contradiction.

Let us put these things a little bit more precisely. Let T be a theory formulated in some language, call it L, whose underlying logic is classical. If T proves a pair of propositions A and $\sim A$, '$\sim$' being classical negation, then T proves all of the propositions of L. In this case, we say that T is trivial. For this reason contradictions must be avoided at all costs in classical logic. The most serious consequence of a contradiction in a classical theory is not the violation of the principle of non-contradiction, but rather the trivialization of the theory. The central point of paraconsistency is that these two things, contradiction and triviality, do not need to be the same thing.

Although classical logic has become the standard approach, several alternative accounts of logical consequence have been proposed and studied. At the beginning of the twentieth century, Brouwer (1907), motivated by considerations regarding the nature of mathematical knowledge, established the basis for intuitionistic logic, a different account of logical consequence. The principle of excluded middle, a so-called fundamental law of thought, together with the principle of non-contradiction, is not valid in intuitionistic logic.

Intuitionistic logic, later formalized by Arend Heyting (1956), a former student of Brouwer, has an epistemological character in clear opposition to Frege's realism. In brief, starting from the assumption that mathematical objects are not discovered but rather created by the human mind, the intuitionists' motivation for rejecting the excluded middle is the existence of mathematical problems for which there are no known solutions. The usual example is the Goldbach conjecture: every even number greater than 2 is the sum of two prime numbers. Let us call this proposition *G.* Until now, there is no proof of *G*, nor is there a counterexample of an even number greater than 2 that is not the sum of two primes. The latter would be a proof of *not G.* For this reason, the intuitionist maintains that we cannot assert the relevant instance of excluded middle, *G or not G.* Doing so would commit us to a Platonic, supersensible realm of previously given mathematical objects.

So-called classical logic is based on principles that, when rejected, may give rise to alternative accounts of logical consequence. Two accepted principles of classical logic are the aforementioned laws of excluded middle and non-contradiction. The principle of non-contradiction can be interpreted from an ontological point of view as a principle about reality, according to which facts, events, and mathematical objects cannot be in contradiction with each other. Its formulation in first order logic, $\forall x \sim(Px \wedge \sim Px)$, says that it cannot be the case that an object simultaneously has and does not have a given property *P*. Note (and this is a point we want to emphasize), that the principle of explosion can be understood as a still more incisive way of saying that there can be no contradiction in reality – otherwise everything is the case, and we know this cannot be so. Explosion is thus a stronger way of expressing an idea usually attributed to non-contradiction. In paraconsistent logics the law of explosion is not valid. Thus, the central question for an intuitive interpretation for paraconsistency is the following: what does it mean to accept a pair of contradictory propositions A and $\neg A$?

It is a fact that contradictions appear in a number of contexts of reasoning, among which three are worth mentioning: (i) computational databases; (ii) semantic and set theoretic paradoxes; (iii) scientific theories. In the first case, contradictions invariably have a provisional character, as an indication of an error to be corrected. We will make only few remarks with regard to the second case, because an analysis of this problem requires a technical exposition that cannot be gone into here. The third case will be discussed below.

Naive set theory, which yields set theoretic paradoxes, has been revised and corrected. Up to the present time, there is no indication that *ZFC* (and its variants) is not consistent – in fact, all indications are to the contrary. The moral we can draw from this is that our intuitive conception of set is defective, a 'product of thought', so to speak, that yields contradictions. Nothing more should be concluded. With respect to semantic paradoxes, they are results about languages with certain characteristics. It is worth noting that the diagonal lemma, essential in the formalization of both the Liar and Curry's paradoxes, is a result about language itself. Thus we believe that the step from paradoxes to the claim that there are true contradictions, in the sense that *reality* is contradictory, is too speculative to be taken seriously.

Here we will concentrate on the third case, the occurrence of contradictions in scientific theories. We are not going to throw away these theories if they are successful in predicting results and describing phenomena. Although mathematicians do their work based on the assumption that mathematics is free of contradictions, in empirical sciences contradictions seem to be unavoidable, and the presence of contradictions is not a sufficient condition for throwing away an interesting theory. It is a fact that contradictory theories exist; the question is how to deal with them. The problem we have on our hands is that of how to formulate an account of logical consequence capable of identifying, in such contexts, the inferences that are allowed, and distinguishing them from those that must be blocked.

2. *mbC*: a minimal logic of formal inconsistency

To begin, let us recall some useful definitions. A theory is a set of propositions closed under logical consequence. This means that everything that is a logical consequence of the theory is also part of the theory. For instance, everything that is a consequence of the basic principles of arithmetic, for instance, '5 + 7 = 12', is also part of arithmetic. We say that a theory T is:

(i) *contradictory* if and only if there is a proposition A in the language of T such that T proves A and $\sim A$ (i.e., T proves a contradiction);

(ii) *trivial* if and only if for any proposition A in the language of T, T proves A (i.e., T proves everything);

(iii) *explosive* if and only if T is trivialized when exposed to a pair of contradictory formulas – i.e.: for all A and B, $T \cup \{A, \sim A\} \vdash B$.

