On Some Cognitive Features of Clifford Algebraic Quantum Mechanics and the Origin of Indeterminism in this Theory: A Derivation of Heisenberg Uncertainty Principle by Using the Clifford Algebra

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Abstract: The main thesis of this paper is that quantum mechanics is a theoretical formulation that necessarily includes also the cognitive function as it is shown by using a Clifford algebraic formulation of this theory. Therefore, quantum mechanics is also a "physical" theory of cognitive processes and of the profound existing link between cognitive dynamics and physical reality per se. Rather recently we gave proof of quantum interference by using only Clifford algebra. In this paper we give proof of the Heisenberg's uncertainty principle by using the same algebraic structure. Therefore, the origins of the most fundamental quantum phenomena as indeterminism and quantum interference lie in traditional quantum physics as well as in its algebraic Clifford formulation.

1. A Clifford algebraic rough scheme of quantum mechanics

Let us state a proper definition of Clifford algebra.

The Clifford (geometric) algebra $Cl_{3,0}$ is an associative algebra generated by three vectors $e_1, e_2,$ and e_3 that satisfy the orthonormality relation

$$e_j e_k + e_k e_j = 2\delta_{jk}$$

for $j,k \in [1,2,3]$.

That is,

$$e_j^2 = 1 \quad \text{and} \quad e_j e_k = -e_k e_j \quad \text{for} \quad j \neq k$$

Let **a** and **b** be two vectors spanned by the three unit spatial vectors in $Cl_{3,0}$. By the orthonormality relation the product of these two vectors is given by the well known identity: $ab = a \cdot b + i(a \times b)$

where $i = e_1 e_2 e_3$ is an imaginary number that commutes with vectors.

Quantum mechanics may be formulated wholly using the Clifford algebraic structure [1].

Let us introduce a rough scheme of quantum mechanics [1].

For Clifford algebraic delineation we will utilize and follow the work that, starting with 1981, was developed by Y. Ilamed and N. Salingaros [2], using sometimes the same technique that these authors introduced in their work.

Let us anticipate that only two basic assumptions, quoted as (a) and (b), are required in order to formulate such rough scheme of quantum mechanics.

Let us consider three abstract basic elements, e_i, with $i = 1,2,3$, and let us admit the following two assumptions:

a) it exists the scalar square for each basic element:

$$e_1 e_1 = k_1 \ , \ e_2 e_2 = k_2 \ , \ e_3 e_3 = k_3 \quad \text{with} \quad k_i \in \Re \ . \qquad (1.1)$$

In particular we have also that

$$e_0 e_0 = 1 .$$

b) The basic elements e_i are anticommuting elements, that is to say:

$$e_1 e_2 = -e_2 e_1 \ , \ e_2 e_3 = -e_3 e_2 \ , \ e_3 e_1 = -e_1 e_3 . \qquad (1.2)$$

In particular it is

$$e_i e_0 = e_0 e_i = e_i .$$

Note that, owing to the axioms (a) and (b), we consider the given basic elements $e_i \, (i = 1,2,3)$ as abstract entities that we call potentialities since do not exist actual numerical entities satisfying both the (1) and the (2) simultaneously. In detail, by the (1.1), the e_i have the potentiality to simultaneously assume the numerical values $\pm k_i^{1/2}$. According to [2], let us introduce the necessary and the sufficient conditions to derive all the basic features of the algebra that we have just introduced. To give proof, let us consider the general multiplication of the three basic elements e_1, e_2, e_3, using scalar coefficients $\omega_k, \lambda_k, \gamma_k$ pertaining to some field:

$$e_1 e_2 = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \ ; \ e_2 e_3 = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \ ; \ e_3 e_1 = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 . \qquad (1.3)$$

