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The Normalization Theorem for the First-Order Classical Natural Deduction with Disjunctive Syllogism *

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【Abstract】 In the present paper, we prove the normalization theorem and the consistency of the first-order classical logic with disjunctive syllogism. First, we propose the natural deduction system S_{CD} for classical propositional logic having rules for conjunction, implication, negation, and disjunction. The rules for disjunctive syllogism are regarded as the rules for disjunction.

After we prove the normalization theorem and the consistency of S_{CD}, we extend S_{CD} to the system S_{PCD} for the first-order classical logic with disjunctive syllogism. It can be shown that S_{PCD} is conservative extension to S_{CD}. Then, the normalization theorem and the consistency of S_{PCD} are given.

【Key Words】 Classical logic, Consistency, Disjunctive syllogism, Inversion corollary, Normalization theorem

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1 Introduction

There are two most important results of Gerhard Gentzen; his Hauptsatz and the consistency proof for arithmetic. The Hauptsatz is often called the cut-elimination theorem or the normalization theorem. It states that every derivation can be transformed into a derivation that contains no unnecessary detour.[1] One of the main consequences of the normalization theorem is the consistency of a system. Although Gentzen (1935) first introduced natural deduction system, he thought that natural deduction system was not suitable for proving the normalization theorem in the case of classical logic. His classical natural deduction system consists of rules for intuitionistic logic with the law of excluded middle. Gentzen (1935, 1936, 1938) therefore invented another logical calculus, called sequent calculus, and proved the normalization theorem and the consistency of arithmetic.

Unlike Gentzen, Prawitz (1965, 1971) formalized a natural deduction system for classical logic as the system employing rules for minimal logic with the rule for *classical reductio ad absurdum*. In this case, the rule for *ex contradictione quodlibet* is regarded as a special instance of the rule for *classical reductio*. He gave the normalization theorem for the classical natural deduction system. His classical system has no rules for disjunction and existential quantification. So to speak, in his system, disjunction and existential quantification are not primitive logical operators.

Gunnar Stålmarck (1991) and Yuuki Andou (1995) proved the

[1]The stated normalization theorem is one of the so called the *weak normalization*. The normalization theorem is divided into two directions, such as the strong and the weak normalization theorem. The *strong normalization* theorem says that every derivation can be transformed into the unique derivation that contains no unnecessary detour regardless of the order in which transforming methods are applied. In the present paper, we only consider the weak normalization theorem

normalization theorem for the first-order classical natural deduction system with full logical operators. To prove the normalization theorem, the system should have proper reduction procedures that eliminate unnecessary detours in the target derivation. Their reduction procedures for a derivation using the rule for *classical reductio* are different from the reductions suggested by Prawitz (1965). For instance, Prawitz's reduction transforms the conclusion of the rule for *classical reductio* into a subformula of the conclusion which can be reduced to an atomic formula.[2] On the other hand, Stålmark and Andou's procedures transform the conclusion of the rule into a consequence of the conclusion which may not always be atomic. The difference between their reduction methods obscures the point on whether the normalization theorem straightforwardly implies the consistency of the system.

Although Stålmark and Andou proved the normalization theorem for the first-order classical system with *full* logical operators, their proofs are complicated in comparison with Prawitz's. Moreover, there seems to be a lot more thing to prove the consistency of the system from their normalization theorem. For instance, it should be proved that there is no derivation of an absurdity whose last step is the rule for *classical reductio*. Since an absurdity operator is regarded as an atomic formula, it is not enough to show that every conclusion of the rule for *classical reductio* is convertible into an atomic formula.

In this paper, we propose an alternative proof of the normalization theorem which straightforwardly implies the consistency of the

[2]Roughly put, a subformula of the conclusion is the formula consisting of the conclusion. An atomic formula is the formula having no logical operators. A precise definition of them will be introduced in the next section.

system. We prove the normalization theorem for the first-order classical natural deduction system with disjunctive syllogism. In stead of using the standard or-rules, we will introduce rules for *disjunctive syllogism* as the rules for disjunction and prove the weak normalization theorem for the first-order classical natural deduction with disjunctive syllogism. Our result has direct consequences, such as the consistency of the system.

Section 2 introduces a natural deduction system S_{CD} for classical propositional logic with disjunctive syllogism. Unlike Prawitz's system for weak classical logic which only has conjunction, implication, and negation, our system has whole rules for logical operators including disjunction. Moreover, Section 3 shows the normalization theorem for S_{CD} and the consistency of it as the direct consequence of the normalization. In Section 4, we extend our results to the first-order classical logic with disjunctive syllogism.

2 Natural Deduction S_{CD} for Classical Propositional Logic with Disjunctive Syllogism

In this section, we introduce a natural deduction system S_{CD} for classical propositional logic with disjunctive syllogism. In Section 2.1, we borrow some terminologies, standard natural deduction rules for classical propositional logic, and reduction procedures from Prawitz (1965, 1971). Section 2.2 briefly investigates the problem of reduction procedures for derivations employing rules for *classical reductio* with disjunction. Then, we propose the natural deduction system S_{CD} for classical propositional logic with disjunctive syllogism.

2.1 Preliminaries: Rules for Weak Classical Propositional Logic

To begin with, we will introduce some terminologies and natural deduction rules. Our language has constants and quantifiers, $\wedge$, $\vee$, $\rightarrow$, $\perp$, $\neg$, $\forall$, and $\exists$ for conjunction, disjunction, implication, absurdity, negation, universal quantification, and existential quantification respectively. Let x, y be free variables and s, t be closed terms. We use φ, ψ, and σ for arbitrary formula. For any formula φ, φ is an *atomic* formula if φ is $\perp$ or a formula having no logical operators; otherwise, it is a *complex* formula. Each formula consists of its subformulas. The notion of *subformula* is defined inductively by (1) φ is a subformula of φ, (2) if $\psi \circ \sigma$ is a subformula of φ then so are ψ, σ where $\circ$ is $\vee$ or $\wedge$ or $\rightarrow$, (3) if $\circ\psi$ is a subformula of φ, then so is $\psi[x/t]$ where $\circ$ is $\forall$ or $\exists$ or $\neg$.[3] Let $\mathfrak{D}$ be a derivation of a given natural deduction system, used in the same manner as 'deduction' in Prawitz (1965). Following Prawitz, we shall use the following conventions: if a derivation $\mathfrak{D}$ ends with a formula φ, we write as shown on the left below and φ is called an 'end-formula.' If it depends on a formula ψ, we write as shown on the right below.

$$\begin{array}{ccc} & & \psi \\ \mathfrak{D} & & \mathfrak{D} \\ \varphi & & \varphi\,. \end{array}$$

Then, we have the natural deduction system S_C for weak classical propositional logic which has rules only for $\wedge$, $\rightarrow$, and *classical reductio* styled by Prawitz (1965). In S_C, $\varphi \vee \psi$ can be taken to be the formula $\neg\varphi \rightarrow \psi$ and $\vee$ is not a primitive logical operator.