There is a difference between a theory being contradictory and being trivial. Contradictoriness entails triviality only when the principle of explosion holds. Without explosion, we may have contradictions without triviality.

In books on logic, we find two different but classically equivalent notions of consistency. Hunter (1973, p. 78ff) calls them *simple* and *absolute* consistency. A deductive system S with a negation $\sim$ is simply consistent if and only if there is no formula A such that $\vdash_S A$ and $\vdash_S \sim A$. The other notion of consistency, absolute consistency, says that a system S is consistent if and only if there is a formula B such that $\nvdash_S$ B. In other words, S does not prove everything. The latter notion is tantamount to saying that S is not trivial, while the former is tantamount to saying that S is non-contradictory. From the point of view of classical logic, both notions are equivalent because the principle of explosion holds. The proof is simple and easy. Suppose *S* is trivial. Hence, it proves everything, including a pair of propositions *A* and ~*A*. Now suppose *S* is contradictory, that is, it proves a pair of propositions *A* and ~*A*. If the principle of explosion holds in *S*, as is the case in classical logic, then *S* proves everything, hence *S* is trivial. Paraconsistent logics separate triviality from inconsistency, restricting the principle of explosion.

In logics of formal inconsistency the principle of explosion is not valid in general, that is, it is not the case that from *any* pair of contradictory propositions everything follows. These logics contain a non-explosive negation, represented here by '¬', such that for some *A* and *B*,

$$A, \neg A \nvdash B.$$

In addition, there is a unary consistency connective called 'ball': '∘*A*' means informally that *A* is consistent. We can thus isolate contradictions in such a way that the application of the law of explosion is restricted to consistent propositions only, thus avoiding triviality even in the presence of one or more contradictions. Therefore, a contradictory theory may be non-trivial. However, as we will see, we can go further and separate the concept of inconsistency from that of contradictoriness. Consistency may be taken as a primitive notion, its meaning being elucidated from outside the formal system. We will return to this point later on.

It is worth mentioning here a common misunderstanding about paraconsistent logics in general. It is true that in the majority of paraconsistent logics the principle of non-contradiction is not valid. However, this does not mean that *systems* of paraconsistent logic have contradictions as theorems. This is a mistake similar to saying that intuitionistic logic proves the negation of excluded middle.

The fundamental distinction between classical logic and paraconsistent logics occurs at the sentential level. In what follows, we will present an axiomatic system of a sentential logic of formal inconsistency, *mbC*.

Let *L* be a language with sentential letters, the set of logical connectives $\{\vee, \wedge, \rightarrow, \neg, \circ\}$, and parentheses. Notice that the consistency operator '∘', mentioned above, is a primitive symbol. The set of formulas of L is obtained recursively in the usual way. Consider the following axiom-schemas:

Ax. 1. $A \to (B \to A)$

Ax. 2. $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

Ax. 3. $A \to (B \to (A \wedge B))$

Ax. 4. $(A \wedge B) \to A$

Ax. 5. $(A \wedge B) \to B$

Ax. 6. $A \to (A \vee B)$

Ax. 7. $B \to (A \vee B)$

Ax. 8. $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$

Ax. 9. $A \vee (A \to B)$

Ax. 10. $A \vee \neg A$

Ax. bc1. $\circ A \to (A \to (\neg A \to B))$

Inference rule: *modus ponens*

The definition of a proof of A from premises Γ ($\Gamma \vdash_{\mathrm{mbC}} A$) is the usual one: a sequence of formulas $B_1 \ldots B_n$ such that A is B_n and each B_i ($1 \leq i \leq n$) is an axiom, a formula that belongs to Γ, or the result of modus ponens. Thus, monotonicity holds:

(2) if $\Gamma \vdash_{\mathrm{mbC}} B$, then $\Gamma, A \vdash_{\mathrm{mbC}} B$, for any A.

A theorem is a formula proved from the empty set of premises. Ax.1 and Ax.2, plus modus ponens, give the deduction theorem:

(3) if $\Gamma, A \vdash_{\mathrm{mbC}} B$, then $\Gamma \vdash_{\mathrm{mbC}} A \to B$.

We thus have inferences that correspond to introduction and elimination rules of implication in a natural deduction system. Since monotonicity holds, we also have the converse of (3), and the deduction theorem holds in both directions.

(4) $\Gamma, A \vdash_{\mathrm{mbC}} B$ if and only if $\Gamma \vdash_{\mathrm{mbC}} A \to B$.

Axioms 3-5 and 6-8 correspond, respectively, to the introduction and elimination rules of conjunction and disjunction in a natural deduction system. Axioms 1-9 form a complete system of positive classical logic, i.e., a system that proves all tautologies that can be formed with $\rightarrow$, $\vee$ and $\wedge$. Negation shows up in Ax.10 and Ax.11. Excluded middle for paraconsistent negation holds, and explosion is restricted to consistent formulas. Consistency is a primitive notion; it is not definable in terms of non-contradiction, that is, $\neg(A \wedge \neg A)$ and $\circ A$ are not equivalent. It is clear that in a system whose consequence relation $\vdash$ enjoys the deduction theorem in both directions, the law of explosion

(1) $A, \neg A \vdash B$

and the axiom schema

(5) $\vdash A \rightarrow (\neg A \rightarrow B)$

are equivalent. To make things simpler, we call both the law of explosion.