Let us introduce left and right alternation:

$$e_1 e_1 e_2 = (e_1 e_1) e_2 \ ; \ e_1 e_2 e_2 = e_1 (e_2 e_2) \ ; \ e_2 e_2 e_3 = (e_2 e_2) e_3 \ ; \ e_2 e_3 e_3 = e_2 (e_3 e_3) \ ; e_3 e_3 e_1 = (e_3 e_3) e_1 \ ;$$
$$e_3 e_1 e_1 = e_3 (e_1 e_1) . \qquad (1.4)$$

Using the (1.4) in the (1.3) it is obtained that

$$k_1 e_2 = \omega_1 k_1 + \omega_2 e_1 e_2 + \omega_3 e_1 e_3 ; \qquad k_2 e_1 = \omega_1 e_1 e_2 + \omega_2 k_2 + \omega_3 e_3 e_2 ;$$
$$k_2 e_3 = \lambda_1 e_2 e_1 + \lambda_2 k_2 + \lambda_3 e_2 e_3 ; \qquad k_3 e_2 = \lambda_1 e_1 e_3 + \lambda_2 e_2 e_3 + \lambda_3 k_3 ;$$
$$k_3 e_1 = \gamma_1 e_3 e_1 + \gamma_2 e_3 e_2 + \gamma_3 k_3 ; \qquad k_1 e_3 = \gamma_1 k_1 + \gamma_2 e_2 e_1 + \gamma_3 e_3 e_1 \ . \tag{1.5}$$

From the (1.5), using the assumption (b), we obtain that

$$\frac{\omega_1}{k_2} e_1 e_2 + \omega_2 - \frac{\omega_3}{k_2} e_2 e_3 = \frac{\gamma_1}{k_3} e_3 e_1 - \frac{\gamma_2}{k_3} e_2 e_3 + \gamma_3 ;$$
$$\omega_1 + \frac{\omega_2}{k_1} e_1 e_2 - \frac{\omega_3}{k_1} e_3 e_1 = -\frac{\lambda_1}{k_3} e_3 e_1 + \frac{\lambda_2}{k_3} e_2 e_3 + \lambda_3 ;$$
$$\gamma_1 - \frac{\gamma_2}{k_1} e_1 e_2 + \frac{\gamma_3}{k_1} e_3 e_1 = -\frac{\lambda_1}{k_2} e_1 e_2 + \lambda_2 + \frac{\lambda_3}{k_2} e_2 e_3 \tag{1.6}$$

For the principle of identity , we have that it must be

$$\omega_1 = \omega_2 = \lambda_2 = \lambda_3 = \gamma_1 = \gamma_3 = 0 \tag{1.7}$$

and

$$-\lambda_1 k_1 + \gamma_2 k_2 = 0 \qquad \gamma_2 k_2 - \omega_3 k_3 = 0 \qquad \lambda_1 k_1 - \omega_3 k_3 = 0 \tag{1.8}$$

The (1.8) is an homogeneous system admitting non trivial solutions since its determinant $\Lambda = 0$, and the following set of solutions is given:

$$k_1 = -\gamma_2 \omega_3 , \ k_2 = -\lambda_1 \omega_3 \ , k_3 = -\lambda_1 \gamma_2 \tag{1.9}$$

Admitting $k_1 = k_2 = k_3 = +1$, it is obtained that

$$\omega_3 = \lambda_1 = \gamma_2 = i \tag{1.10}$$

In this manner, using the (1.3), a theorem, showing the existence of such algebra, is proven. The basic features of this algebra are given in the following manner

$$e_1 e_2 = -e_2 e_1 = i e_3 \ ; \ e_2 e_3 = -e_3 e_2 = i e_1 ; \ e_3 e_1 = -e_1 e_3 = i e_2 \ ; i = e_1 e_2 e_3 \tag{1.11}$$

The content of this theorem is thus established: given three abstract basic elements as defined in (a) and (b), an algebraic structure is established with four generators (e_0, e_1, e_2, e_3).

Of course, as counterpart, the (1.11) are well known also in quantum mechanics and the isomorphism with Pauli's matrices at various orders is well known and discussed in detail in [2].

