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GRUNDLAGEN §64: AN ALTERNATIVE STRATEGY TO ACCOUNT FOR SECOND-ORDER ABSTRACTION

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Abstract. A famous passage in Section 64 of Frege's *Grundlagen* may be seen as a justification for the truth of abstraction principles. The justification is grounded in the procedure of *content recarving* which Frege describes in the passage. In this paper I argue that Frege's procedure of content recarving while possibly correct in the case of first-order equivalence relations is insufficient to grant the truth of second-order abstractions. Moreover, I propose a possible way of justifying second-order abstractions by referring to the operation of content recarving and I show that the proposal relies to a certain extent on the Basic Law V. Therefore, if we are to justify the truth of second-order abstractions by invoking the content recarving procedure we are committed to a special status of some instances of the Basic Law V and thus to a special status of extensions of concepts as abstract objects.

Keywords: Abstraction principles • Basic Law V • content recarving • higher-order logic

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1. Introduction

The name *abstraction principle* is traditionally used to refer to a special kind of implicit definitions where conditions of identity between instances of a certain concept are defined in terms of a correspondent equivalence relation. For instance, being R an equivalence relation, a and b singular terms, and σ the correspondent *abstraction operator*, the abstraction principle associated with σ is expressed by the equivalence:

$$Rab \leftrightarrow \sigma(a) = \sigma(b) \qquad \textbf{(AP)}$$

where all terms defined by means of the abstraction operator are considered as instances of the same concept that is introduced by means of (**AP**).

The most interesting cases of abstraction principles involve second-order equivalence relations; being $\mathscr{R}$ a second-order equivalence relation and Fx, Gx two conceptual expressions, a second-order abstraction principle has the following expression:

$$\mathscr{R}_x(Fx, Gx) \leftrightarrow \sigma_x(Fx) = \sigma_x(Gx) \qquad \textbf{(AP2)}$$

where σ is an operator that forms a singular term when applied to a conceptual expression, i.e. stands for a function from concepts to objects. When we consider as second-order equivalence relation the relation of *equinumerousity* — i.e. the relation that holds between two concepts when their instances can be put into one-to-one correspondence — the resulting abstraction principle is traditionally called *Hume's principle* and the associated abstraction operator is the cardinality operator 'N', i.e. the operator that given a concept Fx, returns the cardinal number of Fx:

$$(Fx \approx_x Gx) \leftrightarrow (N_x(Fx) = N_x(Gx)) \qquad \textbf{(HP)}$$

where the symbol '$\approx$' abbreviates the expression of the relation of equinumerousity in second-order language. Another famous (and notorious) example of second-order abstraction principle is Frege's *Basic Law V*, i.e. the abstraction principle obtained by considering the relation of co-extensionality between two concepts as equivalence relation, and the *extension (or value-range) operator* 'ε' as abstraction operator:

$$\forall x(Fx \leftrightarrow Gx) \leftrightarrow (\varepsilon_x(Fx) = \varepsilon_x(Gx)) \qquad \textbf{(BLV)}$$

Why should we believe that definitions of the sort of (**AP**) and (**AP2**) are true? After all, the fact that we know that (**BLV**) — an instance of (**AP2**) — is inconsistent represents a strong reason to doubt both consistency and truth of any other definition of this kind. What we need is an explanation of the fact that the two limbs of a definition by abstraction have the same truth-value.

Frege in his early attempt to exploit this new kind of definitions presented what may be considered as the sketch of a justification for the truth of a bi-conditional of the form of (**AP**). This is what he is apparently doing in the famous section 64 of the *Grundlagen* (Frege 1950). In a language that is surprisingly obscure and vague for Frege's standard, Frege argues that the two limbs of a first-order abstraction principle share *the same content*, for the content of the right limb may be obtained by re-organizing, re-conceptualizing, recarving the content of the left limb. As it is well-known, Frege has in view the application of this operation of content recarving to the case of Hume's Principle in order to justify the truth of the implicit definition of cardinal numbers. However, few sections later, he gave up this strategy, due to the so-called *Caesar problem*. Hence we are left with no textual evidence on how the procedure of content recarving described in section 64 shall be understood in detail as well as no textual evidence tells us whether such a procedure may be applied to higher-order abstractions so straightforwardly as it seems.

The purpose of this paper is to improve the understanding of Frege's procedure of content recarving in order to asses to what extent it may be used to justify the truth of second-order abstraction principles. More specifically, I will attempt a *partial rational reconstruction* (or partial explication) of the recarving procedure, integrating some fundamental points that may be extracted from Frege's few remarks on the topic with additional considerations which are not directly based on Frege's texts. Given that Frege does not provide sufficiently detailed explanations of the recarving procedure, the result of the analysis to be presented cannot be considered an interpretation in the strict sense, and for this reason "rational reconstruction" is the label that more adequately fits both the purpose and the methodology of the present paper. This rational reconstruction has been presented as partial for I will not offer a complete and general theory of content and its related operations, yet I will limit myself to the explication of some aspects of the recarving operation in the case of abstraction principles. On the grounds of this rational reconstruction I will argue that the application of the procedure of content recarving to the case of second-order equivalence relations does not yield an identity of abstracts as in the first-order case. Moreover, I will show that there seems to be only one strategy for adapting the content recarving procedure to the second-order case, and this strategy relies to a certain extent on the Basic Law V.

In section 2 I will propose my understanding of the operation of content recarving applied to equivalence relations and I will show why this operation is not sufficient to justify the truth of second-order abstractions. In section 3 I will present a sketch of the strategy that may be adopted to improve the content recarving procedure in such a way that a justification of second-order abstraction may be framed and I will make explicit the indispensable use of the Basic Law V. This leads me to argue in favour of a special status of the Basic Law V which will turn out to be more a primitive principle rather than a second-order abstraction principle among others. In section 4 I will consider some objections to my proposal whose responses give me the chance to analyze the impact of the proposal on some philosophical issues related to definitions by abstractions. In section 5 I will develop some conclusive remarks.