The system is not explosive with respect to *non-consistent formulas*. For this reason, we say that it is *gently explosive*. If we simultaneously have contradiction and consistency, the system explodes, becoming trivial. The point, however, is precisely this: *there cannot be consistency and contradiction, simultaneously and with respect to the same formula.* If explosion in classical logic means that there can be no contradiction at all, the restricted principle of explosion may be understood as a more refined way of stating the same basic idea: there can be no contradiction with respect to consistent propositions.

A remarkable feature of *mbC* is that it can be seen as an extension of classical logic. We can define a bottom particle as follows:

(6) $\perp \stackrel{\text{def}}{=} \circ A \wedge A \wedge \neg A$.

Now, as an instance of Ax.9, we get

(7) $A \vee (A \rightarrow \perp)$.

We also have that

(8) $\vdash \perp \rightarrow B$,

since, by Ax.bc1,

(9) $\circ A \wedge A \wedge \neg A \vdash B$.

Axioms 1-8 plus (7) and (8) give us classical logic.[4]

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[4] A complete axiomatization for classical sentential logic, in a language in which $\sim A$ is defined as $A \rightarrow \perp$, is given by axioms 1-8 plus (*) $\sim\sim A \rightarrow A$ (see Robbin 1997, chapter 1). We leave it as an exercise to the reader to prove (*) in the system given by axioms 1-8 plus (7) and (8).

A point that has certainly already occurred to the reader regards the interpretation of the system above. Three questions that pose themselves are:

(i) How can a semantics be given for the formal system above?

(ii) What is the intuitive meaning of the consistency operator?

(iii) How can an intuitive interpretation for contradictory formulas be provided?

Paraconsistent logics were initially introduced in proof-theoretical terms, a procedure that fits the epistemological character that we claim here is appropriate to them. In Carnielli et al. (2007, pp. 38ff), we find a bivalued non-truth-functional semantics that is complete and correct for *mbC*. An *mbC-valuation* is a function that attributes values 0 and 1 to formulas of *L*, satisfying the following clauses:

(i) $v(A \wedge B) = 1$ if and only if $v(A) = 1$ and $v(B) = 1$

(ii) $v(A \vee B) = 1$ if and only if $v(A) = 1$ or $v(B) = 1$

(iii) $v(A \rightarrow B) = 1$ if and only if $v(A) = 0$ or $v(B) = 1$

(iv) $v(\neg A) = 0$ implies $v(A) = 1$

(v) $v(\circ A) = 1$ implies $v(A) = 0$ or $v(\neg A) = 0$

We say that a valuation v is a model of Γ if and only if every proposition of Γ receives the value 1 in v. The notion of logical consequence is defined as usual: $\Gamma \vDash_{mbC} A$ if and only if for every valuation v, if v is a model of Γ, $v(A) = 1$.

Now, we suggest that the values 0 and 1 not be understood as, respectively, false and true *simpliciter*, but rather as expressing the *existence of evidence*: $v(A) = 1$ means that there is evidence that A is the case, and $v(A) = 0$ means that there is no evidence that A is the case. The following passage from da Costa (1982 pp. 9-10) helps to elucidate what we mean by evidence.

> Let us suppose that we want to define an operational concept of negation, at least for the negation of some atomic sentences. ¬A, where A is atomic, is to be true if, and only if, the clauses of an appropriate criterion *c* are fulfilled, clauses that must be empirically testable; i.e., we have an empirical criterion for the truth of the negation of A. Naturally, the same must be valid for the atomic proposition A, for the sake of coherence. Hence, there exists a criterion *d* for the truth of A. But clearly it may happen that the criteria *c* and *d* be such that they entail, under certain critical circumstances, the truth of both A and ¬A.

Although da Costa in the passage above is talking about a criterion of truth, it seems to us that it is much more reasonable, in a situation such as the one there described, not to draw the conclusion that A and $\neg A$ are both *true*. It is better to be more careful and to take the contradictory data only as a sort of a provisional state. Accordingly, the criteria c and d constitute reasons to believe that respectively $\neg A$ and A are true, but they do not establish conclusively that both are true

In *mbC*, excluded middle holds with respect to the paraconsistent negation (axiom 10). Hence, a situation where there is no evidence at all for both A and $\neg A$ is excluded. Indeed, the semantic clause (iv) forbids that A and $\neg A$ simultaneously receive the value 0. The validity of excluded middle may be justified when we by default attribute evidence for $\neg A$ when there is no evidence at all. This happens, for instance, in a criminal investigation in which one starts by considering everyone (in some group of people) not guilty until there is proof to the contrary. Whether or not excluded middle must be valid depends on the reasoning scenario we want to represent. Of course we may also devise a logic of formal inconsistency in which excluded middle is recovered once some information has been added, in a way similar to the axiom *bc1*.[5]

Since excluded middle holds, we have the following three possible scenarios, together with the respective values attributed to A and $\neg A$:

A1.