We may now add some comments to the previous formulation.

Let us attempt to identify the phenomenological counterpart of the algebraic structure given in (1.1), (1.2), and (1.11) with

$$e_1^2 = 1 \ , \ e_2^2 = 1 , \ e_3^2 = 1 \tag{1.12}$$

A generic member of our algebra is given by

$$x = \sum_{i=0}^{3} x_i e_i \tag{1.13}$$

with x_i pertaining to some field $\Re$ or C. The (1.12) evidences that the e_i are abstract potential entities, having the potentiality that we may attribute them the numerical values, or ± 1. Admitting to be $p_1(+1)$ the probability to attribute the value $(+1)$ to e_1 and $p_1(-1)$ the probability to attribute (-1), considering the same corresponding notation for the two remaining basic elements, we may introduce the following mean values:

$$< e_1 >= (+1) p_1(+1) + (-1) p_1(-1) \quad , \quad < e_2 >= (+1) p_2(+1) + (-1) p_2(-1) ,$$
$$< e_3 >= (+1) p_3(+1) + (-1) p_3(-1). \tag{1.14}$$

Selected the generic element of the algebra, given in (1.13), its mean value results

$$< x >= x_1 < e_1 > + x_2 < e_2 > + x_3 < e_3 > \tag{1.15}$$

Let us call

$$a = x_1^2 + x_2^2 + x_3^2 \tag{1.16}$$

so that

$$-a \le x_1 < e_1 > + x_2 < e_2 > + x_3 < e_3 > \le a \tag{1.17}$$

and

$$-1 \le \langle e_i \rangle \le +1 \quad i = (1,2,3) \tag{1.18}$$

The (1.17) must hold for any real number x_i, and, in particular, for

$$x_i = \langle e_i \rangle$$

so that we have the fundamental relation

$$\langle e_1 \rangle^2 + \langle e_2 \rangle^2 + \langle e_3 \rangle^2 \le 1 \tag{1.19}$$

See details of this proof in ref. [3] for quantum mechanics in simple matrix form and for its extension in Clifford algebra in ref. [1]

Let us observe some important things:

1) The (1.19), owing to the (1.14), says that probabilities for basic elements e_i are not independent and this is of basic importance to acknowledge the essential features of a rough quantum mechanical scheme.
2) The (1.19) still says that also mean values of e_i are not independent. In detail, the (1.19) may be considered to represent a general principle of ontic potentialities. We have here a formulation of a basic, irreducible, ontic randomness. In particular, it affirms that we never can attribute simultaneously, definite numerical values to two basic elements e_i. Let us consider, as example, $\langle e_3 \rangle = +1$, that is to say that $e_3 \to +1$, we have consequently that $\langle e_1 \rangle = \langle e_2 \rangle = 0$, that is to say that e_1 and e_2 are both in a complete condition of randomness. The values are equally probable, there is full indetermination. We have a condition of ontic potentiality.

In conclusion, by using only the axioms (a) and (b), by the (1.11), the (1.14) and the (1.19), we have delineated a rough scheme of quantum theory using only an algebraic structure. Let us observe that the elective role in our formulation is performed in particular from the axiom (b) that relates non commutativity of the basic elements. In this algebraic scheme some principles of the basic quantum theoretical framework result to be represented. In particular, this algebraic structure reflects the intrinsic indetermination and the ontic potentiality that are basic components of quantum mechanics. This means that, in absence of a direct numerical attribution, such basic elements are abstract entities that act having an intrinsic, irreducible, indetermination, an ontic randomness, an ontic potentiality. Therefore, by using such rough quantum mechanical scheme, we may explore what is the actual role of potentiality in nature, what is its manner to combine with actual elements of our reality and what is the manner in which potentiality may contribute to the general dynamics of systems in Nature.

