ESSAYS ON THE METAPHYSICS OF QUANTUM MECHANICS

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ABSTRACT OF THE DISSERTATION

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What is the proper metaphysics of quantum mechanics? In this dissertation, I approach the question from three different but related angles. First, I suggest that the quantum state can be understood intrinsically as relations holding among regions in ordinary space-time, from which we can recover the wave function uniquely up to an equivalence class (by representation and uniqueness theorems). The intrinsic account eliminates certain conventional elements (e.g. overall phase) in the representation of the quantum state. It also dispenses with first-order quantification over mathematical objects, which goes some way towards making the quantum world safe for a nominalistic metaphysics suggested in Field (1980, 2016). Second, I argue that the fundamental space of the quantum world is the low-dimensional physical space and not the high-dimensional space isomorphic to the "configuration space." My arguments are based on considerations about dynamics, empirical adequacy, and symmetries of the quantum mechanics. Third, I show that, when we consider quantum mechanics in a time-asymmetric universe (with a large entropy gradient), we obtain new theoretical and conceptual possibilities. In such a model, we can use the low-entropy boundary condition known as the Past Hypothesis (Albert, 2000) to pin down a natural initial quantum state of the universe.

However, the universal quantum state is not a pure state but a mixed state, represented by a density matrix that is the normalized projection onto the Past Hypothesis subspace. This particular choice has interesting consequences for Humean supervenience, statistical mechanical probabilities, and theoretical unity.

Acknowledgements

I came to Rutgers University in the fall of 2013. My plan was to work on the philosophy of language and metaphysics, which I still find fascinating. But it didn't take me long to come to Barry Loewer and David Albert, who taught a series of graduate seminars that year (and every year afterwards) on the philosophy of physics and the philosophy of science. One of the questions they raised was on the metaphysics of quantum mechanics—"what does the wave function represent in the physical world?" (That was pretty much the only thing I thought about during my first year.) Thanks to them, I realized that my true passion lies in the foundations of physics (and they also encouraged me to pursue my other interests such as philosophy of mind, Chinese philosophy, decision theory, and metaphysics, even when they are not always connected to the foundations of physics). I spent the last few years writing up a few papers on the metaphysics of quantum mechanics, some of which became the chapters of this dissertation. I don't know if I have found a good answer yet. But I am grateful to Barry and David, who are the co-directors of this dissertation, for helping me wrestling with this important and interesting question.

I would like to thank Barry for his generosity and patience with me as I tried out many different ideas in my first few years. Even when my ideas conflicted with his own, Barry still helped me without failing and always tried to make my ideas better and my arguments more convincing. If I were fortunate enough to have graduate advisees, I would like to be like Barry (and I might have to borrow some of his jokes and stories).

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I were a long distance away for most of my time at Rutgers, but she has always believed in me and supported my work. She also introduced me to her cats, who are a constant source of distraction. It is fitting that this dissertation is dedicated to Linda and our cats.

Dedication

To Linda, Cōng Cōng, and Dòu Nīu.

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Introduction

Quantum mechanics (and its relativistic cousins) is the most successful physical theory to date. Despite its empirical success, quantum mechanics presents many conceptual and philosophical puzzles. The first problem is the "quantum measurement problem," according to which there is either a logical inconsistency or a fundamental ambiguity in the axioms of measurement. As we have learnt from J. S. Bell, the measurement problem is related to the second problem – the absence of a clear and satisfactory physical ontology (in the standard presentation of textbook quantum mechanics).

Regarding the measurement problem, there has been much progress in the last few decades. Physicists and philosophers have produced three classes of solutions: the de-Broglie Bohm theory (BM), the Everettian theory (EQM), and the Ghirardi-Rimini-Weber and Pearl theories (GRW / CSL).

Regarding the ontology problem, we have learnt much from the solutions to the previous problem. BM postulates an ontology of particles in addition to the quantum wave function, while the original versions of EQM and GRW / CSL postulate only the wave function. However, far from settling the ontology problem, these theories raise new philosophical questions:

1. What is the nature of the wave function?
2. If the wave function "lives on" a high-dimensional space, how does it relate to 3-dimensional macroscopic objects and what Bell calls "local beables?" What, then, is the status of the 3-dimensional physical space?
3. Are local beables fundamental? Is the wave function fundamental?

My dissertation consists in three essays on the metaphysics of quantum mechanics that attempt to make some progress on these questions.

Chapter 1. The Intrinsic Structure of Quantum Mechanics

What is the nature of the wave function? There are two ways of pursuing this question:

1. What is the physical basis for the mathematics used for the wave function? Which mathematical degrees of freedom of the wave function are physically genuine? What is the metaphysical explanation for the merely mathematical or gauge degrees of freedom?

2. What kind of "thing" does the wave function represent? Does it represent a physical field on the configuration space, something nomological, or a *sui generis* entity in its own ontological category?

Chapter 1 addresses the first question. Chapters 2 and 3 bear on the second question.

In Chapter 1, I introduce an intrinsic account of the quantum state. This account contains three desirable features that the standard platonistic account lacks: (1) it does not refer to any abstract mathematical objects such as complex numbers, (2) it is independent of the usual arbitrary conventions in the wave function representation, and (3) it explains why the quantum state has its amplitude and phase degrees of freedom (with the help of new representation and uniqueness theorems). Consequently, this account extends Hartry Field's program outlined in *Science Without Numbers* (1980), responds to David Malament's long-standing impossibility conjecture (1982), and goes towards a genuinely intrinsic and nominalistic account of quantum mechanics.

I also discuss how it bears on the debate about "wave function realism." I suggest that the intrinsic account provides a novel response, on behalf of those that take the wave function to be some kind of physical field, to the objection (Maudlin, 2013) that the field-interpretation of the wave function reifies too many gauge degrees of freedom.

Chapter 2. Our Fundamental Physical Space

Chapter 2 explores the ongoing debate about how our ordinary 3-dimensional space is related to the 3N-dimensional configuration space on which the wave function is defined. Which of the two spaces is our (more) fundamental physical space?

I start by reviewing the debate between the 3N-Fundamentalists (wave function realists) and the 3D-Fundamentalists (primitive ontologists). Instead of framing the debate as putting different weights on different kinds of evidence, I shall evaluate them on how they are overall supported by: (1) the dynamical structure of the quantum theory, (2) our perceptual evidence of the 3D-space, and (3) mathematical symmetries in the wave function. I show that the common arguments based on (1) and (2) are either unsound or incomplete. Completing the arguments, it seems to me, renders the overall considerations based on (1) and (2) *roughly* in favor of 3D-Fundamentalism. A more decisive argument, however, is found when we consider which view leads to a deeper understanding of the physical world. In fact, given the deeper topological explanation from the unordered configurations to the Symmetrization Postulate, we have strong reasons in favor of 3D-Fundamentalism. I therefore conclude that our current overall evidence strongly favors the view that our fundamental physical space in a quantum world is 3-dimensional rather than 3N-dimensional. I also outline future lines of research where the evidential balance can be restored or reversed. Finally, I draw some lessons from this case study to the debate about theoretical equivalence.

Chapter 2 was published in *The Journal of Philosophy* in 2017.

Chapter 3. Quantum Mechanics in a Time-Asymmetric Universe

Chapter 3 departs from the framework of wave function realism. However, it is still about the nature and the reality of the quantum state.

In a quantum universe with a strong arrow of time, we postulate a low-entropy boundary condition (the Past Hypothesis) to account for the temporal asymmetry. In this chapter, I show that the Past Hypothesis also contains enough information to simplify the quantum ontology and define a natural initial condition.

First, I introduce *Density Matrix Realism*, the thesis that the quantum state of the universe is objective and impure. This stands in sharp contrast to *Wave Function Realism*, the thesis that the quantum state of the universe is objective and pure. Second, I suggest that the Past Hypothesis is sufficient to determine a natural density matrix, which is simple and unique. This is achieved by what I call the *Initial Projection Hypothesis*: the initial density matrix of the universe is the (normalized) projection onto the Past Hypothesis subspace (in the Hilbert space). Third, because the initial quantum state is unique and simple, we have a strong case for the *Nomological Thesis*: the initial quantum state of the universe is on a par with laws of nature.

This new package of ideas has several interesting implications, including on the harmony between statistical mechanics and quantum mechanics, theoretical unity of the universe and the subsystems, and the alleged conflict between Humean supervenience and quantum entanglement.

Chapter 3 is forthcoming in *The British Journal for the Philosophy of Science*.

Chapter 1

The Intrinsic Structure of Quantum Mechanics

1.1 Introduction

Quantum mechanics is empirically successful (at least in the non-relativistic domain). But what it means remains highly controversial. Since its initial formulation, there have been many debates (in physics and in philosophy) about the ontology of a quantum-mechanical world. Chief among them is a serious foundational question about how to understand the quantum-mechanical laws and the origin of quantum randomness. That is the topic of the quantum measurement problem. At the time of writing this paper, the following are serious contenders for being the best solution: Bohmian mechanics (BM), spontaneous localization theories (GRW0, GRWf, GRWm, CSL), and Everettian quantum mechanics (EQM and Many-Worlds Interpretation (MWI)).[1]

There are other deep questions about quantum mechanics that have a philosophical and metaphysical flavor. Opening a standard textbook on quantum mechanics, we find an abundance of mathematical objects: Hilbert spaces, operators, matrices, wave functions, and etc. But what do they represent in the physical world? Are they ontologically serious to the same degree or are some merely dispensable instruments that facilitate calculations? In recent debates in metaphysics of quantum mechanics, there is considerable agreement that the universal wave function, modulo some mathematical degrees of freedom, represents something objective — the quantum state of the universe.[2] In contrast, matrices and operators are merely convenient summaries that do

[1]See Norsen (2017) for an updated introduction to the measurement problem and the main solutions.

[2]The universal quantum state, represented by a universal wave function, can give rise to wave functions of the subsystems. The clearest examples are the conditional wave functions in Bohmian mechanics. However, our primary focus here will be on the wave function of the universe.

not play the same fundamental role as the wave function.

However, the meaning of the universal quantum state is far from clear. We know its mathematical representation very well: the universal wave function, which is crucially involved in the dynamics of BM, GRW, and EQM. In the position representation, a scalar-valued wave function is a square-integrable function from the configuration space $\mathbb{R}^{3N}$ to the complex plane $\mathbb{C}$. But what does the wave function really mean? There are two ways of pursuing this question:

1. What kind of "thing" does the wave function represent? Does it represent a physical field on the configuration space, something nomological, or a *sui generis* entity in its own ontological category?

2. What is the physical basis for the mathematics used for the wave function? Which mathematical degrees of freedom of the wave function are physically genuine? What is the metaphysical explanation for the merely mathematical or gauge degrees of freedom?

Much of the philosophical literature on the metaphysics of the wave function has pursued the first line of questions.[3] In this paper, I will primarily pursue the second one, but I will also show that these two are intimately related.

In particular, I will introduce an intrinsic account of the quantum state. It answers the second line of questions by picking out four concrete relations on physical space-time. Thus, it makes explicit the physical basis for the usefulness of the mathematics of the wave function, and it provides a metaphysical explanation for why certain degrees of freedom in the wave function (the scale of the amplitude and the overall phase) are merely gauge. The intrinsic account also has the feature that the fundamental ontology does not include abstract mathematical objects such as complex numbers, functions, vectors, or sets.

The intrinsic account is therefore nominalistic in the sense of Hartry Field (1980). In his influential monograph *Science Without Numbers: A Defense of Nominalism*, Field

[3] See, for example, Albert (1996); Loewer (1996); Wallace and Timpson (2010); North (2013); Ney (2012); Maudlin (2013); Goldstein and Zanghì (2013); Miller (2013); Bhogal and Perry (2015).

advances a new approach to philosophy of mathematics by explicitly constructing nominalistic counterparts of the platonistic physical theories. In particular, he nominalizes Newtonian gravitation theory.[4] In the same spirit, Frank Arntzenius and Cian Dorr (2011) develop a nominalization of differential manifolds, laying down the foundation of a nominalistic theory of classical field theories and general relativity. Up until now, however, there has been no successful nominalization of quantum theory. In fact, it has been an open problem–both conceptually and mathematically–how it is to be done. The non-existence of a nominalistic quantum mechanics has encouraged much skepticism about Field's program of nominalizing fundamental physics and much optimism about the Quine-Putnam Indispensability Argument for Mathematical Objects. Indeed, there is a long-standing conjecture, due to David Malament (1982), that Field's nominalism would not succeed in quantum mechanics. Therefore, being nominalistic, my intrinsic theory of the quantum state would advance Field's nominalistic project and provide (the first step of) an answer to Malament's skepticism.

Another interesting consequence of the account is that it will make progress on the first line of questions about the ontology of quantum mechanics. On an increasingly influential interpretation of the wave function, it represents something physically significant. One version of this view is the so-called "wave function realism," the view that the universal wave function represents a physical field on a high-dimensional (fundamental) space. That is the position developed and defended in Albert (1996), Loewer (1996), Ney (2012), and North (2013). However, Tim Maudlin (2013) has argued that this view leads to an unpleasant proliferation of possibilities: if the wave function represents a physical field (like the classical electromagnetic field), then a change of the wave function by an overall phase transformation will produce a distinct physical possibility. But the two wave functions will be empirically equivalent—no experiments can distinguish them, which is the reason why the overall phase differences are usually regarded as merely *gauge*. Since the intrinsic account of the wave function I offer here is gauge-free insofar as overall phase is concerned, it removes a major obstacle to wave function realism (vis-à-vis Maudlin's objection).

[4] It is not quite complete as it leaves out integration.

In this paper, I will first explain (in §2) the two visions for a fundamental physical theory of the world: the intrinsicalist vision and the nominalistic vision. I will then discuss why quantum theory may seem to resist the intrinsic and nominalistic reformulation. Next (in §3), I will write down an intrinsic and nominalistic theory of the quantum state. Finally (in §4), I will discuss how this account bears on the nature of phase and the debate about *wave function realism*.

Along the way, I axiomatize the quantum phase structure as what I shall call a *periodic difference structure* and prove a representation theorem and a uniqueness theorem. These formal results could prove fruitful for further investigation into the metaphysics of quantum mechanics and theoretical structure in physical theories.

1.2 The Two Visions and the Quantum Obstacle

There are, broadly speaking, two grand visions for what a fundamental physical theory of the world should look like. (To be sure, there are many other visions and aspirations.) The first is what I shall call the intrinsicalist vision, the requirement that the fundamental theory be written in a form without any reference to arbitrary conventions such as coordinate systems and units of scale. The second is the nominalistic vision, the requirement that the fundamental theory be written without any reference to mathematical objects. The first one is familiar to mathematical physicists from the development of synthetic geometry and differential geometry. The second one is familiar to philosophers of mathematics and philosophers of physics working on the ontological commitment of physical theories. First, I will describe the two visions, explain their motivations, and provide some examples. Next, I will explain why quantum mechanics seems to be an obstacle for both programs.

1.2.1 The Intrinsicalist Vision

The intrinsicalist vision is best illustrated with some history of Euclidean geometry. Euclid showed that complex geometrical facts can be demonstrated using rigorous proof on the basis of simple axioms. However, Euclid's axioms do not mention real numbers or coordinate systems, for they were not yet discovered. They are stated with only qualitative predicates such as the equality of line segments and the congruence of angles. With these concepts, Euclid was able to derive a large body of geometrical propositions.

Real numbers and coordinate systems were introduced to facilitate the derivations. With the full power of real analysis, the metric function defined on pairs of tuples of coordinate numbers can greatly speed up the calculations, which usually take up many steps of logical derivation on Euclid's approach. But what are the significance of the real numbers and coordinate systems? When representing a 3-dimensional Euclidean space, a typical choice is to use $\mathbb{R}^3$. It is clear that such a representation has much surplus (or excess) structure: the origin of the coordinate system, the orientation of the axis, and the scale are all arbitrarily chosen (sometimes conveniently chosen for ease of calculation). There is "more information" or "more structure" in $\mathbb{R}^3$ than in the 3-dimensional Euclidean space. In other words, the $\mathbb{R}^3$ representation has gauge degrees of freedom.

The real, intrinsic structure in the 3-dimensional Euclidean space–the structure that is represented by $\mathbb{R}^3$ up to the Euclidean transformations–can be understood as an axiomatic structure of congruence and betweenness. In fact, Hilbert 1899 and Tarski 1959 give us ways to make this statement more precise. After offering a rigorous axiomatization of Euclidean geometry, they prove a representation theorem: any structure instantiates the betweenness and congruence axioms of 3-dimensional Euclidean geometry if and only if there is a 1-1 embedding function from the structure onto $\mathbb{R}^3$ such that if we define a metric function in the usual Pythagorean way then the metric function is homomorphic: it preserves the exact structure of betweenness and congruence. Moreover, they prove a uniqueness theorem: any other embedding function defined on the same domain satisfies the same conditions of homomorphism if and only if it is a Euclidean transformation of the original embedding function: a transformation on $\mathbb{R}^3$ that can be obtained by some combination of shift of origin, reflection, rotation, and positive scaling.

The formal results support the idea that we can think of the genuine, intrinsic features of 3-dimensional Euclidean space as consisting directly of betweenness and congruence relations on spatial points, and we can regard the coordinate system ($\mathbb{R}^3$) and the metric function as extrinsic representational features we bring to facilitate calculations. (Exercise: prove the Pythagorean Theorem with and without real-numbered coordinate systems.) The merely representational artifacts are highly useful but still dispensable.

There are several advantages of having an intrinsic formulation of geometry. First, it eliminates the need for many arbitrary conventions: where to place the origin, how to orient the axis, and what scale to use. Second, in the absence of these arbitrary conventions, we have a theory whose elements *could* stand in one-to-one correspondence with elements of reality. In that case, we can look directly into the real structure of the geometrical objects without worrying that we are looking at some merely representational artifact (or gauge degrees of freedom). By eliminating redundant structure in a theory, an intrinsic formulation gives us a more perspicuous picture of the geometrical reality.

The lessons we learn from the history of Euclidean geometry can be extended to other parts of physics. For example, people have long noticed that there are many gauge degrees of freedom in the representation of both scalar and vector valued physical quantities: temperature, mass, potential, and field values. There has been much debate in philosophy of physics about what structure is physically genuine and and what is merely gauge. It would therefore be helpful to go beyond the scope of physical geometry and extend the intrinsic approach to physical theories in general.

Hartry Field (1980), building on previous work by Krantz et al. (1971), ingeniously extends the intrinsic approach to Newtonian gravitation theory. The result is an elimination of arbitrary choices of zero field value and units of mass. His conjecture is that all physical theories can be "intrinsicalized" in one way or another.

1.2.2 The Nominalist Vision

As mentioned earlier, Field (1980) provides an intrinsic version of Newtonian gravitation theory. But the main motivation and the major achievement of his project is a defense of nominalism, the thesis that there are no abstract entities, and, in particular, no abstract mathematical entities such as numbers, functions, and sets.

The background for Field's nominalistic project is the classic debate between the mathematical nominalist and the mathematical platonist, the latter of whom is ontologically committed to the existence of abstract mathematical objects. Field identifies a main problem of maintaining nominalism is the apparent indispensability of mathematical objects in formulating our best physical theories:

> Since I deny that numbers, functions, sets, etc. exist, I deny that it is legitimate to use terms that purport to refer to such entities, or variables that purport to range over such entities, in our ultimate account of what the world is really like.
>
> This appears to raise a problem: for our ultimate account of what the world is really like must surely include a physical theory; and in developing physical theories one needs to use mathematics; and mathematics is full of such references to and quantifications over numbers, functions, sets, and the like. It would appear then that nominalism is not a position that can reasonably be maintained.[5]

In other words, the main task of defending nominalism would be to respond to the Quine-Putnam Indispensability Argument:[6]

P1 We ought to be ontologically committed to all (and only) those entities that are indispensable to our best theories of the world. [Quine's Criterion of Ontological Commitment]

P2 Mathematical entities are indispensable to our best theories of the world. [The Indispensability Thesis]

C Therefore, we ought to be ontologically committed to mathematical entities.

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[5] Field (2016), Preliminary Remarks, p.1.

[6] The argument was originally proposed by W. V. Quine and later developed by Putnam (1971). This version is from Colyvan (2015).

In particular, Field's task is to refute the second premise–the Indispensability Thesis. Field proposes to replace all platonistic physical theories with attractive nominalistic versions that do not quantify over mathematical objects

Field's nominalistic versions of physical theories would have significant advantages over their platonistic counterparts. First, the nominalistic versions illuminate what exactly in the physical world provide the explanations for the usefulness of any particular mathematical representation. After all, even a platonist might accept that numbers and coordinate systems do not really exist in the physical world but merely represent some concrete physical reality. Such an attitude is consistent with the platonist's endorsement of the Indispensability Thesis. Second, as Field has argued, the nominalistic physics seems to provide better explanations than the platonistic counterparts, for the latter would involve explanation of physical phenomena by things (such as numbers) external to the physical processes themselves.

Field has partially succeeded by writing down an intrinsic theory of physical geometry and Newtonian gravitation, as it contains no explicit first-order quantification over mathematical objects, thus qualifying his theory as nominalistic. But what about other theories? Despite the initial success of his project, there has been significant skepticism about whether his project can extend beyond Newtonian gravitation theory to more advanced theories such as quantum mechanics.

1.2.3 Obstacles From Quantum Theory

We have looked at the motivations for the two visions for what the fundamental theory of the world should look like: the intrinsicalist vision and the nominalistic vision. They should not be thought of as competing against each other. They often converge on a common project. Indeed, Field's reformulation of Newtonian Gravitation Theory is both intrinsic and nominalistic.[7]

Both have had considerable success in certain segments of classical theories. But

[7] However, the intrinsicalist and nominalistic visions can also come apart. For example, we can, in the case of mass, adopt an intrinsic yet platonistic theory of mass ratios. We can also adopt an extrinsic yet nominalistic theory of mass relations by using some arbitrary object (say, my water bottle) as standing for unit mass and assigning comparative relations between that arbitrary object and every other object.

with the rich mathematical structures and abstract formalisms in quantum mechanics, both seem to run into obstacles. David Malament was one of the earliest critics of the nominalistic vision. He voiced his skepticism in his influential review of Field's book. Malament states his general worry as follows:

> Suppose Field wants to give some physical theory a nominalistic reformulation. Further suppose the theory determines a class of mathematical models, each of which consists of a set of "points" together with certain mathematical structures defined on them. Field's nominalization strategy cannot be successful unless the objects represented by the points are appropriately physical (or non-abstract)...But in lots of cases the represented objects *are* abstract. (Malament (1982), pp. 533, emphasis original.)[8]

Given his *general* worry that, often in physical theories, it is abstracta that are represented in the state spaces, Malament conjectures that, in the specific case of quantum mechanics, Field's strategy of nominalization would not "have a chance":

> Here [in the context of quantum mechanics] I do not really see how Field can get started at all. I suppose one can think of the theory as determining a set of models—each a Hilbert space. But what form would the recovery (i.e., representation) theorem take? The only possibility that comes to mind is a theorem of the sort sought by Jauch, Piron, *et al.* They start with "propositions" (or "eventualities") and lattice-theoretic relations as primitive, and then seek to prove that the lattice of propositions is necessarily isomorphic to the lattice of subspaces of some Hilbert space. But of course no theorem of this sort would be of any use to Field. What could be worse than *propositions* (or *eventualities*)? (Malament (1982), pp. 533-34.)

As I understand it, Malament suggests that there are no good places to start nominalizing non-relativistic quantum mechanics. This is because the obvious starting point,

[8]Malament also gives the example of classical Hamiltonian mechanics as another specific instance of the general worry. But this is not the place to get into classical mechanics. Suffice to say that there are several ways to nominalize classical mechanics. Field's nominalistic Newtonian Gravitation Theory is one way. Arntzenius and Dorr (2011) provides another way.

according to Malament and other commentators, is the abstract Hilbert space, $\mathscr{H}$, as it is a state space of the quantum state.

However, there may be other starting points to nominalize quantum mechanics. For example, the configuration space, $\mathbb{R}^{3N}$, is a good candidate. In realist quantum theories such as Bohmian mechanics, Everettian quantum mechanics, and spontaneous localization theories, it is standard to postulate a (normalized) universal wave function $\Psi(\mathbf{x},t)$ defined on the configuration space(-time) and a dynamical equation governing its temporal evolution.[9] In the deterministic case, the wave function evolves according to the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{x},t) = [-\sum_{i=1}^{N}\frac{\hbar^2}{2m_i}\Delta_i + V(x)]\Psi(\mathbf{x},t) := H\Psi(\mathbf{x},t),$$

which relates the temporal derivatives of the wave function to its spatial derivatives. Now, the configuration-space viewpoint can be translated into the Hilbert space formalism. If we regard the wave function (a square-integrable function from the configuration space to complex numbers) as a unit vector $|\Psi(t)\rangle$, then we can form another space—the Hilbert space of the system.[10] Thus, the wave function can be mapped to a state vector, and vice versa. The state vector then rotates (on the unit sphere in the Hilbert space) according to a unitary (Hamiltonian) operator,

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle,$$

which is another way to express the Schrödinger evolution of the wave function.

Hence, there is the possibility of carrying out the nominalization project with the configuration space. In some respects, the configuration-space viewpoint is more friendly to nominalism, as the configuration space is much closely related to physical space than the abstract Hilbert space is.[11] Nevertheless, Malament's worries still remain, because (*prima facie*) the configuration space is also quite abstract, and it is

[9]Bohmian mechanics postulates additional ontologies—particles with precise locations in physical space—and an extra law of motion—the guidance equation. GRW theories postulate an additional stochastic modification of the Schrödinger equation and, for some versions, additional ontologies such as flashes and mass densities in physical space.

[10]This is the Hilbert space $L^2(\mathbb{R}^{3N},\mathbb{C})$, equipped with the inner product $<\psi,\phi>$ of taking the Lebesgue integral of $\psi^*\phi$ over the configuration space, which guarantees Cauchy Completeness.