There is evidence that A is the case;	$v(A) = 1$
There is evidence that A is not the case.	$v(\neg A) = 1$

A2.

There is evidence that A is the case;	$v(A) = 1$
There is no evidence that A is not the case.	$v(\neg A) = 0$

A3.

There is no evidence that A is the case;	$v(A) = 0$
There is evidence that A is not the case.	$v(\neg A) = 1$

[5] In fact, *mbC* may be adapted in order to be able to represent such a situation. The basic idea is to recover excluded middle in the following way: $\circ A \rightarrow (A \vee \neg A)$. A sketch of such a formal system may be found in Carnielli *et al.* 2014b.

Accordingly, we suggest the following meaning for negation:

$v(\neg A) = 1$ means that *there is some evidence that A is not the case*;

$v(\neg A) = 0$ means that *there is no evidence that A is not the case*.

Now, if the evidence is conclusive, we have only two possible (classical) scenarios:

B1. There is conclusive evidence that A is the case.

B2. There is conclusive evidence that A is not the case.

Note that a situation such that there is conclusive evidence for *A* and at the same time non-conclusive evidence for $\neg A$ is not possible. Once the truth of *A* is conclusively established, any evidence for $\neg A$ is cancelled. In this case, we are talking about *truth* and not just evidence. We may have conflicting evidence, but not conflicting truth values. Accordingly,

$v(\circ A) = 1$ means that *the truth value of A has been conclusively established.*

In scenarios B1 and B2 we have $v(\circ A) = 1$. In A1, we must have $v(\circ A) = 0$, but in A2 and A3 we have $v(\circ A) = 0$. In these scenarios, when $v(\circ A)$ turns out to be 1, we get scenarios B1 or B2. A remarkable feature of the above intuitive interpretation is that the simultaneous *truth* of a contradictory pair of propositions is not allowed, on pain of triviality. Under this interpretation, the system *is not neutral* with respect to true contradictions.

It is important here to call attention to the fact that what constitutes evidence for a given proposition *A*, and whether or not such evidence is conclusive and *A* may be established as true, are problems that depend on the specific area of knowledge being dealt with. We may say, roughly speaking, that $\circ A$ says something about the *justification* of *A*, or of $\neg A$, but what exactly it means is not a problem of logic. It is the physicist, the chemist, the mathematician, etc., who is able, with respect to concrete situations, to say what constitutes a conclusive establishment of the truth of a proposition *A*.

Note that paraconsistent negation is weaker than classical negation; it does not have all the properties that classical negation has. Regarding the consistency operator, we want to call attention to the fact that in *mbC* it is not logically equivalent to non-contradiction, that is:

$\circ A \vDash_{mbC} \neg(A \wedge \neg A)$, while $\neg(A \wedge \neg A) \nvDash_{mbC} \circ A$.

This is in accord with a central point of the notion of consistency, namely, that it is polysemic and need not be *defined* from negation. The notion of consistency in natural language and informal reasoning has a number of senses, not always directly related to negation (cf. Carnielli 2011, p. 84). A proposition is often said to be consistent when it is coherent or compatible with previous data, or when it refers to an unchanging or constant situation.

Since consistency is a primitive notion, its meaning is elucidated from outside the formal system. Our suggestion is nothing but one way of interpreting it. It is worth noting that the above interpretation fits in well with the old philosophical idea that truth implies consistency, but consistency does not imply truth.[6] Indeed, $\circ A \nvDash_{\text{mbC}} A$. The consistency of A does not imply the truth of A; but if the truth-value of A has been conclusively established as true, then A is consistent.

Clauses (i)-(iii) are exactly as in classical logic, and therefore all classical tautologies with $\wedge$, $\vee$, and $\rightarrow$ are valid in an *mbC*-valuation. An important feature of the valuation above is that both the negation and the consistency connectives *are not truth-functional*, that is, the value attributed to $\neg A$ and $\circ A$ does not depend only on the value attributed to A. Clause (iv), negation, expresses only a necessary condition. If $v(A) = 0$, we must also have $v(\neg A) = 1$. This was expected, as excluded middle is valid and at least one formula among A and $\neg A$ must receive the value 1. On the other hand, since we do not have a sufficient condition for $v(\neg A) = 0$, it is possible that $v(A) = 1$ and $v(\neg A) = 1$, since $v(A) = 1$ does not make $v(\neg A) = 0$. Since truth-functionality is a special case of compositionality, it is also remarkable that this semantics is not compositional in the sense that the semantic value of the whole expression is functionally determined by the semantic values of its parts and the way they are combined.