2. *Grundlagen* §64 and the truth of abstraction principles

In §64 of Frege (1950), Frege writes:

> The judgment '*line a is parallel to line b*', [...] can be taken as an identity. If we do this, we obtain the concept of direction and say: '*The direction of line a is identical with the direction of line b*'. Thus we replace the symbol $\|$ by the more generic symbol $=$, through removing what is specific in the content of the former and dividing it between a and b. We carve up the content in a way different from the original way, and this yields us a new concept. (Frege 1950, pp.74–5)

Surely this passage is not an example of Frege's usual clarity of exposition. Firstly, it is not clear how to understand the crucial notion of *content* that is involved in the description of the procedure of *recarving*; secondly, it is even less clear how to understand Frege's metaphorical language when he says that "the specific part" of the content of a relation of parallelism is "removed" and successively "divided" to form the content of direction terms. In spite of several attempts to interpret this passage (Hale 1997, Potter 2001, Stirton 2000, Yablo 2008, Ebert 2016), there is no consensus on how to understand the alleged identity of content between two sides of an abstraction principle, and — to my knowledge — the secondary literature is more concerned with the understanding of this relation of identity of content between sentences rather than with the explanation of the operation of content recarving itself.

I will not propose a general theory of Fregean content or a general definition of the operation of content recarving. What I will try to do is to extrapolate some theoretical elements from Frege's wording which are relevant when the justification of abstraction principles is concerned. As it will turn out, I will not manage to explain all the details of the recarving operation; nevertheless, the partial explanation I will propose will be enough to draw some significant conclusions regarding the account of second-order abstraction.

I understand the passage as saying that the same content may admit different internal organizations that result in different syntactic structures. For instance, being C the content of the sentence '$a \parallel b$', we may obtain a different internal organization of C by performing the procedure described in the passage on C as already organized according to the syntactic structure of '$a \parallel b$'. Notice that Frege is suggesting that the operation of recarving is not applied to the content of the entire sentence '$a \parallel b$', yet to the content of the relation of parallelism. In particular, we may remove "what is specific", i.e. the specific part of the content of the relation of parallelism and attach it — in an unspecified way — to the content of 'a' and 'b'. As a result, the "remainder" of the content of the relation of parallelism after removing its "specific content" is the content of the relation of identity; and the outcome of attaching the specific content to 'a' and 'b' is the content of new singular terms, i.e. 'the direction of a' and 'the direction of b'. Thus the two limbs of an abstraction principle have the same content in virtue of the fact that what is removed from one constituent of the content is attached to others. Clearly, in this description there is still a big deal of metaphorical language: it is not clear what words like "part", "remove", "remainder", "split", "attach" should mean in the context of the content of a linguistic expression.

Yet even without a clear understanding of Frege's metaphors it is possible to extrapolate the following fundamental points:

1. The notion of content needed to explain Frege's procedure is something in between sense and truth-value: it is not as coarse grained as truth-value, for the two limbs of an abstraction principle seem to display a certain relevance relation which is stricter than the relation of identity in truth-value; on the

other hand, it is not as fine grained as the notion of sense, for the two limbs of an abstraction principle must be identical in content even if they are concerned with different sort of entities (to this end Hale (1997) spoke of a required notion of 'weak sense');

2. The operation of recarving is performed on the content of the equivalence relation and not directly on the content of the entire sentence in which this relation occurs;
3. The content of an equivalence relation may be seen as the composition of two contents: *the specific part* and *the general part* and the latter coincides with the content of the relation of identity;
4. The procedure of content recarving consists of three nested operations: subtraction, splitting, and attachment. The *subtraction* consists in the removal of the specific content of the equivalence relation; the *splitting* consists in the division of this content (*zerspalten*); the *attachment* consists in the combination of the divided content with the content of the arguments of the equivalence relation. The analysis of this paper is mainly concerned with the subtraction operation and — as I will show — this is sufficient for my main point;
5. The result of the subtraction of content is the content of the relation of identity, while the result of the splitting and the attachment is the content of two new singular terms; it is worth noticing that just the operations of splitting and attachments are responsible for the introduction of novel abstract entities and not the operation of subtraction;
6. The entire procedure is allegedly content preserving insofar as it simply amounts to a redistribution of the same content (what is removed from one "part" is added to other "parts").

The first step to explain away the metaphorical language is to understand in which sense equivalence relations have a specific part in their content or in which sense identity is more general than an equivalence relation, e.g. parallelism. Indeed, Frege explicitly says that we replace a relational symbol "by the more generic symbol" of identity. What he seems to suggest is that what is specific in the content of an equivalence relation *R* is what makes it more specific than identity, i.e. what characterizes the difference in content between *R* and =. In other words, an equivalence relation *R* has a content which includes the content of the relation of identity plus an additional content which is the aforementioned specific part. Thus, in what sense is an equivalence relation *R* more specific than an identity (i.e. has a content including a specific part with respect to the relation of identity)?

Frege does not give any hint regarding such a comparison between equivalence relations in terms of generality and specificity. Thus a hypothetical answer to the previous question must be formulated outside of the context of Frege's texts.

An interesting way of understanding the higher specificity of the relation of parallelism with respect to identity may come from some fundamental algebraic properties of equivalence relations. The idea is the following: an equivalence relation is a congruence with respect to a restricted range of properties — commonly referred to as *invariant properties* — while identity is a congruence with respect to all properties. In other words, two parallel lines are indistinguishable with respect to their *orientational properties*, whereas two identical lines are indistinguishable with respect to all properties (see Wright's interpretation of the notion of parallelism as total congruence for directions in (2001, p.337)). As a consequence, an equivalence relation R may be understood as a relation of *partial or restricted indistinguishability* while an identity is understood as a relation of *total or general indistinguishability*. Identity is more general because is a relation of indistinguishability with respect to all possible contexts, while a distinct equivalence relation R is more specific because is a relation of indistinguishability with respect to a specific range of contexts related to the *R-invariant properties*. Hence the specific content of an equivalence relation R has to do with what determines the restriction to the range of contexts for which R behaves like a relation of indistinguishability. The relation of identity does not express any specific content for there is no restriction on the contexts in which it behaves as a relation of indistinguishability. Whenever just orientational properties are concerned, there is no way of tracing distinctions between parallel lines.

How do these considerations help us to understand the recarving operation? To remove the specific content from the relation of parallelism amounts to remove what determines the restriction to the contexts with respect to which parallelism is a congruence relation: we may call this aspect *the content that selects orientational contexts*. When this content is combined with the content of the term 'line a' ('line b') the result is the content of the new singular term 'the direction of line a' ('the direction of line b') which allegedly determines an abstract object associated with the orientational aspect of line a (line b). The result of removing the content that selects orientational contexts from the content of $\parallel$ is the content of a relation of indistinguishability with no restrictions on the particular context, i.e. the relation of identity.