[11]I should emphasize that, because of its central role in functional analysis, Hilbert space is highly

unclear how to fit it into the nominalistic framework. Therefore, at least *prima facie*, quantum mechanics seems to frustrate the nominalistic vision.

Moreover, the mathematics of quantum mechanics comes with much conventional structure that is hard to get rid of. For example, we know that the exact value of the amplitude of the wave function is not important. For that matter, we can scale it with any arbitrary positive constant. It is true that we usually choose the scale such that we get unity when integrating the amplitude over the entire configuration space. But that is merely conventional. We can, for example, write down the Born rule with a proportionality constant to get unity in the probability function:

$$P(x \in X) = Z \int_X |\Psi(x)|^2 dx,$$

where Z is a normalization constant.

Another example is the overall phase of the wave function. As we learn from modular arithmetic, the exact value of the phase of the wave function is not physically significant, as we can add a constant phase factor to every point in configuration space and the wave function will remain physically the same: producing exactly the same predictions in terms of probabilities.

All these gauge degrees of freedom are frustrating from the point of view of the intrinsicalist vision. They are the manifestation of excess structures in the quantum theory. What exactly is going on in the real world that allows for these gauge degrees of freedom but not others? What is the most metaphysically perspicuous picture of the quantum state, represented by the wave function? Many people would respond that the quantum state is projective, meaning that the state space for the quantum state is not the Hilbert space, but its quotient space: the projective Hilbert space. It can be obtained by quotienting the usual Hilbert space with the equivalence relation $\psi \sim Re^{i\theta}\psi$. But this is not satisfying; the "quotienting" strategy raises a similar question: what

important for fascilitating calculations and proving theorems about quantum mechanics. Nevertheless, we should not regard it as conclusive evidence for ontological priority. Indeed, as we shall see in §3, the configuration-space viewpoint provides a natural platform for the nominalization of the universal wave function. We should also keep in mind that, at the end of the day, it suffices to show that quantum mechanics can be successfully nominalized from *some* viewpoint.

exactly is going on in the real world that allows for quotienting with this equivalence relation but not others?[12] No one, as far as I know, has offered an intrinsic picture of the quantum state, even in the non-relativistic domain.

In short, at least *prima facie*, both the intrinsicalist vision and the nominalist vision are challenged by quantum mechanics.

1.3 An Intrinsic and Nominalistic Account of the Quantum State

In this section, I propose a new account of the quantum state based on some lessons we learned from the debates about wave function realism.[13] As we shall see, it does not take much to overcome the "quantum obstacle." For simplicity, I will focus on the case of a quantum state for a constant number of identical particles without spin.

1.3.1 The Mathematics of the Quantum State

First, let me explain my strategy for nominalizing non-relativistic quantum mechanics.

1. I will start with a Newtonian space-time, whose nominalization is readily available.[14]

2. I will use symmetries as a guide to fundamentality and identify the intrinsic structure of the universal quantum state on the Newtonian space-time. This will

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[12]These questions, I believe, are in the same spirit as Ted Sider's 2016 Locke Lecture (ms.), and especially his final lecture on theoretical equivalence and against what he calls"quotienting by hand." I should mention that both Sider and I are really after gauge-free formulations of physical and metaphysical theories, which are more stringent than merely gauge-independent formulations. For example, modern differential geometry is gauge-independent (coordinate-independent) but not gauge-free (coordinate-free): although manifolds can be defined without privileging any particular coordinate system, their definition still uses coordinate systems (maps and atlas).

[13]Here I'm taking the "Hard Road" to nominalism. As such, my goal is to (1) reformulate quantum mechanics (QM) such that *within the theory* it no longer refers (under first-order quantifiers) to mathematical objects such as numbers, functions, or sets and (2) demonstrate that the platonistic version of QM is conservative over the nominalistic reformulation. To arrive at my theory, and to state and prove the representation theorems, I refer to some mathematical objects. But these are parts of the meta-theory to explicate the relation between my account and the platonistic counterpart and to argue (by *reductio*) against the indispensability thesis. See Field (2016), Preliminary Remarks and Ch. 1 for a clear discussion, and Colyvan (2010) for an up-to-date assessment of the "Easy Road" option. Thanks to Andrea Oldofredi and Ted Sider for suggesting that I make this clear.

[14]It is an interesting question what role Galilean relativity plays in non-relativistic quantum mechanics. I will explore this issue in future work.

be the goal for the remaining part of the paper. (Here we focus only on the quantum state, because it is novel and it seems to resist nominalization. But the theory leaves room for additional ontologies of particles, fields, mass densities supplied by specific interpretations of QM; these additional ontologies are readily nominalizable.)

3. In future work, I will develop nominalistic translations of the dynamical equations and generalize this account to accommodate more complicated quantum theories.

Before we get into the intrinsic structure of the universal quantum state, we need to say a bit more about its mathematical structure. For the quantum state of a spinless system at a time t, we can represent it with a scalar-valued wave function:

$$\Psi_t : \mathbb{R}^{3N} \to \mathbb{C},$$

where N is the number of particles in the system, $\mathbb{R}^{3N}$ is the configuration space of N particles, and $\mathbb{C}$ is the complex plane. (For the quantum state of a system with spin, we can use a vector-valued wave function whose range is the spinor space—$\mathbb{C}^{2^N}$.)

My strategy is to start with a Newtonian space-time (which is usually represented by a Cartesian product of a 3-dimensional Euclidean space and a 1-dimensional time). If we want to nominalize the quantum state, what should we do with the configuration space $\mathbb{R}^{3N}$? As is now familiar from the debate about wave function realism, there are two ways of interpreting the fundamental physical space for a quantum world:

1. $\mathbb{R}^{3N}$ represents the fundamental physical space; the space represented by $\mathbb{R}^3$ only *appears* to be real; the quantum state assigns a complex number to each point in $\mathbb{R}^{3N}$. (Analogy: classical field.)

2. $\mathbb{R}^3$ represents the fundamental physical space; the space represented by $\mathbb{R}^{3N}$ is a mathematical construction—the configuration space; the quantum state assigns a complex number to each region in $\mathbb{R}^3$ that contains N points (i.e. the regions will be irregular and disconnected). (Analogy: multi-field)

Some authors in the debate about wave function realism have argued that given our current total evidence, option (2) is a better interpretation of non-relativistic quantum mechanics.[15] I will not rehearse their arguments here. But one of the key ideas that will help us here is that we can think of the complex-valued function as really "living on" the 3-dimensional physical space, in the sense that it assigns a complex number not to each *point* but each N-element *region* in physical space. We call that a "multi-field."[16]

Taking the wave function into a framework friendly for further nominalization, we can perform the familiar technique of decomposing the complex number $Re^{i\theta}$ into two real numbers: the amplitude R and the phase θ. That is, we can think of the compex-valued multi-field in the physical space as two real-valued multi-fields:

$$R(x_1, x_2, x_3, ..., x_N), \theta(x_1, x_2, x_3, ..., x_N).$$

Here, since we are discussing Newtonian space-time, the $x_1.....x_N$ are simultaneous space-time points. We can think of them as: $(x_{\alpha_1}, x_{\beta_1}, x_{\gamma_1}, x_t)$, $(x_{\alpha_2}, x_{\beta_2}, x_{\gamma_2}, x_t)$, ..., $(x_{\alpha_N}, x_{\beta_N}, x_{\gamma_N}, x_t)$.

Now the task before us is just to come up with a nominalistic and intrinsic description of the two multi-fields. In §3.2 and §3.3, we will find two physical structures (Quantum State Amplitude and Quantum State Phase), which, via the appropriate representation theorems and uniqueness theorems, justify the use of complex numbers and explain the gauge degrees of freedom in the quantum wave function.[17]

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[15] See, for example, Chen (2017) and Hubert and Romano (2018).

[16] This name can be a little confusing. Wave-function "multi-field" was first used in Belot (2012), which was an adaptation of the name "polyfield" introduced by Forrest (1988). See Arntzenius and Dorr (2011) for a completely different object called the "multi-field."

[17] In the case of a vector-valued wave function, since the wave function value consists in 2^N complex numbers, where N is the number of particles, we would need to nominalize 2^{N+1} real-valued functions:

$$R_1(x_1, x_2, x_3, ..., x_N), \theta_1(x_1, x_2, x_3, ..., x_N), R_2(x_1, x_2, x_3, ..., x_N), \theta_2(x_1, x_2, x_3, ..., x_N), ...$$

1.3.2 Quantum State Amplitude

The amplitude part of the quantum state is (like mass density) on the ratio scale, i.e. the physical structure should be invariant under ratio transformations

$$R \to \alpha R.$$

We will start with the Newtonian space-time and help ourselves to the structure of **N-Regions**: collection of all regions that contain exactly N simultaneous space-time points (which are irregular and disconnected regions). We start here because we would like to have a physical realization of the platonistic configuration space. The solution is to identify configuration points with certain special regions of the physical space-time.[18]

In addition to **N-Regions**, the quantum state amplitude structure will contain two primitive relations:

- A two-place relation Amplitude–Geq ($\succeq_A$).
- A three-place relation Amplitude–Sum (S).

Interpretation: $a \succeq_A b$ iff the amplitude of N-Region a is greater than or equal to that of N-Region b; $S(a,b,c)$ iff the amplitude of N-Region c is the sum of those of N-Regions a and b.

Define the following short-hand (all quantifiers below range over only N-Regions):

1. $a =_A b := a \succeq_A b$ and $b \succeq_A a$.
2. $a \succ_A b := a \succeq_A b$ and not $b \succeq_A a$.

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[18]Notes on mereology: As I am taking for granted that quantum mechanics for indistinguishable particles (sometimes called identical particles) works just as well as quantum mechanics for distinguishable particles, I do not require anything more than Atomistic General Extensional Mereology (AGEM). That is, the mereological system that validate the following principles: Partial Ordering of Parthood, Strong Supplementation, Unrestricted Fusion, and Atomicity. See Varzi (2016) for a detailed discussion.

However, I leave open the possibility for adding structures in **N-Regions** to distinguish among different ways of forming regions from the same collection of points, corresponding to permuted configurations of distinguishable particles. We might need to introduce additional structure for mereological composition to distinguish between mereological sums formed from the same atoms but in different orders. This might also be required when we have entangled quantum states of different species of particles. To achieve this, we can borrow some ideas from Kit Fine's "rigid embodiment" and add primitive ordering relations to enrich the structure of mereological sums.

Next, we can write down some axioms for Amplitude–Geq and Amplitude–Sum.[19] Again, all quantifiers below range over only N-Regions. $\forall a, b, c:$

G1 (Connectedness) Either $a \succeq_A b$ or $b \succeq_A a$.

G2 (Transitivity) If $a \succeq_A b$ and $b \succeq_A c$, then $a \succeq_A c$.

S1 (Associativity*) If $\exists x\ S(a,b,x)$ and $\forall x'$ [if $S(a,b,x')$) then $\exists y\ S(x',c,y)$], then $\exists z$ $S(b,c,z)$ and $\forall z'$ [if $S(b,c,z')$) then $\exists w\ S(a,z',w)$] and $\forall f, f', g, g'$ [if $S(a,b,f) \wedge S(f,c,f') \wedge S(b,c,g) \wedge S(a,g,g')$, then $f' \succeq_A g'$].

S2 (Monotonicity*) If $\exists x\ \ S(a,c,x)$ and $a \succeq_A b$, then $\exists y\ \ S(c,b,y)$ and $\forall f, f'$ [if $S(a,c,f) \wedge S(c,b,f')$ then $f \succeq_A f'$].

S3 (Density) If $a \succ_A b$, then $\exists d, x$ $[S(b,d,x)$ and $\forall f$, if $S(b,x,f)$, then $a \succeq_A f]$.

S4 (Non-Negativity) If $S(a,b,c)$, then $c \succeq_A a$.

S5 (Archimedean Property) $\forall a_1, b$, if $\neg S(a_1,a_1,a_1)$ and $\neg S(b,b,b)$, then $\exists a_1, a_2, ..., a_n$ s.t. $b \succ_A a_n$ and $\forall a_i$ [if $b \succ_A a_i$, then $a_n \succeq_A a_i$], where a_i's, if they exist, have the following properties: $S(a_1,a_1,a_2)$, $S(a_1,a_2,a_3)$, $S(a_1,a_3,a_4)$, ..., $S(a_1,a_{n-1},a_n)$.[20]

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[19]Compare with the axioms in Krantz et al. (1971) Defn.3.3: Let A be a nonempty set, $\succeq$ a binary relation on A, B a nonempty subset of $A \times A$, and $\circ$ a binary function from B into A. The quadruple $< A, \succeq, B, \circ >$ is an extensive structure with no essential maximum if the following axioms are satisfied for all $a, b, c \in A$:

1. $< A, \succeq >$ is a weak order. [This is translated as G1 and G2.]
2. If $(a,b) \in B$ and $(a \circ b, c) \in B$, then $(b,c) \in B, (a, b \circ c) \in B$, and $(a \circ b) \circ c \succeq_A a \circ (b \circ c)$. [This is translated as S1.]
3. If $(a,c) \in B$ and $a \succeq b$, then $(c,b) \in B$, and $a \circ c \succeq c \circ b$. [This is translated as S2.]
4. If $a \succ b$, then $\exists d \in A$ s.t. $(b,d) \in B$ and $a \succeq b \circ d$. [This is translated as S3.]
5. If $a \circ b = c$, then $c \succ a$. [This is translated as S4, but allowing N-Regions to have null amplitudes. The representation function will also be zero-valued at those regions.]
6. Every strictly bounded standard sequence is finite, where $a_1, ..., a_n, ...$ is a standard sequence if for $n = 2, .., a_n = a_{n-1} \circ a_1$, and it is strictly bounded if for some $b \in A$ and for all a_n in the sequence, $b \succ a_n$. [This is translated as S5. The translation uses the fact that Axiom 6 is equivalent to another formulation of the Archimedean axiom: $\{n | na$ is defined and $b \succ na\}$ is finite.]

The complications in the nominalistic axioms come from the fact that there can be more than one N-Regions that are the Amplitude-Sum of two N-Regions: $\exists a, b, c, d$ s.t. $S(a,b,c) \wedge S(a,b,d) \wedge c \neq d$. However, in the proof for the representation and uniqueness theorems, we can easily overcome these complications by taking equivalence classes of equal amplitude and recover the amplitude addition function from the Amplitude-Sum relation.

[20]S5 is an infinitary sentence, as the quantifiers in the consequent should be understood as infinite

Since these axioms are the nominalistic translations of a platonistic structure in Krantz et al. (Defn. 3.3), we can formulate the representation and uniqueness theorems for the amplitude structure as follows:

Theorem 1.3.1 (Amplitude Representation Theorem) *¡N-Regions, Amplitude–Geq, Amplitude–Sum¿ satisfies axioms (G1)—(G2) and (S1)—(S5), only if there is a function* $R :$ *N-Regions* $\rightarrow \{0\} \cup \mathbb{R}^{+}$ *such that* $\forall a, b \in$ *N-Regions:*

1. $a \geq_A b \Leftrightarrow R(a) \geq R(b)$;

2. *If* $\exists x$ *s.t.* $S(a, b, x)$, *then* $\forall c$ *[if* $S(a, b, c)$ *then* $R(c) = R(a) + R(b)$*].*

Theorem 1.3.2 (Amplitude Uniqueness Theorem) *If another function* R' *satisfies the conditions on the RHS of the Amplitude Representation Theorem, then there exists a real number* $\alpha > 0$ *such that for all nonmaximal element* $a \in$ *N-Regions,* $R'(a) = \alpha R(a)$.

Proofs: See Krantz et al. (1971), Sections 3.4.3, 3.5, pp. 84-87. Note: Krantz et al. use an addition function $\circ$, while we use a sum relation $S(x, y, z)$, because we allow there to be distinct N-Regions that have the same amplitude. Nevertheless, we can use a standard technique to adapt their proof: we can simply take the equivalence classes

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disjunctions of quantified sentences. However, S5 can also be formulated with a stronger axiom called Dedekind Completeness, whose platonistic version says:

Dedekind Completeness. $\forall M, N \subset A$, if $\forall x \in M, \forall y \in N, y > x$, then $\exists z \in A$ s.t. $\forall x \in M, z > x$ and $\forall y \in N, y > z$.

The nominalistic translation can be done in two ways. We can introduce two levels of mereology so as to distinguish between regions of points and regions of regions of points. Alternatively, as Tom Donaldson, Jennifer Wang, and Gabriel Uzquiano suggest to me, perhaps one can make do with plural quantification in the following way. For example (with $\propto$ for the logical predicate "is one of"), here is one way to state the Dedekind Completeness with plural quantification:

Dedekind Completeness Nom Pl. $\forall mm, nn \in$ N-Regions, if $\forall x \propto mm, \forall y \propto nn, y > x$, then there exists $z \in A$ s.t. $\forall x \propto mm, z > x$ and $\forall y \propto nn, y > z$.

We only need the Archimedean property in the proof. Since Dedekind Completeness is stronger, the proof in Krantz et al. (1971), pp. 84-87 can still go through if we assume Dedekind Completeness Nom Pl. Such strenghthening of S5 has the virtue of avoiding the infinitary sentences in S5. Note: this is the point where we have to trade off certain nice features of first-order logic and standard mereology with the desiderata of the intrinsic and nominalistic account. (I have no problem with infinitary sentences in S5. But one is free to choose instead to use plural quantification to formulate the last axiom as Dedekind Completeness Nom Pl.) This is related to Field's worry in *Science Without Numbers*, Ch. 9, "Logic and Ontology."

N-Regions / $=_A$, where $a =_A b$ if $a \succeq_A b \wedge b \succeq_A a$, on which we can define an addition function with the Amplitude-Sum relation.

The representation theorem suggests that the intrinsic structure of Amplitude-Geq and Amplitude-Sum guarantees the existence of a faithful representation function. But the intrinsic structure makes no essential quantification over numbers, functions, sets, or matrices. The uniqueness theorem explains why the gauge degrees of freedom are the positive multiplication transformations and no further, i.e. why the amplitude function is unique up to a positive normalization constant.

1.3.3 Quantum State Phase

The phase part of the quantum state is (like angles on a plane) of the periodic scale, i.e. the intrinsic physical structure should be invariant under overall phase transformations

$$\theta \to \theta + \phi \text{ mod } 2\pi.$$

We would like something of the form of a "difference structure." But we know that according to standard formalism, just the absolute values of the differences would not be enough, for time reversal on the quantum state is implemented by taking the complex conjugation of the wave function, which is an operation that leaves the absolute values of the differences unchanged. So we will try to construct a signed difference structure such that standard operations on the wave function are faithfully preserved.[21]

We will once again start with **N-Regions**, the collection of all regions that contain exactly N simultaneous space-time points.

The intrinsic structure of phase consists in two primitive relations:

- A three-place relation Phase–Clockwise–Betweenness (C_P),
- A four-place relation Phase–Congruence ($\sim_P$).

[21]Thanks to Sheldon Goldstein for helpful discussions about this point. David Wallace points out (p.c.) that it might be a virtue of the nominalistic theory to display the following choice-point: one can imagine an axiomatization of quantum state phase that involves only absolute phase differences. This would require thinking more deeply about the relationship between quantum phases and temporal structure, as well as a new mathematical axiomatization of the absolute difference structure for phase.

Interpretation: $C_P(a,b,c)$ iff the phase of N-Region b is clock-wise between those of N-Regions a and c (this relation realizes the intuitive idea that 3 o'clock is clock-wise between 1 o'clock and 6 o'clock, but 3 o'clock is not clock-wise between 6 o'clock and 1 o'clock); $ab \sim_P cd$ iff the signed phase difference between N-Regions a and b is the same as that between N-Regions c and d.

The intrinsic structures of Phase–Clockwise–Betweenness and Phase–Congruence satisfy the following axioms for what I shall call a "periodic difference structure":

All quantifiers below range over only N-Regions. $\forall a, b, c, d, e, f$:

C1 At least one of $C_P(a,b,c)$ and $C_P(a,c,b)$ holds; if a,b,c are pair-wise distinct, then exactly one of $C_P(a,b,c)$ and $C_P(a,c,b)$ holds.

C2 If $C_P(a,b,c)$ and $C_P(a,c,d)$, then $C_P(a,b,d)$; if $C_P(a,b,c)$, then $C_P(b,c,a)$.

K1 $ab \sim_P ab$.

K2 $ab \sim_P cd \Leftrightarrow cd \sim_P ab \Leftrightarrow ba \sim_P dc \Leftrightarrow ac \sim_P bd$.

K3 If $ab \sim_P cd$ and $cd \sim_P ef$, then $ab \sim_P ef$.

K4 $\exists h, cb \sim_P ah$; if $C_P(a,b,c)$, then
$\exists d', d''$ s.t. $ba \sim_P d'c$, $ca \sim_P d''b$; $\exists p, q, C_P(a,q,b), C_P(a,b,p)$, $ap \sim_P pb, bq \sim_P qa$.

K5 $ab \sim_P cd \Leftrightarrow [\forall e, f d \sim_P ae \Leftrightarrow fc \sim_P be]$.

K6 $\forall e, f, g, h$, if $fc \sim_P be$ and $gb \sim_P ae$, then $[hf \sim_P ae \Leftrightarrow hc \sim_P ge]$.

K7 If $C_P(a,b,c)$, then $\forall e, d, a', b', c'$ [if $a'd \sim_P ae, b'd \sim_P be, c'd \sim_P ce$, then $C(a',b',c')$].

K8 (Archimedean Property) $\forall a, a_1, b_1$, if $C_P(a,a_1,b_1)$, then
$\exists a_1, a_2, ..., a_n, b_1, b_2, ..., b_n, c_1, ...c_m$ such that $C_P(a,a_1,a_n)$ and $C_P(a,b_n,b_1)$, where $a_n a_{n-1} \sim_P a_{n-1}a_{n-2} \sim_P ... \sim_P a_1 a_2$ and $b_n b_{n-1} \sim_P b_{n-1}b_{n-2} \sim_P ... \sim_P b_1 b_2$, and that $a_1 b_1 \sim_P b_1 c_1 \sim_P ... \sim_P c_n a_1$.[22]

[22] Here it might again be desirable to avoid the infinitary sentences / axiom schema by using plural quantification. See the footnote on Axiom S5.

Axiom (K4) contains several existence assumptions. But such assumptions are justified for a nominalistic quantum theory. We can see this from the structure of the platonistic quantum theory. Thanks to the Schrödinger dynamics, the wave function will spread out continuously over space and time, which will ensure the richness in the phase structure.

With some work, we can prove the following representation and uniqueness theorems:

Theorem 1.3.3 (Phase Representation Theorem) *If*

< N-Regions, Phase–Clockwise–Betweenness, Phase–Congruence> is a periodic difference structure, i.e. satisfies axioms (C1)—(C2) and (K1)—(K8), then for any real number $k > 0$, there is a function f : N-Regions $\to [0, k)$ such that $\forall a, b, c, d \in$ N-Regions:

1. $C_P(c, b, a) \Leftrightarrow f(a) \geq f(b) \geq f(c)$ *or* $f(c) \geq f(a) \geq f(b)$ *or* $f(b) \geq f(c) \geq f(a)$;
2. $ab \sim_P cd \Leftrightarrow f(a) - f(b) = f(c) - f(d) \pmod{k}$.

Theorem 1.3.4 (Phase Uniqueness Theorem) *If another function f' satisfies the conditions on the RHS of the Phase Representation Theorem, then there exists a real number β such that for all element $a \in$ N-Regions , $f'(a) = f(a) + \beta \pmod{k}$.*

Proofs: see Appendix A.

Again, the representation theorem suggests that the intrinsic structure of Phase–Clockwise–Betweenness and Phase–Congruence guarantees the existence of a faithful representation function of phase. But the intrinsic structure makes no essential quantification over numbers, functions, sets, or matrices. The uniqueness theorem explains why the gauge degrees of freedom are the overall phase transformations and no further, i.e. why the phase function is unique up to an additive constant.

Therefore, we have written down an intrinsic and nominalistic theory of the quantum state, consisting in merely four relations on the regions of physical space-time: Amplitude-Sum, Amplitude-Geq, Phase-Congruence, and Phase-Clockwise-Betweenness. As mentioned earlier but evident now, the present account of the quantum state has several desirable features: (1) it does not refer to any abstract mathematical objects such as complex numbers, (2) it is free from the usual arbitrary conventions in the wave function representation, and (3) it explains why the quantum state has its amplitude and phase degrees of freedom.

1.3.4 Comparisons with Balaguer's Account

Let us briefly compare my account with Mark Balaguer's account (1996) of the nominalization of quantum mechanics.

Balaguer's account follows Malament's suggestion of nominalizing quantum mechanics by taking seriously the Hilbert space structure and the representation of "quantum events" with closed subspaces of Hilbert spaces. Following orthodox textbook presentation of quantum mechanics, he suggests that we take as primitives the *propensities* of quantum systems as analogous to probabilities of quantum experimental outcomes.

> I begin by recalling that each quantum state can be thought of as a function from events (A, Δ) to probabilities, i.e., to $[0, 1]$. Thus, each quantum state specifies a set of ordered pairs $< (A, \Delta), r >$. The next thing to notice is that each such ordered pair determines a propensity property of quantum systems, namely, an $r-$strengthed propensity to yield a value in Δ for a measurement of A. We can denote this propensity with "(A, Δ, r)". (Balaguer, 1996, p.218.)

Balaguer suggests that the propensities are "nominalistically kosher." By interpreting the Hilbert space structures as propensities instead of propositions, Balaguer makes some progress in the direction of making quantum mechanics "more nominalistic."