Non-contradiction, as expected, is not valid. Intuitively, it can be the case that there is some non-conclusive evidence both for A and $\neg A$, and this makes it possible to have $v(\neg(A \wedge \neg A))=0$. In addition, it is easy to see that explosion is also not valid: if $v(A) =$

[6] This is the basic argument against truth as coherence. See, for example, Russell (1913), ch. XII.

$v(\neg A) = 1$ and $v(B) = 0$, $v(A\rightarrow(\neg A\rightarrow B)) = 0$. We may have evidence for A and $\neg A$, but have no evidence for B.

Clause (v) also expresses only a necessary condition for attributing the value 1 to $\circ A$: we must have exactly one among A and $\neg A$ with value 0 (by clause (iv) they cannot be both 0). If we have both, $v(A) = v(\neg A) = 1$, we do not have $v(\circ A) = 1$. It is also worth noting that according to clause (v), $\circ A \vDash_{mbC} \neg(A \wedge \neg A)$. Intuitively, if A is consistent, its truth-value has been conclusively established. Hence, it cannot be that we still have evidence for both A and $\neg A$. On the other hand, the converse does not hold: $\neg(A \wedge \neg A) \nvDash_{mbC} \circ A$. We may well have non-conclusive evidence for A and no evidence at all for $\neg A$. In this case $v(\neg(A\wedge\neg A))=1$, but since the evidence for A is non-conclusive, $v(\circ A) = 0$. Non-consistency is independent of contradiction.

There are some inferences that are not allowed once explosion is not valid in general. One is disjunctive syllogism. Actually, disjunctive syllogism and explosion are equivalent in the sense that, added to positive sentential logic (Ax.1-9 plus modus ponens), each one implies the other. But it is easy to see that, according to the interpretation proposed here, disjunctive syllogism is not valid. It can be the case that we have some evidence both for A and $\neg A$, so we may have $v(A \vee B) = v(\neg A) = 1$, but $v(B) = 0$.

Modus ponens holds, but modus tollens (as well as all versions of contraposition) does not hold.

(10) $A \rightarrow B, \neg B \nvDash_{mbC} \neg A$

However, for B consistent,

(11) $\circ B$, $\mathrm{A} \rightarrow B, \neg B \vDash_{mbC} \neg A$.

Is there an intuitive justification for the fact that modus tollens is not valid in *mbC*? The answer is affirmative. In classical logic, modus tollens is valid because the truth of $\neg B$ implies the falsity of B. Hence, as well as truth being preserved by modus ponens, falsity is also preserved by modus tollens: given the truth of $A \rightarrow B$, if B is false, A is false too. This is not the case, however, under the interpretation we are proposing here. $v(\neg B) = 1$ means that we have some evidence that B is not the case. This does not imply that we do not have simultaneous evidence for B, that is, $v(\neg B) = 1$ does not imply $v(B) = 0$. Suppose that we have evidence both for A and B. In this case, $v(A \rightarrow B) = 1$ (notice that there is not required any kind of connection between the meanings of A and B in order to have $v(A \rightarrow B) = 1$). Now, in order to see that modus tollens is not valid, make $v(B) = v(\neg B) = 1$, and $v(A) = 1$, $v(\neg A) = 0$. On the other hand, if $v(\circ B)=1$, we cannot have $v(B) = v(\neg B) = 1$. Hence, (11) is valid.

In *mbC*, as with the majority of logics of formal inconsistency, once the consistency of certain formulas is established, we get classical logic.[7] The schemas below are valid:

$\circ B \vDash_{\text{mbC}} (A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$

$\circ\neg A \vDash_{\text{mbC}} \neg\neg A \rightarrow A$

$\circ A \vDash_{\text{mbC}} A \rightarrow \neg\neg A$

$\circ A, \circ B \vDash_{\text{mbC}} (\mathrm{B} \rightarrow \mathrm{A}) \leftrightarrow (\neg\mathrm{A} \rightarrow \neg\mathrm{B})$

$\circ A, \circ B \vDash_{\text{mbC}} (\neg\mathrm{B} \rightarrow \mathrm{A}) \leftrightarrow (\neg\mathrm{A} \rightarrow \mathrm{B})$

Note that *mbC* is suitable for expressing the central ideas of the intuitive interpretation of paraconsistent negation and the consistency operator we have presented here, namely: (i) a contradiction $A \wedge \neg A$ means that there is conflicting evidence about A, and (ii) $\circ A$ means that the truth-value of A has been conclusively established. Both (i) and (ii) depend on the fact that $\circ\mathrm{A}$ and $\neg(A \wedge \neg A)$ are not logically equivalent.