Clearly, such a preliminary explanation leaves open many philosophical issues, especially regarding the nature of abstract objects and the content of a singular term. Yet those aspects will not concern us for the moment. We may try to fix the fundamental theoretical elements in the proposed explanation by using a suitable notation. The relation of identity may be defined in second-order logic by means of Leibniz's law which says that identity is a relation of *total indistinguishability*: for all objects a and b,

$$a = b \leftrightarrow \forall X(Xa \leftrightarrow Xb) \qquad \textbf{(LL)}$$

Clearly (**LL**) is highly controversial, especially in regard of the right-to-left conditional, i.e. the identity of indiscernibles. Nevertheless, in the context of the present discussion there are mainly two reasons to consider (**LL**) as harmless:

(i) What is being done is a rational reconstruction of Frege's procedure of content recarving based on main theoretical points extrapolated from Frege's account of Logic and Arithmetic; and clearly Frege does accept (**LL**) (it is equivalent to his Basic Law III). Moreover, the acceptance of the identity of indiscernibles is one of the main reason why Frege rejects the Cantorian definition of cardinals by abstraction and thus motivate the urge for a new conception of abstraction not based on the idea of defining units by "stripping away" all features of concrete objects;

(ii) The discussion on the logical foundations of mathematical abstraction does not seem the kind of context in which (**LL**) display its controversial nature. To my knowledge, there is no clear example of qualitative identical but numerically distinct mathematical or logical objects.

Let now R be an equivalence relation. To simplify the definition of R as a relation of partial indistinguishability, we may use the superscript '(R)' on second-order variables to express the fact that we are considering these variables to range over R-invariant properties. This allows us to restrict the second-order quantification in (**LL**) to the contexts selected by the specific content of R and thus to formulate a sort of *partial Leibniz's Law*. For all objects a and b,

$$Rab \leftrightarrow \forall X^{(R)}(X^{(R)}a \leftrightarrow X^{(R)}b) \qquad (\textbf{PLL})$$

With a little — and harmless — abuse of notation, we use the superscript '(R)' on the singular terms 'a' and 'b' to denote the new singular terms expressing the combination of the content of 'a' ('b') with the specific content of R. Therefore, the abstraction principle (**AP**) may be formulated as follows:

$$\forall X^{(R)}(X^{(R)}a \leftrightarrow X^{(R)}b) \leftrightarrow \forall X(Xa^{(R)} \leftrightarrow Xb^{(R)}) \qquad (\textbf{AP'})$$

Where the fact that the superscript '(R)' is transferred from the second-order variables in the expression of R to the singular terms 'a' and 'b' may be seen as a notational representation of the fact that the specific content of R is removed from R — thus yielding an identity — and attached to the contents of 'a' and 'b'.

This preliminary (and partial) explanation of the recarving operation presents an asymmetry: while it seems clear how to understand the operation of removing the specific content from an equivalence relation R, it is not equally clear how to understand the operation of combining this content with the contents of the terms

'a' and 'b'. Yet, we may use (**AP'**) — as far as it goes — as a preliminary reference scheme for understanding the content recarving procedure that should justify the truth of abstraction principles.

Before discussing the case of second-order abstraction it is worth examining a difficulty that emerges with the proposed explanation of the notion of specific content of an equivalence relation and the explanation of the operation of content subtraction. I have proposed that this latter operation may be understood as the removal of a restriction on second-order quantification. The result of this operation is a quantification on all second-order properties (indulging for a while in the metaphysical jargon of "quantifying over properties"). However, what guarantees do we have that each possible property may be meaningfully said to be true or false of the alleged references of new singular terms '$a^{(R)}$' and '$b^{(R)}$'? [1] Consider again the case of lines and directions. After removing the restriction on the quantification over orientational properties we are left with an unrestricted quantification. Yet there are a lot of properties that cannot be meaningfully ascribed to or denied of directions. For instance, there seems to be no defined truth-conditions for the sentence 'The direction of line a is Roman' (at least no definable truth-conditions on the grounds of the content of the relation of parallelism). This difficulty is related to the Caesar Problem and it is commonly called *the Roman Problem* (Fine 2002). It seems that if we are not able to define the truth conditions of any instance of the right side of (**AP'**), we are thereby not able to perform the operation of content subtraction, for some restrictions on second-order quantifiers still may be surreptitiously applied (for instance, when we restrict the range of second-order quantifiers to properties that may be "meaningfully" applied to the references of the terms $a^{(R)}$ and $b^{(R)}$).

The presented difficulty may be even worse if we admit that — and it reasonable to do so — all the properties that we may meaningfully ascribe to or deny of the putative references of the terms $a^{(R)}$ and $b^{(R)}$ are somehow correspondent to just the R-invariant properties. For instance, consider the case of a first-order abstraction principle for cardinality of finite sets saying that two finite sets A and B are equinumerous iff the cardinal number of A is identical to the cardinal number of B. It is easy to show that to all the predicates that may be meaningfully ascribed to finite cardinals (i.e. the arithmetical predicates) correspond just predicates of sets which stand for equinumerousity invariant properties. For instance, suppose that n is the number of A and consider the predicate 'prime'. Say that a set is *primely composed* iff its cardinal number is prime. Now it is evident that if A is primely composed (i.e. if n is prime), then any set B equinumerous to A is also primely composed. Thus 'primely composed' is an equinumerousity-invariant property. A similar reasoning may be applied to any arithmetical predicate. Notice that in these cases, there is no removal of the restriction on the second-order quantifier of the right limb of (**AP'**); in other words, no specific content of the relation of equinumerousity between finite sets has been removed and applied to the singular terms standing for finite sets. If this line of reasoning is correct, then the proposed recarving procedure amounts to a semantically idle grammatical reparsing.

An interesting fact of the proposed explication of the recarving procedure is that it justifies the urge for a solution of the Roman Problem, and thus of the Caesar Problem[2]. Indeed, without a rationale for the meaningful application of all predicates (or at least of a wide variety of them) to the newly introduced singular terms, we cannot say that the first operation of the recarving procedure — viz. content subtraction — has been performed. Hence, given the proposed account, we cannot dismiss the Roman problem as a categorial mistake or as an example of an irrelevant context of use: *the theoretical urge is to provide meaningful truth conditions to sentences of the form 'the direction of a is Roman' grounded in the specific content of the relation of parallelism.*.