However, Balaguer's account faces a problem—it is not clear how Balaguer's account relates to any mainstream realist interpretation of quantum mechanics. This is because the realist interpretations—Bohmian Mechanics, GRW spontaneous collapse theories, and Everettian Quantum Mechanics—crucially involve the quantum state represented by a wave function, not a function from events to probabilities.[23] And once we add

[23]See Bueno (2003) for a discussion about the conflicts between Balaguer's account and the modal

the wave function (perhaps in the nominalistic form introduced in this paper), the probabilities can be calculated (via the Born rule) from the wave function itself, which makes primitive propensities redundant. If Balaguer's account is based on orthodox quantum mechanics, then it would suffer from the dependence on vague notions such as "measurement," "observation," and "observables," which should have no place in the fundamental ontology or dynamics of a physical theory.[24]

1.4 "Wave Function Realism"

The intrinsic and nominalistic account of the quantum state provides a natural response to some of the standard objections to "wave function realism."[25] According to David Albert (1996), realism about the wave function naturally commits one to accept that the wave function is a physical field defined on a fundamentally 3N-dimensional wave function space. Tim Maudlin (2013) criticizes Albert's view partly on the ground that such "naive" realism would commit one to take as fundamental the gauge degrees of freedom such as the absolute values of the amplitude and the phase, leaving empirically equivalent formulations as metaphysically distinct. This "naive" realism is inconsistent with the physicists' attitude of understanding the Hilbert space projectively and thinking of the quantum state as an equivalence class of wave functions ($\psi \sim Re^{i\theta}\psi$). If a defender of wave function realism were to take the physicists' attitude, says the opponent, it would be much less natural to think of the wave function as really a physical *field*, as something that assigns physical properties to each point in the 3N-dimensional space. Defenders of wave function realism have largely responded by biting the bullet and accepting the costs.

But the situation changes given the present account of the quantum state. Given the intrinsic theory of the quantum state, one can be realist about the quantum state by

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interpretation of QM.

[24] Bell (1989), "Against 'Measurement,' " pp. 215-16.

[25] "Wave function realists," such as David Albert, Barry Loewer, Alyssa Ney, and Jill North, maintain that the fundamental physical space for a quantum world is 3N-dimensional. In contrast, primitive ontologists, such as Valia Allori, Detlef Dürr, Sheldon Goldstein, Tim Maudlin, Roderich Tumulka, and Nino Zanghi, argue that the fundamental physical space is 3-dimensional.

being realist about the four intrinsic relations underlying the mathematical and gauge-dependent description of the wave function. The intrinsic relations are invariant under the gauge transformations. Regardless of whether one believes in a fundamentally high-dimensional space or a fundamentally low-dimensional space, the intrinsic account will recover the wave function unique up to the gauge transformations ($\psi \sim Re^{i\theta}\psi$).

I should emphasize that my intrinsic account of the wave function is essentially a version of comparativism about quantities. As such, it should be distinguished from eliminitivism about quantities. Just as a comparativist about mass does not eliminate mass facts but ground them in comparative mass relations, my approach does not eliminate wave function facts but ground them in comparative amplitude and phase relations. My account does not in the least suggest any anti-realism about the wave function.[26]

Therefore, my account removes a major obstacle for wave function realism. One can use the intrinsic account of the quantum state to identify two field-like entities on the configuration space (by thinking of the N-Regions as points in the 3N-dimensional space) without committing to the excess structure of absolute amplitude and overall phase.[27]

1.5 Conclusion

There are many *prima facie* reasons for doubting that we can ever find an intrinsic and nominalistic theory of quantum mechanics. However, in this paper, we have offered an intrinsic and nominalistic account of the quantum state, consisting in four relations on regions of physical space:

1. Amplitude-Sum (S),

2. Amplitude-Geq ($\succeq_A$),

[26] Thanks to David Glick for suggesting that I make this clear.

[27] Unsurprisingly, the present account also provides some new arsenal for the defenders of the fundamental 3-dimensional space. The intrinsic account of the quantum state fleshes out some details in the multi-field proposal.

3. Phase-Congruence ($\sim_P$),

4. Phase-Clockwise-Betweenness (C_P).

This account, I believe, offers a deeper understanding of the nature of the quantum state that at the very least complements that of the standard account. By removing the references to mathematical objects, our account of the quantum state provides a framework for nominalizing quantum mechanics. By excising superfluous structure such as overall phase, it reveals the intrinsic structure postulated by quantum mechanics. Here we have focused on the universal quantum state. As the origin of quantum non-locality and randomness, the universal wave function has no classical counterpart and seems to resist an intrinsic and nominalistic treatment. With the focus on the universal quantum state, our account still leaves room for including additional ontologies of particles, fields, mass densities supplied by specific solutions to the quantum measurement problem such as BM, GRWm, and GRWf; these additional ontologies are readily nominalizable.

Let us anticipate some directions for future research. First, the intrinsic structure of the quantum state at different times is constrained by the quantum dynamics. In the platonistic theory, the dynamics is described by the Schrödinger equation. To nominalize the dynamics, we can decompose the Schrödinger equation into two equations, in terms of amplitude and gradient of phase of the wave function. The key would be to codify the differential operations (which Field has done for Newtonian Gravitation Theory) in such a way to be compatible with our phase and amplitude relations. Second, we have described how to think of the quantum state for a system with constant number of particles. How should we extend this account to accommodate particle creation and annihilation in quantum field theories? I think the best way to answer that question would be to think carefully about the ontology of a quantum field theory. A possible interpretation is to think of the quantum state of a variable number of particles as being represented by a complex valued function whose domain is $\bigcup_{N=0}^{\infty} \mathbb{R}^{3N}$—the union of all configuration spaces (of different number of particles). In that case, the extension of our theory would be easy: (1) keep the axioms as they are and (2) let the quantifiers range over K-regions, where the integer K ranges from zero to infinity. Third, we have considered quantum states for spinless systems. A possible way to extend the present account to accommodate spinorial degrees of freedom would be to use two comparative relations for each complex number assigned by the wave function. That strategy is conceptually similar to the situation in the present account. But it is certainly not the only strategy, especially considering the gauge degrees of freedom in the spin space. Fourth, as we have learned from debates about the relational theories of motion and the comparative theories of quantities, there is always the possibility of a theory becoming indeterministic when drawing from only comparative predicates without fixing an absolute scale.[28] It would be interesting to investigate whether similar problems of indeterminism arise in our comparative theory of the quantum state. Finally, the formal results obtained for the periodic difference structure could be useful for further investigation into the metaphysics of phase.

The nature of the quantum state is the origin of many deeply puzzling features of a quantum world. It is especially challenging given realist commitments. I hope that the account discussed in this paper makes some progress towards a better understanding of it.

[28]See Dasgupta (2013); Baker (2014); Martens (2016), and Field (2016), preface to the second edition, pp. 41-44.

Chapter 2

Our Fundamental Physical Space

> Already in my original paper I stressed the circumstance that I was unable to give a logical reason for the exclusion principle or to deduce it from more general assumptions. I had always the feeling, and I still have it today, that this is a deficiency.
>
> Wolfgang Pauli (1946 Nobel Lecture)

Introduction

This is an essay about the metaphysics of quantum mechanics. In particular, it is about the metaphysics of the quantum wave function and what it says about our fundamental physical space.

To be sure, the discussions about the metaphysics within quantum mechanics have come a long way. In the heyday of the Copenhagen Interpretation, Niels Bohr and his followers trumpeted the instrumentalist reading of quantum mechanics and the principles of complementarity, indeterminacy, measurement recipes, and various other revisionary metaphysics. During the past three decades, largely due to the influential work of J. S. Bell, the foundations of quantum mechanics have been much clarified and freed from the Copenhagen hegemony.[1] Currently, there are physicists, mathematicians, and philosophers of physics working on solving the measurement problem by proposing and analyzing various realist quantum theories, such as Bohmian mechanics (BM), Ghirardi-Rimini-Weber theory (GRW), and Everettian / Many-World Interpretation as well as trying to extend them to the relativistic domains with particle creation and

[1] For good philosophical and historical analyses about this issue, see Bell (2004) for and Cushing (1994).

annihilation. However, with all the conceptual and the mathematical developments in quantum mechanics (QM), its central object—the wave function—remains mysterious.

Recently, philosophers of physics and metaphysicians have focused on this difficult issue. Having understood several clear solutions to the measurement problem, they can clearly formulate their questions about the wave function and their disagreements about its nature. Roughly speaking, there are those who take the wave function to represent some mind-dependent thing, such as our ignorance; there are also people who take it to represent some mind-independent reality, such as a physical object, a physical field, or a law of nature.

In this paper, I shall assume realism about the wave function—that is, what the wave function represents is something real, objective, and physical. (Thus, it is not purely epistemic or subjective).[2] I shall conduct the discussion in non-relativistic quantum mechanics, noting that the lessons we learn here may well apply to the relativistic domain.[3] To disentangle the debate from some unfortunately misleading terminologies, I shall adopt the following convention.[4] I use *the quantum state* to refer to the physical object and reserve the definite description *the wave function* for its mathematical representation, Ψ. In the position representation (which I use throughout this paper), the domain of the wave function is all the ways that fundamental particles can be arranged in space and the codomain is the complex field $\mathbb{C}$.[5] (For convenience and simplicity,

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[2]I take it that there are good reasons to be a realist about the wave function in this sense. One reason is its important role in the quantum dynamics. See Albert (1996) and Ney (2012). The recently proven PBR theorem lends further support for this position. See the original PBR paper Pusey et al. (2012) and Matthew Leifer's excellent review article Leifer (2014).

[3]See Myrvold (2014) for an insightful discussion on the complications when we transfer the discussion to the relativistic domain with particle creation and annihiliation. I take it that a clear ontology of QFT will provide a natural arena for conducting a similar debate. Since a consistent QFT is still an incomplete work-in-progress (although an effective quantum field theory is highly useful and predictively accurate, and there have been proposals with clearer ontology such as Bell-type QFT), I believe that there are many values in conducting the discussion as below—in the non-relativistic QM—although we know that it is only an approximation to the final theory. The clarity of the non-relativistic QM, if for no other purposes, makes many issues more transparent.

[4]I am indebted to Maudlin (2013) for the following distinctions.

[5]It is reasonable to wonder whether our definition applies to "flashy" GRW (GRWf) or mass-density versions of GRW (GRWm). Although particles are not fundamental in these theories, the definitions of GRWf and GRWm and the mathematical structures of these theories can be captured by using wave functions defined on the particle-location configuration space. *When interpreting the physical theory* (either according to the low-dimensional view or the high-dimensional view described below), if we think

I leave out spin when discussing the value of the wave function, noting that it can be added as additional degrees of freedom (spinor values). Hence, $\Psi \in L^2(\mathbb{R}^{3N}, \mathbb{C})$.) The domain of the wave function is often called *the configuration space*, and it is usually represented by $\mathbb{R}^{3N}$, whose dimension is 3 times N, where N is the total number of the fundamental particles in the universe. (I shall represent the configuration space by $\mathbb{R}^{3N}$ instead of $\mathbb{E}^{3N}$, noting that the latter would be more physical than the former.) For the wave function of a two-particle system in $\mathbb{R}^3$, the configuration space is $\mathbb{R}^6$. For the wave function of our universe, whose total number of fundamental particles most likely exceeds 10^{80}, the configuration space has at least as many dimensions as 10^{80}.

Our key question is: given realism about the quantum state, how is the configuration space related to our familiar 3-dimensional space? There are, roughly speaking, two views corresponding to two ways to answer that question.[6]

3N-Fundamentalism $\mathbb{R}^{3N}$ represents the fundamental physical space in quantum mechanics. Our ordinary space, represented by $\mathbb{R}^3$, is less fundamental than and stands in some grounding relations (such as emergence) to $\mathbb{R}^{3N}$.[7]

3D-Fundamentalism $\mathbb{R}^3$ represents the fundamental physical space in quantum mechanics. The configuration space, represented by $\mathbb{R}^{3N}$, is less fundamental than

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of the wave function as representing something *in addition to* the material ontology, we can regard the wave function space as having a structure that is independent from the material ontology (and the configurations of material stuffs). We can call the domain of the wave function "the configuration space," but we do not need to understand it literally as the space of configurations of the material ontology. How the material ontology is connected to the wave function on "the configuration space" is explained by the definitions (such as the mass-density function $m(x,t)$ in GRWm) and the dynamical laws. Interpreted this way, many arguments that rely on the configuration space of particle locations, including those in Sections 2 and 3, apply directly to GRWf and GRWm. The exceptions are those arguments that rely on a *justification* or an *explanation* of the configuration space. Therefore, our arguments in Section 4.2 apply most smoothly to theories with a particle ontology such as Bohmian Mechanics and much less smoothly to GRW theories.

[6] I leave out the middle position between the two views, which says that both the 3N space and the 3D space are fundamental. See Dorr (2009) for an exploration of that view.

[7] This view is often called Wave Function Realism, although we should remember that this is unfortunately misleading terminology, for its opponents are also realists about the wave function. Under my classification, 3N-Fundamentalists include David Albert, Barry Loewer, Alyssa Ney, Jill North, and *arguably*, the time-slice of J. S. Bell that wrote "Quantum Mechanics for Cosmologists:" "*No one can understand this theory until he is willing to think of ψ as a real objective field rather than just a 'probability amplitude'. Even though it propagates not in 3-space but in 3N-space.*" (Bell (1987) p.128, his emphasis.)

> and stands in some grounding relations (such as mathematical construction) to $\mathbb{R}^3$.[8]

3N-Fundamentalism and 3D-Fundamentalism are the targets of the recent debates under the name "The Metaphysics of the Wave Function," "Wave Function Realism versus Primitive Ontology," and "Configuration Space Realism versus Local Beables." (To be sure, these debates often branch into discussions about other important issues, such as the nature of the quantum state: whether Ψ is a field, a law, or a *sui generis* physical object.) In those debates, the most common considerations include: (1) the dynamical structure of the quantum theory, and (2) the role of our ordinary experiences. Usually, the opposing thinkers are taken to assign different weights to these considerations.

For example, Jill North (2013) summarizes the disagreement as follows:

> This brings us to a basic disagreement between wave function space and ordinary space views: how much to emphasize the dynamics in figuring out the fundamental nature of the world. Three-space views prioritize our evidence from ordinary experience, claiming that the world appears three-dimensional because it is fundamentally three-dimensional. Wave function space views prioritize our inferences from the dynamics, claiming that the world is fundamentally high-dimensional because the dynamical laws indicate that it is.[9]

[8]This view is often taken to be a commitment to Primitive Ontology or Primary Ontology. Under my classification, 3D-Fundamentalists include Bradley Monton, Detlef Dürr, Sheldon Goldstein, Nino Zanghì, Roderich Tumulka, James Taylor, Ward Struvye, Valia Allori, Tim Maudlin, Michael Esfeld, CianCarlo Ghirardi, and Angelo Bassi. Depending on how one thinks about the nature of the wave function, there are three ways to flesh out 3D-Fundamentalism: (1) the wave function is a physical object—a multi-field (more later); (2) the wave function is nomological or quasi-nomological; (3) the wave function is a *sui generis* entity in its own category. No matter how one fleshes out the view, the following discussion should be relevant to all, although (1) is the most natural viewpoint to conduct the discussion. In a companion paper, I discuss the viability of (2) in connection to David Lewis's thesis of Humean supervenience. In a future paper, I plan to discuss the viability of (3). In any case, defending (2) and (3) takes much more work and might in the end decrease the plausibility of 3D-Fundamentalism. If the following discussion is right, since (1) is the most plausible competitor with 3N-Fundamentalism with a high-dimensional field, it has been premature for 3D-Fundamentalists to give up the defense of (1). Indeed, a careful examination of the pros and the cons reveals its superiority over 3N-Fundamentalism (and its internal competitors (2) and (3)). I shall not defend this claim here, and for this paper the reader does not need to take sides in this internal debate among 3D-Fundamentalists.

[9]North (2013), p. 196.

However, as I shall argue, the common arguments based on (1) and (2) are either unsound or incomplete. Completing the arguments, it seems to me, render the overall considerations based on (1) and (2) *roughly* in favor of 3D-Fundamentalism. However, there is another relevant consideration: (3) the deep explanations of striking phenomena. Here, we find a more decisive argument in favor of 3D-Fundamentalism as it leads to a deeper understanding of the mathematical symmetries in the wave function, known as the Symmetrization Postulate. Since 3D-Fundamentalism explains the Symmetrization Postulate much better than 3N-Fundamentalism does, we have strong reasons to accept 3D-Fundamentalism.

I shall argue that the considerations (1), (2), and (3), taken together, favors 3D-Fundamentalism over 3N-Fundamentalism. Here is the roadmap. In §1, I shall argue, against the 3N-Fundamentalist, that the common argument from the dynamical structure of quantum mechanics is unsound, and that after proper refinement it no longer provides strong reason in favor of 3N-Fundamentalism. (As a bonus, I will provide a direct answer to David Albert's question to the primitive ontologists: what is their criterion for something to be the fundamental physical space?) In §2, I shall argue, against the typical 3D-Fundamentalist, that the common argument from the Manifest Image can be resisted with spatial functionalism, but such a response can be weakened by examining the details of the functionalist strategies. This suggests that the argument from the Manifest Image, though no longer decisive, still has some force. In §3, I shall explore which position leads to a better understanding of the physical world and formulate a new argument based on the recent mathematical results about the relationship between identical particles and the symmetries in the wave function. The deeper topological explanation offered by 3D-Fundamentalism strongly suggests that our fundamental physical space in a quantum world is 3-dimensional rather than 3N-dimensional.

Finally, I shall offer future directions of research, point out how a 3N-Fundamentalist might be able to respond to the last two considerations, and offer additional explanations to restore or reverse the evidential balance between 3D-Fundamentalism and 3N-Fundamentalism.

As for the scope, this essay is more or less orthogonal to the debate between 3D and 4D views about time and persistence. However, the analysis in this essay should be of interest to participants in those debates as well as to philosophers working on the nature of the quantum state, the fundamentality / emergence of space-time, and mathematical explanation in physics. As we shall see, our case study also motivates a particular way of understanding the equivalence of theories in terms of explanatory equivalence.

2.1 Evidence #1: The Dynamical Structure of Quantum Mechanics

At first glance, 3N-Fundamentalism looks like a highly surprising idea. What can possibly motivate such a revisionary metaphysical thesis? The initial motivation was the reality of the non-local and non-separable quantum state and the phenomena of quantum entanglement—causal influences that are unweakened by arbitrary spatial separation. Perhaps what we see in front of us (the interference patterns on the screen during the double-slit experiment), the idea goes, are merely projections from something like a "Platonic Heaven"—some higher-dimensional reality on which the dynamics is perfectly local.

The initial motivation leads to our first piece of evidence in the investigation of our fundamental physical space. Such evidence is based on the dynamical laws and the dynamical objects in quantum mechanics. This seems to be the strongest evidence in favor of 3N-Fundamentalism. In this section, I suggest that the dynamical structure of the quantum theory, under closer examination, does not support the view that the fundamental physical space in quantum mechanics is 3N-dimensional.

2.1.1 The Argument for 3N-Fundametalism

Here is one way to write down the argument for 3N-Fundamentalism based on considerations of the dynamics.[10]

[10]Strictly speaking, the following are based on considerations of both the kinematics and the dynamics of the quantum theory. Nevertheless, the kinematical structure of the quantum wave function plays an important role in determining the dynamics of the material ontology (Bohmian particles, mass densities,

P1 We have (defeasible) reasons to infer a fundamental structure of the world that matches the dynamical structure of our fundamental physical theory. [The Dynamical Matching Principle]

P2 The dynamical structure of quantum mechanics (a candidate fundamental physical theory) includes the quantum state which is defined over a 3N-dimensional configuration space. [The 3N-Structure of the Quantum State]

C1 We have (defeasible) reasons to infer that our fundamental physical space is 3N-dimensional.

P1—the Dynamical Matching Principle—follows from a more general principle of inference from physical theories to metaphysical theories:

The Matching Principle We have (defeasible) reasons to infer a fundamental structure of the world that matches the structure of our fundamental physical theory. We should infer no more and no less fundamental structure of the world than what is needed to support the structure of our fundamental physical theory.

The Matching Principle can be a highly useful guide for empirically-minded metaphysicians.[11] For example, we can use it to infer that the classical space-time is Galilean rather than Newtonian. This inference agrees with our intuitions. Moreover, the inference is not based on controversial empiricist (perhaps verificationalist) assumptions about eliminating unobservable or unverifiable structure.

2.1.2 The Assumptions in Premise 2

Suppose that something like the Matching Principle is true and the particular application with the Dynamical Matching Principle is justified. Then the argument would rest on **P2**. However, does the quantum state, represented by the wave function Ψ, really

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etc) that move in the physical space.

[11]This principle comes from North (2013) and North (MS) (forthcoming). See also Albert (1996) and Albert (2015) for ideas in a similar spirit.

have a 3N-dimensional structure? In other words, does the wave function have to be defined over a high-dimensional configuration space?

Many people would answer positively. How else can we define the wave function, if not on a high-dimensional configuration space? For surely the wave function has to take into account quantum systems with entangled subsystems that have non-local influences on each other, which have to be represented by wave functions that are non-separable. To take these into account, the usual reasoning goes, the wave function of an N-particle system (say, our universe with 10^{80} particles), in general, has to be defined as a field over on a 3N-dimensional space. Alyssa Ney (2012), for example, endorses this line of reasoning:

> ...[Entangled] states can only be distinguished, and hence completely characterized in a higher-than-3-dimensional configuration space. They are states of something that can only be adequately characterized as inhabiting this higher-dimensional space. This is the quantum wavefunction.[12]

North's reasoning is similar:

> In quantum mechanics, however, we must formulate the dynamics on a high-dimensional space. This is because quantum mechanical systems can be in entangled states, for which the wave function is nonseparable. Such a wave function cannot be broken down into individual three-dimensional wave functions, corresponding to what we think of as particles in three-dimensional space. That would leave out information about correlations among different parts of the system, correlations that have experimentally observed effects. Only the entire wave function, defined over the entire high-dimensional space, contains all the information that factors into the future evolution of quantum mechanical systems.[13]

There are two assumptions that are worth making explicit in the above reasonings (to be sure, North and Ney discuss them in their respective papers):

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[12]Ney (2012), p. 556.

[13]North (2013), p. 190

Ψ-as-a-Field The quantum state, represented by the wave function Ψ, is a field.

Field-Value-on-Points The space that a field "lives on" is the smallest space where it assigns a value for each point in that space.

The two assumptions[14] are explicit in David Albert's argument in his important paper "Elementary Quantum Metaphysics":

> The sorts of physical objects that wave functions are, on this way of thinking, *are* (plainly) fields—which is to say that they are the sorts of objects whose states one specifies by specifying the values of some set of numbers at every point in the space where they live, the sorts of objects whose states one specifies (in this case) by specifying the values of two numbers (one of which is usually referred to as an amplitude, and the other as a phase) at every point in the universe's so-called configuration space.[15]

However, as I shall argue, there are good reasons to reject both assumptions.

To deny **Ψ-as-a-Field**, one might instead propose that Ψ is not strictly speaking a field (in any classical sense). For example, an electromagnetic field assigns a field value to every point in the 3-dimensional space, and the values are meaningful partly because multiplying the field by a constant will result in a different field and different dynamics. (The gauge freedom in the electric potential therefore indicates its relation to something more physical and more fundamental, such as the electric field.[16]) However, the quantum wave function, multiplied by a global phase factor ($\Psi \Rightarrow e^{i\theta}\Psi$), remains physically the same. Insisting on **Ψ-as-a-Field** will lead one to recognize physically indistinguishable features (that play no additional explanatory role) as physically real and meaningful. It is desirable, therefore, that we do not recognize Ψ as a fundamental field.[17]

[14] The qualification "smallest" is added to ensure that the principles render a unique space that a field lives on.

[15] Albert (1996), p. 278

[16] Thanks to Michael Townsen Hicks for alerting me to the connection.

[17] However, it is possible to provide a gauge-free account of the wave function by giving a nominalistic and intrinsic account of quantum mechanics. In a future paper, I will provide the beginning of such an account. But the next point holds regardless of the success of that program.

However, even if one were to embrace the indistinguishable features brought about by **Ψ-as-a-Field**, one can still deny **Field-Value-on-Points**. We usually think, as Albert describes, that the fundamental physical space is the fundamental space that the dynamical object "lives on." In particular, the fundamental space that a field lives on is one in which we specify the state of the field at a time by specifying the values at *each point*. This is indeed the case for the electromagnetic field. But why think that this is an essential feature for a physical field?

Suppose that a fundamental field is mathematically represented by a function from one space to another space. Such a function can be as general as one likes. A special case, of course, is the electromagnetic field that assigns values to each point in $\mathbb{R}^3$. But another one can be a quantum multi-field that assigns values to every N-tuple of points in $\mathbb{R}^3$. That is, the function takes N arguments from $\mathbb{R}^3$. If we think of the electromagnetic field as assigning properties to each point in $\mathbb{R}^3$, we can think of the quantum multi-field as assigning properties to each plurality of N points in $\mathbb{R}^3$ (N-plurality). The multi-field defined over N-pluralities of points in $\mathbb{R}^3$ is, unsurprisingly, equivalent to the quantum wave function defined over each point in $\mathbb{R}^{3N}$. [18]

However, this approach has a disadvantage. On this conception of the multi-field, we lose Albert's very clear criterion for something to be the fundamental physical space (based on the space that the fundamental field "lives on"). Indeed, Albert asks,[19] having given up his criterion, what would be an alternative conception of the fundamental physical space that the 3D-Fundamentalists can give?

No one, as far as I know, has provided an alternative criterion for something to be the fundamental physical space. But would it be terrible if there were none? Some people would not worry, for the following reason. When we work on the metaphysics of physical theories, we have in mind complete physical theories—theories that specify their fundamental physical spaces; that is, for each theory, we do not need to infer

[18]This approach of taking the wave function as a "multi-field" is explained in Forrest (1988) Ch. 6 and Belot (2012). I have heard that for many working mathematical physicists in the Bohmian tradition, this has always been how they think about the wavefunction.