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[3] $\psi[x/t]$ means the substitution of x for t in ψ. Moreover, $\circ$ can be $\forall$ and $\exists$. $\forall$ and $\exists$ will be introduced in Section 4.

$$\cfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ \varphi_1 & \varphi_2\end{matrix}}{\varphi_1 \wedge \varphi_2}\wedge I \quad \cfrac{\varphi_1 \wedge \varphi_2}{\varphi_i}\wedge E_{(i=1,2)} \quad \cfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{matrix}}{\varphi \to \psi}\to I_{,1} \quad \cfrac{\varphi \to \psi \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix}}{\psi}\to E \quad \cfrac{\begin{matrix}[\neg\varphi]^1\\ \mathfrak{D}\\ \bot\end{matrix}}{\varphi}\bot_{C,1}$$

The negation of φ, i.e. $\neg\varphi$, is defined by $\varphi \to \bot$. We call the formulas directly above the line in each rule, 'premise,' and the formula directly below the line, 'conclusion.' *Assumptions* which can be discharged are in the square brackets, e.g. $[\varphi]$. The *open assumptions* of a derivation are the assumptions on which the end-formula depends. A derivation is called *closed* if it contains no open assumptions, otherwise it is called *open*. A *major premise* of the elimination rule for a constant is the premise containing the constant in the elimination rule and all other premises are *minor premises*. The *maximum formula* is the conclusion of an introduction rule or of $\bot_C-$rule at the same time the major premise of an elimination rule. A *cut* in a derivation is a sequence of two rules, such that the first one is an introduction rule (or $\bot_C-$rule) ending with the maximum formula of the second one, which is an elimination rule. Prawitz (1965, pp. 36-38) introduces reduction procedures to remove the cut. We now have Prawitz's reduction procedures for $\bot_C-$, $\wedge-$, and $\to -$rules.

Let us consider any two derivations $\mathfrak{D}_1$ and $\mathfrak{D}_2$ having the same end-formula. We say that a derivation $\mathfrak{D}'$ is an *immediate subderivation* of $\mathfrak{D}_1$ if $\mathfrak{D}'$ is an initial part of $\mathfrak{D}_1$ ending with a premise of the last inference step in $\mathfrak{D}_1$. Let $\mathfrak{D}_1 \rhd \mathfrak{D}_2$ mean that $\mathfrak{D}_1$ *reduces* to $\mathfrak{D}_2$ by applying a single reduction step to an immediate subderivation $\mathfrak{D}'$ of $\mathfrak{D}_1$. For our convenience's sake, we will introduce standard reduction procedures $\rhd_\wedge$ and $\rhd_\rightarrow$ for $\wedge$ and $\rightarrow$ respectively.

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi_1\end{matrix}\quad\begin{matrix}\mathfrak{D}_2\\ \varphi_2\end{matrix}}{\varphi_1\wedge\varphi_2}\wedge I}{\varphi_i}\wedge E_{(i=1,2)} \quad \rhd_\wedge \quad \begin{matrix}\mathfrak{D}_i\\ \varphi_i\end{matrix} \qquad\qquad \dfrac{\dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{matrix}}{\varphi\rightarrow\psi}\rightarrow I_{,1}\quad\begin{matrix}\mathfrak{D}_2\\ \varphi\end{matrix}}{\psi}\rightarrow E \quad \rhd_\rightarrow \quad \begin{matrix}\mathfrak{D}_2\\ \varphi\\ \mathfrak{D}_1\\ \psi\end{matrix}$$

Moreover, Prawitz (1965, p. 40) introduces auxiliary reduction procedures for derivations using $\bot_C$−rule with conjunction and implication in its conclusion. When the maximum formula $\varphi\wedge\psi$ (or $\varphi\rightarrow\psi$) is derived by $\bot_C$−rule from the derivation $\mathfrak{D}$ of $\bot$ from the assumption $[\neg(\varphi\wedge\psi)]$ (or $[\neg(\varphi\rightarrow\psi)]$), it is reduced by the following reduction procedure $\unrhd_{\bot_{C(\wedge)}}$ (or $\unrhd_{\bot_{C(\rightarrow)}}$) as below.

$$\dfrac{\dfrac{\begin{matrix}\dfrac{\dfrac{[\neg\varphi]^2\quad\dfrac{[\varphi\wedge\psi]^1}{\varphi}\wedge E}{\bot}\rightarrow E}{\neg(\varphi\wedge\psi)}\rightarrow I_{,1}\\ \mathfrak{D}\\ \bot\end{matrix}}{\varphi}\bot_{C,2}\quad\dfrac{\begin{matrix}\dfrac{\dfrac{[\neg\psi]^3\quad\dfrac{[\varphi\wedge\psi]^4}{\psi}\wedge E}{\bot}\rightarrow E}{\neg(\varphi\wedge\psi)}\rightarrow I_{,4}\\ \mathfrak{D}\\ \bot\end{matrix}}{\psi}\bot_{C,3}}{\varphi\wedge\psi}\wedge I \qquad \dfrac{\dfrac{\begin{matrix}\dfrac{\dfrac{[\neg\psi]^3\quad\dfrac{[\varphi\rightarrow\psi]^1\quad[\varphi]^2}{\psi}\rightarrow E}{\bot}\rightarrow E}{\neg(\varphi\rightarrow\psi)}\rightarrow I_{,1}\\ \mathfrak{D}\\ \bot\end{matrix}}{\psi}\bot_{C,3}}{\varphi\rightarrow\psi}\rightarrow I_{,2}$$

The main role of these standard reduction procedures is to eliminate cuts. When the derivation has no cut (or maximum formula), we say that it is in *normal form*. Let $\mathbb{R}$ be a set of reduction procedures. A reduction procedure $\begin{matrix}\varphi_1,\dots,\varphi_n\\ \mathfrak{D}\\ \psi\end{matrix} \ \rhd\ \begin{matrix}\varphi_1,\dots,\varphi_n\\ \mathfrak{D}'\\ \psi\end{matrix}$ in $\mathbb{R}$ is *closure under substitution* iff, for any derivation $\begin{matrix}\mathfrak{D}_1 & & \mathfrak{D}_n\\ \varphi_1 & ,\dots, & \varphi_n\end{matrix}$, a reduction procedure

$$\begin{array}{ccc} \mathfrak{D}_1 \quad \mathfrak{D}_n & & \mathfrak{D}_1 \quad \mathfrak{D}_n \\ \varphi_1,\dots,\varphi_n & & \varphi_1,\dots,\varphi_n \\ \mathfrak{D} & & \mathfrak{D}' \\ \psi & \rhd & \psi \end{array}$$

is in $\mathbb{R}$ as well. Every reduction procedure in $\mathbb{R}$ is to be closed under substitution of derivations for open assumptions, and 'normal derivation' and its related notions are defined in the following ways[4]

