The Mereological foundation of Megethology

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April 24, 2015

Abstract

In *Mathematics is megethology* (Lewis, 1993) David K. Lewis proposes a structuralist reconstruction of classical set theory based on mereology. In order to formulate suitable hypotheses about the size of the universe of individuals without the help of set-theoretical notions, he uses the device of Boolos' *plural quantification* for treating second order logic without commitment to set-theoretical entities.

In this paper we show how, assuming the existence of a pairing function on atoms, as the unique assumption non expressed in a mereological language, a mereological foundation of set theory is achievable within first order logic. Furthermore, we show how a mereological codification of ordered pairs is achievable with a very restricted use of the notion of *plurality* without *plural quantification*.

1 Introduction

David Lewis (Lewis, 1993) proposes a reconstruction of classical set theory grounded on mereology and plural quantification (see also (Lewis, 1991)).[1]

Mereology is the theory of parthood relation. One of its first formalizations is Leonard and Goodman *Calculus of individuals* ((Leonard & Goodman, 1940), 33-40).[2] Plural quantification is a reinterpretation of second order monadic logic, proposed by Boolos in (Boolos, 1984), (Boolos, 1985).[3] In Boolos' perspective second order monadic logic is ontologically innocent: contrary to the most accredited view, it doesn't entail any commitment to classes or to properties but only to individuals, as first order logic does. Acccording to Boolos, second order quantification differs from first order quantification only in that it refers to individuals plurally, while the latter refers to individuals singularly.

By combining mereology with plural quantification, Lewis in (Lewis, 1993) introduces *megethology*, a powerful framework in which one can formulate strong assumptions about the size of the universe of individuals (corresponding to the existence of strongly inaccessible cardinals). Within such a framework Lewis develops a structuralist class theory, where the role of classes is played by individuals. Thus, accepting Boolos' thesis about the innocence of second order logic, Lewis achieves an ontological reduction of classes to individuals.

Boolos' view, however, though very attractive, is highly controversial. It

[1]For an analysis of the power of mereology and plural quantification see also Geoffrey Hellman's (Hellman, 1996) and (Hellman, 2009).

[2]For a general survey of mereology see (Varzi, 2012).

[3]A general introduction to plural quantification is in (Linnebo, 2013).

has faced the criticism of several philosophers of mathematics, (Parsons in (Parsons, 1990), Resnik in (Resnik, 1988) and, more recently, Linnebo, for example in (Linnebo, 2003)). Quine's old claim that second order logic is "set theory in disguise" doesn't seem to have lost all its advocates. Be that as it may, we think that, in order to better estimate the foundational power of the mereological component, it is worth trying to develop mgethology *without* the help of plural quantification.

Though in the Lewis' paper plural quantification is used extensively, a careful analysis shows that it plays an essential role only as a device for providing a mereological codification of ordered pairs. This is surely an interesting step. However, the notion of ordered pair is certainly not one of the most problematic notions of set theory. Therefore giving up plural quantification and taking ordered pairs as primitive seems to be a natural move in order to make the mereological conception of set theory more perspicuous.[4]

Accordingly, we will propose a version of megethology within first order logic with identity by using, besides the mereological part-whole relation $\leq$, a primitive *pairing function* $(\; , \;)$ from ordered pairs of atoms to atoms. In this framework we will develop a structuralist interpretation of set theory with proper classes (in the Von Neumann-Goedel-Bernays style). In the last section we will show how a suitable mereological codification of ordered pairs of atoms by atoms is achievable with a restricted use of the notion of plurality that does not involve plural quantification.

[4](Hellman, 1989) and (Feferman & Hellman, 1995) also incorporated primitive pairing in their systems.

2 Mereological axioms

The language $\mathcal{ML}$ of megethology is a first order language with identity. Variables x, y,... range over individuals. The primitive non-logical constants are the binary relation $\leq$ ($x \leq y$ is to be read "x is part of y") and the pairing function (,) from ordered pairs of atoms to atoms.

1.1 Axiom. $\leq$ *is a partial order* (reflexive, antisymmetric, transitive).

$x < y$ (x is a proper part of y) if$_{df.}$ $x \leq y$ and $x \neq y$.

x is an atom if$_{df.}$ it has no proper parts.

1.2 Axiom. *Every individual has an atomic part.*

Fusion schema:

1.3 Axiom. $\exists z \phi \to \exists ! y \forall x (x \text{ } atom \to (x \leq y \leftrightarrow \exists z (\phi \wedge x \leq z))$, where ϕ is a formula with possible parameters and without x, y free.