[19]Albert raised this question during his talk "On Primitive Ontology" at the 2014 Black Forest Summer School in Philosophy of Physics.

its fundamental space from the other parts of the theory. For each complete theory, the fundamental space is given. Our task is to decide which one (among the complete theories) is the best theory, by considering their theoretical virtues and vices (especially their super-empirical virtues such as simplicity, parsimony, explanatory depth, and so on). Albert's question is not an urgent one, because it is about a different feature of theories: how we arrive at them in the first place. We could wonder whether we have inferred in the right way when we say that the fundamental space of the theory is 3-dimensional. But we could also just ask which theory is more plausible: the quantum theory defined on $\mathbb{R}^3$ or the quantum theory defined on $\mathbb{R}^{3N}$.

However, we might construe Albert's question in a different way, as asking about a distinctive theoretical virtue, namely, internal coherence. In contrast to the previous construal, this perspective would make Albert's worry highly relevant to the interpretive projects in the metaphysics of quantum mechanics. As I think of it, internal coherence is distinct from logical consistency. A metaphysical / physical theory can be logically consistent without being internally coherent, in the sense that it might contain theoretical parts that are in serious tension (though no logical incompatibility) with each other. What the tension is can be precisified according to the types of the theory. For example, a theory with a field-like entity defined not on individual points but on non-trivial regions in the fundamental space has theoretical parts that are in tension. Hence, one way for a theory to lack internal coherence is for it to fail Albert's criterion of **Field-Value-on-Points**.

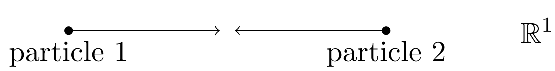
Construed in that way, **Field-Value-on-Points** is a precisification of the theoretical virtue of internal coherence. However, I shall argue that Albert's argument against 3D-Fundamentalism will not work even if internal coherence is a theoretical virtue, for there is another equally valid way to precisify internal coherence.

Let us consider the following proposal for a first pass:

Field-Value-on-Pluralities The space that a field "lives on" is the smallest space where it assigns values for pluralities of points in that space.

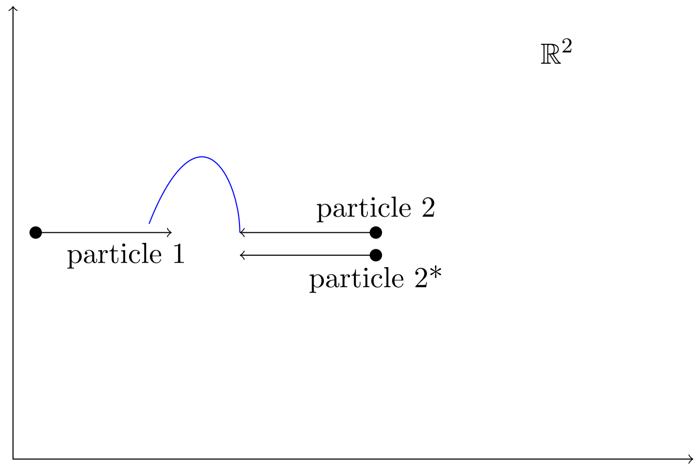
The modified criterion no longer requires the field to take on values at each point in the space. But this seems to lead to another disastrous consequence for the 3D-Fundamentalist who holds the multi-field view, for the smallest space where it assigns values for a plurality of points is not $\mathbb{R}^3$ but $\mathbb{R}^1$ (a linear subspace of $\mathbb{R}^3$) where 3N fundamental particles move around (in a 1-dimensional space).[20] Should the 3D-Fundamentalist, based on the above criterion, accept $\mathbb{R}^1$ as the fundamental physical space?

That initial worry dissolves itself if we take the particles to be indistinguishable and physically identical, and we take the configuration space to be an unordered configuration space (we will return to this proposal in §3). If the fundamental physical space is $\mathbb{R}^1$, then two particles cannot pass through each other (see diagram below). For two particles to pass through each other (which we would like to make possible given the degrees of freedom of the particles in the 3D space and the complexity of our world), there must be a moment in time where they occupy the same point, which means that the wave function will be undefined at that moment. (Recall that we represent an unordered configuration as a set instead of an ordered tuple, and that the wave function is defined as functions from sets of N points in $\mathbb{R}^3$ to complex numbers. At the instant where the two particles occupy the same point (as the following diagram in $\mathbb{R}^1$ shows), the configuration will lose at least one point, dropping down to a configuration of 3N-1 points. The wave function would therefore be undefined.) However, things would be different as soon as we go to a bigger space, say, $\mathbb{R}^2$. A particle can "pass through" another by missing it by a small distance (such as particle 2* in the second diagram below) or going around it, perhaps in a non-straight line, with arbitrarily small positive distance ϵ (such as particle 2).[21] Thus, the bigger space $\mathbb{R}^2$ provides more "expressive power" than $\mathbb{R}^1$.



[20] Thanks to David Albert for pointing out this worry.

[21] Thanks to Sheldon Goldstein and Nino Zanghì for pointing this out to me.



Why not, then, take $\mathbb{R}^2$ instead of $\mathbb{R}^3$ as our fundamental physical space, since it is also a linear subspace of $\mathbb{R}^3$? There are two reasons. First, we do not know whether the total number of apparent particles in 3D space is even, which is required if the fundamental dynamics is given by 3N/2 many particles in $\mathbb{R}^2$. However, we do know (as the reader can easily verify) that the number of fundamental particles times the dimensions of the fundamental space is a multiple of 3. It seems plausible that, at least in the quantum theory, the fundamental physical space should remain neutral about whether there are $2k$ or $2k - 1$ particles. Second, even more than the dynamics of 3N particles in $\mathbb{R}^1$, the dynamics of 3N/2 many particles, especially the Hamiltonian function, (even if we assume N is even) is unlikely to be natural.[22] It is appropriate to add a neutrality constraint and a naturality constraint to the previous criterion of the fundamental physical space:

Field-Value-on-Pluralities* The space that a field "lives on" is the smallest and most natural space that is neutral to the parity of the total number of fundamental particles and that the field assigns values for pluralities of points in that space.

If we regard Albert's question as tracking the theoretical virtue of what I call "internal coherence," then I take Field-Value-on-Pluralities* as an equally attractive way of precisifying internal coherence. Given this precisification, 3D-Fundamentalism wins, as $\mathbb{R}^3$

[22]There may be many other reasons why the 3-dimensional space is special. In §3.2, we consider the case of identical particles, for which spaces with fewer than 3 dimensions would allow fractional statistics that correspond to anyons, in addition to fermions and bosons.

satisfies this criterion and is the space where the quantum multi-field lives on. Without offering strong reasons to favor his precisification, Albert's argument falls short of delivering the conclusion.

In this section, I have examined and improved the Argument against 3D-Fundamentalism and still found its key premise unjustified. Therefore, our first piece of evidence from the dynamical structure of quantum mechanics underdetermines our choice between 3N-Fundamentalism and 3D-Fundamentalism.

2.2 Evidence #2: Our Ordinary Perceptual Experiences

In this debate, many people have pointed out the obvious—our Manifest Image—the tables, chairs, and experimental devices in front of us. How can these apparently 3-dimensional objects exist if reality is vastly high-dimensional? If the high-dimensional quantum mechanics could not explain the apparent reality of pointer readings in 3 dimensions, from which we come to be justified in believing in quantum mechanics, the theory would surely be self-undermining.

Hence it is common for 3D-Fundamentalists to use the second piece of evidence—our ordinary perceptual experiences—to argue in favor of their view. In this section, I suggest that although the evidence seems to support their view over the alternative—3N-Fundamentalism, the evidential force depends crucially on the fate of functionalism.

2.2.1 The Argument for 3D-Fundamentalism

Here is one way to write down the argument for 3D-Fundamentalism based on considerations of the Manifest Image of 3-dimensional observers and pointers. (I shall use relatively weak assumptions to generate a relatively strong argument.)

P3 If we cannot locate the Manifest Image of human observers and pointer readings in an empirical (fundamental) scientific theory, then we have (defeasible) reasons against that theory.

P4 We cannot locate the Manifest Image of human observers and pointer readings in the quantum theory with 3N-Fundamentalism, which is an empirical (fundamental) scientific theory.

C2 We have (defeasible) reasons against the quantum theory on 3N-Fundamentalism.

The first premise—**P3**—does not depend on an insistence of respecting the common sense or the reality of the Manifest Image. Instead, it follows from a more general principle:

Self-Undermining If we arrive at an empirical (fundamental) theory T based solely on evidence E, and either we see that T entails that E is false without providing a plausible error theory, or we see that T entails that our acquiring E is produced by an unreliable process, or we see an ineliminable explanatory gap from T to the truth of E (or some nearby propositions), then we have (defeasible) reasons against T.

To see how **P3** follows from **Self-Undermining**, we need to rewrite **P3** in more precise terms.

P3* If an empirical (fundamental) scientific theory contains an ineliminable explanatory gap for the true claims (or some nearby propositions) about human observers' pointer readings after experiments, then we have (defeasible) reasons against that theory.

P3* seems highly plausible.[23] The key idea is that our belief in a fundamental scientific theory is not justified in a void, or properly basic, or merely coherent with respect to the rest of our metaphysical beliefs. Rather, we are justified in believing in it because we (or the scientific community as a whole) have acquired sufficient empirical evidence through the pointer readings connected to particular experimental set-ups. These pointer readings can be in principle reduced to the positions of entities in a 3-dimensional space. And we acquire such crucial pieces of evidence through perception

[23]This is similar to the demand for "empirical coherence" in Barrett (1999) and Healey (2002). The principle, I take it, is weaker than the often cited principle in Maudlin (2007b) that the physical theory makes contact with evidence via "local beables." Huggett and Wüthrich (2013) discuss, in light of the recent developments of quantum gravity and string theory, the complications of that requirement.

and testimony. Each person in the society may not acquire the evidence in a direct way. However, there has to be some direct perception of the 3-dimensional recordings of the positions that justifies our beliefs in the scientific theory. That is, we come to believe in the scientific theory on the basis of our observations about the arrangement of macroscopic objects in the 3-dimensional space. Now, if the scientific theory cannot explain the truth of such observations of pointer readings, or if it turns out that we would be systematically wrong about the arrangements of these macroscopic objects in the 3-dimensional space, then we acquire an undermining defeater: the theory suggests that our evidence for the theory is false. This is especially objectionable for scientific theories that attempt to give a comprehensive explanation for the empirical world that includes the human observers, pointers, and laboratory set-ups.

2.2.2 The Assumptions in Premise 4

If we accept **P3***, then the success of the argument depends on the second premise—**P4**. **P4** is the claim that we cannot find the Manifest Image of human observers and pointer readings in the quantum theory on 3N-Fundamentalism. In other words, there is an ineliminable explanatory gap for the true claims (or some nearby propositions) about human observers' position readings of the experimental set-ups.

Many people sympathetic to 3D-Fundamentalism take this premise to be straightforward.[24] However, 3N-Fundamentalists have offered a proposal about how to close the explanatory gap. Their general strategy goes by the name of "functionalism."[25]

In our context here, "functionalism" refers to a formulation of the sufficient condition for what counts as emergent and real objects:

Functionalism If we have a fundamental theory of the world in which we can define a mapping from the fundamental degrees of freedom to $Q_1, Q_2, ...Q_n$, and

[24]See, for example, Allori (2013) and Maudlin (2013).

[25]It is different from functionalism in philosophy of mind, as it is concerned not with minds but macroscopic physical objects, but the two theories have interesting similarities that are worth exploring further.

$Q_1, Q_2, ...Q_n$ have the same counterfactual profile of what we take to be the constituents of the Manifest Image, then $Q_1, Q_2, ...Q_n$ are real, emergent entities from that fundamental theory and our discourse about the Manifest Image is made true by the fundamental theory via the functionalist mapping.[26]

More specifically, we take the 3N-Fundamentalist to be adopting the following sufficient condition:

Functionalism about pointers If we have a fundamental theory of the world in which we can define a mapping from the fundamental degrees of freedom to $Q_1, Q_2, ...Q_n$, and $Q_1, Q_2, ...Q_n$ have the same counterfactual profile of what we take to be the 3-dimensional pointers in the Manifest Image, then $Q_1, Q_2, ...Q_n$ are real, emergent entities from that fundamental theory and our discourse about pointer readings is made true by the fundamental theory via the functionalist mapping.

Three questions arise. First, can we find such a mapping? Answer: yes, it is available to the 3N-Fundamentalist. The most obvious solution would be to choose just the three degrees of freedom associated with each apparent particle from which we construct the configuration space. The 3-dimensional particles are not fundamental, but we recognize that there is a mapping from the 3N-dimensional space to N particles in the 3-dimensional space. Therefore, our discourse about the pointers and human observers that are made out of particles in the 3-dimensional space is grounded via the functionalist mapping from the fundamental picture (in the Bohmian 3N picture: one Marvelous Point moving in a high-dimensional space).

Second, is the mapping unique? Answer: it depends. We can, for example, use the dynamical structure to privilege one set of mappings. Albert (1996) takes this strategy

[26]Thanks to an anonymous referee, I realize that this definition of functionalism leaves open the question how to understand the notion of "counterfactual profiles" in certain theories. Take GRWm for example: when we evaluate a counterfactual situation by changing the mass densities within a bounded spatial region R, it is not at all clear how we should change the mass densities outside of R or how we should change the wave function to reflect the changes in the mass densities as the mappings are many-to-one. There may be some ways to handle this worry (and we invite the defenders of 3N-Fundamentalism to address this worry), but the referee is right that it raises another sort of worry on the viability of the functionalist program.

and suggests that there is a uniquely preferred mapping because of the contingent structure of the Hamiltonian of the universe that governs the dynamics of the fundamental degrees of freedom:

$$H = \sum_{i}^{\Omega} \frac{p_i^2}{2m_i} + \sum_{k,j=1;k\neq j}^{\Omega/3} V_{j,k}([(x_{3j-2} - x_{3k-2})^2 + (x_{3j-1} - x_{3k-1})^2 + (x_{3j} - x_{3k})^2])$$

The idea is that the fundamental degrees of freedom are contingently grouped into triples in the Hamiltonian potential term, instantiating a relation just like the familiar Pythagorean distance relation, which gives rise to the 3-dimensional Euclidean metric. Each triple is what we take to be a particle in the 3-dimensional space. Even in a classical world, the Pythagorean distance relation gives rise to a structure that is invariant under rotation, translation, and reflection symmetries, and grounds our claims about the 3-dimensional space. In a relativisitic world, the Minkowski metric, according to many people, gives rise to a genuinely 4-dimensional spacetime:

$$d(a,b)^2 = -(c\delta t)^2 + (\delta x)^2 + (\delta y)^2 + (\delta z)^2$$

Therefore, the reasoning goes, the mathematical structure in the Hamiltonian potential term gives rise to an emergent, distinguished, and invariant structure of 3-dimensionality—$\mathbb{R}^3$—that grounds our discourse about pointer readings and human observers. Moreover, the Hamiltonian, in a unique way, puts the fundamental degrees of freedom into $\Omega/3$ triples.

Third, isn't the sufficient condition too permissive? If what suffices to be emergent and real is just to stand in a mathematical mapping relation to the fundamental dynamical objects, then (as Tim Maudlin[27] and John Hawthorne[28] observe independently) there will be many more emergent objects than we realize, a consequence that is highly implausible. Under **Functionalism**, simply by defining a mathematical mapping from the location of every object in the world to three feet north of that object, there will be emergent objects interacting with other emergent objects unbeknown to

[27] Personal communication February 2015.

[28] See Hawthorne (2010), pp.147-152.

all of us! Since the 3-feet-north mapping is completely arbitrary, there will be infinitely many mappings, each of which will realize a different set of emergent entities. As a consequence, just by defining trivial mappings like these, we can create metaphysical monstrosities of an unimaginable scale.

The 3-feet-north objection is a special case of a general objection to structuralism and functionalism: the mere existence of certain structure, counterfactual dependence, or causal relations is sometimes insufficient ground for establishing the reality of the emergent entities. In our context, the 3-feet-north objection suggests that the functionalist criterion is insufficient, and hence the proposed mapping from the fundamental degrees of freedom in 3N-Fundamentalism does not *explain* the pointers or observers in the 3-dimensional space.

I see two potential responses. First, the 3N-Fundamentalist can bite the bullet, embrace the seemingly absurd consequence of the functionalist criterion, and include in her ontology infinitely many sets of emergent entities. To show that the consequences are tolerable, she can give an argument that shows that all sets of emergent entities are in fact equivalent in some way. One obvious argument makes use of relationalism about space. If relationalism is true, then there is no container (the substantial space) in addition to spatial relations among objects. Take any set of emergent entities (for example, the emergent entities that are 3 feet north of everything in the world), the spatial relations instantiated among them are the same spatial relations instantiated in any other set (for example, the emergent entities that are 5 feet west of everything in the world). The sets of emergent entities related by such mappings are just different ways of describing the same relational world. Call this **The Relationalist Approach**.

The Relationalist Approach, I think, is the most promising response on behalf of the 3N-Fundamentalist. However, it faces a problem. **Functionalism**, if true, seems to be a necessary truth. Relationalism, on the other hand, does not seem to be a necessary truth. The mere possibility of the failure of relationalism implies that there are possible worlds in which the functionalist criterion generates too many emergent entities. Since those possible worlds are not too remote, our modal intuitions seem robust enough to account for them. The metaphysical monstrosity seems highly implausible. So it seems that **Functionalism** is still false. At this point, the 3N-Fundamentalist can reply that **Functionalism** is in fact a contingent truth that holds only among relational worlds. It would take future work to show why this is true or how it follows from other commitments of **Functionalism**.

Instead of suggesting that the thesis is contingent, the 3N-Fundamentalist can reply that there are ways of restricting the functionalist criterion such that it does not lead to the absurd consequences. One way to restrict the criterion is to say that the functionalist mapping has to be an identity mapping.[29] The reason that projection mappings from $\mathbb{R}^{3N}$ to $\mathbb{R}^3$ are insufficient is that a projection map composed with a spatial translation is another a projection map; since the spatial translamtion can be completely arbitrary, there will be an infinite number of projection mappings that are equally good. However, if we let the functionalist relation to be the identity mapping, we can eliminate this problem. Applied to our case, the relation takes the triplets of degrees of freedom in the configuration space as *identical* with the 3-dimensional particles. For example, take the Bohmian "marvelous point" in the 3N-dimensional configuration space: its coordinates x_1, y_1, z_1 just are a 3-dimensional particle. Identity is a strict relation that is not preserved by arbitrary mappings. Neither does the strategy rely on relationalism about space. Call this **The Identity Approach**.

However, not only does it have difficulty extending to GRW theories, the Identity Approach gets rid of the metaphysical monstrosities at the expense of eliminating the Manifest Image altogether. To see this, let us borrow some idioms from the grounding literature. The grounding relation, as commonly conceived,[30] is an umbrella term for several kinds of metaphysical dependence relations. One widely-accepted feature of grounding is that if a set of facts or entities, Σ, grounds another set of facts or entities Γ, then it is not the case that Γ grounds Σ. Such an asymmetry is a defining feature of the grounding relation. If the particles and their electromagnetic interactions ground the existence of a table, it is not the case that the table's existence grounds the particles and their electromagnetic interactions. Let us examine the suggestion that the functionalist

[29] Barry Loewer suggests this in personal communication, but he does not necessarily endorse it.

[30] For example, see Rosen (2010).

mapping has to be an identity mapping. As I understand it, the functionalist mapping is a metaphysical dependence relation that falls under the umbrella of the grounding relation. Hence, the mapping has to be asymmetric. However, the identity relation is symmetric. We have arrived at a contradiction. Therefore, one of the assumptions has to go. Since the functionalist mapping is supposed to metaphysically explain the emergence of 3-dimensional objects such as pointers and observers, it is best understood as a metaphysical explanation relation that is asymmetric. So it seems to me that the 3N-Fundamentalist should reject the suggestion that the functionalist criterion only allows identity mappings.

There is another way to restrict the functionalist criterion. Instead of counting mathematical mappings of any sort, a 3N-Fundamentalist can restrict to mappings between different spaces. For this smaller class of mathematical mappings, the 3-feet-north counterexamples do not arise, for the dynamics and causal behaviors on different levels are likely to be quite different. Call this **The Different Space Approach**. In addition to being obviously *ad hoc* and unprincipled, the Different Space Approach does not block composite mappings. The composite mappings first from $\mathbb{R}^{3N}$ to $\mathbb{R}^3$ then from $\mathbb{R}^3$ to itself are not ruled out by the restriction and still generate the same counterexamples.

The above discussions of different approaches of functionalism—the Relationalist Approach, the Identity Approach, and the Different Space Approach—suggest that the current functionalist criterion is too permissive and leads to disastrous results for the 3N-Fundamentalists. Thus, we are right to doubt whether there can be any principled way to close the apparent explanatory gap in 3N-Fundamentalism. However, **Functionalism**, just like **Structuralism**, is still being developed, and its application here is novel and potentially promising. Given the recent work on the emergence of space-time and structural realism, there may well be future work that suggests better proposals than the ones we have considered. Anecdotal evidence suggests that many philosophers are not deterred by the above counterexamples and are still searching for some version of **Functionalism** that closes the explanatory gap in a principled way.

So far our discussions have focused on the two main considerations in the literature: the dynamical structure of the quantum theory and the successful explanation of the Manifest Image. If I am right, then the common arguments based on these considerations are either unsound or in need of refinement. After locating the crucial premises, I show how they can be resisted on either side. While the 3D-Fundamentalist can respond line by line by giving an alternative criterion of fundamental physical space, the 3N-Fundamentalist cannot provide a satisfactory functionalist criterion. So far, the considerations based on the dynamical structure and the Manifest Image are roughly in favor of 3D-Fundamentalism. However, anecdotal evidence suggests that this has not persuaded many 3N-Fundamentalists to switch sides, as they are often inclined to bite the bullet and swallow the costs of counter-intuitiveness.

I take this to suggest that the most common philosophical arguments do not fully settle the debate between the two views. It looks like we have reached a stalemate. However, I suggest that we have not. In the next section, by looking into which view leads to a deeper understanding of the quantum world, I argue that we can find a new class of powerful arguments that favor 3D-Fundamentalism over 3N-Fundamentalism.

2.3 Evidence #3: Mathematical Symmetries in the Wave Function

Having seen that the common arguments from the dynamical structure and ordinary experiences do not fully settle the debate between 3D-Fundamentalism and 3N-Fundamentalism, I suggest we look at the debate from a different angle. As we are examining scientifically motivated metaphysical positions, we would like to see which one leads to a better or deeper understanding of the scientific phenomena. In this section, I offer an argument for 3D-Fundamentalism on the basis that it provides a deeper mathematical explanation of certain symmetries in the quantum theory. Since I do not take this to be the final word on the issue, I offer this as a first step toward an open research program to explore various mathematical explanations[31] in the quantum theories.

[31] Here I do not take sides whether such mathematical explanations are non-causal explanations.

2.3.1 Another Argument for 3D-Fundamentalism

P5 If a fundamental theory T explains S while T' does not, and S should be explained rather than postulated (other things being equal), then we have (defeasible) reasons to infer that T is more likely than its alternative to be the fundamental theory.

P6 3D-Fundamentalism explains the Symmetrization Postulate but 3N-Fundamentalism does not.

P7 The Symmetrization Postulate should be explained rather than postulated.

C3 We have (defeasible) reasons to infer that 3D-Fundamentalism is more likely to be the fundamental theory than 3N-Fundamentalism.

The first premise—**P5**—seems to me highly plausible. When we compare two fundamental theories, we measure them not just by empirical adequacy and internal coherence but also by explanatory depth. Say that a theory T provides a deeper explanation for X than T' does if T explains X from its axioms and T' postulates X as an axiom.

For example, if a fundamental physical theory T says that the **Symmetrization Postulate** is true (there are two groups of fundamental particles and one group has symmetric wave functions and the other group has anti-symmetric wave functions), if (other things being equal) we would like to understand why that is the case, and if we find out that its alternative S provides an explanation for that fact, then we have defeasible reasons to favor S over T.

2.3.2 Justifying Premise 6

If we accept **P5**, then the success of the argument depends on **P6** and **P7**. It is important to note that **P6** is not the unique premise that delivers the conclusion. There probably are many mathematical explanations that favor 3D-Fundamentalism over 3N-Fundamentalism, including Lorentz symmetry.[32] (Indeed, I take it to be an

[32] For example, see Allori (2013), pp. 72-73.

open research question whether there are *good* mathematical explanations that support 3N-Fundamentalism. To be sure, what counts as a good explanation and what counts as something that should be explained can be controversial.)

In any case, **P6** focuses on a particular mathematical explanation that arises from the mathematical study of identical particles and the nature of their configuration space.[33] [34]

Roughly speaking, identical particles share the same intrinsic physical properties, that is, the same qualitative properties recognized by the physical theory. For example, all electrons can be regarded as identical (which in this context means *physically indistinguishable*) since they have the same charge, mass, and spin. (To be sure, their different positions in space cannot correspond to their intrinsic properties, for if nothing else they would not be able to move.) Suppose we take the ordinary configuration space $\mathbb{R}^{3N}$ for N electrons. Then the configuration space is ordered:

$$(x_1, y_1, z_1, x_2, y_2, z_2, ..., x_n, y_n, z_n)$$

Each point in the configuration space is an ordered 3N-tuple and can be read as the x, y, z coordinates of electron 1, those of electron 2, those of electron 3, and all the way up to the x, y, z coordinates of electron N. The numeral labels provide names (and thus distinguishability) to the electrons. Therefore, the configuration in which electron 1 and electron 2 are exchanged will be a different configuration and hence will be represented by a different point in the configuration space.

However, the distinct points resulting from permutations of the electrons do not give rise to any real physical differences recognized by the physical laws. If we were to have the simplest ontology supporting the laws of quantum mechanics (or indeed any atomistic theory, including the Democritean atomism and classical mechanics), we should have a physical space representing all and only the real physical differences, excluding differences resulting from permutations of identical particles. Therefore, we

[33]In writing this section, I am grateful for many extensive discussions with Sheldon Goldstein and Roderich Tumulka about their papers on the Symmetrization Postulate.