Definition 2.1. A sequence $< \mathfrak{D}_1, \dots, \mathfrak{D}_i, \mathfrak{D}_{i+1}, \dots >$ of derivations is a *reduction sequence* relative to $\mathbb{R}$ iff $\mathfrak{D}_i \rhd \mathfrak{D}_{i+1}$ relative to $\mathbb{R}$ where $1 \leqslant i$ for any natural number i. A derivation $\mathfrak{D}_1$ is *reducible* to $\mathfrak{D}_i$ ($\mathfrak{D}_1 \succ \mathfrak{D}_i$) relative to $\mathbb{R}$ iff there is a sequence $< \mathfrak{D}_1, \mathfrak{D}_2, \dots, \mathfrak{D}_i >$ relative to $\mathbb{R}$ where for each $j < i$, $\mathfrak{D}_j \rhd \mathfrak{D}_{j+1}$; $\mathfrak{D}_1$ is *irreducible* relative to $\mathbb{R}$ iff there is no derivation $\mathfrak{D}'$ to which $\mathfrak{D}_1 \rhd \mathfrak{D}'$ relative to $\mathbb{R}$ except $\mathfrak{D}_1$ itself. A derivation $\mathfrak{D}$ is *normal* (or in *normal form*) relative to $\mathbb{R}$ iff $\mathfrak{D}$ is irreducible relative to $\mathbb{R}$, i.e. $\mathfrak{D}$ has no maximum formula.

Let $\mathbb{R}_C$ be a set of reduction procedures having $\rhd_{\rightarrow}$, $\rhd_{\wedge}$, $\trianglerighteq_{\bot_{C(\rightarrow)}}$, and $\trianglerighteq_{\bot_{C(\wedge)}}$. Prawitz (1965, Ch. 3) first showed that every conclusion of $\bot_C$−rule can be transformed into an atomic formula, and then he proved the weak normalization theorem that, for every derivation $\mathfrak{D}$ in S_C, there is a normal derivation $\mathfrak{D}'$ in S_C such that $\mathfrak{D} \succ \mathfrak{D}'$ relative to $\mathbb{R}_C$.[5] His weak normalization theorem has direct consequences, such as the inversion corollary and the consistency of S_C. The inversion corollary is the result that every closed derivation in

[4] In Definition 2.1, for any term x and y, let $x \leqslant y$ mean that x is less than or equal to y. For our convenience's sake, we drop the 'relative to $\mathbb{R}$ in the suggested notions if there is no misunderstanding.

[5] More precisely, Prawitz (1965, Ch.3) proved the weak normalization theorem for S_C with $\forall$−rules. However, in this section we only consider a system S_C for weak classical propositional logic.

S_C can be reduced to one using an introduction rule in the last step, as a closed normal derivation is of exactly that form. No introduction rule derives $\perp$. The consistency of S_C is readily proved by the inversion corollary. These results were not satisfactory because they were proved in S_C for weak classical logic. Although Stålmarck (1991) and Andou (1995) proved the normalization theorem for full first ordwer classical logic, since their reductions transform the maximum formula ending with $\perp_C-$rule into a consequence of the maximum formula, it is unclear whether their results directly imply the inversion corollary and the consistency of their systems. In other words, their reductions transform a maximum formula derived by $\perp_C-$rule into a conclusion of the elimination rule of the maximum formula. The conclusion of the elimination rule is not always a subformula of the maximum formula. When it is assured that the conclusion is a subformula of the maximum formula, it is easily proved that every conclusion of $\perp_C-$rule can be reduced to an atomic formula. Then, it should be established that $\perp_C-$rule does not have $\perp$ as its conclusion in the reduced derivation. They did not prove the inversion corollary and the consistency of the full first-order classical logic, which requires further work.

Instead of proving the inversion corollary and the consistency from Stålmarck (1991) and Andou (1995), we introduce an alternative system for classical propositional system with disjunctive syllogism. Then, unlike Prawitz's weak classical logic, the system has a logical operator for disjunction. We can not only prove the normalization theorem for the system but also its direct consequences, i.e. the inversion and the consistency of the system.

2.2 Rules for Classical Propositional Logic with Disjunctive Syllogism

Prawitz's proof of the normalization theorem for weak classical propositional logic S_C uses auxiliary reduction procedures $\trianglerighteq_{\perp_{C(\wedge)}}$ and $\trianglerighteq_{\perp_{C(\to)}}$ which apply derivations having a maximum formula derived by $\perp_C$-rule. When the conclusion of $\perp_C-$rule is a maximum formula of the form $\varphi \wedge \psi$ (or $\varphi \to \psi$), the reduction procedure $\trianglerighteq_{\perp_{C(\wedge)}}$ (or $\trianglerighteq_{\perp_{C(\to)}}$) makes it into a derivation of a subformula of $\varphi \wedge \psi$ (or $\varphi \to \psi$). His weak classical logic neither employed the rules for disjunction (and existential quantification) nor regarded them as primitive logical operators. He might have difficulty reducing the disjunctive formula to its subformula. The following is the reduction procedure for $\vee$.

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \varphi_i\end{matrix}}{\varphi_1 \vee \varphi_2}\vee I_{i=1,2} \quad \begin{matrix}[\varphi_1]^1\\ \mathfrak{D}_2\\ \psi\end{matrix} \quad \begin{matrix}[\varphi_2]^2\\ \mathfrak{D}_3\\ \psi\end{matrix}}{\psi}\vee E_{,1,2} \qquad \rhd_{\vee} \qquad \begin{matrix}\mathfrak{D}_1\\ \varphi_i\\ \mathfrak{D}_{i+1}\\ \psi\end{matrix}$$

Unlike the reduction procedures for $\wedge$ and $\to$, the conclusion of the derivation reduced by $\rhd_{\vee}$ is not always a subformula of $\varphi_1 \vee \varphi_2$. Conclusions of $\wedge E-$ and $\to E-$rules are subformulas of their major premises whereas the conclusion of $\vee E-$rule is the consequence of its major premise which is not always a subformula of its major premise. When $\varphi_1 \vee \varphi_2$ is a maximum formula derived by $\perp_C-$rule, these characteristics of $\vee E-$rule make it difficult to present a reduction procedure in which the conclusion of $\perp_C-$rule is a subformula of $\varphi_1 \vee \varphi_2$.