A complete and satisfactory analysis of the Caesar Problem goes far beyond the purpose of this paper; yet I will propose a sketch of the sort of solution that is required. To my mind, Crispin Wright's approach to the difficulty is the one that mostly fit the theoretical demand of the proposed account of abstraction (The first formulation of Writght's solution is in Wright (1983, section xiv); further refinements and replies to objections are found in the article *To bury Caesar* included in (Hale and Wright 2001)). According to Wright's proposal it is possible to decide whether the direction of line a falls under a certain sortal concept S on the grounds of the content of the relation of parallelism. The general idea — which for lack of space cannot be completely spelled out — is that every sortal concept has a defined and specific criterion of identity for its instances. Thus, assuming that 'Roman' is sortal, the direction of line a is Roman only if directions are the kind of entities that may be identified by means of the criterion of identity for Roman citizens. However, the criterion for identity and difference among directions is framed in terms of parallelism between lines, and Roman citizens do not seem to be the sort of entities which are identified by means of the relation of parallelism and this — according to Wright — is a good reason to conclude that the direction of line a is not Roman. Here it is important to grasp that the fact that the sortal concept of direction has identity conditions framed in terms of parallism between lines — and thus in terms of properties invariant with respect to the relation of parallelism — does not exclude that directions may also have a general criterion of identity defined by Leibniz's Law: indeed, by using Wright's train of thought it is in principle possible to determine the truth conditions of the right limb of (**LL**) (i.e. to solve the Roman Problem). Clearly, this proposal has several open issues, some of them related to Wright's account in itself, some others to the adaptation of Wright's solution to the theoretical framework of the explications provided in this paper.

Let us consider now the application of (**AP'**) to the case of second-order abstraction. Let $\mathscr{R}$ be a second-order equivalence relation; we want to define $\mathscr{R}$ as a partial (second-order) indistinguishability. To do this, we need third-order quantification: for all concepts F, G,

$$\mathscr{R}_x(Fx, Gx) \longleftrightarrow \forall \Phi^{(\mathscr{R})}[\Phi_x^{(\mathscr{R})}(Fx) \longleftrightarrow \Phi_x^{(\mathscr{R})}(Gx)] \qquad \textbf{(PLL2)}$$

where the superscript '$(\mathscr{R})$' is applied to the third-order variable Φ in the same way as '(R)' was used on second-order variables. We can now formulate a similar reference scheme for second-order abstraction principles:

$$\forall \Phi^{(\mathscr{R})}[\Phi_x^{(\mathscr{R})}(Fx) \longleftrightarrow \Phi_x^{(\mathscr{R})}(Gx)] \longleftrightarrow \forall \Phi[\Phi_x((Fx)^{(\mathscr{R}}) \longleftrightarrow \Phi_x((Gx)^{(\mathscr{R})}))] \qquad \textbf{(AP2')}$$

How (**AP2'**) is to be understood? The right limb represents the relation of second-order indistinguishability, i.e. interchangeability of conceptual expressions salva veritate in all contexts. This relation is applied to two — unspecified — conceptual expressions '$(Fx)^{(\mathscr{R})}$' and '$(Gx)^{(\mathscr{R})}$' obtained by combining the specific content of $\mathscr{R}$ respectively with the contents of the conceptual expressions 'Fx' and 'Gx'. Again, I offer no account for the operation of combination of contents. I can only say that when we remove the specific content of a second-order equivalence relation $\mathscr{R}$ (i.e. a second-order partial indistinguishability) what we obtain is a second-order relation of total indistinguishability. As a consequence, the result of combining such content with the content of the relata of $\mathscr{R}$ cannot yield the content of two singular terms on pain of not being possible to compose this new content with the remainder of the content of $\mathscr{R}$ after the removal of its specific part. More simply, the content of a second-order equivalence relation does not include the content of the relation of identity, yet the content of its second-order counterpart whose expression appears in the right limb of (**AP2'**). This fact will interest us in this paper, whatever the expressions '$(Fx)^{(\mathscr{R})}$' and '$(Gx)^{(\mathscr{R})}$' may stand for.

One may object that given that I said nothing on the operation of combining the specific content of $\mathscr{R}$ with the conceptual expressions 'Fx' and 'Gx', I cannot draw the conclusion that '$(Fx)^{(\mathscr{R})}$' and '$(Gx)^{(\mathscr{R})}$' are not singular terms. However, this is not the source of the difficulty I am presenting related to second-order abstraction. Suppose that — by virtue of an unspecified operation of content combination — '$(Fx)^{(\mathscr{R})}$' and '$(Gx)^{(\mathscr{R})}$' are singular terms. In this case, the combination of the specific content of $\mathscr{R}$ with a conceptual expression corresponds to the application of a second-order abstraction operator. Consider now the operation of content subtraction: according to the provided explication, such an operation amounts to the removing of the restriction on third order quantifiers in the left limb of (**AP2'**). Clearly this operation yields the expression of the relation of second-order indistinguishability which cannot receive two singular terms as arguments. In order to have a well-formed formula as a result of the procedure of content recarving we should need a further content preserving operation: the operation that converts the content of the relation of second-order indistinguishability into the content of the first-order relation of identity. Assuming that the former relation is just the second-order version of the latter, what we need here is a general principle for lowering the orders. In the next section I will show that such a principle is very close to the infamous Basic Law V. Yet what matters for the argument I will present is that, in any case, second-order abstraction by recarving requires a further principle of lowering orders (which is my main point).

Moreover, if we assume that the recarving procedure — as Frege described — consists just of the three operations of content subtraction, division, and addition, then we must require that expressions of the sort of $(Fx)^{(\mathscr{R})}$ are conceptual expressions, on pain of obtaining an ill-formed sentence. This because — as I have already clarified — the result of content subtraction (which amount to the removal of the restriction on third-order quantifiers) is a second-order relation and thus must receive conceptual expressions as arguments.