[34]I do not include this explanation in the "inferences from the dynamics" category, for it concerns the mathematical construction of the theory.

can introduce the notion of an unordered configuration space for N particles in 3-dimensional space as:

$$^N\mathbb{R}^3 := \{S \subset R^3 | \text{ cardinality}(S) = N\}$$

So a point in this configuration space would be:

$$\{(x_1, y_1, z_1), (x_2, y_2, z_2), ..., (x_n, y_n, z_n)\}$$

Notice that the points are not 3N-tuples but sets of N elements, which are unordered (given the extensionality axiom of ZF set theory). In fact, it is a set of tuples, but the ordering corresponds to the coordinatization of the 3-dimensional space, which can be ultimately eliminated by using $\mathbb{E}^3$ instead of $\mathbb{R}^3$.

Even in classical mechanics, we can use the unordered configuration space $^N\mathbb{R}^3$ (or the relevant unordered phase space) instead of the ordered configuration space $\mathbb{R}^{3N}$. Such a choice would not lead to any physical differences, as the two spaces share the same *local* properties. They differ, however, in their *global* topological properties, which become essential in quantum mechanics. In particular, $^N\mathbb{R}^3$ (like the circle S^1) is topologically non-trivial in that it is not simply-connected, as not every closed loop is contractible to a point, while $\mathbb{R}^{3N}$ (like the real line $\mathbb{R}^1$) is simply connected—a much more trivial topology. As we will explain below, their global topological differences allow us to derive the deep and useful principle known as the Symmetrization Postulate.

Conventionally, a quantum theory with wave functions defined over an ordered configuration space needs to postulate an additional requirement—the Symmetrization Postulate—to rule out certain types of particles. It says that there are only two kinds of particles: symmetric wave functions for bosons and anti-symmetric wave functions for fermions. The "symmetry" and "anti-symmetry" refer to the behavior of the wave functions under position exchange. Stated for particles in 3-dimensions:

Symmetrization Postulate: There are only two kinds of wave functions:

$$\text{(Bosons)} \qquad \psi_B(\mathbf{x}_{\sigma(1)}, ..., \mathbf{x}_{\sigma(N)}) = \psi_B(\mathbf{x}_1, ..., \mathbf{x}_N),$$

$$\text{(Fermions)} \qquad \psi_F(\mathbf{x}_{\sigma(1)}, ..., \mathbf{x}_{\sigma(N)}) = (-1)^{\sigma}\psi_F(\mathbf{x}_1, ..., \mathbf{x}_N),$$

where σ is a permutation of $\{1, 2, ..., N\}$ in the permutation group S_N, $(-1)^\sigma$ denotes the sign of σ, $\mathbf{x}_i \in \mathbb{R}^3$ for $i = 1, 2, ..., N$.

This is an extremely deep and highly useful principle. To have a more intuitive grasp of it, we can recall the more familiar Pauli Exclusion Principle, which says that no two electrons can occupy the same quantum state (usually characterized by quantum numbers). For one direction of implication, consider two electrons (fermions) with a wave function that is totally anti-symmetric under position exchange. It follows that they cannot occupy the same position, that is:

$$\forall x_1, x_2 \in \mathbb{R}^3, \psi(x_1, x_2) = -\psi(x_2, x_1) \Longrightarrow \forall x \in \mathbb{R}^3, \psi(x, x) = 0$$

Surprisingly, the Symmetrization Postulate, being deep and useful, emerges as a result of a beautiful mathematical analysis about the topological properties of the unordered configuration space.[35] Here we sketch only an outline of the derivation; we refer interested readers to the Appendix B for more technical details.

Let us take Bohmian Mechanics (BM) as the background theory, in which there really are particles with precise trajectories, guided by a universal wave function. (Let us use scalar-valued wave functions to avoid further technicalities. We will later return to the question whether such an explanation is available on the Copenhagen intepretation and GRW theories.) In BM, the universal wave function evolves according to the Schrödinger equation, and particles move according to the universal wave function and the guidance equation. The configuration space for N identical particles in $\mathbb{R}^3$ is ${}^N\mathbb{R}^3$, not $\mathbb{R}^{3N}$. However, they are intimately related. In fact, $\mathbb{R}^{3N}$ is what is called the "universal covering space" of ${}^N\mathbb{R}^3$. There is a natural projection mapping from $\mathbb{R}^{3N}$ to ${}^N\mathbb{R}^3$ that forgets the ordering of particles. Since the physical configuration lies in ${}^N\mathbb{R}^3$, the velocity field needs to be defined there, whereas the wave function can still very well be defined on $\mathbb{R}^{3N}$, as we can project its velocity field from $\mathbb{R}^{3N}$ to the ${}^N\mathbb{R}^3$. However, not every velocity field can be so projected. To ensure that it can , we impose

[35] For the classic papers in the mathematical physics literature, see Dowker (1972) and Leinaas and Myrheim (1977). But Dürr et al. (2006) and Dürr et al. (2007) carry out the explanation much more thoroughly and successfully in Bohmian mechanics. The outline in the next paragraph is a summary of their more technical derivation in the case of scalar wave functions.

a natural "periodicity condition" on the wave function on $\mathbb{R}^{3N}$:

$$\forall \hat{q} \in \mathbb{R}^{3N}, \sigma \in S_N, \psi(\sigma\hat{q}) = \gamma_\sigma \psi(\hat{q}).$$

Since the fundamental group S_N is a finite group, the topological factor γ_σ has to be a group character (see **Unitarity Theorem** in Appendix B). But S_N has only two characters: (1) the trivial character $\gamma_\sigma = 1$ and (2) the alternating character $\gamma_\sigma = sign(\sigma) = 1$ or -1 depending on whether $\sigma \in S_N$ is an even or an odd permutation. The former corresponds to the symmetric wave functions of bosons and the latter to the anti-symmetric wave functions of fermions. Since any other topological factors are banned, we have ruled out other types of particles such as anyons. This result is equivalent to the Symmetrization Postulate.

We have seen that given some natural assumptions about identical particles in $\mathbb{R}^3$, the unordered configuration space ${}^N\mathbb{R}^3$, and a globally well-defined velocity field, we have very easily arrived at an explanation for the Symmetrization Postulate. With Bohmian Mechanics in the background, the derivation is well-motivated at each step. The rigorous mathematical work is done by Bohmian researchers in two beautiful papers Dürr et al. (2006) and Dürr et al. (2007), of which the above discussion is a simplification (of the simple case of scalar-valued wave functions). (For technical details, please see Appendix B.)

The above explanation might still work in the context of standard textbook quantum mechanics and the Copenhagen interpretation, according to which we should not take it seriously that particles are things that have precise locations and velocities simultaneously. Usually, in this context, we are told to act as if there are particles and to construct various mathematical objects from the degrees of freedom of the particles (such as the use of the configuration space). From a foundational point of view, it is not clear why the configuration space should be useful in such a theory. But if we play the usual game of acting *as if* there are particles, perhaps we can also act *as if* they are really identical, *as if* their states are given by a point in the unordered configuration space, and *as if* the above assumptions in the Bohmian theory are also justified in this theory. Indeed, this seems to be the implicit attitude taken in two pioneer studies—Dowker (1972) and Leinaas and Myrheim (1977)—about quantum mechanics for identical particles and unordered configuration space.

What about other solutions to the measurement problem, such as GRWm (a spontaneous collapse theory with a mass-density ontology)? Since its fundamental material ontology consists in a mass-density field (not particles) in the 3-dimensional space, it is unclear why we should use the configuration space of particle locations. However, usually the wave function and the mass-density field are defined with the help of the configuration space. [36] Nevertheless, it seems inconsistent with the spirit of GRWm to act as if there are fundamental particles or as if the above-mentioned Bohmian assumptions hold in the theory. In particular, there do not seem to be any compelling reasons to consider ${}^N\mathbb{R}^3$, a crucial piece in the above derivation, unlike in the case of BM. Hence, we do not think that the previous argument applies smoothly to the case of GRW theories or the Many-Worlds interpretation with a mass-density ontology.[37] Unlike the arguments in the previous sections, the argument we offer here does depend on the specific interpretation of quantum mechanics, namely Bohmian Mechanics. Whether we can make similar topological arguments in the context of GRW and MWI would require thinking more deeply about the structures of their state spaces. (This stands in contrast with the situation in §1 and §2, where the arguments do not require a justification of the structure of the configuration space.)

Returning to the construction of the unordered configuration space ${}^N\mathbb{R}^3$, we observe that it stands in a special relation to $\mathbb{R}^3$, namely, mathematical construction. In this sense, a quantum system of N-identical-particles in $\mathbb{R}^3$ naturally give rise to ${}^N\mathbb{R}^3$, which in turn naturally give rise to the mathematical explanation of the Symmetrization Postulate.

[36] For example, when defining the mass-density function at a time t, we start with a $|\Psi_t|^2$-distribution in the particle-location configuration space $\mathbb{R}^{3N}$, and obtain the value by using the marginal distribution weighted by the particle masses and summing over all particle degrees of freedom:

$$\forall x \in \mathbb{R}^3, m(x,t) = \sum_{i=1}^{N} m_i \int_{\mathbb{R}^{3N}} d^3x_1...d^3x_N \delta^3(x_i - x)|\Psi_t(x_1,...,x_N)|^2$$

[37] We thank an anonymous referee for helping us make this clear.

However, this explanation is not available for the quantum theory on 3N-Fundamentalism. Although in the above explanation we use $\mathbb{R}^{3N}$ in defining the wave function (as it is the universal covering space of ${}^N\mathbb{R}^3$), simply starting with $\mathbb{R}^{3N}$ gets us nowhere near to the characters of the permutation group. Starting with a wave function and a "marvelous point" (the only fundamental particle) on $\mathbb{R}^{3N}$, we do not have any motivation to consider ${}^N\mathbb{R}^3$, the natural configuration space for N particles in $\mathbb{R}^3$. Consequently, we cannot make use of its explanatory power in the derivation of the Symmetrization Postulate. (That is because, given a particle moving in $\mathbb{R}^{3N}$, we do not have any motivation to use the covering-space construction to derive the topological factors of the wave function, and we do not have an explanation for the Symmetrization Postulate.)

Therefore, we have justified **P6**—the second premise in the argument.

2.3.3 Deep versus Shallow Explanations

What about **P7**, the idea that the Symmetrization Postulate should be explained rather than postulated?[38] Here are some compelling reasons to endorse it:

- Its form is very different from the other laws of QM such as the Schrödinger equation. It is a direct restriction on the possible quantum states of systems with indistinguishable particles.
- It has high explanatory power. The Symmetrization Postulate explains Pauli's Exclusion Principle and the Periodic Table. It also explains why there are exactly two kinds of particles in QM.
- In the words of Pauli, at the occasion of his 1946 Nobel Lecture, as quoted in the epigraph: "Already in my original paper I stressed the circumstance that I was

[38]**P7** shows the difference between my argument here and Maudlin's argument in Maudlin (2013) pp.140-42. The difference in our arguments lies in our use of different explananda. Maudlin suggests that for, say, an 8-particle universe, the natural configuration space is ${}^8\mathbb{R}^3$ instead of $\mathbb{R}^{24}$, but that cries out for an explanation: why is this particular space ${}^8\mathbb{R}^3$ the fundamental space instead of some other space with a different factorization of 24 (such as ${}^4\mathbb{R}^6$)? This is an insightful argument, but I do not think it would *convince* many defenders of 3N-Fundamentalism, as on their view the fundamental space does not need further explanation. In contrast, our argument is quite different: we are not asking for an explanation of the fundamental space, but for an explanation of the Symmetrization Postulate, something that everyone agrees to be a deep and puzzling symmetry in quantum mechanics. Thus, I include Pauli's quote as it beautifully conveys the deep puzzlement.

> unable to give a logical reason for the exclusion principle or to deduce it from more general assumptions. I had always the feeling, and I still have it today, that this is a deficiency."[39]

Granted, one might still deny **P7** at this point. But it seems to me (and to many working physicists) that **P7** is much more likely to be true than its denial. The above reasons suggest that an explanation of **P7** would point us to something deep in the physical nature of the world. It is true that in an ultimately axiomatic theory of physics, the explanatory relations can be reversed (just as logical theories can be axiomatized in different ways corresponding to different explanatory relations). However, I think it is reasonable to believe that the final quantum theory ought not to postulate the Symmetrization Postulate as fundamental but rather as something that follows from simpler axioms.

2.3.4 What about ${}^{N}\mathbb{R}^3$-Fundamentalism?

In this essay, I have assumed that the 3N-Fundamentalist is really a $\mathbb{R}^{3N}$-Fundamentalist. She can of course modify her thesis and believe instead in a fundamentally multiply-connected physical space, ${}^{N}\mathbb{R}^3$. The 3D-Fundamentalist can respond by charging her opponent's move as *ad hoc* or unnatural, since ${}^{N}\mathbb{R}^3$ is "clearly" about N identical particles in $\mathbb{R}^3$. But similar criticisms have not prevented anyone from taking $\mathbb{R}^{3N}$ as fundamental, even though in some sense it is "clearly" about N particles in $\mathbb{R}^3$.

3D-Fundamentalism	3N-Fundamentalism
$\mathbb{R}^3$+distinguishable particles	$\mathbb{R}^{3N}$
$\mathbb{R}^3$+indistinguishable particles	${}^{N}\mathbb{R}^3$

There is, however, another argument against the 3N-Fundamentalist's maneuver that is even less controversial. My main goal in this paper has been to evaluate

[39] Pauli et al. (1994), p.171.

the evidential support for 3D-Fundamentalism and 3N-Fundamentalism. Given 3N-Fundamentalism, the fundamental physical space has two main possibilities: (1) a simply-connected space $\mathbb{R}^{3N}$ and (2) a multiply-connected space that is a quotient space of $\mathbb{R}^{3N}$ by the permutation group of k objects where k divides $3N$. I believe that (1) is intrinsically more likely than (2). But suppose we grant them equal probability. Still, (2) comes in many further possibilities, and ${}^N\mathbb{R}^3$ is intrinsically as likely as any other quotient space. So we should assign equal probability to each possibility in (2). Given a large N, ${}^N\mathbb{R}^3$ will receive a very small probability compared to that of $\mathbb{R}^{3N}$.[40] In the table above, the ratio of the two areas for $\mathbb{R}^{3N}$ and ${}^N\mathbb{R}^3$ roughly corresponds to the relative confidence that we have in them.

Given 3D-Fundamentalism, there is only one main candidate for the fundamental physical space–$\mathbb{R}^3$ and two main candidates for the configuration space–$\mathbb{R}^{3N}$ and ${}^N\mathbb{R}^3$. Although an ontology of identical particles is simpler than that of non-identical particles, for the sake of the argument, we can give these two possibilities equal weight. Hence, in the table above, the two candidates divide the area roughly in half.

If we take as a datum that the Symmetrization Postulate is true, assume that ${}^N\mathbb{R}^3$ is the correct route to explain it, and we allow explanatory strength to come in degrees, then overall (considering all versions) 3D-Fundamentalism explains it better than 3N-Fundamentalism does. So although both 3D-Fundamentalism and 3N-Fundamentalism contain versions of them that explain the Symmetrization Postulate, 3D-Fundamentalism is better supported by the successful explanation. Therefore, the evidence "disconfirms" 3N-Fundamentalism over 3D-Fundamentalism. (That is, if we assume a standard way of thinking about update and Bayesian confirmation theory.[41]) In fact, we can reformulate the original argument by replacing **P6** with a weaker premise:

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[40]Thanks to Gordon Belot for helpful discussions here.

[41]In a different context (the problem of evil) at the Rutgers Religious Epistemology Workshop in May 2014, Lara Buchak discussed a different approach to rational updating—updating by conditionals. It is up to the defender of 3N-Fundamentalism to develop and apply that approach here. Even if that were to succeed, however, I think it would be a significant concession that 3N-Fundametnalism is disconfirmed in the "Bayesian" sense.

P6* 3D-Fundamentalism explains the Symmetrization Postulate significantly better than 3N-Fundamentalism.

Therefore, even if ${}^{N}\mathbb{R}^3$-Fundamentalism is an option for the 3N-Fundamentalism, her view is still not well supported by the explanation. The above argument fleshes out our intuition that the 3N-Fundamentalist maneuver is *ad hoc* or unnatural.

However, this should reflect only our current state of knowledge and should not be taken as the final word on the issue. Indeed, there are several ways that future research can go:

1. We discover more mathematical explanations only available to 3D-Fundamentalism, which gives us more reason to favor 3D-Fundamentalism over 3N-Fundamentalism.

2. We discover some mathematical explanations only available to 3N-Fundamentalism, which restores or reverses the evidential balance between 3D-Fundamentalism and 3N-Fundamentalism.

3. We discover that there exists a mathematical explanation of the Symmetrization Postulate from other resources in 3N-Fundamentalism that is independent from the topological considerations as discussed above, which restores the evidential balance between 3D-Fundamentalism and 3N-Fundamentalism.

Considerations	3D-Fundamentalism	3N-Fundamentalism
Dynamics	0/+	+
Manifest Image	+	-/0
Explanatory Depth	+	-/0

2.4 Conclusion

Based on the above evaluation of the three kinds of evidence (see Table 1, a summary of pros and cons, where "+" means "strongly in favor," "-" means "strongly against," and "0" means "roughly neutral."), I conclude that, given our current knowledge, it is more likely that the fundamental physical space in quantum mechanics is 3-dimensional rather than 3N-dimensional. However, as noted above, there are future directions of research where the 3N-Fundamentalists can restore or even reverse the evidential balance.

Our debate here is related to the discussions about theoretical equivalence; we observe the following:

1. 3N-Fundamentalism and 3D-Fundamentalism are highly equivalent in theoretical structure–the mathematics used is exactly the same;

2. The debate between 3N-Fundamentalists and 3D-Fundamentalists almost reaches a stalemate and hence might be considered as non-substantive;

3. An important new piece of evidence lies not in the inferences from the dynamics or ordinary experiences but in the mathematical symmetries in the wave function. What breaks the tie is the fact that 3N-Fundamentalism and 3D-Fundamentalism are explanatorily inequivalent, and the latter explains the Symmetrization Postulate better than the former.

Therefore, the discussion about the wave function provides another useful case for the ongoing debate about theoretical equivalence and structure. The connection should prove fruitful for future research.[42]

[42] Thanks to David Schroeren for discussing this last point with me.

Chapter 3

Quantum Mechanics in a Time-Asymmetric Universe

3.1 Introduction

In the foundations of quantum mechanics, it has been argued that the wave function (pure state) of the universe represents something objective and not something merely epistemic. Let us call this view *Wave Function Realism*. There are many realist proposals for how to understand the wave function. Some argue that it represents things in the ontology, either a physical field propagating on a fundamental high-dimensional space, or a multi-field propagating on the three-dimensional physical space. Others argue that it is in the "nomology"—having the same status as laws of nature. Still others argue that it might belong to a new ontological category.[1]

Wave Function Realism has generated much debate. In fact, it has been rejected by many people, notably by quantum Bayesians, and various anti-realists and instrumentalists. As a scientific realist, I do not find their arguments convincing. In previous papers, I have assumed and defended Wave Function Realism. Nevertheless, in this paper I want to argue for a different perspective, for reasons related to the origin of time-asymmetry in a quantum universe.

To be sure, realism about the universal wave function is highly natural in the context of standard quantum mechanics and various realist quantum theories such as Bohmian

[1]See Albert (1996); Loewer (1996); Wallace and Timpson (2010); Ney (2012); North (2013); Maudlin (2013); Goldstein and Zanghì (2013); Miller (2014); Esfeld (2014); Bhogal and Perry (2015); Callender (2015); Esfeld and Deckert (2017); Chen (2018a, 2017, 2016); Hubert and Romano (2018). For a survey of this literature, see Chen (2018b). Notice that this is not how Albert, Loewer, or Ney characterizes wave function realism. For them, to be a wave function realist is to be a realist about the wave function and a fundamental high-dimensional space—the "configuration space." For the purpose of this paper, let us use *Wave Function Realism* to designate just the commitment that the wave function represents something objective.

mechanics (BM), GRW spontaneous collapse theories, and Everettian quantum mechanics (EQM). In those theories, the universal wave function is indispensable to the kinematics and the dynamics of the quantum system. However, as I would like to emphasize in this paper, our world is not just quantum-mechanical. We also live in a world with a strong arrow of time (large entropy gradient). There are thermodynamic phenomena that we hope to explain with quantum mechanics and quantum statistical mechanics. A central theme of this paper is to suggest that quantum statistical mechanics is highly relevant for assessing the fundamentality and reality of the universal wave function.

We will take a close look at the connections between the foundations of quantum statistical mechanics and various solutions to the quantum measurement problem. When we do, we realize that we do not *need* to postulate a universal wave function. We need only certain "coarse-grained" information about the quantum macrostate, which can be represented by a density matrix. A natural question is: can we understand the universal quantum state as a density matrix rather than a wave function? That is, can we take an "ontic" rather than an "epistemic" attitude towards the density matrix?

The first step of this paper is to argue that we can. I call this view *Density Matrix Realism*, the thesis that the actual quantum state of the universe is objective (as opposed to subjective or epistemic) and impure (mixed). This idea may be unfamiliar to some people, as we are used to take the mixed states to represent our *epistemic uncertainties of the actual pure state* (a wave function). The proposal here is that the density matrix directly represents the actual quantum state of the universe; there is no further fact about which is the actual wave function. In this sense, the density matrix is "fundamental." In fact, this idea has come up in the foundations of physics.[2] In the first step, we provide a systematic discussion of Density Matrix Realism by reformulating Bohmian mechanics, GRW theories, and Everettian quantum mechanics in terms of a *fundamental* density matrix.

[2]See, for example, Dürr et al. (2005); Maroney (2005), Wallace (2011, 2012), and Wallace (2016).

The second step is to point out that Density Matrix Realism allows us to combine quantum ontology with time-asymmetry in a new way. In classical and quantum statistical mechanics, thermodynamic time-asymmetry arises from a special boundary condition called the Past Hypothesis.[3] I suggest that the information in the *Past Hypothesis* is sufficient to determine a natural density matrix. I postulate the *Initial Projection Hypothesis*: the quantum state of the universe at t_0 is given by the (normalized) projection onto the Past Hypothesis subspace, which is a particular low-dimensional subspace in the total Hilbert space. The conjunction of this hypothesis with Density Matrix Realism pins down a *unique* initial quantum state. Since the Initial Projection Hypothesis is as simple as the Past Hypothesis, we can use arguments for the simplicity of the latter (which is necessary for it to be a law of nature) to argue for the simplicity of the former. We can thus infer that the initial quantum state is very *simple*.

The third step is to show that, because of the simplicity and the uniqueness of the initial quantum state (now given by a fundamental density matrix), we have a strong case for the *Nomological Thesis*: the initial quantum state of the world is on a par with laws of nature. It is a modal thesis. It implies that the initial quantum state of our world is nomologically necessary; it could not have been otherwise.

As we shall see, this package of views has interesting implications for the reduction of statistical mechanical probabilities to quantum mechanics, the dynamic and kinematic unity of the universe and the subsystems, the nature of the initial quantum state, and Humean supervenience in a quantum world.

Here is the roadmap of the paper. First, in §2, I review the foundations of quantum mechanics and quantum statistical mechanics. In §3, I introduce the framework of Density Matrix Realism and provide some illustrations. In §4, I propose the Initial Projection Hypothesis in the framework of Density Matrix Realism. In §5, I discuss their implications for statistical mechanics, dynamic unity, and kinematic unity. In §6, I suggest that they provide a strong case for the Nomological Thesis and a new solution to the conflict between quantum entanglement and Humean supervenience.

[3]For an extended discussion, see Albert (2000).

3.2 Foundations of Quantum Mechanics and Statistical Mechanics

In this section, we first review the foundations of quantum mechanics and statistical mechanics. As we shall see in the next section, they suggest an alternative to Wave Function Realism.

3.2.1 Quantum Mechanics

Standard quantum mechanics is often presented with a set of axioms and rules about measurement. Firstly, there is a quantum state of the system, represented by a wave function ψ. For a spin-less N-particle quantum system in $\mathbb{R}^3$, the wave function is a (square-integrable) function from the configuration space $\mathbb{R}^{3N}$ to the complex numbers $\mathbb{C}$. Secondly, the wave function evolves in time according to the the Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \tag{3.1}$$

Thirdly, the Schrödinger evolution of the wave function is supplemented with collapse rules. The wave function typically evolves into superpositions of macrostates, such as the cat being alive and the cat being dead. This can be represented by wave functions on the configuration space with disjoint macroscopic supports X and Y. During measurements, which are not precisely defined processes in the standard formalism, the wave function undergoes collapses. Moreover, the probability that it collapses into any particular macrostate X is given by the Born rule:

$$P(X) = \int_X |\psi(x)|^2 dx \tag{3.2}$$

As such, quantum mechanics is not a candidate for a fundamental physical theory. It has two dynamical laws: the deterministic Schrödinger equation and the stochastic collapse rule. What are the conditions for applying the former, and what are the conditions for applying the latter? Measurements and observations are extremely vague concepts. Take a concrete experimental apparatus for example. When should we treat it as part of the quantum system that evolves linearly and when should we treat it as an "observer," i.e. something that stands outside the quantum system and collapses the wave function? That is, in short, the quantum measurement problem.[4]

Various solutions have been proposed regarding the measurement problem. Bohmian mechanics (BM) solves it by adding particles to the ontology and an additional guidance equation for the particles' motion. Ghirardi-Rimini-Weber (GRW) theories postulate a spontaneous collapse mechanism. Everettian quantum mechanics (EQM) simply removes the collapse rules from standard quantum mechanics and suggest that there are many (emergent) worlds, corresponding to the branches of the wave function, which are all real. My aim here is not to adjudicate among these theories. Suffice it to say that they are all quantum theories that remove the centrality of observations and observers.