Stålmarck (1991) and Andou (1995) proposed an alternative reduction procedure applying to derivations of a disjunctive formula derived by $\bot_C-$rule.

$$\dfrac{\dfrac{\begin{array}{c}[\neg(\varphi_1\vee\varphi_2)]^1\\ \mathfrak{D}_1\\ \bot\end{array}}{\varphi_1\vee\varphi_2}\,\bot_{C,1} \quad \begin{array}{c}[\varphi_1]^1\\ \mathfrak{D}_2\\ \psi\end{array} \quad \begin{array}{c}[\varphi_2]^2\\ \mathfrak{D}_3\\ \psi\end{array}}{\psi}\,\vee E_{,1,2} \qquad \rhd_\vee \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{[\neg\psi]^4 \quad \dfrac{[\varphi_1\vee\varphi_2]^1 \quad \begin{array}{c}[\varphi_1]^2\\ \mathfrak{D}_2\\ \psi\end{array} \quad \begin{array}{c}[\varphi_2]^3\\ \mathfrak{D}_3\\ \psi\end{array}}{\psi}\,\vee E_{,2,3}}{\bot}\,\to E}{\neg(\varphi_1\vee\varphi_2)}\,\to I_{,1}\\ \mathfrak{D}_1\\ \bot\end{array}}{\psi}\,\bot_{C,4}$$

Their reduction procedure transforms the derivation of the maximum formula $\varphi_1\vee\varphi_2$ into the derivation of the consequence ψ of $\varphi_1\vee\varphi_2$. If Prawitz wanted to have a process reducing the maximum formula to its subformula, Stålmark and Andou's proposal is unsatisfactory. There is no guarantee that ψ is a subformula of $\varphi_1\vee\varphi_2$.

Following Gentzen (1935, 1936, 1938), Prawitz (1965, 1971, 2015) proved the normalization theorem for weak classical logic and its extension to first-order arithmetic. The main purpose of the normalization theorem is to prove the consistency of the intended systems. In order to have the consistency proof, the normalization theorem must imply the inversion corollary that, for every closed derivation $\mathfrak{D}$ in a system S, there is a derivation $\mathfrak{D}'$ in S such that the last step of $\mathfrak{D}$ is an introduction or $\bot_C-$rule having an atomic formula as its conclusion. Plus, it should be established that there is no closed normal derivation $\mathfrak{D}$ of $\bot$ derived by $\bot_C-$rule in the system. The conclusion of $\to E-$ and $\wedge E-$rules are subformulas of their maximum formula, and there is no I-rule deriving $\bot$. It is natural to think that there is no closed normal derivation of $\bot$ in S.

On the contrary, Stålmark and Andou's reduction transforms the maximum formula derived by $\bot_C-$rule into the consequence of the maximum formula. The problem is that the atomic formula $\bot$ can be a conclusion of $\vee E-$rule. Although, in their systems, every conclusion of $\bot_C-$rule can be converted into an atomic formula, it should be further proved that such conclusion is not $\bot$. Prawitz's style reduction procedures for $\bot_C-$rule do not raise the problem because they concern elimination rules whose conclusions are subformulas of their major premises. If there are rules for disjunction whose elimination rules derive only subformulas of their major premise, Prawitz's style reductions for $\bot_C-$rule can be applied, and it will be directly proved the inversion corollary and the consistency from the normalization theorem.

For the alternative proof of the normalization theorem for classical propositional logic regarding disjunction as a primitive logical operator, we introduce rules for disjunctive syllogism as the rules for disjunction, $\veebar$, and the reduction procedures for $\veebar$.

$$\dfrac{\begin{matrix}[\neg\varphi_1]^1 & [\neg\varphi_2]^2\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ \varphi_2 & \varphi_1\end{matrix}}{\varphi_1 \veebar \varphi_2}\veebar I_{,1,2} \qquad \dfrac{\begin{matrix} & \mathfrak{D}_3\\ \varphi_1 \veebar \varphi_2 & \neg\varphi_1\end{matrix}}{\varphi_2}\veebar E_1 \qquad \dfrac{\begin{matrix} & \mathfrak{D}_4\\ \varphi_1 \veebar \varphi_2 & \neg\varphi_2\end{matrix}}{\varphi_1}\veebar E_2$$

$$\dfrac{\dfrac{\begin{matrix}[\neg\varphi_1]^1 & [\neg\varphi_2]^2\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ \varphi_2 & \varphi_1\end{matrix}}{\varphi_1 \veebar \varphi_2}\veebar I_{,1,2} \quad \begin{matrix}\mathfrak{D}_3\\ \neg\varphi_1\end{matrix}}{\varphi_2}\veebar E_1 \quad \rhd_{\veebar_1} \quad \begin{matrix}\mathfrak{D}_3\\ \neg\varphi_1\\ \mathfrak{D}_1\\ \varphi_2\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}[\neg\varphi_1]^1 & [\neg\varphi_2]^2\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ \varphi_2 & \varphi_1\end{matrix}}{\varphi_1 \veebar \varphi_2}\veebar I_{,1,2} \quad \begin{matrix}\mathfrak{D}_4\\ \neg\varphi_2\end{matrix}}{\varphi_1}\veebar E_2 \quad \rhd_{\veebar_2} \quad \begin{matrix}\mathfrak{D}_4\\ \neg\varphi_2\\ \mathfrak{D}_2\\ \varphi_1\end{matrix}$$

$\veebar E-$rules have the form of disjunctive syllogism. Disjunctive syllogism is one of the main inference rules for disjunction. We may accept $\veebar-$rules as the alternative rules for disjunction. Then, $\veebar$ is regarded as a principal operator for disjunction. As the conclusion of

$\veebar E-$rule is a subformula of its major premise, we have a Prawitz's style auxiliary reduction procedure $\trianglerighteq_{\bot_{C(\veebar)}}$ for $\bot_C-$rule whose conclusion is $\varphi_1 \veebar \varphi_2$.

$$\dfrac{\begin{matrix}[\neg(\varphi_1 \veebar \varphi_2)]^1\\ \mathfrak{D}\\ \bot\end{matrix}}{\varphi_1 \veebar \varphi_2}\,\bot_{C,1} \quad \trianglerighteq_{\bot_{C(\veebar)}} \quad \dfrac{\dfrac{\begin{matrix}\dfrac{\dfrac{[\neg\varphi_2]^3 \quad \dfrac{[\varphi_1 \veebar \varphi_2]^1 \quad [\neg\varphi_1]^2}{\varphi_2}\,\veebar E_1}{\bot}\,\to E}{\neg(\varphi_1 \veebar \varphi_2)}\,\to I_{,1}\\ \mathfrak{D}\\ \bot\end{matrix}}{\varphi_2}\,\bot_{C,3} \qquad \dfrac{\begin{matrix}\dfrac{\dfrac{[\neg\varphi_1]^5 \quad \dfrac{[\varphi_1 \veebar \varphi_2]^4 \quad [\neg\varphi_2]^6}{\varphi_1}\,\veebar E_2}{\bot}\,\to E}{\neg(\varphi_1 \veebar \varphi_2)}\,\to I_{,4}\\ \mathfrak{D}\\ \bot\end{matrix}}{\varphi_1}\,\bot_{C,5}}{\varphi_1 \veebar \varphi_2}\,\veebar I_{,2,6}$$

Let S_{CD} be an extension of S_C by adding $\veebar-$rules, and let $\mathbb{R}_{CD}$ be an extension of $\mathbb{R}_C$ by adding $\triangleright_{\veebar}$ and $\trianglerighteq_{\bot_{C(\veebar)}}$.[6] Then, we call S_{CD} the natural deduction system for classical propositional logic with disjunctive syllogism. In the next section, we will prove the normalization theorem and the consistency of S_{CD}.