Let us add still some other feature of the scheme that we have in consideration. Let us consider two generic elements of our algebra, given as in the (1.13), and let us indicate them by x and y. Owing to the (1.11), they will result in general not commutative, that is to say

$$xy \ne yx \tag{1.20}$$

However, under suitable conditions, non-commutativity may fail and such abstract entities return to be $xy = yx$

Starting with 1974, [2] we introduced a theorem showing that necessary and sufficient condition for two given algebraic elements, x and y, to be commutative is that

$$xy = yx \leftrightarrow x_j = \lambda\, y_j \;, \forall \lambda \,(\, j = 1,2,3\,) \tag{1.21}$$

The algebraic structure given in (1.1), (1.2), (1.11), and (1.19) admits idempotents. Let us consider two of such idempotents:

$$\psi_1 = \frac{1+e_3}{2} \quad \text{and} \quad \psi_2 = \frac{1-e_3}{2} \tag{1.22}$$

It is easy to verify that $\psi_1^2 = \psi_1$ and $\psi_2^2 = \psi_2$.

Let us examine now the following algebraic relations:

$$e_3\psi_1 = \psi_1 e_3 = \psi_1 \tag{1.23}$$

$$e_3\psi_2 = \psi_2 e_3 = -\psi_2 \tag{1.24}$$

Similar relations hold in the case of e_1 or e_2. The relevant result is that the (1.23) establishes that the given algebraic structure, with reference to the idempotent ψ_1, attributes to e_3 the numerical value of $+1$ while the (1.24) establishes that, with reference to ψ_2, the numerical value of -1 is attributed to e_3.

The conclusion is very important: the conceptual counter part of the (1.23) and (1.24) is that we are in presence of a self-referential process. On the basis of such self-referential process, as given in (1.23) and in (1.24), this algebraic structure is able to attribute a precise numerical value to its basic elements. Each of the three basic elements may " transitate" from the condition of pure potentiality to a condition of actualisation, that is to say, in mathematical terms, from the pure, symbolic representation of their being abstract elements to that one of a real number. Let us remember that, on the basis of the (1.19), this self-referential process may regard each time one and only one of the three basic elements. It is well known that self-referential processes relate the basic phenomenology of our mind and consciousness.

In conclusion, for the first time we have an algebraic structure that represents a rough quantum mechanical scheme and that, at the same time, evidences, on the basis of a self-referential process, that it is possible a transition from potentiality to actualisation. Other features of our formulation are given in [1]. It remains to evidence that a profound link exists between the idempotents prospected as example in the (1.22) and the traditional wave function that is introduced in standard quantum mechanics.

Let us consider the mean values of (1.22). We have that

$$2 < \psi_1 > = 1 + < e_3 > \quad \text{and} \quad 2 < \psi_2 > = 1 - < e_3 > \tag{1.25}$$

Using the last equation in (1.14) we obtain that

$$p_3(+1) = \frac{1 + < e_3 >}{2} \quad \text{and} \quad p_3(-1) = \frac{1 - < e_3 >}{2} \tag{1.26}$$

Therefore, considering the (1.22), we have that

$$p_3(+1) = < \psi_1 > \quad \text{and} \quad p_3(-1) = < \psi_2 > \tag{1.27}$$

The same result holds obviously when considering the basic elements e_1 or e_2. Considering that in quantum mechanics (Born probability rule), given the wave function $\varphi_{+,-}$, we have

$$\left|\varphi_{+,-}\right|^2 = p_{+,-} \tag{1.28}$$

we conclude that

$$\varphi_3(+) = \sqrt{< \psi_1 >} e^{i\vartheta_1} \quad \text{and} \quad \varphi_3(-) = \sqrt{< \psi_2 >} e^{i\vartheta_2} \tag{1.29}$$

In this manner we have evidenced that our rough scheme of quantum mechanics foresees the existence of wave functions as exactly traditional quantum mechanics makes.