Such y is the *fusion* of the ϕ's. (in words: if there are some ϕ's, there is the fusion of all ϕ's).

From 1.2 and 1.3 it follows that every individual is the fusion of its atoms.

Besides it follows that:

1.4 Extensionality. *Two individuals are equal iff they have the same atoms.*

1.5 Complementation. *Every individual has a complement.*

If all ϕ's have some part in common, we define the *product* of the ϕ's as the fusion of all their common parts.

1.6 Axiom for the pairing function. $(x, y) = (u, z) \leftrightarrow x = u \wedge y = z$, where x, y, u, z are atoms as well as the codes of the ordered pairs.

3 Axioms on the size of the universe

Using the pairing function, we can correlate atoms by means of individuals. An *atom-relation* (briefly *a-relation*) is a fusion of ordered pairs of atoms.

In particular, a function is a *many-one-a-relation.* Such a function can be extended in a natural way to non atomic individuals by mapping an individual into the fusion of the images of its atoms.

Following (Lewis, 1991) we say that an individual is *small* if there is no *1-1-a-relation* between its atoms and all atoms of the universe V.

An individual x is (Dedekind) infinite if there is a *1-1-a-relation* between its atoms and the atoms of a proper part of x.

The following axioms assure that V (the fusion of all atoms) is very big:

2.1 Axiom. *Some small individual is infinite.*

Let r be an *a-relation.* We say that the atom x *r-represents* the individual y if, for all atoms z, $z \leq y$ *iff* $(z, x) \leq r$.

The following axiom of representability enables us to represent small individuals by atoms:

2.2 Axiom. There is an *a-relation* r *such that every small individual has a unique* r*-representative.*

Such an *a-relation* will be called a *representing a-relation.*

2.3 Axiom. *If x is small and r is a representing a-relation, then the fusion of all r-representatives of parts of x is small.*

It is worth noting that Axioms 2.2 and 2.3 are independent of the preceding ones, as the following counter-examples show:

About 2.2. Suppose that the size of the universe is a cardinal α, with $\aleph_0 < \alpha < \mathfrak{c}$ (the cardinality of the continuum), what is compatible with the failure of the *continuum hypothesis*. Then 2.1 holds because any countable individual is small. But, in order to represent by atoms the parts of a denumerable individual, $\mathfrak{c}$-many atoms are needed; so 2.2 fails.

About 2.3. Consider set theory without the power axiom. This theory can be interpreted in a universe of sets, where all infinite sets are denumerable. Interpreting small individuals as sets and atoms as singletons, 2.1 and 2.2 hold, but 2.3 fails, because the singletons of the parts of a denumerable set do not form a set.

2.4 *Axiom of choice.* There is an a-function that maps each representative atom into an atom of the represented individual.

4 Interpretation of set theory

Let r be a representing *a-relation.* Let u be an atom that is not an *r-representative* (since V is infinite, we can assume that such an atom exists). Relative to r and u we define a model of set theory. We shall refer to the Kelley-Morse axiomatization of class theory, as formulated in (Kelley & Stone, 1955).

Let U be the mereological product of all individuals x such that (i) $u \leq x$ and (ii) the *r-representative* of any small part of x is a part of x.

In other words, U is the smallest individual, including u, and closed under r-representation.

Take as *classes* the parts of U and use capital letters X, Y, $Z, \dots$ as variables for classes. Define *membership* as follows:

$X \in Y$ if$_{df.}$ X is small and its *r-representative* is part of Y.

A class X is a *set* if$_{df.}$, for some Y, $X \in Y$.

It follows immediately that a class is a set *iff* it is small.

We will prove that all the main axioms of the Kelley-Morse class theory hold.

3.1 Lemma. *Every atom of U, different from u, is an r-representative.*

Proof. Let y be an atom, different from u, which is not an r-representative. The fusion x of all atoms of U, different from y, satisfies conditions (i), (ii) of the definition of U. Hence y is not part of U. ■

So u is the unique class without members and will be denoted by 0.

3.2 Extensionality. *If two classes have the same members, they are identical.*

Proof. It follows from the lemma that classes with the same members are fusions of the same atoms.■

3.3 Comprehension. *Let ϕ be a formula of class theory, whose free variables are among* X, $\mathrm{Z}_1,\dots,\mathrm{Z}_n$. *For all classes* $\mathrm{Z}_1,\dots,\mathrm{Z}_n$*, there is a class* Y *whose members are just the sets* X *such that* ϕ*.*

We introduce the usual notation $Y = \{X\colon \phi\}$.