3 Proofs of the Normalization Theorem and the Consistency of S_{CD}

In this section, we shall prove the (weak) normalization theorem for S_{CD} in Section 3.1 and the consistency of S_{CD} in Section 3.2.

Theorem 3.1 (Normalization for S_{CD})**.** *For every closed derivation $\mathfrak{D}$ in S_{CD}, there is a closed normal derivation $\mathfrak{D}'$ in S_{CD} such that $\mathfrak{D} \succ \mathfrak{D}'$ relative to $\mathbb{R}_{CD}$.*

Corollary 3.2 (Consistency of S_{CD})**.** *There is no closed normal derivation of $\bot$ in S_{CD}.*

[6] A system S' is an *extension* of S if S' is S itself or results from S by adding further rules. A set $\mathbb{R}'$ is an *extension* of $\mathbb{R}$ if $\mathbb{R}'$ results from $\mathbb{R}$ by adding reduction procedures which are closed under substitution in $\mathbb{R}'$.

3.1 The Proof of the Normalization Theorem for S_{CD}.

For the proof of theorem 3.1, we first introduce some primary definitions, and then we prove the result.

Definition 3.3 (The Degree of a Formula)**.** The *degree* $d(\varphi)$ of a formula φ is defined by $d(\bot) = 0$, $d(\alpha)$ for an atomic formula α, $d(\varphi \circ \psi) = d(\varphi) + d(\psi) + 1$ for binary operators $\circ$, $d(\circ\varphi) = d(\varphi) + 1$ for unary operator $\circ$. The *degree* $d(\mathfrak{D})$ of a derivation $\mathfrak{D}$ is defined by $max\{d(\varphi) \mid \varphi$ is a maximum formula in $\mathfrak{D}\}$.

$d(\varphi)$ is the number of operators in φ. $d(\mathfrak{D})$ is the maximal degree of maximum formulas in $\mathfrak{D}$. $d(\mathfrak{D}) = 0$ if there is no maximum formula in $\mathfrak{D}$. We say that φ is a *maximal maximum formula* in $\mathfrak{D}$ if $d(\varphi) = d(\mathfrak{D})$.

Definition 3.4 (The Cut Degree of a Derivation)**.** Let $\mathbb{M}^{\mathfrak{D}}$ be a set of maximal maximum formulas in $\mathfrak{D}$, i.e. $\mathbb{M}^{\mathfrak{D}} = \{\varphi | \varphi$ is a maximal maximum formula in $\mathfrak{D}$ and $d(\varphi) = d(\mathfrak{D})\}$. Then, the *cut degree* $cd(\mathfrak{D})$ of $\mathfrak{D}$ is defined by an ordered pair $(d(\mathfrak{D}), |\mathbb{M}^{\mathfrak{D}}|)$.[7]

$cd(\mathfrak{D})$ is the maximal degree of a maximum formula with the number of such formulas. If $\mathfrak{D}$ has no maximum formulas, put $cd(\mathfrak{D}) = (0,0)$. We will systematically lower the cut degree of a derivation until all maximum formulas have been eliminated. The ordering on $cd(\mathfrak{D})$ is lexicographic: $(d(\mathfrak{D}), |\mathbb{M}^{\mathfrak{D}}|) < (d(\mathfrak{D}'), |\mathbb{M}^{\mathfrak{D}'}|) =^{def} (d(\mathfrak{D}) < d(\mathfrak{D}')) \vee (d(\mathfrak{D}) = d(\mathfrak{D}') \wedge |\mathbb{M}^{\mathfrak{D}}| < |\mathbb{M}^{\mathfrak{D}'}|)$.

Lemma 3.5. *Let Φ be any formula in S_{CD}. Let $\mathfrak{D}$ be any closed derivation of Φ in S_{CD} where Φ is the conclusion of $\bot_C$−rule. Suppose that $d(\mathfrak{D}) = d(\Phi) > 0$ and all other conclusions of $\bot_C$−rule in*

[7] For any set $\mathbb{M}$ of formulas, $|\mathbb{M}|$ means the cardinality of $\mathbb{M}$, i.e. the number of formulas in $\mathbb{M}$.

$\mathfrak{D}$ have degree less than $d(\Phi)$. Then, there is a closed derivation $\mathfrak{D}'$ of Φ in S_{CD} such that $\mathfrak{D} \succ \mathfrak{D}'$ relative to $\mathbb{R}_{CD}$ and $d(\Phi) = 0$.

Proof. Let Σ be a subderivation of $\mathfrak{D}$ which has the form

$$\dfrac{\begin{array}{c}[\neg\Phi]^1\\ \Sigma\\ \bot\end{array}}{\Phi}\,\bot_{C,1}$$

Then, Φ has one of the forms $\varphi \wedge \psi, \varphi \to \psi, \neg\varphi$, and $\varphi \veebar \psi$. The application of reduction procedures in $\mathbb{R}_{CD}$ reduces the degree of Φ. Hence, by successively repeating the application of reduction procedures, we finally get a derivation $\mathfrak{D}'$ of Φ in which $d(\Phi) = 0$. □

Lemma 3.6. *Let Φ be any formula in S_{CD}. Let $\mathfrak{D}$ be any closed derivation of S_{CD} having a maximal maximum formula Φ in the last inference rule. Suppose that $d(\mathfrak{D}) = d(\Phi) = n$ and all other maximum formulas in $\mathfrak{D}$ have degree less than n. Then, there is a derivation $\mathfrak{D}'$ in S_{CD} such that $\mathfrak{D} \succ \mathfrak{D}'$ relative to $\mathbb{R}_{CD}$ and $d(\mathfrak{D}') < n$.*

Proof. We consider all possible maximum formulas in the last inference rule and check the degrees of the derivations after the applications of reduction procedures. Since $\mathfrak{D}$ is not normal, there is no case that $d(\mathfrak{D}) = 0$. Also, by Lemma 3.5, every maximum formula derived by $\bot_C-$rule is convertible into an atomic formula.

If Φ has the form of $\varphi \to \psi$, $\mathfrak{D}$ has the form below left and it reduces to the derivation $\mathfrak{D}'$ below right by $\rhd_{\to}$.

$$\dfrac{\dfrac{\begin{array}{c}[\varphi]^1\\ \mathfrak{D}_1\\ \psi\end{array}}{\varphi\to\psi}\to I_{,1} \quad \begin{array}{c}\mathfrak{D}_2\\ \varphi\end{array}}{\psi}\to E \qquad \rhd_{\to} \quad \begin{array}{c}\mathfrak{D}_2\\ \varphi\\ \mathfrak{D}_1\\ \psi\end{array}$$

Since $d(\mathfrak{D}_1)$ and $d(\mathfrak{D}_2)$ are less than n, $d(\mathfrak{D}') < n$. The case that Φ has the form of $\neg\varphi$ is similar to the case of $\varphi \to \psi$. It is easy to prove the case that Φ has the form of $\varphi \wedge \psi$.