We have seen that classical logic may be recovered within *mbC*. Thus, *mbC* is a formal system able to deal simultaneously with preservation of evidence and preservation of truth. Hence, it seems to us that there is no question of whether or not *mbC* is an account of logical consequence, unless logic is strictly considered a theory of preservation of truth, nothing more than that. But we endorse a concept of logic according to which it is not restricted to the idea of truth preservation. Logical consequence is indeed the central notion of logic, but the task of logic is to tell us which conclusions can be drawn from a given set of premises, under certain conditions, in concrete situations of reasoning. And sometimes it may be the case that it is not only truth that is at stake. Nevertheless, although this

[7] A *derivability adjustment theorem*, which recovers classical consequence once some information is added, holds for *mbC*. See details in Carnielli and Coniglio (2014)

discussion about the concept of logic is a very relevant subject, and closely related to the development of non-classical logics, it cannot be further developed here

The fact is that there is still a lot of work to be done on logics of formal inconsistency. As has been shown in Carnielli et al. (2007) and in Carnielli & Marcos (2002), several systems may be formulated, with very subtle differences among them. Aside from the technical problems of finding complete and correct semantics, and of proving metatheorems, there are still a lot of open questions related to the intuitive interpretation of these systems and to the philosophical concepts expressed by them. In addition, it is very likely that the intuitive interpretation we have presented here can be improved. However, the point we want to emphasize here is that *mbC*, an economical and elegant formal system, has the resources to express the basic ideas (i) and (ii) mentioned above.

3. On the nature of logic

In this section we shall present a philosophical interpretation for the logics of formal inconsistency. We start by calling attention to the fact that there is a perennial philosophical question about the nature of logic, namely, whether the main character of logic is epistemological, ontological, or linguistic. We call attention to the epistemological character of intuitionistic logic, which is in clear opposition to the realist (and, we claim, ontological) view of logic found in Frege's works. Our main argument depends on the duality between the rejection of explosion by *mbC* and the rejection of excluded middle by intuitionistic logic, both of which are due to epistemological reasons.

In building formal systems we deal with several logical principles, and it may justly be asked what these are principles about. This is a central issue in philosophy of logic. Here we follow Popper (1963, pp. 206ff), who presents the problem in a very clear way. The question is whether the rules of logic are:

(I.a) laws of thought in the sense that they describe how we actually think;

(I.b) laws of thought in the sense that they are normative laws, i.e., laws that tell us how we should think;

(II) the most general laws of nature, i.e., laws that apply to any kind of object;

(III) laws of certain descriptive languages.

We thus have three basic options, which are not mutually exclusive: the laws of logic have (I) epistemological, (II) ontological, or (III) linguistic character. With respect to (I), they may be (I.a) or descriptive (I.b) normative. Let us illustrate the issue with some examples.

Aristotle, defending the principle of non-contradiction, makes it clear that it is a principle about reality, "the most certain principle of all things" (*Metaphysics* 1005b11). Worth mentioning also is the well-known passage, "the same attribute cannot at the same time belong and not belong to the same subject in the same respect" (*Metaphysics* 1005b19-21), which is a claim about objects and their properties.

On the other hand, a very illustrative example of the epistemological side of logic can be found in the so-called logic of Port-Royal, where we read:

> Logic is the art of conducting reasoning well in knowing things, as much to instruct ourselves about them as to instruct others.
>
> This art consists in reflections that have been made on the four principal operations of mind: *conceiving, judging, reasoning,* and *ordering*.
>
> (…) [T]his art does not consist in finding the means to perform these operations, since nature alone furnishes them in giving us reason, but in reflecting on what nature makes us do, which serves three purposes.
>
> The first is to assure us that we are using reason well …
>
> The second is to reveal and explain more easily the errors or defects that can occur in mental operations.
>
> The third purpose is to make us better acquainted with the nature of the mind by reflecting on its actions. (Arnauld, A. & Nicole, 1996, p. 23)

Logic is conceived as having a normative character. So far so good. But logic is also conceived as a tool for analyzing mental processes of reasoning. This analysis, when further extended by different approaches to logical consequence, as is now done by some non-classical logics, shows that there can be different standards of correct reasoning in different situations. This aspect of logic, however, has been put in a secondary place by Frege's attack on psychologism. Frege wanted to eliminate everything subjective from logic. For Frege, laws of logic cannot be obtained from concrete reasoning practices.

Basically, his argument is the following. From the assumption that truth is not relative, it follows that the basic criterion for an inference to be correct, namely, the preservation of truth, should be the same for everyone. When different people make different inferences, we must have a criterion for deciding which one is correct. Combined with Frege's well-known Platonism, the result is a conception of logic that emphasizes the ontological (and realist) aspect of classical logic.

> Our conception of the laws of logic is necessarily decisive for our treatment of the science of logic, and that conception in turn is connected with our understanding of the word 'true'. It will be granted by all at the outset that the laws of logic ought to be guiding principles for thought in the attainment of truth, yet this is only too easily forgotten, and here what is fatal is the double meaning of the word 'law'. In one sense a law asserts what is; in the other it prescribes what ought to be. Only in the latter sense can the laws of logic be called 'laws of thought'(…) If being true is thus independent of being acknowledged by somebody or other, then the laws of truth are not psychological laws: they are boundary stones set in an eternal foundation, which our thought can overflow, but never displace. (Frege 1893, (1964) p. 13).