Proof. Given $Z_1,\dots, Z_n$, if there is some set X such that ϕ, then Y is the fusion of the *r-representatives* of all sets satisfying ϕ; otherwise $Y = 0$.■

Standard class-theoretical notions are defined as usual.

$X \subset Y$ means that X is a subclass of Y, including the case that $X = Y$.

$2^X =_{df.} \{Y\colon\ Y \subset X\}$.

From axiom 2.3 we get immediately

3.4 Subsets. *If X is a set, so is* 2^X.

$\{X, Y\} =_{df.} \{Z\colon Z = X \vee Z = Y\}$.

3.5 Pairs. $\{X, Y\}$ *is a set.*

Proof. $\{X, Y\}$ is 0, if both X and Y are proper classes. Otherwise it is the fusion of at most two atoms, so it is small.■

Ordered pairs are defined in the usual way. Relations are (possibly proper) classes of ordered pairs. Functions are many-one relations. If the domain of a function is small, so is clearly its range. Thus

3.6 Substitution. *If the domain of a function is a set, so is its range.*

3.7 Union. *If X is a set, so is $\bigcup X$.*

Proof. Suppose, by reduction, that $\bigcup X$ is a proper class. Then it has the same size as U and we can suppose $U = \bigcup X$. For any class Y, the intersections of Y with the members of X are (by 3.6) the members of a set $Y*$. By mapping Y into $Y*$, we get a bijection between classes and sets. This leads to an absurdity, arguing as in Cantor's theorem.■

3.8 Regularity. *If $X \neq 0$, there is a member Y of X such that $X \cap Y = 0$.*

Proof. Let x be the fusion of all atoms n satisfying the following condition: for every individual y such that $n \leq\ y$, there is an atom $m \leq y$ such that, for no atom $p \leq y\ ,\ (p\ ,\ m\ ) \leq r$. It is easily seen that x satisfies conditions (i), (ii) of the definition of U. So $U \leq x$ and the theorem follows.■

By using the foregoing set-theoretical axioms, one can define ordinal numbers and show that they are the members of a proper class, which is well-ordered by membership. Since proper classes have the same size as U, there is a well-order of U and the axiom of choice follows:

3.9 Choice. *There is a choice function that assigns to every set one of its members.*

5 Categoricity

Our model U of class theory is relative to the arbitrary choice of r and u. We want to show that the models obtained by changing r and u are isomorphic.

Categoricity. *Let r', u' be an a-relation and an atom satisfying the same conditions as r, u and let U' be the relative model of class theory. Then there is an a-function h (extended to individuals in the natural way) through which U and U' are isomorphic.*

Proof. We will define, by transfinite induction, for every ordinal α of U, a $1-1-$ a-function h_α from a part of U to a part of U' such that

(i) $h_\alpha\ (u) = u'$;

(ii) If $h_\alpha\ (a) = a'$ where a, a' are atoms, then either $a = u$ and $a' = u'$, or a is the r-representative of x, a' the r'-representative of x' and $h_\alpha(x) = x'$.

Put:

(1) $h_0\ (u) = u'$;

(2) $h_{\alpha+1}$ extends h_α as follows: for every x, x' such that $h_\alpha(x) = x'$, put $h_{\alpha+1}(a) = a$', where a is the r-representative of x and a' the r'-representative of x'.

(3) $h_\alpha = \cup_{\beta<\alpha}\ h_\beta$, for α limit.

One recognizes, by transfinite induction, that all functions are 1-1 and satisfy clauses (i) and (ii).

We claim that the fusion h of all h_α is the required isomorphism.

Let D be the domain of h and, for each α, D_α the domain of h_α. Of course, $D \leq U$. In order to show that $D = U$, we have to prove that D is closed under r-representation. Observe that all D_α are small. For, if D_α is small, so is $D_{\alpha+1}$, by 2.3; besides, the fusion of a bounded sequence of small individuals is small. Hence, by induction, every D_α is small. Now, let x be a small part of D. Since the atoms of x are few and each of them occurs in some D_α, x is part of some D_α and its representative is part of $D_{\alpha+1}$. Thus D is closed under r-representation; and hence $D = U$.