If Φ has the form of $\varphi \veebar \psi$, $\mathfrak{D}$ has the form below left and it reduces to the derivation $\mathfrak{D}'$ below right by $\rhd_{\veebar_1}$.

$$\dfrac{\dfrac{\begin{array}{c}[\neg\varphi_1]^1\\ \mathfrak{D}_1\\ \varphi_2\end{array} \quad \begin{array}{c}[\neg\varphi_2]^2\\ \mathfrak{D}_2\\ \varphi_1\end{array}}{\varphi_1\veebar\varphi_2}\veebar I_{,1,2} \quad \begin{array}{c}\mathfrak{D}_3\\ \neg\varphi_1\end{array}}{\varphi_2}\veebar E_1 \qquad \rhd_{\veebar_1} \quad \begin{array}{c}\mathfrak{D}_3\\ \neg\varphi_1\\ \mathfrak{D}_1\\ \varphi_2\end{array}$$

If $d(\neg\varphi_1) < D(\varphi_1 \veebar \varphi_2)$, then $d(\mathfrak{D}') < d(\mathfrak{D})$. If $d(\neg\varphi_1) = d(\varphi_1 \veebar \varphi_2)$ and $\neg\varphi_1$ in $\mathfrak{D}'$ is not a maximum formula, then $d(\mathfrak{D}') < d(\mathfrak{D})$. If $d(\neg\varphi_1) = d(\varphi_1 \veebar \varphi_2)$ and $\neg\varphi_1$ in $\mathfrak{D}'$ is a maximum formula, then $\mathfrak{D}'$ has the form left below. We apply $\rhd_{\to}$ to $\mathfrak{D}'$ and have the derivation $\mathfrak{D}''$ below right.

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}[\varphi_1]^i\\ \vdots\\ \bot\end{array}}{\neg\varphi_1}\to I_{,i} \quad \begin{array}{c}\vdots\\ \varphi_1\end{array}}{\bot}\to E\\ \vdots\\ \varphi_2\end{array} \qquad \rhd_{\to} \quad \begin{array}{c}\vdots\\ \varphi_1\\ \vdots\\ \varphi_2\end{array}$$

Then, $d(\mathfrak{D}'') < d(\mathfrak{D})$. The case of $\rhd_{\veebar_2}$ is similar. By all cases, the proof is done. $\square$

Lemma 3.7. *For any closed derivation $\mathfrak{D}$ in S_{CD}, if $(0,0) < cd(\mathfrak{D})$, then there is a closed derivation $\mathfrak{D}'$ such that $\mathfrak{D} \succ \mathfrak{D}'$ relative to $\mathbb{R}_{CD}$ and $cd(\mathfrak{D}') < cd(\mathfrak{D})$.*

Proof. Suppose that $(0,0) < cd(\mathfrak{D})$. Then, there is a maximum formula in $\mathfrak{D}$. Choose a maximum formula Φ that $d(\mathfrak{D}) = d(\Phi)$ and all other maximum formulas in $\mathfrak{D}$ have degree less than $d(\Phi)$. By Lemma 3.5 and 3.6, there is a closed derivation $\mathfrak{D}'$ in S_{CD} such that $\mathfrak{D} \succ \mathfrak{D}'$ relative to $\mathbb{R}_{CD}$ and $d(\mathfrak{D}') < d(\mathfrak{D})$. If $\mathfrak{D}$ has the only one maximum formula, $d(\mathfrak{D}') < d(\mathfrak{D})$ or $|\mathbb{M}^{\mathfrak{D}'}| = 0$. In both cases, $cd(\mathfrak{D})$ gets smaller, i.e. $cd(\mathfrak{D}') < cd(\mathfrak{D})$. $\square$

Now, we prove the normalization theorem for S_{CD}, i.e. Theorem 3.1. By Lemma 3.7, for any derivation $\mathfrak{D}$ in S_{CD}, $cd(\mathfrak{D})$ can be lowered to (0.0) in a finite number of steps. Therefore, all derivations in S_{CD} have their normal derivation.

3.2 The Consistency of S_{CD}.

In order to prove the consistency of S_{CD} as the corollary of Theorem 3.1, we need to prove the inversion corollary (Corollary 3.10). For our purpose of proving the inversion corollary, we introduce some definitions. We say that a *top-formula* in a derivation $\mathfrak{D}$ is a formula occurrence that does not stand immediately below any formula occurrence in $\mathfrak{D}$. An *end-formula* in $\mathfrak{D}$ is the formula occurrence in $\mathfrak{D}$ that does not stand immediately above any formula occurrence in $\mathfrak{D}$. It is natural to think that, for any closed derivation $\mathfrak{D}$ in S_{CD}, a top-formula in $\mathfrak{D}$ is an assumption. Then, we have the following definitions.

Definition 3.8. A *thread* is a sequence of formulas $\varphi_0, ..., \varphi_n$ in a derivation $\mathfrak{D}$ such that φ_0 is a top-formula in $\mathfrak{D}$, φ_i is a premise immediately above φ_{i+1} where $0 \leqslant i \leqslant n-1$, and φ_n is the end-formula of $\mathfrak{D}$. A *branch* is an initial part of a thread in a derivation $\mathfrak{D}$ which stops at the first minor premise or at the end-formula of $\mathfrak{D}$. A *main branch* is a branch which stops at the end-formula of $\mathfrak{D}$ and contains no minor premise.

A branch can only pass through introduction rules and through major premises of elimination rules. We prove two lemmas for the inversion corollary and the consistency of S_{CD}.

Lemma 3.9. *Let $\mathfrak{D}$ be any closed normal derivation in S_{CD}, there is no application of $I-$rule or $\bot_C-$rule which precedes the application of $E-$rule in a branch of $\mathfrak{D}$.*

Proof. Suppose an application of $I-$rule, in short, an $I-$application, precedes an application of $E-$rule, in short, an $E-$application, in a branch of $\mathfrak{D}$. Then, there is a last $I-$application that precedes the first $E-$application. However, since $\mathfrak{D}$ is normal, it is clearly impossible. □

Corollary 3.10 (Inversion Corollary)**.** *For every closed derivation $\mathfrak{D}$ in S_{CD}, there is a derivation $\mathfrak{D}'$ in S_{CD} such that $\mathfrak{D} \succ \mathfrak{D}'$ relative to $\mathbb{R}_{CD}$ and the last step of $\mathfrak{D}'$ is an $I-$rule or $\bot_C-$rule having an atomic formula as its conclusion.*

Proof. By Theorem 3.1, every closed derivation $\mathfrak{D}$ in S_{CD} has its normal derivation $\mathfrak{D}'$. Suppose that the last step of $\mathfrak{D}'$ is an $E-$rule.