> (…) [O]ne can very well speak of laws of thought too. But there is an imminent danger here of mixing different things up. Perhaps the expression "law of thought" is interpreted by analogy with "law of nature" and the generalization of thinking as a mental occurrence is meant by it. A law of thought in this sense would be a psychological law. And so one might come to believe that logic deals with the mental process of thinking and the psychological laws in accordance with which it takes place. This would be a misunderstanding of the task of logic, for truth has not been given the place which is its due here. (Frege 1918, (1997) p. 325).

For Frege, logic is normative, but in a secondary sense. Along with truths of arithmetic, the logical relations between propositions are already given, eternal. This is not surprising at all. Since he wanted to prove that arithmetic could be obtained from purely logical principles, truths of arithmetic would inherit, so to speak, the realistic character of the logical principles from which they were obtained. Logic thus has an ontological character; it is part of reality, as are mathematical objects.

It is very interesting to contrast Frege's realism with Brouwer's intuitionism, whose basic ideas can be found for the first time in his doctoral thesis, written at the very beginning of twentieth century. The approaches are quite opposed.

> *Mathematics can deal with no other matter than that which it has itself constructed*. In the preceding pages it has been shown for the fundamental parts of mathematics how they can be built up from units of perception (Brouwer 1907, (1975) p. 51)
>
> The words of your mathematical demonstration merely accompany a mathematical *construction* that is effected without words …
>
> While thus mathematics is independent of logic, logic does depend upon mathematics: in the first place *intuitive logical reasoning* is that special kind of mathematical reasoning which remains if, considering mathematical structures, one restricts oneself to relations of whole and part (Brouwer 1907, (1975) p. 73-74).

It is remarkable that Brouwer's doctoral thesis (1907) was written between the two above-quoted works by Frege (1893 and 1919). Brouwer, like Frege himself, is primarily interested in mathematics. For Brouwer, however, the truths of mathematics are not discovered but rather constructed. Mathematics is not a part of logic, as Frege wanted to prove. Quite the contrary, logic is abstracted from mathematics. It is, so to speak, a description of human reasoning in constructing correct mathematical proofs. Mathematics is a product of the human mind, mental constructions that do not depend on language or logic. The raw material for these constructions is the intuition of time (this is the meaning of the phrase 'built up from units of perception').

This epistemological motivation is reflected in intuitionistic logic, which was formalized by Heyting (1956). Excluded middle is rejected precisely because mathematical objects are considered mental constructions. Accordingly, to assert an instance of excluded middle related to an unsolved mathematical problem (for instance, Goldbach's conjecture), would be a commitment to a Platonic realm of abstract objects, an idea rejected by Brouwer and his followers.

With respect to the linguistic aspects of logic, we shall make just a few comments. According to a widespread opinion, a linguistic conception of logic prevailed during the last century. From this perspective, logic has to do above all with the structure and functioning of certain languages. Indeed, sometimes logic is defined as a mathematical study of formal languages. There is no consensus for this view, however, and it is likely that it is not prevalent today.[8] Even though we cannot completely separate the linguistic from the epistemological aspects – i.e., separate language from thought –, we endorse the view that logic is primarily a theory about reality and thought, and that the linguistic aspect is secondary.[9]

At first sight, it might seem that Frege's conception according to which there is only one logic, that is, only one account of logical consequence, is correct. Indeed, for Frege, Russell, and Quine, *the* logic is classical logic. From a realist point of view, this fits well with the perspective of the empirical sciences: excluded middle and bivalence have a strong appeal. Ultimately, reality will decide between *A* and *not A*, which is the same as deciding between the truth and falsity of *A*.

The identification of an intuitionistic notion of provability with truth was not successful. As is shown by Raatikainen (2004), in the works of Brouwer and Heyting we find some attempts to formulate an explanation of the notion of truth in terms of provability, but all of them produce counterintuitive results.

On the other hand, the basic intuitionistic argument that rejects a supersensible realm of abstract objects is philosophically motivated – and it is notable that this argument usually seems rather convincing to students of philosophy. As Velleman & Alexander (2002, pp. 91ff) put it, realism seems to be compelling when we consider a proposition like *every star has at least one planet orbiting it*. However, when we pass from this example to Goldbach's conjecture, the situation changes quite a bit. In the former case, it is very reasonable to say that reality is one way or the other; but if we say with regard to the latter

[8] A rejection of the linguistic conception of logic, and an argument that logic is above all a theory with ontological and epistemological aspects, can be found in the *Introduction* to Chateaubriand (2001).

[9] Notice that this is in line with Chateaubriand's opinion, as manifested in Chateaubriand (2001, p. 16): "the fundamental character of logic is metaphysical, not linguistic. On the one hand I see it as an ontological theory that is part of a theory of the most general and universal features of reality; of being *qua* being, as Aristotle said. On the other hand I see it as an epistemological theory that is part of a general theory of knowledge."

case that 'the world of mathematical numbers' is one way or the other, there is a question to be faced: where is this world?