To simplify the discussions, I will use BM as a key example.[5] In BM, in addition to the wave function that evolves unitarily according to the Schrödinger equation, particles have precise locations, and their configuration $Q = (Q_1, Q_2, ..., Q_N)$ follows the guidance equation:

$$\frac{dQ_i}{dt} = \frac{\hbar}{m_i} \mathrm{Im} \frac{\nabla_i \psi(q)}{\psi(q)} (q = Q) \tag{3.3}$$

Moreover, the initial particle distribution is given by the quantum equilibrium distribution:

$$\rho_{t_0}(q) = |\psi(q, t_0)|^2 \tag{3.4}$$

By equivariance, if this condition holds at the initial time, then it holds at all times. Consequently, BM agrees with standard quantum mechanics with respect to the Born rule predictions (which are all there is to the observable predictions of quantum mechanics). For a universe with N particles, let us call the wave function of the universe the *universal wave function* and denote it by $\Psi(\boldsymbol{q_1}, \boldsymbol{q_2}, ...\boldsymbol{q_N})$.

3.2.2 Quantum Statistical Mechanics

Statistical mechanics concerns macroscopic systems such as gas in a box. It is an important subject for understanding the arrow of time. For concreteness, let us consider

[4] See Bell (1990) and Myrvold (2017) for introductions to the quantum measurement problem.

[5] See Dürr et al. (1992) for a rigorous presentation of BM and its statistical analysis.

a quantum-mechanical system with N fermions (with $N > 10^{20}$) in a box $\Lambda = [0, L]^3 \subset \mathbb{R}^3$ and a Hamiltonian $\hat{H}$. I will first present the "individualistic" view followed by the "ensemblist" view of quantum statistical mechanics (QSM).[6] I will include some brief remarks comparing QSM to classical statistical mechanics (CSM), the latter of which may be more familiar to some readers.

1. Microstate: at any time t, the microstate of the system is given by a normalized (and anti-symmetrized) wave function:

$$\psi(\boldsymbol{q_1}, ..., \boldsymbol{q_N}) \in \mathscr{H}_{total} = L^2(\mathbb{R}^{3N}, \mathbb{C}^k) \ , \ \parallel \psi \parallel_{L^2} = 1, \tag{3.5}$$

   where $\mathscr{H}_{total} = L^2(\mathbb{R}^{3N}, \mathbb{C}^k)$ is the total Hilbert space of the system. (In CSM, the microstate is given by the positions and the momenta of all the particles, represented by a point in phase space.)

2. Dynamics: the time dependence of $\psi(\boldsymbol{q_1}, ..., \boldsymbol{q_N}; t)$ is given by the Schrödinger equation:

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi. \tag{3.6}$$

   (In CSM, the particles move according to the Hamiltonian equations.)

3. Energy shell: the physically relevant part of the total Hilbert space is the subspace ("the energy shell"):

$$\mathscr{H} \subseteq \mathscr{H}_{total} \ , \ \mathscr{H} = \text{span}\{\phi_\alpha : E_\alpha \in [E, E + \delta E]\}, \tag{3.7}$$

   This is the subspace (of the total Hilbert space) spanned by energy eigenstates ϕ_α whose eigenvalues E_α belong to the $[E, E + \delta E]$ range. Let $D = \dim\mathscr{H}$, the number of energy levels between E and $E + \delta E$.

   We only consider wave functions ψ in $\mathscr{H}$.

4. Measure: the measure μ is given by the normalized surface area measure on the unit sphere in the energy subspace $\mathscr{S}(\mathscr{H})$.

[6] Here I follow the discussions in Goldstein et al. (2010a) and Goldstein and Tumulka (2011).

5. Macrostate: with a choice of macro-variables (suitably "rounded" *à la* Von Neumann (1955)), the energy shell $\mathscr{H}$ can be orthogonally decomposed into macro-spaces:

$$\mathscr{H} = \oplus_\nu \mathscr{H}_\nu \ , \ \sum_\nu \dim \mathscr{H}_\nu = D \tag{3.8}$$

Each $\mathscr{H}_\nu$ corresponds more or less to small ranges of values of macro-variables that we have chosen in advance. (In CSM, the phase space can be partitioned into sets of phase points. They will be the macrostates.)

6. Non-unique correspondence: typically, a wave function is in a superposition of macrostates and is not entirely in any one of the macrospaces. However, we can make sense of situations where ψ is (in the Hilbert space norm) very close to a macrostate $\mathscr{H}_\nu$:

$$\langle\psi| P_\nu |\psi\rangle \approx 1, \tag{3.9}$$

where P_ν is the projection operator onto $\mathscr{H}_\nu$. This means that almost all of $|\psi\rangle$ lies in $\mathscr{H}_\nu$. (In CSM, a phase point is always entirely within some macrostate.)

7. Thermal equilibrium: typically, there is a dominant macro-space $\mathscr{H}_{eq}$ that has a dimension that is almost equal to D:

$$\frac{\dim \mathscr{H}_{eq}}{\dim \mathscr{H}} \approx 1. \tag{3.10}$$

A system with wave function ψ is in equilibrium if the wave function ψ is very close to $\mathscr{H}_{eq}$ in the sense of (3.9): $\langle\psi| P_{eq} |\psi\rangle \approx 1$.

Simple Example. Consider a gas consisting of $n = 10^{23}$ atoms in a box $\Lambda \subseteq \mathbb{R}^3$. The system is governed by quantum mechanics. We orthogonally decompose the Hilbert space $\mathscr{H}$ into 51 macro-spaces: $\mathscr{H}_0 \oplus \mathscr{H}_2 \oplus \mathscr{H}_4 \oplus ... \oplus \mathscr{H}_{100}$, where $\mathscr{H}_\nu$ is the subspace corresponding to the macrostate such that the number of atoms in the left half of the box is between $(\nu - 1)\%$ and $(\nu + 1)\%$ of n. In this example, $\mathscr{H}_{50}$ has the overwhelming majority of dimensions and is thus the equilibrium macro-space. A system whose wave function is very close to $\mathscr{H}_{50}$ is in equilibrium (for this choice of macrostates).

8. Boltzmann Entropy: the Boltzmann entropy of a quantum-mechanical system with wave function ψ that is very close to a macrostate $\mathscr{H}_\nu$ is given by:

$$S_B(\psi) = k_B \log(\dim \mathscr{H}_\nu), \tag{3.11}$$

where $\mathscr{H}_\nu$ denotes the subspace containing almost all of ψ in the sense of (3.9). The thermal equilibrium state thus has the maximum entropy:

$$S_B(eq) = k_B \log(\dim \mathscr{H}_{eq}) \approx k_B \log(D), \tag{3.12}$$

where $\mathscr{H}_{eq}$ denotes the equilibrium macrostate. (In CSM, Boltzmann entropy of a phase point is proportional to the logarithm of the volume measure of the macrostate it belongs to.)

9. Low-Entropy Initial Condition: when we consider the universe as a quantum-mechanical system, we postulate a special low-entropy boundary condition on the universal wave function—the quantum-mechanical version of the *Past Hypothesis*:

$$\Psi(t_0) \in \mathscr{H}_{PH} \ , \ \dim \mathscr{H}_{PH} \ll \dim \mathscr{H}_{eq} \approx \dim \mathscr{H} \tag{3.13}$$

where $\mathscr{H}_{PH}$ is the Past Hypothesis macro-space with dimension much smaller than that of the equilibrium macro-space.[7] Hence, the initial state has very low entropy in the sense of (3.11). (In CSM, the Past Hypothesis says that the initial microstate is in a low-entropy macrostate with very small volume.)

10. A central task of QSM is to establish mathematical results that demonstrate (or suggest) that μ−most (maybe even all) wave functions of small subsystems, such as gas in a box, will approach thermal equilibrium.

Above is the individualistic view of QSM in a nutshell. In contrast, the ensemblist view of QSM differs in several ways. First, on the ensemblist view, instead of focusing on the wave function of an individual system, the focus is on an ensemble of systems

[7] We should assume that $\mathscr{H}_{PH}$ is finite-dimensional, in which case we can use the normalized surface area measure on the unit sphere as the typicality measure for # 10. It remains an open question in QSM about how to formulate the low-entropy initial condition when the initial macro-space is infinite-dimensional.

that have the same statistical state $\hat{W}$, a density matrix.[8] It evolves according to the von Neumann equation:

$$i\hbar \frac{d\hat{W}(t)}{dt} = [\hat{H}, \hat{W}]. \tag{3.14}$$

The crucial difference between the individualistic and the ensemblist views of QSM lies in the definition of thermal equilibrium. On the ensemblist view, a system is in thermal equilibrium if:

$$W = \rho_{mc} \text{ or } W = \rho_{can}, \tag{3.15}$$

where ρ_{mc} is the microcanonical ensemble and ρ_{can} is the canonical ensemble.[9]

For the QSM individualist, if the microstate ψ of a system is close to some macro-space $\mathscr{H}_\nu$ in the sense of (3.9), we can say that the macrostate of the system is $\mathscr{H}_\nu$. The macrostate is naturally associated with a density matrix:

$$\hat{W}_\nu = \frac{I_\nu}{dim \mathscr{H}_\nu}, \tag{3.17}$$

where I_ν is the projection operator onto $\mathscr{H}_\nu$. $\hat{W}_\nu$ is thus a representation of the macrostate. It can be decomposed into wave functions, but the decomposition is not unique. Different measures can give rise to the same density matrix. One such choice is $\mu(d\psi)$, the uniform distribution over wave functions:

$$\hat{W}_\nu = \int_{\mathscr{S}(\mathscr{H}_\nu)} \mu(d\psi) \left|\psi\right\rangle \left\langle\psi\right|. \tag{3.18}$$

In (3.18), $\hat{W}_\nu$ is defined with a choice of measure on wave functions in $\mathscr{H}_\nu$. However, we should not be misled into thinking that the density matrix is derivative of wave functions. What is intrinsic to a density matrix is its geometrical meaning in the Hilbert space. In the case of $\hat{W}_\nu$, as shown in the canonical description (3.17), it is just a normalized projection operator.[10]

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[8]Ensemblists would further insist that it makes no sense to talk about the thermodynamic state of an individual system.

[9]The microcanonical ensemble is the projection operator onto the energy shell $\mathscr{H}$ normalized by its dimension. The canonical ensemble is:

$$\rho_{can} = \frac{\exp(-\beta\hat{H})}{Z}, \tag{3.16}$$

where $Z = \text{tr} \exp(-\beta\hat{H})$, and β is the inverse temperature of the quantum system.

[10]Thanks to Sheldon Goldstein for helping me appreciate the intrinsic meaning of density matrices. That was instrumental in the final formulation of the Initial Projection Hypothesis in §4.2.

3.3 Density Matrix Realism

According to Wave Function Realism, the quantum state of the universe is objective and pure. On this view, Ψ is both the microstate of QSM and a dynamical object of QM.

Let us recall the arguments for Wave Function Realism. Why do we attribute objective status to the quantum state represented by a wave function? It is because the wave function plays crucial roles in the realist quantum theories. In BM, the wave function appears in the fundamental dynamical equations and guides particle motion. In GRW, the wave function spontaneously collapses and gives rise to macroscopic configurations of tables and chairs. In EQM, the wave function is the whole world. If the universe is accurately described by BM, GRW, or EQM, then the wave function is an active "agent" that makes a difference in the world. The wave function cannot represent just our ignorance. It has to be objective, so the arguments go. But what is the nature of the quantum state that it represents? As mentioned in the beginning of this paper, there are several interpretations: the two field interpretations, the nomological interpretation, and the *sui generis* interpretation.

On the other hand, we often use W, a density matrix, to represent our ignorance of ψ, the actual wave function of a quantum system. W can also represent a macrostate in QSM.[11]

Is it possible to be a realist about the density matrix of the universe and attribute objective status to the quantum state it represents? That depends on whether we can write down realist quantum theories directly in terms of W. Perhaps W does not have enough information to be the basis of a realist quantum theory. However, if we can formulate quantum dynamics directly in terms of W instead of Ψ such that W guides Bohmian particles, or W collapses, or W realizes the emergent multiverse, then we will have good reasons for taking W to represent something objective in those theories. At the very least, the reasons for that will be on a par with those for Wave Function

[11]In some cases, W is easier for calculation than Ψ, such as in the case of GRW collapse theories where there are multiple sources of randomness. Thanks to Roderich Tumulka for discussions here.

Realism in the Ψ-theories.

However, can we describe the quantum universe with W instead of Ψ? The answer is yes. Dürr et al. (2005) has worked out the Bohmian version. In this section, I describe how. Let us call this new framework *Density Matrix Realism*.[12] I will use W-Bohmian Mechanics as the main example and explain how a fundamental density matrix can be empirically adequate for describing a quantum world. We can also construct W-Everett theories and W-GRW theories (which have not appeared in print so far). Similar to Wave Function Realism, Density Matrix Realism is open to several interpretations. At the end of this section, I will provide three field interpretations of W. In §6, I discuss and motivate a nomological interpretation.

3.3.1 W-Bohmian Mechanics

First, we illustrate the differences between Wave Function Realism and Density Matrix Realism by thinking about two different Bohmian theories.

In standard Bohmian mechanics (BM), an N-particle universe at a time t is described by $(Q(t), \Psi(t))$. The universal wave function guides particle motion and provides the probability distribution of particle configurations. Given the centrality of Ψ in BM, we take the wave function to represent something objective (and it is open to several realist interpretations).

It is somewhat surprising that we can formulate a Bohmian theory with only W and Q. This was introduced as W-Bohmian Mechanics (W-BM) in Dürr et al. (2005). The fundamental density matrix $W(t)$ is governed by the von Neumann equation (3.14). Next, the particle configuration $Q(t)$ evolves according to an analogue of the guidance equation (W-guidance equation):

$$\frac{dQ_i}{dt} = \frac{\hbar}{m_i}\mathrm{Im}\frac{\nabla_{q_i} W(q, q', t)}{W(q, q', t)}(q = q' = Q), \tag{3.19}$$

[12] The possibility that the universe can be described by a fundamental density matrix (mixed state) has been suggested by multiple authors and explored to various extents (see Footnote #2). What is new in this paper is the combination of Density Matrix Realism with the Initial Projection Hypothesis (§4) and the argument for the Nomological Thesis (§6) based on that. However, Density Matrix Realism is unfamiliar enough to warrant some clarifications and developments, and the GRW versions are new.

(Here we have set aside spin degrees of freedom. If we include spin, we can add the partial trace operator $\text{tr}_{\mathbb{C}^k}$ before each occurrence of "W.") Finally, we can impose an initial probability distribution similar to that of the quantum equilibrium distribution:

$$P(Q(t_0) \in dq) = W(q, q, t_0)dq. \tag{3.20}$$

The system is also equivariant: if the probability distribution holds at t_0, it holds at all times.[13]

With the defining equations—the von Neumann equation (3.14) and the W-guidance equation (3.19)—and the initial probability distribution (3.20), we have a theory that directly uses a density matrix $W(t)$ to characterize the trajectories $Q(t)$ of the universe's N particles. If a universe is accurately described by W-BM, then W represents the fundamental quantum state in the theory that guides particle motion; it does not do so via some other entity Ψ. If we have good reasons to be a wave function realist in BM, then we have equally good reasons to be a density matrix realist in W-BM.

W-BM is empirically equivalent to BM with respect to the observable quantum phenomena, that is, pointer readings in quantum-mechanical experiments. By the usual typicality analysis (Dürr et al. (1992)), this follows from (3.20), which is analogous to the quantum equilibrium distribution in BM. With the respective dynamical equations, both BM and W-BM generate an equivariant Born-rule probability distribution over all measurement outcomes.[14]

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[13]Equivariance holds because of the following continuity equation:

$$\frac{\partial W(q, q, t)}{\partial t} = -\text{div}(W(q, q, t)v),$$

where v denotes the velocity field generated via (3.19). See Dürr et al. (1992, 2005).

[14]Here I am assuming that two theories are empirically equivalent if they assign the same probability distribution to all possible outcomes of experiments. This is the criterion used in the standard Bohmian statistical analysis (Dürr et al. (1992)). Empirical equivalence between BM and W-BM follows from the equivariance property plus the quantum equilibrium distribution. Suppose W-BM is governed by a universal density matrix W and suppose BM is governed by a universal wave function chosen at random whose statistical density matrix is W. Then the initial particle distributions on both theories are the same: $W(q, q, t_0)$. By equivariance, the particle distributions will always be the same. Hence, they always agree on what is typical. See Dürr et al. (2005). This is a general argument. In Chen (2019), I present the general argument followed by a subsystem analysis of W-BM, in terms of conditional density matrices.

3.3.2 W-Everettian and W-GRW Theories

W-BM is a quantum theory in which the density matrix is objective. In this theory, realism about the universal density matrix is based on the central role it plays in the laws of a W-Bohmian universe: it appears in the fundamental dynamical equations and it guides particle motion. (In §3.3, we will provide three concrete physical interpretations of W.) What about other quantum theories, such as Everettian and GRW theories? Is it possible to "replace" their universal wave functions with universal density matrices? We show that such suggestions are also possible.[15] First, let us define *local beables* (à la Bell (2004)). Local beables are the part of the ontology that is localized (to some bounded region) in physical space. Neither the total energy function nor the wave function is a local beable. Candidate local beables include particles, space-time events (flashes), and matter density ($m(x,t)$).

For the Everettian theory with no local beables (S0), we can postulate that the fundamental quantum state is represented by a density matrix $W(t)$ that evolves unitarily by the von Neumann equation (3.14). Let us call this theory W-Everett theory (W-S0). Since there are no additional variables in the theory, the density matrix represents the entire quantum universe. The density matrix will give rise to many branches that (for all practical purposes) do not interfere with each other. The difference is that there will be (in some sense) more branches in the W-Everett quantum state than in the Everett quantum state. In the W-Everett universe, the world history will be described by the undulation of the density matrix.[16]

It is difficult to find tables and chairs in a universe described only by a quantum state. One proposal is to add "local beables" to the theory in the form of a mass-density ontology $m(x,t)$. The wave-function version was introduced as Sm by Allori et al. (2010). The idea is that the wave function evolves by the Schrödinger equation

[15] Thanks to Roderich Tumulka, Sheldon Goldstein, and Matthias Lienert for discussions here. The W-GRW formalism was suggested first in Allori et al. (2013).

[16] W-S0 is a novel version of Everettian theory, one that will require more mathematical analysis to fully justify the emergence of macroscopic branching structure. It faces the familiar preferred-basis problem as standard Everett does. In addition, on W-S0 there will be some non-uniqueness in the decompositions of the Hilbert space into macrospaces. I leave the analysis for future work.

and determines the shape of the mass density. This idea can be used to construct a density-matrix version (W-Sm). In this theory, $W(t)$ will evolve unitarily by the von Neumann equation. Next, we can define the mass-density function directly in terms of $W(t)$:

$$m(x,t) = \mathrm{tr}(M(x)W(t)), \tag{3.21}$$

where x is a physical space variable, $M(x) = \sum_i m_i \delta(Q_i - x)$ is the mass-density operator, which is defined via the position operator $Q_i \psi(q_1, q_2, ...q_n) = q_i \psi(q_1, q_2, ...q_n)$. This allows us to determine the mass-density ontology at time t via $W(t)$.

For the density-matrix version of GRW theory with just a quantum state (W-GRW0), we need to introduce the collapse of a density matrix. Similar to the wave function in GRW0, between collapses, the density matrix in W-GRW0 will evolve unitarily according to the von Neumann equation. It collapses randomly, where the random time for an N-particle system is distributed with rate $N\lambda$, where λ is of order 10^{-15} s^{-1}. At a random time when a collapse occur at "particle" k at time T^-, the post-collapse density matrix at time T^+ is the following:

$$W_{T^+} = \frac{\Lambda_k(X)^{1/2} W_{T^-} \Lambda_k(X)^{1/2}}{\mathrm{tr}(W_{T^-}\Lambda_k(X))}, \tag{3.22}$$

with X distributed by the following probability density:

$$\rho(x) = \mathrm{tr}(W_{T^-}\Lambda_k(x)), \tag{3.23}$$

where W_{T^+} is the post-collapse density matrix, W_{T^-} is the pre-collapse density matrix, X is the center of the actual collapse, and $\Lambda_k(x)$ is the collapse rate operator.[17]

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[17] A collapse rate operator is defined as follows:

$$\Lambda_k(x) = \frac{1}{(2\pi\sigma^2)^{3/2}} e^{-\frac{(Q_k - x)^2}{2\sigma^2}},$$

where Q_k is the position operator of "particle" k, and σ is a new constant of nature of order 10^{-7} m postulated in current GRW theories. Compare W-GRW to Ψ-GRW, where collapses happen at the same rate, and the post-collapse wave function is the following:

$$\Psi_{T^+} = \frac{\Lambda_k(X)^{1/2}\Psi_{T^-}}{\|\Lambda_k(X)^{1/2}\Psi_{T^-}\|}, \tag{3.24}$$

with the collapse center X being chosen randomly with probability distribution $\rho(x) = \|\Lambda_k(x)^{1/2}\Psi_{T^-}\|^2 dx$.

For the GRW theory (W-GRWm) with both a quantum state $W(t)$ and a mass-density ontology $m(x,t)$, we can combine the above steps: $W(t)$ evolves by the von Neumann equation that is randomly interrupted by collapses (3.22) and $m(x,t)$ is defined by (3.21). We can define GRW with a flash-ontology (W-GRWf) in a similar way, by using $W(t)$ to characterize the distribution of flashes in physical space-time. The flashes are the space-time events at the centers (X) of the W-GRW collapses.

To sum up: in W-S0, the entire world history is described by $W(t)$; in W-Sm, the local beables (mass-density) is determined by $W(t)$; in W-GRW theories, $W(t)$ spontaneously collapses. These roles were originally played by Ψ, and now they are played by W. In so far as we have good reasons for Wave Function Realism based on the roles that Ψ plays in the Ψ-theories, we have equally good reasons for Density Matrix Realism if the universe is accurately described by W-theories.

3.3.3 Field Intepretations of W

Realism about the density matrix only implies that it is objective and not epistemic. Realism is compatible with a wide range of concrete interpretations of what the density matrix represents. In this section, I provide three field interpretations of the density matrix. But they do not exhaust all available options. In §6, I motivate a nomological interpretation of the density matrix that is also realist.

In debates about the metaphysics of the wave function, realists have offered several interpretations of Ψ. Wave function realists, such as Albert and Loewer, have offered a concrete physical interpretation: Ψ represents a physical field on the high-dimensional configuration space that is taken to be the fundamental physical space.[18]

Can we interpret the density matrix in a similar way? Let us start with a mathematical representation of the density matrix $W(t)$. It is defined as a positive, bounded, self-adjoint operator $\hat{W}: \mathscr{H} \rightarrow \mathscr{H}$ with $\mathrm{tr}\hat{W} = 1$. For W-BM, the configuration space $\mathbb{R}^{3N}$, and a density operator $\hat{W}$, the relevant Hilbert space is $\mathscr{H}$, which is a subspace

[18] In Chen (2017), I argue against this view and suggest that there are many good reasons—internal and external to quantum mechanics—for taking the low-dimensional physical space-time to be fundamental.

of the total Hilbert space, i.e. $\mathscr{H} \subseteq \mathscr{H}_{total} = L^2(\mathbb{R}^{3N}, \mathbb{C})$. Now, the density matrix $\hat{W}$ can also be represented as a function

$$W : \mathbb{R}^{3N} \times \mathbb{R}^{3N} \to \mathbb{C} \tag{3.25}$$

(If we include spin, the range will be the endomorphism space $\mathrm{End}(\mathbb{C}^k)$ of the space of linear maps from $\mathbb{C}^k$ to itself. Notice that we have already used the position representation in (3.19) and (3.20).)

This representation enables three field interpretations of the density matrix. Let us use W-BM as an example. First, the fundamental space is represented by $\mathbb{R}^{6N}$, and W represents a field on that space that assigns properties (represented by complex numbers) to each point in $\mathbb{R}^{6N}$. In the Bohmian version, W guides the motion of a "world particle" like a river guides the motion of a ping pong ball. (However, the world particle only moves in a $\mathbb{R}^{3N}$ subspace.) Second, the fundamental space is $\mathbb{R}^{3N}$, and W represents a multi-field on that space that assigns properties to every ordered pair of points (q, q') in $\mathbb{R}^{3N}$. The world particle moves according to the gradient taken with respect to the first variable of the multi-field. Third, the fundamental space is the physical space represented by $\mathbb{R}^3$, and the density matrix represents a multi-field that assigns properties to every ordered pair of N-regions, where each N-region is composed of N points in physical space. On this view, the density matrix guides the motion of N particles in physical space.[19]

These three field interpretations are available to the density matrix realists. In so far as we have good grounds for accepting the field interpretations of wave function realism, we have equally good grounds for accepting these interpretations for the W-theories. These physical interpretations, I hope, can provide further reasons for wave function realists to take seriously the idea that density matrices *can* represent something physically significant. In §6, we introduce a new interpretation of W as something nomological, and we will motivate that with the new Initial Projection Hypothesis. That, I believe, is the most interesting realist interpretation of the universal density matrix all things considered.

[19] For discussions about the multi-field interpretation, see Forrest (1988); Belot (2012), Chen (2017), Chen (ms.) section 3, and Hubert and Romano (2018).

3.4 The Initial Projection Hypothesis

W-quantum theories are alternatives to Ψ-quantum theories. However, all of these theories are time-symmetric, as they obey time-reversal invariance $t \to -t$.

In statistical mechanics, a fundamental postulate is added to the time-symmetric dynamics: the Past Hypothesis, which is a low-entropy boundary condition of the universe. In this section, we will first discuss the wave-function version of the Past Hypothesis. Then we will use it to pick out a special density matrix. I call this the *Initial Projection Hypothesis*. Finally, we point out some connections between the Initial Projection Hypothesis and Penrose's Weyl Curvature Hypothesis.