Then, by Lemma 3.9, no $I-$rule (or $\perp_C$-rule) precedes the $E-$rule in $\mathfrak{D}'$. Since a top-formula in $\mathfrak{D}'$ is an assumption discharged by $I-$rule, $\mathfrak{D}'$ is to be an open derivation, which is contrary to the fact that $\mathfrak{D}'$ is a closed normal derivation. $\square$

By the inversion corollary, every closed derivation in S_{CD} can be transformed into one using $I-$rule or $\perp_C-$rule in the last step, as a closed normal derivation has the same form. When the last step of the derivation is a $\perp_C-$rule, then we need an additional lemma.

Lemma 3.11. *There is no closed normal derivation of $\perp$ derived by $\perp_C-$rule in S_{CD}.*

Proof. Suppose the opposite. Then, there should be a normal derivation $\mathfrak{D}$ of $\perp$ from the assumption $[\neg\perp]$ which does not apply $\perp_C-$rule. Since $\neg\perp$ is a tautology, $\mathfrak{D}$ is to be a closed normal derivation of $\perp$. By the inversion corollary, the last step of $\mathfrak{D}$ is an $I-$rule. However, no $I-$rule derives $\perp$. Hence, there is no such $\mathfrak{D}$. $\square$

By Lemma 3.11, $\perp_C-$rule in S_{CD} does not derive $\perp$. Also, if the last step is an $I-$rule, since no $I-$rule derive $\perp$, there is no closed normal derivation of $\perp$. Therefore, the consistency of S_{CD}, i.e. Corollary 3.2, is proved.

In the next section, we extend our results to the first-order classical logic with disjunctive syllogism.

4 An Extension to the First-Order Classical Logic with Disjunctive Syllogism

When Prawitz (1965) proved the normalization theorem for weak classical logic, his system does not contain rules for $\vee$ and $\exists$. Like

$\vee E-$rule, the conclusion of $\exists E-$rule is not always a subformula of its major premise. In order to have Prawitz-style reduction procedure for $\perp_C-$rule concerning universal and existential quantifications, we define existential quantification in terms of $\veebar$. Also, the universal quantification can be defined in terms of $\wedge$.

It is often said that existential quantification can be defined by finite disjunctions and universal quantification can be defined by finite conjunctions. Likewise, we define $\exists x\varphi x$ in terms of $\veebar$ and $\forall x\varphi x$ in terms of $\wedge$. At first, we introduce abbreviations of finite conjunctions and disjunctions as below.

$$\bigwedge_{i\leqslant 0}\varphi_i =^{def} \varphi_0,\ and \bigwedge_{i\leqslant n+1}\varphi_i =^{def} \bigwedge_{i\leqslant n}\varphi_i \wedge \varphi_{n+1} \tag{1}$$

$$\bigvee_{i\leqslant 0}\varphi_i =^{def} \varphi_0,\ and \bigvee_{i\leqslant n+1}\varphi_i =^{def} \bigvee_{i\leqslant n}\varphi_i \veebar \varphi_{n+1} \tag{2}$$

Now, for a unary predicate φ and an $n-$ary predicate ψ, we define $\forall x\varphi x$ and $\exists x\varphi x$ in terms of $\wedge$ and $\veebar$ respectively.

$$\forall x\varphi x =^{def} \bigwedge_{i\leqslant l}\varphi(t_i),\ and$$

$$\forall x_1\forall x_2...\forall x_n\psi(x_1,...,x_n) =^{def} \bigwedge_{i\leqslant l}\forall x_2...\forall x_n\psi(t_i,x_2,...,x_n) \tag{3}$$

$$\exists x\varphi x =^{def} \bigvee_{i\leqslant l}\varphi(t_i),\ and$$

$$\exists x_1\exists x_2...\exists x_n\psi(x_1,...,x_n) =^{def} \bigvee_{i\leqslant l}\exists x_2...\exists x_n\psi(t_i,x_2,...,x_n) \tag{4}$$

Then, $\exists I-$ and $\exists E-$rules, and reduction procedures for $\exists$ are stated as follows.

$$
\dfrac{\begin{array}{c}[\neg\bigvee\varphi(t_{n-1})]^1\\ \mathfrak{D}_1\\ \varphi(t_n)\end{array} \qquad \begin{array}{c}[\neg\varphi(t_n)]^2\\ \mathfrak{D}_2\\ \bigvee\varphi(t_{n-1})\end{array}}{\exists x\varphi x}\exists I_{,1,2}
\qquad
\dfrac{\exists x\varphi x \quad \begin{array}{c}\mathfrak{D}_3\\ \neg\varphi(t_n)\end{array}}{\bigvee\varphi(t_{n-1})}\exists E_1
\qquad
\dfrac{\exists x\varphi x \quad \begin{array}{c}\mathfrak{D}_4\\ \neg\bigvee\varphi(t_{n-1})\end{array}}{\varphi(t_n)}\exists E_2
$$

$$
\dfrac{\dfrac{\begin{array}{c}[\neg\bigvee\varphi(t_{n-1})]^1\\ \mathfrak{D}_1\\ \varphi(t_n)\end{array} \qquad \begin{array}{c}[\neg\varphi(t_n)]^2\\ \mathfrak{D}_2\\ \bigvee\varphi(t_{n-1})\end{array}}{\exists x\varphi x}\exists I_{,1,2} \qquad \begin{array}{c}\mathfrak{D}_3\\ \neg\varphi(t_n)\end{array}}{\bigvee\varphi(t_{n-1})}\exists E_1
\quad \rhd_{\exists E_1} \quad
\begin{array}{c}\mathfrak{D}_3\\ \neg\varphi(t_n)\\ \mathfrak{D}_2\\ \bigvee\varphi(t_{n-1})\end{array}
$$

$$
\dfrac{\dfrac{\begin{array}{c}[\neg\bigvee\varphi(t_{n-1})]^1\\ \mathfrak{D}_1\\ \varphi(t_n)\end{array} \qquad \begin{array}{c}[\neg\varphi(t_n)]^2\\ \mathfrak{D}_2\\ \bigvee\varphi(t_{n-1})\end{array}}{\exists x\varphi x}\exists I_{,1,2} \qquad \begin{array}{c}\mathfrak{D}_4\\ \neg\bigvee\varphi(t_{n-1})\end{array}}{\varphi(t_n)}\exists E_2
\quad \rhd_{\exists E_2} \quad
\begin{array}{c}\mathfrak{D}_4\\ \neg\bigvee\varphi(t_{n-1})\\ \mathfrak{D}_1\\ \varphi(t_n)\end{array}
$$

Moreover, we can find an auxiliary reduction procedure for $\perp_C$−rule concerning $\exists$.

$$
\dfrac{\begin{array}{c}[\neg\exists x\varphi x]^1\\ \mathfrak{D}\\ \perp\end{array}}{\exists x\varphi x}\perp_{C,1}
\quad \unrhd_{\perp_{C(\exists)}} \quad
\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{[\neg\varphi t_n]^3 \quad \dfrac{[\exists\varphi x]^1 \quad [\neg\bigvee\varphi t_{n-1}]^2}{\varphi t_n}\exists E_2}{\perp}\to E}{\neg\exists x\varphi x}\to I_{,1}\\ \mathfrak{D}\\ \perp\end{array}}{\varphi t_n}\perp_{C,3}
\qquad
\dfrac{\begin{array}{c}\dfrac{\dfrac{[\neg\bigvee\varphi t_{n-1}]^5 \quad \dfrac{[\exists\varphi x]^4 \quad [\neg\varphi t_n]^6}{\bigvee\varphi t_{n-1}}\exists E_1}{\perp}\to E}{\neg\exists x\varphi x}\to I_{,4}\\ \mathfrak{D}\\ \perp\end{array}}{\bigvee\varphi t_{n-1}}\perp_{C,5}}{\exists x\varphi x}\exists I_{,2,6}
$$

Rules for $\forall$ and reduction procedures $\rhd_\forall$ and $\unrhd_{\perp_{C(\forall)}}$ for $\forall$ are given in a similar way to the standard rules and reductions for $\forall$ suggested by Prawitz (1965).