What is the moral to be taken from this? That classical and intuitionistic logic are not talking about the same thing. The former is connected to reality through a realist notion of truth; the latter is not about truth, but rather about reasoning. In our view, assertability based on the intuitionistic notion of constructive proof is what is expressed by intuitionistic logic.

Now, one may ask what all of this has to do with logics of formal inconsistency and paraconsistent logics in general. The question concerning the nature of logic is a perennial problem of philosophy. We believe that it has no solution in the sense of some conclusive argument in defense of one or the other view. This is so, first, because logic is simultaneously about thought, and reality, and, second, because different accounts of logical consequence may be more appropriate for expressing one view than another. We also claim that this is precisely the case with, on the one hand, classical logic and its ontological motivations and, on the other hand, with the epistemological approach of intuitionism and logics of formal inconsistency.

The reader may have already noticed a duality between intuitionistic and paraconsistent logics. In the latter, we may have two sentences A and $\neg A$ with value 1; non-contradiction does not hold. In the former, we may have two sentences A and $\neg A$ (now '$\neg$' is intuitionistic negation) with a value 0; excluded middle, the dual of non-contradiction, does not hold. If we stop to think about them for a moment, and put aside any realist bias, we see that we are facing two analogous situations. Let us take a look at this point.

At first sight, it really seems that an inference principle like *modus tollens* is valid whether we want it to be or not, that its validity is not a matter of any kind of choice or context whatever. This is indeed correct if we are talking about truth in the realist sense, a framework in which classical logic works well. However, we have just seen the reason why *modus tollens* is not valid if we are reasoning in a context with a non-explosive negation. Something analogous happens in intuitionistic logic. Once we have endorsed a constructive notion of proof, we cannot carry out a proof by cases using excluded middle because the result might not be a constructive proof. When we say that logics of formal inconsistency accept some contradictions without exploding the system, this does not mean that these contradictions are *true*. We may compare this with Kripke models for intuitionistic logic, where it can happen that a pair of sentences *A* and *not A* (intuitionistic negation) receive the value 0 in some stage (i.e., possible world). Such a stage would be a refutation of excluded middle. This does not mean that *A* and *not A* are *false*, however, but rather that neither has been proved yet. Analogously, as suggested above, when two sentences *A* and *not A* receive 1 in an *mbC*-valuation it means that we have evidence for both, not that both are simultaneously true.

The position defended here with respect to paraconsistency thus differs fundamentally from dialetheism. According to Graham Priest and his collaborators,

> A dialetheia is a sentence, A, such that both it and its negation, ¬A, are true (…) Dialetheism is the view that there are dialetheias. (…) dialetheism amounts to the claim that there are true contradictions." (Priest & Berto 2013, introduction)

One of the motivations for dialetheism is indeed the paradoxes. But Priest tries to extend it these motivations empirical phenomena:

> [T]he paradoxes of self-reference are not the only examples of dialetheias that have been mooted. Other cases involve contradictions affecting concrete objects and the empirical world (Priest & Berto 2013, sec. 3.3)
> [T]here are, if not conclusive, then at least plausible reasons for supposing that these [discrete temporal changes] may produce dialetheias. (Priest 2006, p. 159)

As well as the problem of the nature of logic mentioned above, the question of whether or not there are real contradictions, facts, and/or events in contradiction with each other, is a perennial problem in philosophy. It seems to us that this problem has no perspective of a conclusive solution, given the state of science at the present time. We agree, however, that the thesis that there are real contradictions, a view endorsed by Hegel and, according to some interpreters, Heraclitus, has an important place in the history of philosophy. On the other hand, it is not a contentious issue that contradictions do appear in the process of acquiring knowledge and dealing with data. In taking contradictions epistemologically, we have tried to show here that an intuitive justification for paraconsistency not committed to dialetheism is possible with respect to logics of formal inconsistency.

4. Final Remarks

We hope we have been successful here in showing that to have available a logical formalism capable of dealing with contradictions may have good philosophical motivations, and is not the same as having some kind of philosophical sympathy for contradictions. We still believe that trying to avoid contradictions is an indispensable criterion of rationality. In order to do this, however, we need a logic that does not collapse in the face of a pair of propositions *A* and *not A*. A contradiction may be taken as a provisional state, a kind of excess of information or excessively optimistic attitude that should, at least in principle, be eliminated by means of further investigation.

Finally, a few words about a semantics for logics of formal inconsistency. The task of finding intuitively acceptable semantics for non-classical logics is indeed a major philosophical challenge. The bivalued semantics presented here is appropriate for expressing the basic features of the intuitive meanings of the consistency operator and paraconsistent negation when we are dealing with the notions of truth and evidence. However, we believe that both the formal system and the semantics could be improved. There two possible routes: either (i) we keep the idea of a two-valued semantics but modify both the deductive system and the semantics clauses, or (ii) we try to find another kind of semantics, suitable to the idea of dealing simultaneously with truth and evidence. With respect to the latter, it is not unlikely that we could find a better way of expressing the relationship between truth and evidence by means of possible-translations semantics (Carnielli 2000). However, the task of investigating such options will be left for another time.

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