3.4.1 The Past Hypothesis

The history of the Past Hypothesis goes back to Ludwig Boltzmann.[20] To explain time asymmetry in a universe governed by time-symmetric equations, Botlzmann's solution is to add a boundary condition: the universe started in a special state of very low-entropy. Richard Feynman agrees, "For some reason, the universe at one time had a very low entropy for its energy content, and since then the entropy has increased."[21] Such a low-entropy initial condition explains the arrow of time in thermodynamics.[22]

David Albert (2000) has called this condition the *Past Hypothesis* (PH). However, his proposal is stronger than the usual one concerning a low-entropy initial condition. The usual one just postulates that the universe started in some low-entropy macrostate. It can be any of the many macrostates, so long as it has sufficiently low entropy. Albert's PH postulates that there is a *particular* low-entropy macrostate that the universe starts in—the one that underlies the reliability of our inferences to the past. It is the task of cosmology to discover that initial macrostate. In what follows, I refer to the strong

[20]For an extended discussion, see Boltzmann (2012), Albert (2000), and Callender (2011).

[21]Feynman et al. (2015), 46-8.

[22]See Lebowitz (2008); Ehrenfest and Ehrenfest (2002) and Penrose (1979) for more discussions about a low-entropy initial condition. See Earman (2006) for worries about the Past Hypothesis as an initial condition for the universe. See Goldstein et al. (2016) for a discussion about the possibility, and some recent examples, of explaining the arrow of time without the Past Hypothesis.

version of PH unless indicated otherwise.[23]

In QSM, PH takes the form of §2.2 #9. That is, the microstate (a wave function) starts in a particular low-dimensional subspace in Hilbert space (the PH-subspace). However, it does not pin down a unique microstate. There is still a continuous infinity of possible microstates compatible with the PH-subspace.

It is plausible to think that, for PH to work as a successful explanation for the Second Law, it has to be on a par with other fundamental laws of nature. That is, we should take PH to be a law of nature and not just a contingent initial condition, for otherwise it might be highly unlikely that our past was in lower entropy and that our inferences to the past are reliable. Already in the context of a weaker version of PH, Feynman (2017) suggests that the low-entropy initial condition should be understood as a law of nature. However, PH by itself is not enough. Since there are anti-thermodynamic exceptions even for trajectories starting from the PH-subspace, it is crucial to impose another law about a uniform probability distribution on the subspace. This is the quantum analog of what Albert (2000) calls the Statistical Postulate (SP). It corresponds to the measure μ we specified in §2.2 #4. We used it to state the typicality statement in #10. Barry Loewer calls the joint system—the package of laws that includes PH and SP in addition to the dynamical laws of physics—the Mentaculus Vision.[24]

3.4.2 Introducing the Initial Projection Hypothesis

The Past Hypothesis uses a low-entropy macrostate (PH-subspace) to constrain the microstate of the system (a state vector in QSM). This is natural from the perspective of Wave Function Realism, according to which the state vector (the wave function)

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[23] In Chen (2018d), a companion paper, I discuss different versions of the Past Hypothesis—the strong, the weak, and the fuzzy—as well as their implications for the uniqueness of the initial quantum state that we will come to soon. The upshot is that in all cases it will be sufficiently unique for eliminating statistical mechanical probabilities.

[24] For developments and defenses of the nomological account of the Past Hypothesis and the Statistical Postulate, see Albert (2000); Loewer (2007); Wallace (2011, 2012) and Loewer (2016). Albert and Loewer are writing mainly in the context of CSM. The Mentaculus Vision is supposed to provide a "probability map of the world." As such, it requires one to take the probability distribution very seriously.

To be sure, the view that PH is nomological has been challenged. See discussions in Price (1997); Sklar (1995), and Callender (2004). However, those challenges are no more threatening to IPH being a law than PH being a law. We will come back to this point after introducing IPH.

represents the physical degrees of freedom of the system. The initial state of the system is described by a normalized wave function $\Psi(t_0)$. $\Psi(t_0)$ has to lie in the special low-dimensional Hilbert space $\mathscr{H}_{PH}$ with $dim\mathscr{H}_{PH} \ll dim\mathscr{H}_{eq}$. Moreover, there are many different choices of initial wave functions in $\mathscr{H}_{PH}$. That is, PH is compatible with many different low-entropy wave functions. Furthermore, for stating the typicality statements, we also need to specify a measure μ on the unit sphere of $\mathscr{H}_{PH}$. For the finite-dimensional case, it is just the normalized surface area measure on the unit sphere.

Density Matrix Realism suggests an alternative way to think about the low-entropy boundary condition. PH pins down the initial macrostate $\mathscr{H}_{PH}$, a special subspace of the total Hilbert space. Although $\mathscr{H}_{PH}$ is compatible with many density matrices, there is a natural choice—the normalized projection operator onto $\mathscr{H}_{PH}$. Just as in (3.17), we can specify it as:

$$\hat{W}_{IPH}(t_0) = \frac{I_{PH}}{dim\mathscr{H}_{PH}}, \tag{3.26}$$

where t_0 represents a temporal boundary of the universe, I_{PH} is the projection operator onto $\mathscr{H}_{PH}$, dim counts the dimension of the Hilbert space, and $dim\mathscr{H}_{PH} \ll dim\mathscr{H}_{eq}$. Since the quantum state at t_0 has the lowest entropy, we call t_0 the initial time. We shall call (3.26) the *Initial Projection Hypothesis* (IPH). In words: the initial density matrix of the universe is the normalized projection onto the PH-subspace.

I propose that we add IPH to any W-quantum theory. The resultant theories will be called W_{IPH}-theories. For example, here are the equations of W_{IPH}-BM:

(A) $\hat{W}_{IPH}(t_0) = \frac{I_{PH}}{dim\mathscr{H}_{PH}}$,

(B) $P(Q(t_0) \in dq) = W_{IPH}(q, q, t_0)dq$,

(C) $i\hbar\frac{\partial \hat{W}}{\partial t} = [\hat{H}, \hat{W}]$,

(D) $\frac{dQ_i}{dt} = \frac{\hbar}{m_i}\text{Im}\frac{\nabla_{q_i} W_{IPH}(q,q',t)}{W_{IPH}(q,q',t)}(q = q' = Q)$.

(A) is IPH and (B)—(D) are the defining equations of W-BM. (Given the initial quantum state $\hat{W}_{IPH}(t_0)$, there is a live possibility that for every particle at t_0, its velocity is zero. However, even in this possibility, as long as the initial quantum state "spreads out" later, as we assume it would, the particle configuration will typically start moving at a later time. This is true because of equivariance.[25])

Contrast these equations with BM formulated with wave functions and PH (not including SP for now), which will be called Ψ_{PH}-BM:

(A') $\Psi(t_0) \in \mathscr{H}_{PH}$,

(B') $P(Q(t_0) \in dq) = |\Psi(q, t_0)|^2 dq$,

(C') $i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$,

(D') $\frac{dQ_i}{dt} = \frac{\hbar}{m_i} \mathrm{Im} \frac{\nabla_{q_i} \Psi(q,t)}{\Psi(q,t)}(Q)$.

IPH (A) in W_{IPH}-BM plays the same role as PH (A') in Ψ_{PH}-BM. Should IPH be interpreted as a law of nature in W_{IPH}-theories? I think it should be, for the same reason that PH should be interpreted as a law of nature in the corresponding theories. The reason that PH should be interpreted as a law[26] is because it is a particularly simple and informative statement that accounts for the widespread thermodynamic asymmetry in time. PH is simple because it characterizes a simple macrostate $\mathscr{H}_{PH}$, of which the initial wave function is a vector. PH is informative because with PH the dynamical equations predict time asymmetry and without PH the dynamical equations cannot. Similarly, IPH is simple because it provides crucial resources for explaining the arrow of time. IPH is informative because it is essential for explaining the time asymmetry in a quantum universe described by a density matrix. (This is in addition to the fact that IPH helps determine the W_{IPH}-version of the guidance equation (D).) To be sure, PH and IPH as laws face the same worries: both are statements about boundary conditions but we usually think of laws as dynamical equations. However, these worries are no more threatening to IPH being a law than PH being a law.

Let us make three remarks about IPH. Firstly, IPH defines a unique initial quantum state. The quantum state $\hat{W}_{IPH}(t_0)$ is informationally equivalent to the constraint that

[25]Thanks to Sheldon Goldstein and Tim Maudlin for discussions here.

[26]See, for example, Feynman (2017); Albert (2000); Loewer (2007) and Loewer (2016).

PH imposes on the initial microstates. Assuming that PH selects a unique low-entropy macrostate, $\hat{W}_{IPH}(t_0)$ is singled out by the data in PH.[27]

Secondly, on the universal scale, we do not need to impose an additional probability or typicality measure on the Hilbert space. $\hat{W}_{IPH}(t_0)$ is mathematically equivalent to an integral over projection onto each normalized state vectors (wave functions) compatible with PH *with respect to a normalized surface area measure* μ. But here we are not defining $\hat{W}_{IPH}(t_0)$ in terms of state vectors. Rather, we are thinking of $\hat{W}_{IPH}(t_0)$ as a geometric object in the Hilbert space: the (normalized) projection operator onto $\mathscr{H}_{PH}$. That is the *intrinsic* understanding of the density matrix.[28]

Thirdly, $\hat{W}_{IPH}(t_0)$ is simple. Related to the first remark, IPH defines $\hat{W}_{IPH}(t_0)$ explicitly as the normalized projection operator onto $\mathscr{H}_{PH}$. There is a natural correspondence between a subspace and its projection operator. If we specify the subspace, we know what its projection operator is, and vice versa. Since the projection operator onto a subspace carries no more information than that subspace itself, the projection operator is no more complex than $\mathscr{H}_{PH}$. This is different from Ψ_{PH}, which is confined by PH to be a vector inside $\mathscr{H}_{PH}$. A vector carries more information than the subspace it belongs to, as specifying a subspace is not sufficient to determine a vector. For example, to determine a vector in an 18-dimensional subspace of a 36-dimensional vector space, we need 18 coordinates in addition to specifying the subspace. The higher the dimension of the subspace, the more information is needed to specify the vector.

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[27] The weaker versions of PH are vague about the exact initial low-entropy macrostate. It is vague because, even with a choice of macro-variables, there may be many subspaces that can play the role of a low-entropy initial condition. It would be arbitrary, from the viewpoint of wave-function theories, to pick a specific subspace. In contrast, it would not be arbitrary from the viewpoint of W_{IPH}-theories, as the specific subspace defines W_{IPH}, which determines the dynamics.

[28] After writing the paper, I discovered that David Wallace has come to a similar idea in a forthcoming paper. There are some subtle differences. He proposes that we can reinterpret probability distributions in QSM as actual mixed states. Consequently, the problem of statistical mechanical probability is "radically transformed" (if not eliminated) in QSM. Wallace's proposal is compatible with different probability distributions and hence different mixed states of the system. It does not require one to choose a particular quantum state such as (3.26). In contrast, I propose a particular, natural initial quantum state of the universe based on the PH subspace—the normalized projection onto the PH subspace (3.26). As we discuss in §5.1, this also leads to the elimination of statistical mechanical probability, since the initial state is fixed in the theory. Moreover, as we discuss below, the natural state inherits the simplicity of the PH subspace, which has implications for the nature of the quantum state. For a more detailed comparison, see Chen (2018d).

If PH had fixed Ψ_{PH} (the QSM microstate), it would have required much more information and become a much more complex posit. PH as it is determines Ψ_{PH} only up to an equivalence class (the QSM macrostate). As we shall see in §6, the simplicity of $\hat{W}_{IPH}(t_0)$ will be a crucial ingredient for a new version of the nomological interpretation of the quantum state.

3.4.3 Connections to the Weyl Curvature Hypothesis

Let us point out some connections between our Initial Projection Hypothesis (IPH) and the Weyl Curvature Hypothesis (WCH) proposed by Penrose (1979). Thinking about the origin of the Second Law of Thermodynamics in the early universe with high homogeneity and isotropy, and the relationship between space-time geometry and entropy, Penrose proposes a low-entropy hypothesis:

> I propose, then, that there should be complete lack of chaos in the initial *geometry*. We need, in any case, some kind of low-entropy constraint on the initial state. But thermal equilibrium apparently held (at least very closely so) for the *matter* (including radiation) in the early stages. So the 'lowness' of the initial entropy was not a result of some special matter distribution, but, instead, of some very special initial spacetime geometry. The indications of [previous sections], in particular, are that this restriction on the early geometry should be something like: *the Weyl curvature C_{abcd} vanishes at any initial singularity*. (Penrose (1979), p.630, emphasis original)

The Weyl curvature tensor C_{abcd} is the traceless part of the Riemann curvature tensor R_{abcd}. It is not fixed completely by the stress-energy tensor and thus has independent degrees of freedom in Einstein's general theory of relativity. Since the entropy of the matter distribution is quite high, the origin of thermodynamic asymmetry should be due to the low entropy in geometry, which corresponds very roughly to the vanishing of the Weyl curvature tensor.

WCH is an elegant and simple way of encoding the initial low-entropy boundary condition in the classical spacetime geometry. If WCH could be extended to a quantum theory of gravity, presumably it would pick out a simple subspace (or subspaces) of the total Hilbert space that corresponds to $C_{abcd} \to 0$. Applying IPH to such a theory, the initial density matrix will be the normalized projection onto that subspace (subspaces).[29]

3.5 Theoretical Payoffs

W_{PH}-quantum theories, the result of applying IPH to W-theories, have two theoretical payoffs, which we explore in this section. These are by no means decisive arguments in favor of the density-matrix framework, but they display some interesting differences with the wave-function framework.

3.5.1 Harmony between Statistical Mechanics and Quantum Mechanics

In W_{PH}-quantum theories, statistical mechanics is made more harmonious with quantum mechanics. As we pointed out earlier, standard QM and QSM contain the wave function in addition to the density matrix, and they require the addition of both the Past Hypothesis (PH) and the Statistical Postulate (SP) to the dynamical laws. In particular, we have two kinds of probabilities: the quantum-mechanical ones (Born rule probabilities) and the statistical mechanical ones (SP). The situation is quite different in our framework. This is true for all the W_{PH}-theories. We will use W_{PH}-BM ((A)—(D)) as an example.

W_{PH}-BM completely specifies the initial quantum state, unlike Ψ_{PH}-BM. For Ψ_{PH}-BM, because of time-reversal invariance, some initial wave functions compatible with PH will evolve to lower entropy. These are called *anti-entropic* exceptions. However, the uniform probability distribution (SP) assigns low probability to these exceptions. Hence, we expect that with overwhelming probability the actual wave function is entropic. For W_{PH}-BM, in contrast, there is no need for something like SP, as there

[29]There is another connection between the current project and Penrose's work. The W-Everettian theory that we considered in §3.2 combined with the Initial Projection Hypothesis is a theory that satisfies *strong determinism* (Penrose (1989)). This is because the entire history of the W_{PH}-Everettian universe described by $W_{IPH}(t)$, including its initial condition, is fixed by the laws.

is only one initial density matrix compatible with IPH—$W_{IPH}(t_0)$. It is guaranteed to evolve to future states that have entropic behaviors. Therefore, on the universal scale, W_{PH}-BM eliminates the need for SP and thus the need for a probability/typicality measure that is in addition to the quantum-mechanical measure (B). This is a nice feature of W_{PH}-theories, as it is desirable to unify the two sources of randomness: quantum-mechanical and statistical-mechanical. Of course, wave functions and statistical-mechanical probabilities are still useful to analyze subsystems such as gas in a box, but they no longer play fundamental roles in W_{PH}-theories. Another strategy to eliminate SP has been explored in the context of GRW jumps by Albert (2000). Wallace (2011, 2012) has proposed a replacement of SP with a non-probabilistic constraint on the microstate, giving rise to the *Simple Dynamical Conjecture*. These are quite different proposals, all of which deserve further developments.

3.5.2 Descriptions of the Universe and the Subsystems

W_{PH}-quantum theories also bring more unity to the kinematics and the dynamics of the universe and the subsystems.

Let us start with a quantum-mechanical universe U. Suppose it contains many subsystems. Some of them will be interacting heavily with the environment, while others will be effectively isolated from the environment. For a universe that contain some quasi-isolated subsystems (interactions with the environment effectively vanish), the following is a desirable property:

Dynamic Unity The dynamical laws of the universe are the same as the effective laws of most quasi-isolated subsystems.

Dynamic Unity is a property that can come in degrees, rather than an "on-or-off" property. Theory A has more dynamic unity than Theory B, if the fundamental equations in A are valid in more subsystems than those in B. This property is desirable, but not indispensable. It is desirable because law systems that apply both at the universal level and at the subsystem level are unifying and explanatory.

W-BM has more dynamic unity than BM formulated with a universal wave function.

For quantum systems without spin, we can always follow Dürr et al. (1992) to define *conditional wave functions* in BM. For example, if the universe is partitioned into a system S_1 and its environment S_2, then for S_1, we can define its conditional wave function:

$$\psi_{cond}(q_1) = C\Psi(q_1, Q_2), \tag{3.27}$$

where C is a normalization factor and Q_2 is the actual configuration of S_2. $\psi_{cond}(q_1)$ always gives the velocity field for the particles in S_1 according to the guidance equation. However, for quantum systems with spin, this is not always true. Since BM is described by $(\Psi(t), Q(t))$, it does not contain actual values of spin. Since there are no actual spins to plug into the spin indices of the wave function, we cannot always define conditional wave functions in an analogous way. Nevertheless, in those circumstances, we can follow Dürr et al. (2005) to define a *conditional density matrix* for S_1, by plugging in the actual configuration of S_2 and tracing over the spin components in the wave function associated with S_2.[30] The conditional density matrix will guide the particles in S_1 by the W-guidance equation (the spin version with the partial trace operator).

In W-BM, the W-guidance equation is always valid for the universe and the subsystems. In BM, sometimes subsystems do not have conditional wave functions, and thus the wave-function version of the guidance equation is not always valid. In this sense, the W-BM equations are valid in more circumstances than the BM equations. However, this point does not rely on IPH.

What about Everettian and GRW theories? Since GRW and Everettian theories do not have fundamental particles, we cannot obtain conditional wave functions for subsystems as in BM. However, even in the Ψ-versions of GRW and Everett, many subsystems will not have pure-state descriptions by wave functions due to the prevalence

[30]The conditional density matrix for S_1 is defined as:

$$W_{cond}{}^{s_1}_{s_1'}(q_1, q_1') = \frac{1}{N}\sum_{s_2} \Psi^{s_1 s_2}(q_1, Q_2)\Psi^*_{s_1 s_2}(q_1', Q_2), \tag{3.28}$$

with the normalizing factor:

$$N = \int_{Q_1} dq_1 \sum_{s_1 s_2} \Psi^{s_1 s_2}(q_1, Q_2)\Psi^*_{s_1 s_2}(q_1', Q_2). \tag{3.29}$$

of entanglement. Most subsystems can be described only by a mixed-state density matrix, even when the universe as a whole is described by a wave function. In contrast, in W_{PH}-Everett theories and W_{PH}-GRW theories, there is more uniformity across the subsystem level and the universal level: the universe as a whole as well as most subsystems are described by the same kind of object—a (mixed-state) density matrix. Since state descriptions concern the kinematics of a theory, we say that W-Everett and W-GRW theories have more *kinematic unity* than their Ψ-counterparts:

Kinematic Unity The state description of the universe is of the same kind as the state descriptions of most quasi-isolated subsystems.

So far, my main goal has been to show that Density Matrix Realism + IPH is a viable position. They have theoretical payoffs that are interestingly different from those in the original package (Wave Function Realism + PH). In the next section, we look at their relevance to the nature of the quantum state.

3.6 The Nomological Thesis

Combining Density Matrix Realism with IPH gives us W_{PH}-quantum theories that have interesting theoretical payoffs. We have also argued that the initial quantum state in such theories would be simple and unique. In this section, we show that the latter fact lends support to the nomological interpretation of the quantum state:

The Nomological Thesis: The initial quantum state of the world is nomological.

However, "nomological" has several senses and has been used in several ways in the literature. We will start with some clarifications.

3.6.1 The Classical Case

We can clarify the sense of the"nomological" by taking another look at classical mechanics. In classical N-particle Hamiltonian mechanics, it is widely accepted that the Hamiltonian function is nomological, and that the ontology consists in particles with positions and momenta. Their state is given by $X = (\boldsymbol{q_1}(t), ..., \boldsymbol{q_N}(t); \boldsymbol{p_1}(t), ..., \boldsymbol{p_n}(t))$, and the Hamiltonian is $H = H(X)$. Particles move according to the Hamiltonian equations:

$$\frac{d\boldsymbol{q_i}(t)}{dt} = \frac{\partial H}{\partial \boldsymbol{p_i}}, \frac{d\boldsymbol{p_i}(t)}{dt} = -\frac{\partial H}{\partial \boldsymbol{q_i}}. \tag{3.30}$$

Their motion corresponds to a trajectory in phase space. The velocity field on phase space is obtained by taking suitable derivatives of the Hamiltonian function H. The equations have the form:

$$\frac{dX}{dt} = F(X) = F^H(X) \tag{3.31}$$

Here, $F^H(X)$ is $H(q,p)$ with suitable derivative operators. The Hamiltonian equations have a simple form, because H is simple. H can be written explicitly as follows:

$$H = \sum_i^N \frac{p_i^2}{2m_i} + V, \tag{3.32}$$

where V takes on this form when we consider electric and gravitational potentials:

$$V = \frac{1}{4\pi\epsilon_0} \sum_{1 \leq j \leq k \leq N} \frac{e_j e_k}{|q_j - q_k|} + \sum_{1 \leq j \leq k \leq N} \frac{G m_j m_k}{|q_j - q_k|}, \tag{3.33}$$

That is, the RHS of the Hamiltonian equations, after making the Hamiltonian function explicit, are still simple. H is just a convenient shorthand for (3.32) and (3.33). Moreover, H is also fixed by the theory. A classical universe is governed by the dynamical laws plus the fundamental interactions. If H were different in (3.31), then we would have a different physical theory (though it would still belong to the class of theories called classical mechanics). For example, we can add another term in (3.33) to encode another fundamental interaction, which will result in a different theory.

Consequently, it is standard to interpret H as a function in (3.30) that does not represent things or properties of the ontology. Expressed in terms of H, the equations of motion take a particularly simple form. The sense that H is nomological is that (i) it generates motion, (ii) it is simple, (iii) it is fixed by the theory (nomologically necessary), and (iv) it does not represent things in the ontology. In contrast, the position and momentum variables in (3.30) are "ontological" in that they represent things and properties of the ontology, take on complicated values, change according to H, and are not completely fixed by the theory (contingent).

3.6.2 The Quantum Case

It is according to the above sense that Dürr et al. (1996); Goldstein and Teufel (2001), and Goldstein and Zanghì (2013) propose that the universal wave function in BM is nomological (and governs things in the ontology). With the guidance equation, Ψ generates the motion of particles. It is of the same form as above:

$$\frac{dX}{dt} = F(X) = F^{\Psi}(X). \tag{3.34}$$

Why is it simple? Generic wave functions are not simple. However, they observe that, in some formulations of quantum gravity, the universal wave function satisfies the Wheeler-DeWitt equation and is therefore stationary. To be stationary, the wave function does not have time-dependence and probably has many symmetries, in which case it *could* be quite simple. The Bohmian theory then will explicitly stipulate what the universal wave function is. Therefore, in these theories, provided that Ψ is sufficiently simple, we can afford the same interpretation of Ψ as we can for H in classical mechanics: both are nomological in the above sense.

W_{IPH}-BM also supports the nomological interpretation of the quantum state but via a different route. With the W-guidance equation, W_{IPH} generates the motion of particles. It is of the same form as above:

$$\frac{dX}{dt} = F(X) = F^{W_{IPH}}(X). \tag{3.35}$$

Why is it simple? Here we do not need to appeal to specific versions of quantum gravity, which are still to be worked out and may not guarantee the simplicity of Ψ. Instead, we can just appeal to IPH. We have argued in §4.2 that IPH is simple and that $W_{IPH}(t_0)$ is simple. Since the quantum state evolves unitarily by the von Neumann equation, we can obtain the quantum state at any later time as:

$$\hat{W}_{IPH}(t) = e^{-i\hat{H}t/\hbar}\hat{W}_{IPH}(t_0)e^{i\hat{H}t/\hbar} \tag{3.36}$$

Since $W_{IPH}(t)$ is a simple function of the time-evolution operator and the initial density matrix, and since both are simple, $W_{IPH}(t)$ is also simple. So we can think of $W_{IPH}(t)$ just as a convenient shorthand for (3.36). (This is not true for $|\Psi(t)\rangle = \hat{H}|\Psi(t_0)\rangle$, as generic $|\Psi(t_0)\rangle$ is not simple at all.)

The "shorthand" way of thinking about $W_{IPH}(t)$ implies that the equation of particle motion has a time-dependent form $F^{W_{IPH}}(X,t)$. Does time-dependence undercut the nomological interpretation? It does not in this case, as the $F^{W_{IPH}}(X,t)$ is still simple even with time-dependence. It is true that time-independence is often a hallmark of a nomological object, but it is not always the case. In this case, we have simplicity without time-independence. Moreover, unlike the proposal of Dürr et al. (1996); Goldstein and Teufel (2001), and Goldstein and Zanghì (2013), we do not need time-independence to argue for the simplicity of the quantum state.