The consequence of the given remarks on the extension of the proposed explanation of Frege's operation of content recarving to the second-order case is that we can no more justify the truth of second-order abstractions of the form of (**AP2**), for now the right limb of our equivalence is not an identity between new singular terms. Our first conclusion is that if Frege's procedure of content recarving is to be understood according to the reference scheme (**AP'**), then such a procedure cannot be straightforwardly used to justify the truth of second-order abstraction. By referring to the context of Frege's attempted (and rejected) argument, the extension of the operations described in §64 of the *Grundlagen* from the case of directions (first-order) to the case of cardinal numbers (second-order) is unhelpful and at best incomplete.

In the first-order case we have seen that the result of combining the specific content of R with a and b is a singular term having as expression the application of an operator σ to a or b. In other words, the term $a^{(R)}$ is syntactically rendered as $\sigma(a)$, where σ is the abstraction operator associated to R. In the second-order case, we cannot formalize alleged conceptual expressions such as '$(Fx)^{(\mathscr{R})}$' by means of an abstraction operator, for what would be required is a second-order operator that whenever applied to a conceptual expression returns another expression of the same syntactic category. In other words, this operator should stand for a function from concepts to concepts which is not definable given that there is no available relation of identity between concepts. Thus there is no rigorous formalization of (**AP2'**).

Another issue is the determination of '$(Fx)^{(\mathscr{R})}$' and '$(Gx)^{(\mathscr{R})}$'. Assuming that they stand for concepts, what sort of concepts $(Fx)^{(\mathscr{R})}$ and $(Gx)^{(\mathscr{R})}$ are? Consider, for instance, Hume's Principle: it says that given two equinumerous concepts F and G, there are two concepts that are indistinguishable. Perhaps these concepts — designated by means of our notation by '$(Fx)^{(\approx)}$' and '$(Gx)^{(\approx)}$' — are "special representatives" of a certain equivalence class of equinumerousity, i.e. certain paradigmatic concepts of a fixed cardinality. Nevertheless, such a hypothesis depends on a deeper analysis of both the notion of content and the operation of content merging which is not carried out in this paper.

It is worth highlighting that the problem of the indeterminacy of concepts such as $(Fx)^{(\approx)}$ is not a problem specifically engendered by our reading of the recarving operation. Even in the first-order case there is an indeterminacy problem regarding the singular terms flanking the identity sign. We denote the terms introduced by a first-order abstraction by means of a syntactic operator 'σ', yet the recarving procedure — as understood — does not tell us which function among all combinatorially possible functions satisfying the abstraction principle is the denotation of 'σ' . As in the first-order case we don't know how to univocally pick out the reference of abstract terms, in the second-order case we are unable to define the conceptual expressions appearing in the right limb of (**AP2'**).

Let's take a stock. I have proposed a particular understanding of Frege's procedure of content recarving based on an interesting way of comparing equivalence relations. In particular, I have argued that an equivalence relation may be conceived as a relation of partial indistinguishability, i.e. a relation of interchangeability salva veritate in a restricted range of contexts (a congruence with respect to these contexts). The specific part of the content of an equivalence relation is what determines such a restriction on contexts. Given that the relation of identity is an unrestricted interchangeability salva veritate, it has no specific content. To remove the specific part of the content of an equivalence relation *R* is perform a generalization on the contexts with respect to which *R* is a congruence, i.e. to transform it into a relation of general congruence. To combine the specific content of *R* with the relata of *R* is to introduce entities associated with the specific aspects that determine the contexts with respect to which *R* is a congruence. In the first-order case this procedure yields an identity between novel singular terms; in the second-order case, the procedure yields either a relation of second-order indistinguishability between novel conceptual expressions or requires further principles (more on this in the next section). Therefore, the procedure does not provide a straightforward justification for the truth of second-order abstractions. In the next section, I will show a possible way in which a justification for second-order abstractions may be formulated on the ground of the procedure of content recarving.

3. The missing link: Basic Law V

We have seen that by recarving the content of a second-order equivalence relation by no means we obtain an identity of abstracts, yet a relation of higher-order indistinguishability between two unspecified concepts. Therefore the procedure described by Frege in *Grundlagen* §64 interpreted according to my proposal is insufficient to frame a justification for the alleged truth of a second-order abstraction principle of the form of (**AP2**). In this section I will show a possible strategy to fill the gap in the recarving procedure.

According to the proposed account of the recarving procedure, given a second-order equivalence relation $\mathscr{R}$, we obtain the following principle: for all concepts F, G, there are two concepts $(Fx)^{(\mathscr{R})}$ and $(Gx)^{(\mathscr{R})}$ obtained by combining the specific content of $\mathscr{R}$ respectively with the contents of F and G such that

$$\mathscr{R}_x(Fx, Gx) \longleftrightarrow ((Fx)^{(\mathscr{R})} \equiv_x (Gx)^{(\mathscr{R})}) \qquad \textbf{(GLA64)}$$

where the symbol '≡' is used to abbreviate the expression of the relation of second-order indistinguishability in higher-order logic.

In order to argue in favour of the truth of the abstraction principle (**AP2**), we need an argument to the effect that:

$$((Fx)^{(\mathscr{R})} \equiv_x (Gx)^{(\mathscr{R})}) \longleftrightarrow \sigma_x(Fx) = \sigma_x(Gx) \qquad (*)$$

In other words, we need an argument that justifies the possibility of *lowering the orders* by converting a second-order indistinguishability between two concepts into an identity of objects. In order to formulate an account of second-order abstraction that may be adapted to all cases, it is desirable that such an argument does not appeal to particular features of the concepts $(Fx)^{(\mathscr{R})}, (Gx)^{(\mathscr{R})}$ as well as of the abstraction operator σ. A general argumentative strategy may appeal to the relation of second-order indistinguishability itself; more precisely, we may invoke *a general principle of objectification of concepts* saying that whenever two concepts are indistinguishable, their objectifications are identical. Formally, we may express such a principle for two concepts A, B as follows:

$$(Ax \equiv_x Bx) \longleftrightarrow (o_x(Ax) = o_x(Bx)) \qquad \textbf{(PO)}$$

We call (**PO**) the principle of objectification. This principle has two virtues: 1) it is — to a certain extent — a general principle that may be used in the justification of different second-order abstractions, 2) it proposes a general strategy of lowering the orders. Considered an abstraction principle of the form of (**AP2**), each instance may be justified by considering that:

$$((Fx)^{(\mathscr{R})} \equiv_x (Gx)^{(\mathscr{R})}) \longleftrightarrow o_x((Fx)^{(\mathscr{R})}) = o_x((Gx)^{(\mathscr{R})})$$

Hence every instance of (**AP2**) should amount to the following equivalence:

$$\mathscr{R}_x(Fx, Gx) \longleftrightarrow o_x((Fx)^{(\mathscr{R})}) = o_x((Gx)^{(\mathscr{R})}) \qquad \textbf{(AP2'')}$$

Thus it is possible to make explicit the train of thought that might lead to a justification of second-order abstraction by recarving the content of a second-order equivalence relation. Firstly, the content of $\mathscr{R}$ is recarved so that a second-order indistinguishability between two special concepts obtains. Successively, by invoking the principle of objectification we introduce the objectual representatives of these new concepts which turn out to be the abstract objects introduced by the considered abstraction principle. Clearly, this strategy requires the truth of (**PO**) which is a controversial matter. Hence it is necessary to discuss this principle.