We need here to make an important digression. Quantum mechanics runs usually about some fixed axioms. States of physical systems are represented by vectors in Hilbert spaces : historically, theoretical physicists as Planck, Bohr, Heisenberg, Pauli, Born, Dirac, established the rather general and consistent quantum mechanics in the form that is presently known to day. The question on the manner in which systems behave sometimes like particles and sometimes like waves as well as the question about the exact meaning of the complex wave functions are usually retained to represent examples of open question in the theory. In our opinion there is often no matter for such questions, and this is evidenced in our formulation about the rough quantum mechanical scheme by Clifford algebra. We consider the quantum wave function as the first evidence of the strong link existing between cognitive performance and linked physical description at some stages of our reality. Of course, we retain that superposition and interference effects by wave functions play a key role. We support that wave intensities and probability densities are not a matter of simple interpretation. It is often retained that that such interpretation is added to quantum mechanics as it may be established evaluating that the Born probability rule was in fact introduced and thus added to quantum mechanics for purposes of probabilistic interpretation of quantum theory. It is no matter of a so simple Born interpretation. There is instead a precise theorem, proved and published well before quantum mechanics, that shows the fundamental role of the superposition principle and the profound link existing between quantum wave functions and probability densities. The theorem was published in 1915 by Fejer and by Riesz [4]. There is an excellent paper by F.H. Frohner that, time ago, properly evidenced the profound existing link between probability theory and quantum mechanics [5]. For any purpose, we retain of importance to report here this theorem that states

$$0 \le \rho(x) \equiv \sum_{l=-n}^{n} c_l e^{ilx} \equiv \left|\sum_{k=0}^{n} a_k e^{ikx}\right|^2 \equiv \left|\psi(x)\right|^2$$

where the complex Fourier polynomial $\psi(x)$ has not restrictions, where instead to the Fourier polynomial $\rho(x)$ is imposed the requirement of its reality and non-negativity.

So, in conclusion, such required link exists and it is mathematically established. This is the matter in spite of the continuous claims that in quantum mechanics such link holds only on the basis of a given Born's interpretation.

Let us look now to another link existing between standard quantum mechanics and our rough quantum mechanical scheme. It is well known the central role that is developed in traditional quantum mechanics from density matrix operator. In our scheme of quantum mechanics, we have the corresponding algebraic member that is given in the following manner

$$\rho = a + be_1 + ce_2 + de_3 \tag{1.30}$$

with

$$a=\frac{|c_1|^2}{2}+\frac{|c_2|^2}{2},\ b=\frac{c_1^*c_2+c_1c_2^*}{2},\ c=\frac{i(c_1c_2^*-c_1^*c_2)}{2},\quad d=\frac{|c_1|^2-|c_2|^2}{2} \tag{1.31}$$

where in matrix notation, e_1, e_2, and e_3 are the well known Pauli matrices. The complex coefficients c_i ($i=1,2$) are the well known probability amplitudes for the considered quantum state

$$\psi=\begin{pmatrix}c_1\\ c_2\end{pmatrix} \quad \text{and} \quad |c_1|^2+|c_2|^2=1 \tag{1.32}$$

For a pure state in quantum mechanics it is $\rho^2=\rho$. In our scheme a theorem may be demonstrated that

$$\rho^2=\rho \leftrightarrow a=\frac{1}{2} \quad \text{and} \quad a^2=b^2+c^2+d^2 \tag{1.33}$$

The details of this our theorem are given in [6]. Written in matrix form we have also $Tr(\rho)=2a=1$. In this manner we have the necessary and sufficient conditions for ρ to represent a potential state or, in traditional quantum mechanics, to have a superposition of states.
We have to examine now quantum time evolution.
It is clear that the quantum like scheme we are discussing is based on the previously mentioned Clifford algebra. Of course, generally speaking, we may consider our quantum rough in analogy with Heisenberg quantum description.
In standard quantum theory, given the operator α connected to a given observable A, the mean value at a given time t will be given as