Since $W_{IPH}(t_0)$ is fixed by IPH, $F^{W_{IPH}}$ is also fixed by the theory. Let us expand (3.35) to make it more explicit:

$$\frac{dQ_i}{dt} = \frac{\hbar}{m_i}\mathrm{Im}\frac{\nabla_{q_i} W_{IPH}(q,q',t)}{W_{IPH}(q,q',t)}(Q) = \frac{\hbar}{m_i}\mathrm{Im}\frac{\nabla_{q_i}\langle q|\, e^{-i\hat{H}t/\hbar}\hat{W}_{IPH}(t_0)e^{i\hat{H}t/\hbar}\,|q'\rangle}{\langle q|\, e^{-i\hat{H}t/\hbar}\hat{W}_{IPH}(t_0)e^{i\hat{H}t/\hbar}\,|q'\rangle}(q=q'=Q) \tag{3.37}$$

The initial quantum state (multiplied by the time-evolution operators) generates motion, has a simple form, and is fixed by the boundary condition (IPH) in W_{IPH}-BM. Therefore, it is nomological. This is of course a modal thesis. The initial quantum state, which is completely specified by IPH, could not have been different.

Let us consider other W_{IPH}-theories with local beables. In W_{IPH}-Sm, the initial quantum state has the same simple form and is fixed by IPH. It does not generate a velocity field, since there are no fundamental particles in the theory. Instead, it determines the configuration of the mass-density field on physical space. This is arguably different from the sense of nomological that H in classical mechanics displays. Nevertheless, the mass-density field and the Bohmian particles play a similar role—they are "local beables" that make up tables and chairs, and they are governed by the quantum state. In W_{IPH}-GRWm and W_{IPH}-GRWf, the initial quantum state has the same simple form and is fixed by IPH. It does not generate a velocity field, and it evolves stochastically. This will determine a probability distribution over configurations of local beables—mass densities or flashes—on physical space. The initial quantum state in these theories can be given an *extended nomological interpretation*, in the sense that condition (i) is extended such that it covers other kinds of ontologies and dynamics: (i') the quantum state determines (deterministically or stochastically) the configuration of local beables.

The W_{IPH}-theories with local beables support the nomological interpretation of the initial quantum state. It can be interpreted in non-Humean ways and Humean ways. On the non-Humean proposal, we can think of the initial quantum state as an additional nomological entity that *explains* the distribution of particles, fields, or flashes. On the Humean proposal, in contrast, we can think of the initial quantum state as something that *summarizes* a separable mosaic. This leads to reconciliation between Humean supervenience and quantum entanglement.

3.6.3 Humean Supervenience

Recall that according to Humean supervenience (HS), the "vast mosaic of local matters of particular fact" is a *supervenience base* for everything else in the world, the *metaphysical ground floor* on which everything else depends. On this view, laws of physics are nothing over and above the "mosaic." They are just the axioms in the simplest and most informative summaries of the local matters of particular fact. A consequence of HS is that the complete physical state of the universe is determined by the properties and spatiotemporal arrangement of the local matters (suitably extended to account for vector-valued magnititudes) of particular facts. It follows that there should not be any state of the universe that fails to be determined by the properties of individual space-time points.[31] Quantum entanglement, if it were in the fundamental ontology, would present an obstacle to HS, because entanglement is not determined by the properties of space-time points. The consideration above suggests a strong *prima facie* conflict between HS and quantum physics. On the basis of quantum non-separability, Tim Maudlin has proposed an influential argument against HS.[32]

W_{IPH}-theories with local beables offer a way out of the conflict between quantum entanglement and Humean supervenience. A Humean can interpret the laws (including the IPH) as the axioms in the best system that summarize a separable mosaic. Take

[31]This is one reading of David Lewis. Tim Maudlin (2007a) calls this thesis "Separability."

[32]See Maudlin (2007a), Chapter 2.

W_{IPH}-BM as an example:

The $\mathbf{W}_{IPH}$-BM mosaic: particle trajectories $Q(t)$ on physical space-time.

The $\mathbf{W}_{IPH}$-BM best system: four equations—the simplest and strongest axioms summarizing the mosaic:

(A) $\hat{W}_{IPH}(t_0) = \frac{I_{PH}}{dim \mathscr{H}_{PH}}$

(B) $P(Q(t_0) \in dq) = W_{IPH}(q, q, t_0)dq,$

(C) $i\hbar \frac{\partial \hat{W}}{\partial t} = [\hat{H}, \hat{W}],$

(D) $\frac{dQ_i}{dt} = \frac{\hbar}{m_i} \mathrm{Im} \frac{\nabla_{q_i} W_{IPH}(q,q',t)}{W_{IPH}(q,q',t)} (q = q' = Q).$

Notice that (A)—(D) are simple and informative statements about $Q(t)$. They are expressed in terms of $\hat{W}_{IPH}(t)$, which via law (C) can be expressed in terms of $\hat{W}_{IPH}(t_0)$. We have argued previously that the initial quantum state can be given a nomological interpretation. The Humean maneuver is that the law statements are to be understood as axioms of the best summaries of the mosaic. The mosaic described above is completely separable, while the best system, completely specifying the quantum state and the dynamical laws, contains all the information about quantum entanglement and superpositions. The entanglement facts are no longer fundamental. As on the original version of Humean supervenience, the best system consisting of (A)—(D) supervenes on the mosaic. Hence, this proposal reconciles Humean supervenience with quantum entanglement. As it turns out, the above version of Quantum Humeanism also achieves more theoretical harmony, dynamical unity, and kinematic unity (§5), which are desirable from the Humean best-system viewpoint. We can perform similar "Humeanization" maneuvers on the density matrix in other quantum theories with local beables—W-GRWm, W-GRWf, and W-Sm (although such procedures might not be as compelling).

This version of Quantum Humeanism based on W_{IPH}-theories is different from the other approaches in the literature: Albert (1996); Loewer (1996); Miller (2014); Esfeld (2014); Bhogal and Perry (2015); Callender (2015) and Esfeld and Deckert (2017). In contrast to the high-dimensional proposal of Albert (1996) and Loewer (1996), our version preserves the fundamentality of physical space.

The difference between our version and those of Miller (2014); Esfeld (2014); Bhogal and Perry (2015); Callender (2015), and Esfeld and Deckert (2017) is more subtle. They are concerned primarily with Ψ-BM. We can summarize their views as follows (although they do not agree on all the details). There are several parts to their proposals. First, the wave function is merely part of the best system. It is more like parameters in the laws such as mass and charge. Second, just like the rest of the best system, the wave function supervenes on the mosaic of particle trajectories. Third, the wave function does not have to be very simple. The Humean theorizer, on this view, just needs to find the simplest and strongest summary of the particle histories, but the resultant system can be complex *simpliciter*. One interpretation of this view is that the best system for Ψ_{PH}-BM is just (A')—(D') in §4.2 (although they do not explicitly consider (A')), such that neither the mosaic nor the best system specifies the exact values of the universal wave function. In contrast, our best system completely specifies the universal quantum state. The key difference between our approaches is that their interpretation of the wave function places much weaker constraints than our nomological interpretation does. It is much easier for something to count as being part of the best system on their approach than on ours. While they do not require the quantum state to be simple, we do. For them, the Bohmian guidance equation is likely very complex after plugging in the actual wave function Ψ_{PH} on the RHS, but Ψ_{PH} can still be part of their best system.[33] For us, it is crucial that the equation remains simple after plugging in $W_{IPH}(t_0)$ for it to be in the best system. Consequently, $W_{IPH}(t_0)$ is nomological in the sense spelled out in §6.1, and we can give it a Humean interpretation similar to that of the Hamiltonian function in CM. Generic Ψ_{PH}, on the other hand, cannot be nomological in our sense. But that is ok for them, as their best-system interpretation does not require the strong nomological condition that we use. Here we do not attempt to provide a detailed comparison; we do that in Chen (2018c).

[33]See Dewar (2017) §5 for some worries about the weaker criterion on the best system.

3.7 Conclusion

I have introduced a new package of views: Density Matrix Realism, the Initial Projection Hypothesis, and the Nomological Thesis. In the first two steps, we introduced a new class of quantum theories—W_{PH}-theories. In the final step, we argue that it is a theory in which the initial quantum state *can* be given a nomological interpretation. Each is interesting in its own right, and they do not need to be taken together. However, they fit together quite well. They provide alternatives to standard versions of realism about quantum mechanics, a new way to get rid of statistical-mechanical probabilities, and a new solution to the conflict between quantum entanglement and Humean Supervenience. To be sure, there are many other features of W_{PH}-theories in general and the nomological interpretation in particular that are worth exploring further.

The most interesting feature of the new package, I think, is that it brings together the foundations of quantum mechanics and quantum statistical mechanics. In W_{PH}-theories, the arrow of time becomes intimately connected to the quantum-mechanical phenomena in nature. It is satisfying to see that nature is so unified.

Appendix A

Proofs of Theorems 1.3.3 and 1.3.4 of Chapter 1

Step 1. We begin by enriching < N-Regions, Phase–Clockwise–Betweenness, Phase–Congruence> with some additional structures.

First, to simplify the notations, let us think of N-Regions as a set of regions, and let us now only consider $\Omega :=$ N-Regions / $=_P$, the set of "equal phase" equivalence classes by quotienting out $=_P$. ($a =_P b$ if they form phase intervals the same way: $\forall c \in S, ac \sim_P bc$.)

Second, we fix an arbitrary $A_0 \in \Omega$ to be the "zero phase equivalence class."

Third, we define a non-inclusive relation C on Ω according to C_P on N-Regions. ($\forall A, B, C \in \Omega$, $C(A, B, C)$ iff A, B, C are pairwise distinct and $\forall a \in A, \forall b \in B, \forall c \in C$, $C(a, b, c)$.)

Fourth, we define an addition function $\circ : \Omega \times \Omega \to \Omega$. $\forall A, B \in \Omega$, $C = A \circ B$ is the unique element in Ω such that $CB \sim AA_0$, which is guaranteed to exist by (K4) and provably unique as elements in Ω form a partition over N-Regions.

Step 2. We show that the enriched structure $< \Omega, \circ, C >$ with identity element A_0 satisfies the axioms for a periodic extensive structure defined in Luce (1971).

Axiom 0. $< \Omega, \circ >$ is an Abelian semigroup.

First, we show that $\circ$ is closed: $\forall A, B \in \Omega$, $A \circ B \in \Omega$.

This follows from (K4).

Second, we show that $\circ$ is associative: $\forall A, B, C \in \Omega$, $A \circ (B \circ C) = (A \circ B) \circ C$.

This follows from (K6).

Third, we show that $\circ$ is commutative: $\forall A, B \in \Omega$, $A \circ B = B \circ A$.

This follows from (K2).

$\forall A, B, C, D \in \Omega$:

Axiom 1. Exactly one of $C(A, B, C)$ or $C(A, C, B)$ holds.

This follows from C1.

Axiom 2. $C(A, B, C)$ implies $C(B, C, A)$.

This follows from C2.

Axiom 3. $C(A, B, C)$ and $C(A, C, D)$ implies $C(A, B, D)$.

This follows from C2.

Axiom 4. $C(A, B, C)$ implies $C(A \circ D, B \circ D, C \circ D)$ and $C(D \circ A, D \circ B, D \circ C)$.

This follows from (K7).

Axiom 5. If $C(A_0, A, B)$, then there exists a positive integer n such that $C(A_0, A, nA)$ and $C(A_0, nB, B)$.

This follows from (K8).

Therefore, the enriched structure $< \Omega, \circ, C >$ with identity element A_0 satisfies the axioms for a periodic extensive structure defined in Luce (1971).

Step 3. We use the homomorphisms in Luce (1971) to find the homomorphisms for ¡ N-Regions, Phase–Clockwise–Betweenness, Phase–Congruence¿.

Since $< \Omega, \circ, C >$ satisfy the axioms for a periodic structure, Corollary in Luce (1971) says that for any real $K > 0$, there is a unique function ϕ from Ω into $[0, K)$ s.t. $\forall A, B, C \in \Omega$:

1. $C(C, B, A) \Leftrightarrow \phi(A) > \phi(B) > \phi(C)$ or $\phi(C) > \phi(A) > \phi(B)$ or $\phi(B) > \phi(C) > \phi(A)$;
2. $\phi(A \circ B) = \phi(A) + \phi(B) \pmod K$;
3. $\phi(A_0) = 0$.

Now, we define f : N-Regions $\to [0, K)$ as follows: $f(a) = \phi(A)$, where $a \in A$. So we have $C_P(c, b, a) \Leftrightarrow f(a) \geq f(b) \geq f(c)$ or $f(c) \geq f(a) \geq f(b)$ or $f(b) \geq f(c) \geq f(a)$.

We can also define ψ : N-Regions $\times$ N-Regions $\to [0, K)$ as follows: $\psi(a, b) = \phi(A) - \phi(B) \pmod K$, where $a \in A$ and $b \in B$. Hence, $\forall a, b \in$ N-Regions, $\psi(a, b) = f(a) - f(b) \pmod K$.

Moreover, given (K5), $\forall a \in A, b \in B, c \in C, d \in D$, $ab \sim_P cd$

$\Leftrightarrow AB \sim CD$

$\Leftrightarrow A \circ D = B \circ C$

$\Leftrightarrow \phi(A \circ D) = \phi(B \circ C)$

$\Leftrightarrow \phi(A) + \phi(D) = \phi(B) + \phi(C) \pmod K$

$\Leftrightarrow \forall a \in A, b \in B, c \in C, d \in D,\ f(a) + f(d) = f(b) + f(c) \pmod K$

$\Leftrightarrow \forall a \in A, b \in B, c \in C, d \in D,\ f(a) - f(b) = f(c) - f(d) \pmod K$

Therefore, we have demonstrated the existence of homomorphisms.

Step 4. We prove the uniqueness theorem.

If another function f' : N-Regions $\to [0, K)$ with the same properties exists, then it satisfies the same homomorphism conditions. Suppose (for *reductio*) f' and f differ by more than a constant mod K.

Let $D(a, b)$ by the function that measures the differences between f and f' on N-regions:

$$D(a, b) = [f(a) - f(b)] - [f'(a) - f'(b)] \text{ mod } K.$$

Without loss of generality, let us suppose that there exist two regions x, y where $D(x, y) \neq 0$. By (K8), there will be a sequence of pairs of regions that are phase-congruent to x, y: $xy \sim_P ya_1 \sim_P a_1a_2 \sim_P \ldots \sim_P a_nx$. Since by assumption both f and f' preserve the structure of phase-congruence, we have (mod K):

$$f(x) - f(y) = f(y) - f(a_1) = \ldots = f(a_n) - f(x),$$

$$f'(x) - f'(y) = f'(y) - f'(a_1) = \ldots = f'(a_n) - f'(x).$$

Consequently:

$$D(x, y) = [f(x) - f(y)] - [f'(x) - f'(y)] = D(y, a_1) = \ldots = D(a_n, x)$$

Hence, since the above D's are not zero, they must be either all positive or all negative. If they are all positive, then the sum of them will be positive:

$$D(x, y) + D(y, a_1) + \ldots + D(a_n, x) > 0$$

However, expanding them in terms of f and f' we have a telescoping sum:

$$[f(x) - f(y)] - [f'(x) - f'(y)] + [f(y) - f(a_1)] - [f'(y) - f'(a_1)]$$

$$+ \ldots + [f(a_n) - f(x)] - [f'(a_n) - f'(x)] = 0.$$

Contradiction. The same argument works for the case when all the D's are negative. Therefore, $D(x, y) = 0$ for all N-regions. Let a_0 be where f assigns zero. Then

$$f'(a) - f'(a_0) \bmod K = f(a) - f(a_0) \bmod K = f(a),$$

which entails that

$$f'(a) = f(a) + \beta \bmod K,$$

with the constant $\beta = f'(a_0)$. QED.

Appendix B

A Topological Explanation of the Symmetrization Postulate in Chapter 2

Here, we provide the technical details of our argument in §2.3.2 by following the mathematical derivations in Dürr et al. (2006) and Dürr et al. (2007) regarding the case of scalar-valued wave functions. We omit the case of vector-valued wave functions as it is mathematically more complicated but conceptually similar to the scalar case.

Our goal is to arrive at all possible Bohmian dynamics for N identical particles moving in $\mathbb{R}^3$, guided by a scalar wave function. We want to show that, given some natural assumptions, there are only two possible dynamics, corresponding to the symmetric wave functions for bosons and the anti-symmetric wave functions for fermions. We will use the covering-space construction and examine the permissible topological factors.[1]

The natural configuration space for N identical particles in $\mathbb{R}^3$ is:

$$^{N}\mathbb{R}^3 := \{S \subset R^3 | \text{ cardinality}(S) = N\}$$

Since it is a multiply-connected topological space, we follow the usual covering-space construction to define the dynamics on its universal covering space and project it down to $^{N}\mathbb{R}^3$. Its universal covering space is, unsurprisingly, the simply-connected $\mathbb{R}^{3N}$. Let $Cov(\mathbb{R}^{3N}, ^{N}\mathbb{R}^3)$ denote the covering group of the base space $^{N}\mathbb{R}^3$ and its universal covering space $\mathbb{R}^{3N}$. Given a map $\gamma : Cov(\mathbb{R}^{3N}, ^{N}\mathbb{R}^3) \to \mathbb{C}$, we define a wave function

[1] This argument, of course, is not intended as a mathematical demonstration, as we appeal to considerations of naturalness and simplicity. But these assumptions are relatively weak, and they are guided by strong intuitions of the practicing mathematical physicists. Even better, in our case of $^{N}\mathbb{R}^3$, we can prove the **Unitarity Theorem**—that the topological factor in the periodicity condition has to be a character of the fundamental group.

on this space: $\psi : \mathbb{R}^{3N} \to \mathbb{C}$ with the following periodicity condition associated with γ:

$$\forall \hat{q} \in \mathbb{R}^{3N}, \forall \sigma \in Cov(\mathbb{R}^{3N}, {}^N\mathbb{R}^3), \psi(\sigma\hat{q}) = \gamma_\sigma \psi(\hat{q}).$$

First, we show that if the wave function does not identically vanish (which is a natural assumption), the topological factor γ is a representation. Given any $\sigma_1, \sigma_1 \in Cov(\mathbb{R}^{3N}, {}^N\mathbb{R}^3)$:

$$\gamma_{\sigma_1\sigma_2}\psi(\hat{q}) = \psi(\sigma_1\sigma_2\hat{q}) = \gamma_{\sigma_1}\psi(\sigma_2\hat{q}) = \gamma_{\sigma_1}\gamma_{\sigma_2}\psi(\hat{q}).$$

Hence, $\gamma_{\sigma_1\sigma_2} = \gamma_{\sigma_1}\gamma_{\sigma_2}$. Therefore, γ is a structure-preserving map and a representation of the covering group.

It is a well-known fact that the covering group is canonically isomorphic to the fundamental group $\pi_1(Q, q)$, where Q is the base space. In our case, the fundamental group $\pi_1({}^N\mathbb{R}^3, q)$ is S_N, the group of permutations of N objects. It has only two characters: (1) the trivial character $\gamma_\sigma = 1$ and (2) the alternating character $\gamma_\sigma = sign(\sigma) = 1$ or -1 depending on whether $\sigma \in S_N$ is an even or an odd permutation. The former corresponds to the symmetric wave functions of bosons and the latter to the anti-symmetric wave functions of fermions. If we can justify the use of the periodicity condition and a ban on any other topological factors in the periodicity condition, we would be able to rule out other types of particles such as anyons. This result would be equivalent to the Symmetrization Postulate.

The periodicity condition is the most natural condition to require if we want a projectable velocity field given by the wave function on the covering space. The periodicity condition implies that $\nabla\psi(\sigma\hat{q}) = \gamma_\sigma\sigma^*\nabla\psi(\hat{q})$, where σ^* is the push-forward action of σ on tangent vectors. Hence, if we define the velocity field on $\mathbb{R}^{3N}$ in the usual way:

$$\hat{v}^\psi(\hat{q}) := \hbar \mathrm{Im} \frac{\nabla\psi}{\psi}(\hat{q}),$$

then it is related to other levels of the covering space:

$$\hat{v}^\psi(\sigma\hat{q}) = \sigma^*\hat{v}^\psi(\hat{q}).$$

This makes $\hat{v}$ projectable to a well-defined velocity field on the base space ${}^N\mathbb{R}^3$:

$$v^\psi(q) = \pi^*\hat{v}^\psi(\hat{q}),$$

where $\hat{q}$ is an arbitrary point in $\mathbb{R}^{3N}$ that projects down to q via the projection map π.

Moreover, the periodicity condition is natural because it is preserved by the Schrödinger evolution. Therefore, given an initial point q_0 in the unordered configuration space ${}^N\mathbb{R}^3$, we can choose an arbitrary point $\hat{q_0}$ in the ordered configuration space $\mathbb{R}^{3N}$ that projects to q_0 via π, evolve $\hat{q_0}$ by the usual guidance equation in $\mathbb{R}^{3N}$ until time t, and get the final point $q_t = \pi(\hat{q_t})$. The final point $q_t \in^N \mathbb{R}^3$ is independent of the choice of the initial $\hat{q_0} \in \mathbb{R}^{3N}$.

We explain why it is crucial for the topological factor to be not only a representation of the fundamental group but also unitary, as it is required to ensure that the probability distribution is equivariant. Given the periodicity condition, we have

$$|\psi(\sigma\hat{q})|^2 = |\gamma_\sigma|^2 |\psi(\hat{q})|^2.$$

To make $|\psi(\hat{q})|^2$ projectable to a function on ${}^N\mathbb{R}^3$, we require that $\forall \sigma \in S_N, |\gamma_\sigma|^2 = 1$. This is equivalent to γ being a character (a unitary representation) for the fundamental group. Given the Schrödinger equation on $\mathbb{R}^{3N}$ and the projection of $|\psi(\hat{q})|^2$ to $|\psi|^2(q)$, we have:

$$\frac{\partial |\psi_t|^2(q)}{\partial t} = -\nabla \cdot (|\psi_t|^2(q) v^{\psi_t}(q)).$$

Compare this with the transport equation for a probability density ρ on ${}^N\mathbb{R}^3$:

$$\frac{\partial \rho_t(q)}{\partial t} = -\nabla \cdot (\rho_t(q) v^{\psi_t}(q)).$$

Therefore, if $\rho_{t_0}(q) = |\psi_{t_0}|^2(q)$ at the initial time t_0, then $\rho_t(q) = |\psi_t|^2(q)$ at all later times. We have arrived at equivariance.

The above argument works for the general case of multiply-connected spaces as well as the particular case of ${}^N\mathbb{R}^3$. In our case of ${}^N\mathbb{R}^3$, we can prove the following simple theorem that the topological factor γ must be a unitary representation, i.e. a group character.

Unitarity Theorem *Let* $\sigma \in Cov(\mathbb{R}^{3N}, {}^N\mathbb{R}^3)$, $\gamma : Cov(\mathbb{R}^{3N}, {}^N\mathbb{R}^3) \to \mathbb{C}$. *If* $\gamma_\sigma \neq 0$ *and* $\gamma_{\sigma_1\sigma_2} = \gamma_{\sigma_1}\gamma_{\sigma_2}$, $\forall \sigma_1, \sigma_2 \in Cov(\mathbb{R}^{3N}, {}^N\mathbb{R}^3)$, *then* $|\gamma_\sigma| = 1$.

Proof The fundamental group of ${}^N\mathbb{R}^3$ is the permutation group S_N, which has $N!$ elements. We obtain that $\gamma_{Id} = 1$, because $\gamma_\sigma = \gamma_{(\sigma * Id)} = \gamma_\sigma \gamma_{Id}$ and $\gamma_\sigma \neq 0$. It is a general fact that in a finite group with k elements, every element σ satisfies $\sigma^k = Id$. Therefore, rewriting $\gamma_\sigma = Re^{i\theta}$, we have $1e^{0i} = 1 = \gamma_{Id} = \gamma_{\sigma^k} = (\gamma_\sigma)^k = R^k e^{ik\theta}$. So we have: $|\gamma_\sigma| = 1$, which makes γ_σ a unitary representation of the covering group, which is a character of the fundamental group. □

Therefore, the periodicity condition associated with the topological factor:

$$\forall \hat{q} \in \mathbb{R}^{3N}, \forall \sigma \in S_N, \psi(\sigma\hat{q}) = \gamma_\sigma \psi(\hat{q})$$

is a highly natural and simple condition that guarantees well-defined dynamics on ${}^N\mathbb{R}^3$, and the topological factors are the characters of the fundamental group.[2] In the case of ${}^N\mathbb{R}^3$, the fundamental group is the permutation group of N objects: S_N. Recall that it has only two characters: (1) the trivial character $\gamma_\sigma = 1$ and (2) the alternating character $\gamma_\sigma = sign(\sigma) = 1$ or -1 depending on whether $\sigma \in S_N$ is an even or an odd permutation. This leads to two possible dynamics corresponding to the symmetric and the anti-symmetric wave functions (and no more):

$$\text{(Bosons)} \qquad \psi_B(\mathbf{x}_{\sigma(1)}, ..., \mathbf{x}_{\sigma(N)}) = \psi_B(\mathbf{x}_1, ..., \mathbf{x}_N),$$

$$\text{(Fermions)} \qquad \psi_F(\mathbf{x}_{\sigma(1)}, ..., \mathbf{x}_{\sigma(N)}) = (-1)^\sigma \psi_F(\mathbf{x}_1, ..., \mathbf{x}_N),$$

where σ is a permutation of $\{1, 2, ..., N\}$ in the permutation group S_N, $(-1)^\sigma$ denotes the sign of σ, $\mathbf{x}_i \in \mathbb{R}^3$ for $i = 1, 2, ..., N$. Therefore, we have arrived at the statement of the Symmetrization Postulate for the 3-dimensional physical space. Interestingly, the same argument would predict that there are more possibilities than fermions and bosons if we go to a smaller space. For N identical particles in a 2-dimensional space, there is the additional possibility of fractional statistics, corresponding to anyons.[3]

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[2]In our particular case of ${}^N\mathbb{R}^3$, we have the **Unitarity Theorem** to explain why the topological factors have to be characters of the fundamental group. In the general case, even without such a theorem there are still many good reasons why we should restrict the topological factors to the group characters. See Dürr et al. (2007) §9 "The Character Quantization Principle."

[3]Perhaps this is yet another reason why the 3-dimensionality of the physical space is a distinguished feature of reality.

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