The first problem with (**PO**) regards its inconsistency. In fact, if we take (**PO**) as holding for all concepts A, B, it requires the existence of as many objects as indistinguishable concepts; given that there are more intensional concepts than extensions and given that there cannot be as many extensions as objects, (**PO**) requires the existence of too many objects. A possible way of avoiding paradoxes is to consider only certain instances of (**PO**), i.e. limiting the validity of (**PO**) to all concepts A, B such that the existence of their objectifications does not lead to paradoxes; yet it is not clear how to understand a criterion of discrimination between well and ill-behaved concepts. A second difficulty is related to the sort of objects that objectified concepts should be. Are they properties intended as first-order entities? Is then (**PO**) an acceptable criterion of identity for properties?

These difficulties may be mitigated by considering the relation of second-order indistinguishability as a relation of co-extensionality and by identifying objectified concepts with their extensions, i.e. set-like entities. Yet how could we justify this theoretical move?

The relation of second-order indistinguishability is a relation framed just in terms of logical vocabulary. Therefore, it depends on both the underlying logic and the significant range of third-order quantifiers. If we are interested in assessing the truth of second-order abstraction in mathematical contexts or in philosophical contexts that have to do with the foundations of mathematics, we may reasonably assume that the underlying logic and the relevant contexts of higher-order predication are both extensional (Linnebo 2006, Shapiro 1991, p.17). In other words, we may consider co-extensional concepts as interchangeable salva veritate in all relevant contexts.

The consequence of this assumption is that the relation of second-order indistinguishability is reduced to the relation of co-extensionality and the Basic Law V (or at least some of its instances) is taken as a principle of objectification of concepts:

$$(Ax \equiv_x Bx) \longleftrightarrow \forall x(Ax \longleftrightarrow Bx) \longleftrightarrow (\varepsilon_x(Ax) = \varepsilon_x(Bx))$$

As a result, (**AP2”**) has the following expression:

$$\mathscr{R}_x(Fx, Gx) \longleftrightarrow (\varepsilon_x((Fx)^{(\mathscr{R})}) = \varepsilon_x((Gx)^{(\mathscr{R})}) \qquad \textbf{(AP2*)}$$

(**AP2***) represents the proposed account of second-order abstraction. Given a second-order equivalence relation $\mathscr{R}$ and two concepts F, G, by recarving the content of $\mathscr{R}$ a relation of second-order indistinguishability between two new concepts $F^{(\mathscr{R})}, G^{(\mathscr{R})}$ is obtained. Provided that the relation $\mathscr{R}$ is considered only in extensional contexts, we may take a second-order indistinguishability to be a relation of co-extensionality. Thus by the correspondent instance of the Basic Law V — and under the assumption that $F^{(\mathscr{R})}$ and $G^{(\mathscr{R})}$ are well-behaved concepts — the sentence saying that F and G are related through $\mathscr{R}$ may be taken as an identity between the extensions of the concepts $F^{(\mathscr{R})}$ and $G^{(\mathscr{R})}$. As a consequence, the abstract of the concept F associated to the equivalence relation $\mathscr{R}$ is the extension of the concept $F^{(\mathscr{R})}$ obtained by combining the specific content of $\mathscr{R}$ with the content of F.

Before dealing with the possible objections to the argument I have proposed, it is worth highlighting its main conclusions:

1. It has been shown that if the procedure of content recarving is to be understood as a conversion of a relation of partial indistinguishability into a total indistinguishability, then the procedure is not straightforwardly applicable to the case of second-order abstraction, for it yields a second-order indistinguishability between concepts and not an identity of objects,
2. The application of the procedure to the second-order case requires a principle of objectification of concepts which — under the given assumptions — may be taken to be the Basic Law V,
3. The Basic Law V shall not be considered as a second-order abstraction principle among others, yet as a primitive principle of objectification of concepts (or of lowering orders). This conclusion may be drawn by considering that if the Basic Law V is regarded as a second-order abstraction on a par with the others, then the proposed account will be circular,
4. All abstracts introduced via consistent second-order abstraction principles are set-like entities, i.e. extensions of some special category of concepts.

I consider 1 and 2 as the negative outcome of the analysis and 3 and 4 as the positive one. Indeed, 1 and 2 show the limitations of the account of the content recarving procedure described in *Grundlagen* §64: it is impossible to obtain an identity of abstracts by a mere decomposition of the content of a second-order equivalence relation and without relying on a primitive principle of objectification. 3 and 4 show how the procedure of content recarving may be completed by appealing to allegedly consistent instances of the Basic Law V and thus along which general lines a justification of second-order abstraction may be formulated without jettisoning the recarving procedure.

I will come back to these points in the following sections while dealing with three fundamental objections to the proposed account of abstraction.

4. Objections and difficulties

The proposed account of second-order abstraction relies to a certain extent on the Basic Law V. This fact gives rise to two three intertwined charges:

1. *the charge of circularity:* the Basic Law V is commonly considered a second-order abstraction principle, for it amounts to the "conversion" of a second-order equivalence relation into an identity. However, the proposed account of second-order abstraction is framed in terms of the Basic Law V. Therefore, the proposed account is circular.
2. *the charge of inconsistency:* if the justification of all second-order abstraction principles appeals to an inconsistent principle — i.e. the Basic Law V — then either the proposed justification fails to account for the truth of second-order abstraction or it simply shows the inconsistency of all second-order abstraction principles. In the former case, the proposed account does not achieve its goal — i.e. a general strategy for justifying second-order abstraction — in the latter case, seems to conflict with the plausible assumption that there are consistent second-order abstraction principles. Thus the proposal is either useless or contradictory,
3. *the charge of incompleteness:* It has been said that the truth of a second-order abstraction principle may be justified by appealing to the operation of content recarving in conjunction with a principle of objectification of concepts. Assuming that the content recarving operation is sound, what guarantees do we have that the principle of objectification is true? In other words, the truth of a second-order abstraction principle depends upon the truth of a principle of objectification of concept for which no convincing justification has been provided; thus the proposed account of the truth of second-order abstraction is at best incomplete.