Similarly, the codomain of h is the whole U'. For, observe that the height of the ordinals of U' is the same as that of the ordinals of U, since it is determined by the size of the universe of atoms.

Besides, it is plain that h preserves membership, so that it is an isomorphism between U and U'.■

It is worth noticing that our result of categoricity presupposes an absolute sense of the mereological notions of *part* and of *fusion*. In virtue of that non-standard models are ruled out.

Besides, notice that categoricity shows the independence of the set theoretic structure of the choice of the representing relation and the non-representing atom, but the structure still depends on the size of the universe of atoms. Changing the universe of atoms one gets a result of *quasi-categoricity*: one of the models is isomorphic to a segment of the other.[5]

Thus, the mereological structure of individuals determines, up to isomorphism, a model of class theory.

We can conclude that the conception of a big universe of individuals, mereologically structured, provides, with the help of any given pairing function, an adequate framework for a structuralist foundation of class theory.

[5]The situation is very similar to Zermelo's *quasi-categoricity*. See on that (Hellman, 2002).

6 Consistency of megethology

We want to observe that megethology is, in turn, interpretable in Kelley-Morse class theory, so that it is consistent relatively to the latter.

For, take non-empty classes as individuals, singletons of sets as atoms, subclasses as parts. Define the pairing function by putting

$(x,\ y) =_{df.} \{\{x\},\ \{x\ ,y\}\}$, where x, y are singletons.

As it is known, it is provable from Kelley-Morse axioms that a class is proper iff it is equinumerous with the total class U (see (Fraenkel, Bar-Hillel, & Levy, 1973, p.137)). So sets are small classes and we can define a representing a-relation r as follows: $(x\ ,\ y) \leq r$ *iff*, for some set z, x is the singleton of a member of z and y the singleton of z.

All axioms of megethology are straightforwardly verifiable.

7 A mereological codification of ordered pairs

Our development of megethology, without plural quantification, with the help of a primitive pairing function, puts forward that Lewis' use of plural quantification serves only the purpose of supplying a mereological codification of ordered pairs. We want to show that even for this purpose plural quantification is overwhelming.

Lewis codifies ordered pairs of atoms as biatoms, relative to certain given pluralities of individuals. In this way he can identify a binary relation with a certain plurality of biatoms. So quantification over relations is understood as plural quantification. We will show that, with a suitable modification of Lewis' strategy, ordered pairs of atoms can be codified by single atoms (instead of biatoms). In this way an a-relation can be understood as a fusion of atoms, i.e. as a single individual, and quantification over relations is reduced to first order quantification.

Following a Burgess' idea (Burgess, Hazen, & Lewis, 1991), assume that the universe U of atoms is divided in three microcosms U_1, U_2, U_3 of the same cardinality of U and pairwise non-overlapping. Let f be a 1-1 map from U_3 to U_1 and g a 1-1-map from U_3 to U_2, understood as pluralities of biatoms. So every atom of U_3 has a counterpart in U_1 and a counterpart in U_2. Let A be a plurality of three-atoms satisfying the following conditions:

(i) for every x in U_1, y in U_2 there is a unique z in U_3 such that $x + y + z$ is in A;

(ii) every atom in U_3 can occur in a unique three-atom of A.

Relative to the pluralities f, g and A, we can codify the ordered pairs of U_3: if x and y are atoms of U_3, take as ordered pair (x, y) the unique z in U_3 such that $f(x) + g(y) + z$ is in A. In this way the ordered pairs of atoms of U_3 are defined mereologically, relative to a finite number of fixed pluralities, i.e. f, g, and A. Of course, the restriction of ordered pairs to U_3 is sufficient for the reconstruction of megethology: since the size of U_3 is the same as that of U, once ordered pairs are defined, one can build megethology on the universe U_3.

8 Conclusions

In this paper we have shown that – provided a machinery for pairing is taken as primitive – mereology is adequate to a reformulation of megethology in first order logic with identity.

Besides, we have proved that, in order to develop megethology, ordered pairs can be restricted to atoms and can be codified by means of finitely many pluralities without involving plural quantification.

We have proved in (Carrara & Martino, 2014) that, in turn, plural quantification is adequate to a reformulation of megethology without the help of mereology. The two results show that Lewis' framework, endorsing both mereology and plural quantification, is overwhelming and stress the power of each of the two components separately.

9 Acknowledgments

We would like to thank the referees of the *JPL* for the helpful comments and suggestions.

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