$$<\alpha_t>=(\psi_0,U^{-1}\alpha U\psi_0)$$

with U time evolution operator.
It is well known that we have

$$i\eta\frac{d<\alpha>}{dt}=i\eta<\frac{\partial\alpha}{\partial t}+\frac{1}{i\eta}[\alpha,H]> \tag{1.34}$$

and

$$\frac{d\alpha}{dt}=\frac{\partial\alpha}{\partial t}+\frac{1}{i\eta}[\alpha,H] \tag{1.35}$$

where H is the Hamiltonian of the system. It is well known that members of Clifford algebra transform according to

$$e_i^{'}=U^+e_iU \quad , \quad U^+U=1 \tag{1.36}$$

In [1] we give a rigorous proof of the (1.34) and of the (1.35) using the Clifford algebra.
Still we have to remember here that in the past there were attempts to go beyond the linear Schrodinger equation [7], but, as well as we know, nobody tried to do the same thing in the Heisenberg's picture. It is very important to outline here that in the non linear case, such two, Heisenberg and Schrodinger, representations, no more result to be equivalent.
We have in fact that

$$U=\exp(-i\eta Ht)=1-\frac{i}{\eta}Ht+(\frac{iH}{\eta})^2\frac{t^2}{2!}-(\frac{iH}{\eta})^3\frac{t^3}{3!}+........................ \tag{1.37}$$

and

$$\frac{dU}{dt}=-\frac{iH}{\eta}\left[1-\frac{i}{\eta}Ht+(\frac{iH}{\eta})^2\frac{t^2}{2!}+..........\right] \tag{1.38}$$

By using the Clifford rough scheme of quantum mechanics we are in the condition to take account also for such possible non linear processes in Heisenberg like quantum representation. Still, the manner in which such Hamiltonian may be constructed for psychological states in the Clifford algebra framework is given in [8].

This completes our brief exposition on a rough scheme of quantum mechanics.
We have now to investigate two basic statements.
First of all, we have to legitimate the reason, the importance, the advantage to use a Clifford algebraic formulation of quantum mechanics.

The second point is that we must give at least a qualitative explanation on the basic question because we retain that logic, language and thus human cognitive apparatus play a so fundamental role in quantum mechanics, while in classical mechanics they play only an auxiliary one.

In the (1.29) we have established a profound link between Clifford rough scheme of quantum mechanics and standard quantum theory formulated, as it is well known, by using the language of the wave functions, and, in a more axiomatic elaboration, by the well known formalism of Hilbert spaces and of linear hermitean operators. By using the theorem that was given in 1915 by Fejer and by Riesz and the excellent presentation given by F.H. Frohner, we have also expressed the link existing between wave function and probabilistic interpretation of quantum mechanics.

Here is the decisive point of our elaboration.

Let us abandon now our rough scheme of quantum mechanics and let us consider standard quantum theory.

Let us follow an excellent argument developed by Y. Orlov [9].

Let $\left|\varphi_n\right\rangle$, $n = 1,2,........$, be eigenstates of an observable(or of a number of commuting observables), K, considered here as discrete, with all φ_n different, and let $\left|\psi\right\rangle$ be an arbitrary state of a physical system with

$$\left|\psi\right\rangle = \sum_n a_n \left|\varphi_n\right\rangle .$$

An exact measurement of K with result $K = k_i$ leads to the reduction $\left|\psi\right\rangle \to \left|\varphi_i\right\rangle$. It is generally accepted in quantum mechanics that reduced states describe only conditions of physical systems subsequent to measurements.

Orlov shows the following Lemma.

The truth of a logical statement about a numerical value of a physical observable is itself an observable, and is represented in quantum mechanics by the pure state density matrix or by a sum of such matrices.

Ergo: owing the well known isomorphism between matrix representations and Clifford algebraic structure in our rough scheme of quantum mechanics, it follows that

The truth of a logical statement about a numerical value of a physical observable is itself an observable, and is represented in quantum mechanics by the pure state density matrix or by a sum of such matrices. And this is to say that it is represented in quantum mechanics by a Clifford algebraic structure.