Regarding 1, I will not dispute the details of what should be taken as a definition of an abstraction principle; I will only show that the Basic Law V cannot be taken as a result of the recarving operation and thus is different from all other abstraction principles. Under the considered assumptions on the underlying logic and the context of third-order quantification, the relation of co-extensionality has been taken as a relation of second-order indistinguishability. Such a relation has a special status among second-order equivalence relations as identity has a special status among first-order ones: it is the most general equivalence relation, it is a congruence with respect to all contexts, it expresses *no specific content*. If the relation of co-extensionality has no specific content, then it is not possible to apply the recarving procedure of §64. As a consequence, whatever may be a justification for the truth of the consistent instances of the Basic Law V, it would not be based on an operation of recarving the content of the relation of co-extensionality as described in section 2. As in the case of identity, co-extensionality is the kind of equivalence relation whose content cannot be divided into a general and a specific part — since there is no specific part — and thus there is no content to be removed from the relation of co-extensionality by means of which we may form new abstract terms.

If the relation of co-extensionality cannot be recarved as described, then the Basic Law V is not the sort of principle that may be justified in the way attempted in this paper. Hence, the Basic Law V is a different sort of principle: it is not the kind of definition that may be called *an abstraction by content recarving*. It is clear that in this paper when I speak of second-order abstraction principles I am referring to those which may be accounted for by the recarving operation. The Basic Law V is not the result of an abstraction by recarving and it is used to justify the truth of all principles resulting from an abstraction by recarving: there is no circularity.

Regarding 2, I remark that the use of the Basic Law V does not condemn all second-order abstractions to inconsistency, for this depends on the instances of the Basic Law V that are taken into account. In fact, nowhere we have used the Basic Law V as a general (i.e. universally quantified) principle. Consider the general expression of (**AP2***): for all concepts A, B there are two concepts $A^{(\mathscr{R})}, B^{(\mathscr{R})}$ obtained by combining the specific content of $\mathscr{R}$ respectively with the content of A and B such that,

$$\mathscr{R}_x(Ax, Bx) \longleftrightarrow (\varepsilon_x((Ax)^{(\mathscr{R})}) = \varepsilon_x((Bx)^{(\mathscr{R})}))$$

By endorsing this account of second-order abstraction we are not requiring that all concepts A, B must have an extension, yet we require the existence of the extensions only of the concepts of the sort of $(Ax)^{(\mathscr{R})}, (Bx)^{(\mathscr{R})}$. If among all concepts obtained by the application of the specific content of $\mathscr{R}$ there is at least one that is ill-behaved — i.e. for which the assumption that it has an extension leads to contradiction — it is a matter that cannot be discussed without specifying what sort of concepts are $(Ax)^{(\mathscr{R})}, (Bx)^{(\mathscr{R})}$.

It is worth noting that the proposed account of abstraction suggests an interesting view on the problem of the consistency of second-order abstraction principles. I have shown that the formulation of a second-order abstraction principle may be achieved (and understood) in two steps: the first step amounts to the recarving operation which returns the indistinguishability of two concepts (i.e. their co-extensionality):

$$\mathscr{R}_x(Ax, Bx) \longleftrightarrow \forall x[(Ax)^{(\mathscr{R})} \longleftrightarrow (Bx)^{(\mathscr{R})}x] \qquad (\mathbf{1})$$

the second step is represented by the objectification of the concepts appearing in the right limb of (**1**) by means of the correspondent instance of the Basic Law V:

$$\forall x[A^{(\mathscr{R})}x \longleftrightarrow B^{(\mathscr{R})}x] \longleftrightarrow [\varepsilon_x((Ax)^{(\mathscr{R})}) = \varepsilon_x((Bx)^{(\mathscr{R})})] \qquad (\mathbf{2})$$

The first stage of the procedure simply requires the existence of as many equivalence classes of co-extensionality of concepts of the sort of $A^{(\mathscr{R})}$ as $\mathscr{R}$-equivalence classes. Thus (**1**) requires the existence of at least as many extensional concepts as $\mathscr{R}$-equivalence classes of concepts. Under the assumption that the relation of co-extensionality is a second-order indistinguishability, this condition is always met. The second stage of the procedure requires the existence of as many objects as equivalence classes of co-extensionality of concepts of the form $A^{(\mathscr{R})}$. By (**1**), there are as many equivalence classes of co-extensionality of concepts of the form $A^{(\mathscr{R})}$ as $\mathscr{R}$-equivalence classes. Thus (**2**) is requiring the existence of as many objects as $\mathscr{R}$-equivalence classes of concepts. It is only at this point that the inconsistency may arise, for instance when there are more $\mathscr{R}$-equivalence classes of concepts than objects in the first-order domain. Hence, the proposed account of second-order abstraction helps us to understand the fundamental reason for the possible inconsistency of a second-order abstraction: any possible contradiction following from a second-order abstraction is not engendered by the assumption that the content of a second-order equivalence relation is recarved in the way represented by (**1**), yet by the extent to which every second-order abstraction requires the objectification of concepts. Recarving the content of $\mathscr{R}$ as in (**1**) is a harmless operation; objectifying concepts may be dangerous, insofar as it may appeal to the idea that every concept has an objectual representative, which contradicts Cantor's theorem. This is the reason why we don't expect first-order abstraction principles to be inconsistent: they are justified solely by the recarving procedure, they do not require the objectification of any concept.