Let S_{PCD} be a natural deduction system which is an extension of S_{CD} by adding rules for $\exists$ and $\forall$. Let $\mathbb{R}_{PCD}$ be a set of reductions that is an extension of $\mathbb{R}_{CD}$ by adding $\rhd_{\exists}$, $\rhd_{\forall}$, $\trianglerighteq_{\perp_{C(\exists)}}$, and $\trianglerighteq_{\perp_{C(\forall)}}$. We say that a system S' is a *conservative extension* of S if every formula derivable in S' is already derivable in S. Then, the following theorems, and corollary are established.

Theorem 4.1. *S_{PCD} is a conservative extension of S_{CD}.*

Theorem 4.2 (Normalization for S_{PCD})**.** *For every closed derivation $\mathfrak{D}$ in S_{PCD}, there is a closed normal derivation $\mathfrak{D}'$ in S_{PCD} such that $\mathfrak{D} \succ \mathfrak{D}'$ relative to $\mathbb{R}_{PCD}$.*

Corollary 4.3 (Consistency of S_{PCD})**.** *There is no closed normal derivation of $\perp$ in S_{PCD}.*

S_{PCD} is the system for first-order classical logic with disjunctive syllogism. Since S_{PCD} is a conservative extension of S_{CD}, by Theorem 3.1 and Corollary 3.2, Theorem 4.2 and Corollary 4.3 can be easily proved.

5 Conclusion

The systems S_{PCD} for the first-order classical logic with disjunctive syllogism has whole logical operators $\wedge, \veebar, \rightarrow, \neg, \forall$, and $\exists$. Also, if a vacuous discharge is permissible for $\veebar I-$ and $\perp_C-$rule, the standard $\vee I-$ and $\vee E-$rules are definable in terms of $\veebar-$rules.

$$\dfrac{\begin{array}{c}\mathfrak{D}\\ \varphi\end{array}}{\varphi \vee \psi}\vee I \quad \textit{is defined by} \quad \dfrac{\dfrac{\dfrac{[\neg\varphi]^1 \quad \begin{array}{c}\mathfrak{D}\\ \varphi\end{array}}{\perp}\rightarrow E}{\psi}\perp_{C,\emptyset} \qquad \begin{array}{c}\mathfrak{D}\\ \varphi\end{array}}{\varphi \veebar \psi}\veebar I_{,1,\emptyset}$$

$$\dfrac{\varphi\vee\psi \quad \begin{matrix}[\varphi]^1\\ \mathfrak{D}_2\\ \sigma\end{matrix} \quad \begin{matrix}[\psi]^2\\ \mathfrak{D}_3\\ \sigma\end{matrix}}{\sigma}\vee E_{,1,2} \quad \textit{is defined by} \quad \dfrac{\dfrac{[\neg\sigma]^3 \quad \dfrac{\dfrac{\begin{matrix}[\psi]^4\\ \mathfrak{D}_3\\ \sigma\end{matrix}}{\psi\to\sigma}\to I_{,4} \quad \dfrac{\varphi\,\underline{\vee}\,\psi \quad \dfrac{\dfrac{[\neg\sigma]^3 \quad \dfrac{\dfrac{\begin{matrix}[\varphi]^1\\ \mathfrak{D}_2\\ \sigma\end{matrix}}{\varphi\to\sigma} I_{,1} \quad [\varphi]^2}{\sigma}\to E}{\bot}\neg I_{,2}}{\neg\varphi}\neg I_{,2}}{\psi}\underline{\vee} E_1}{\sigma}\to E}{\bot}\neg E}{\sigma}\bot_{C,3}$$

One might be tempted to reject $\underline{\vee} I-$rule on the ground that it does not satisfy the complexity condition proposed by Michael Dummett (1991, p. 258).

> ... the minimal demand we should make on an introduction rule intended to be self-justifying is that its form be such as to guarantee that, in any application of it, the conclusion will be of higher logical complexity that any of the [premises] and than any discharged hypothesis

The complexity condition depends on Dummett's philosophical ground that $I-$rule should be self-justifying and exhaustively determine the meaning of a principal logical operator. Peter Milne (1994) argues that it is not clear that there is such proof-theoretic ground. So to speak, some are not obliged to confer meaning on the formula introduced. He had claimed that it is impossible for $\to I-$rule (or $\neg I-$rule) to determine the meaning of $\neg$ without circularity. Moreover, Milne (2002) thought that the complexity condition is, in his own phrase, "exorbitant." Whether $I-$rule should satisfy the complexity condition is still an open question.

In the present paper, we have proved the normalization and the consistency of S_{PCD} which is the system for the first-order classical logic with disjunctive syllogism. Unlike Prawitz's, our proofs deal with whole logical operators. That is, disjunction and existential quantification are not considered to be logical operators in Prawitz's proof of normalization for weak classical logic. Whereas Stålmarck (1991) and Andou (1995) did not prove the inversion corollary and the consistency of their system, we have proved both results by using Prawitz's style reduction procedures with $\veebar-$rules. Further researches are required if the results are extended to the first-order arithmetic.

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선언적 삼단논법을 지닌 1차 고전 자연연역 체계의 정형화 정리

최 승 락

이 글에서 우리는 선언적 삼단논법을 지닌 1차 고전 자연연역 체계의 정형화 정리와 일관성을 증명할 것이다. 먼저, 우리는 선언적 삼단논법에 관한 규칙을 선언에 관한 규칙으로 고려하여 연언, 선언, 조건언, 부정언을 지니는 고전 명제논리 체계 S_{CD}를 제시할 것이다.

S_{CD}의 정형화 정리와 일관성 정리 증명을 제시하고 우리는 S_{CD}를 1차 고전논리에 관한 자연연역 체계 S_{PCD}로 확장할 것이다. S_{PCD}는 S_{CD}의 보존적 확장임이 보여질 것이며 S_{PCD}의 정형화 정리와 일관성도 증명될 것이다.

주요어: 고전논리, 일관성, 선언적 삼단논법, 전도 따름정리, 정형화 정리