From the historical point of view, the proposed account offers an interesting conjecture regarding the fact that the possible inconsistency of second-order abstraction has escaped Frege's analysis. Frege may have overlooked the fact that a second-order abstraction principle — like Hume's principle — may be inconsistent precisely because he may have overlooked the difference in applying the recarving procedure between first and second-order abstraction. In (Frege 1950), Frege does contemplate the possible inconsistency of abstraction. In §65 he exposes what he calls "the second doubt" on the acceptability of the abstraction principle of directions: the possibility that by recarving the content of a first-order relation R as an identity of abstracts we enter in conflict with the laws of identity. And, when the recarving procedure is concerned, this might really be the *only reason* that may give rise to contradictions. Yet Frege rapidly dismisses this doubt, by considering that if R is an equivalence relation, there is no conflict with the laws of identity. Given that Frege is concerned only with the outcome of the recarving procedure, he sees no possible inconsistency in abstraction principles precisely because there is no possible inconsistency in the recarving procedure. What he overlooked is that when second-order abstraction is concerned, the recarving procedure is insufficient to justify an identity of abstracts and what we need is a general principle of objectification of concepts which is where possible inconsistencies may lurk.

Regarding 3, it has to be said that the objection does not harm the proposed account of second-order abstraction. In fact, my purpose was not to provide a conclusive argument in favour of the truth of second-order abstraction principles. My purpose is to show that there is an alternative strategy to attempt the justification of second-order abstraction based on the content recarving operation and that such a strategy requires the use of certain instances of a principle of objectification of concepts which has been identified with the Basic Law V. Suppose that there were an argument to the effect that every instance of the Basic Law V is false: then I would have shown that it is impossible to justify the truth of second-order abstraction principles by invoking a content recarving procedure as understood in section 2. In other words, my purpose was not to show that there are no limitations in proposing an account of second-order abstraction by recarving, yet that possible limitations depend upon the need for a principle of objectification of concepts and not upon the recarving procedure itself.

However, the third charge turns our attention on an interesting issue, i.e. the extent to which the generality of a second-order abstraction principle is limited by a certain underlying theory of extensions, e.g. a theory based on some consistent revised version of the Basic Law V (Boolos 1986, Shapiro 2003). Consider the case of the formulation of Hume's Principle according to which the second-order variables are meant to range over all concepts. As shown in Boolos (1987), such an abstraction principle is relatively consistent (equiconsistent with second-order arithmetic). Yet another issue is whether the universally quantified formulation of Hume's Principle is true and this is precisely what my proposal is meant to assess. Given a concept F, call the concept $F^{(\approx)}$ — obtained by combining the specific content of the relation of equinumerousity with the content of F — *the cardinality concept associated with* F. Consider now the concept 'self-identical'; under a set theoretic account of extensions, the assumption that this concept forms a set leads to a contradiction. Given that I have said nothing on the nature of cardinality concepts, one may suppose that whenever a concept does not form a set, also its associated cardinality concept fails to form a set; hence there is no object such as the extension of the cardinality concept associated with the concept 'self-identical' and thus there is no cardinal number of this concept. As a consequence, the cardinal number of the universe — *the anti-zero* as Boolos (1998) calls it — does not exist and it is not true that *for every* two concepts F, G, F and G are equinumerous iff they have the same number. Under the given assumptions on the behaviour of cardinality concepts, the proposed account says that Hume's Principle is true just in case we consider well-behaved concepts, i.e. concepts that form a set[3]. This example tells us that according to the proposed account of second-order abstraction, due to its dependence on a principle of objectification of concepts, the truth of an abstraction principle may be justifiable only by operating certain restrictions on the range of the second-order variables , even though the universally quantified formulation of the principle is consistent.

Another consequence related to the third objection has to do with the status of the chosen principle of objectification of concepts. One may argue that such principle is not a logical principle and thus that the operation of objectifying a concept is not a logical operation but perhaps something lying within the scope of a certain basic mathematical theory of set-like objects. As a consequence, second-order abstractions configure themselves as mathematical principles and not as logical definitions. This fact may engender some tensions with the idea that we may appeal to second-order abstraction principles — e.g. for cardinal and real numbers — to provide foundations to mathematical concepts from a logical perspective. For instance, Wright's Neo-logicist program (Wright 1983) may be an example of such a foundationalist approach. I cannot see to what extent this may be regarded as an undesirable outcome of my proposal; perhaps we may be in a situation very close to that in which Frege was: after attempting to understand abstraction more deeply, he concluded its inadequacy for a logicist foundation of mathematics. However, any possible tension between my proposal and the Neo-logicist program relies on the status of the notion of extension of a concept (i.e. concept objectified), precisely on the assumption that this notion belongs to the realm of mathematics and thus shall not qualify as a logical notion. On the other hand, the Neo-Logicist may conceive my proposal as part of a defense of a set-theoretic logicism. Surely to evaluate this issue will take us too far from the scope and the purpose of the present paper.

5. Conclusion

In this paper I have attempted the proposal of an alternative account of second-order abstraction principles starting from a partial explication of Frege's procedure of content recarving described in §64 of the *Grundlagen*. In particular, I have shown that while first-order abstraction principles may be justified solely on the ground of the operation of recarving the content of an equivalence relation, in the case of second-order abstraction an additional operation is required, i.e. concept objectification which ascribes a central role to the Basic Law V in the proposed account of second-order abstraction.

One of the main consequences of my proposal is that all abstracts introduced via second-order abstraction are extensions of concepts; the main limitation of the proposal is that I did not specify which sort of concepts define those extensions for a given second-order equivalence relation. Hence, the presented account of second-order abstraction has a residual indeterminacy. This fact may be seen as a higher-order version of the Caesar Problem: as in the first-order case we are not able to say which sort of objects e.g. directions should be, in the second-order case we are not able to say the extensions of what sort of concepts cardinal numbers should be. All we can say is that they are extensions of concepts for they are introduced by objectifying the concepts resulting from the content recarving operation.

Perhaps it is possible to overcome this limitation and to develop a detailed account of the operation of content recarving that may spell out what it means to combine the specific content of a second-order equivalence relation with the content of a conceptual expression; perhaps we cannot, and the indeterminacy of reference of abstract terms is a problem that cannot be solved by appealing to the recarving operation (as we may suppose Frege have thought). In any case, it seems that an attempt to complete my proposal requires a general theory of a certain notion of content which looks more like a general research program in philosophical logic rather than a specific investigation on the meaning of abstraction.

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Notes

[1]I thank an anonymous reviewer for this remark.
[2]If Leibniz's Law for identity is accepted, then the Caesar Problem and the Roman Problem entail each other.
[3]Wright explicitly considers the possibility of restricting the second-order variables in the expression of Hume's Principle only to well-behaved concepts.