This is the reason because our rough quantum scheme by Clifford algebra is so important. It relates directly the truth values of logical statements, this is to say the logic, the language and thus the cognitive performance of human subjects.

Orlov gives proof of this Lemma.

Consider the logical statement Λ_{φ_l} :

"The system is in state $\left|\varphi_l\right\rangle$ ***"*** *or, in short,*

$$"K = k_l"$$

The statement Λ_{φ_l} can be either true (the numerical value 1) or "false" (the value 0); these are the two values of the classical logical truth of statement Λ_{φ_l} .

Statements corresponding to different k_l 's (let us call them elementary) are mutually exclusive.

Λ_{φ_l} is true if the measurement of K results in k_l, and false otherwise.

So there is the important conclusion: Λ_{φ_l} is an observable and, like any other observable in quantum mechanics, it must be represented by a Hermitean operator, $\hat{\Lambda}_{\varphi_l}$, the truth operator of statement Λ_{φ_l}. The truth of Λ_{φ_l} is measured simultaneously with K and it is an observable, a quantum observable, a "cognitive like" observable which can be called a logical observable in Orlov definition.

For it we have (see the (1.22), (1.23), (1.24) of our Clifford elaboration)

$$\left(\hat{\Lambda}_{\varphi_l}\right)^2 = \hat{\Lambda}_{\varphi_l}$$

K and $\hat{\Lambda}_{\varphi_l}$ have a common set of eigenvectors $\left|\varphi_j\right\rangle$, $j = 1,2,........$, and we have

$$\hat{\Lambda}_{\varphi_l}\left|\varphi_j\right\rangle = \delta_{lj}\left|\varphi_j\right\rangle$$

where the eigenvalue (1) is for true and the eigenvalue (0) is for false, otherwise.

It is concluded that $\hat{\Lambda}_{\varphi_l}$ must be identified with the density matrix of the pure state $\left|\varphi_l\right\rangle$, and this is to say that

$$\hat{\Lambda}_{\varphi_l} = \left|\varphi_l\right\rangle\left\langle\varphi_l\right|$$

The lemma is shown. Since our choice of K was arbitrary, a density matrix of any pure state $\left|\psi\right\rangle$ defined in the Hilbert space of the physical system

$$\hat{\Lambda}_{\psi} = \left|\psi\right\rangle\left\langle\psi\right|$$

with

$$\hat{\Lambda}_{\psi}\left|\psi\right\rangle = \left|\psi\right\rangle$$

and

$$\hat{\Lambda}_{\overline{\psi}}\left|\overline{\psi}\right\rangle = 0$$

being $\left|\overline{\psi}\right\rangle$ any state orthogonal to $\left|\psi\right\rangle$, represents the logical statement

"The system is in state $\left|\psi\right\rangle$"

and the truth of such a statement depends on whether the physical system is really in state $\left|\psi\right\rangle$.

Orlov also shows that t every non elementary statement about numerical values of commuting physical observables is represented by a corresponding sum of density matrices. If there is a degeneracy such that the same state corresponds to $k_{l_1}, k_{l_2}, ...$ then the statement that the system is in this state is represented by summing the corresponding truth operators.

In conclusion the proof of the Lemma shows that logic and language have a fundamental role in quantum mechanics. We evidence that this role is described by using an algebraic Clifford framework. A profound role of logic and of language means only one thing: one cannot escape to consider the cognitive performance in quantum mechanics.

Of course, the possibility for quantum mechanics to have a role in cognitive functions of humans was for the first time advanced in detail by Y.A. Khrennikov [10] who evidenced the possible role of quantum interference in concrete cognitive experiments. We gave experimental proof of the correctness of such proposal[8,11] by experiments involving the perception and cognition of ambiguous figures in human subjects. We were also engaged in theoretical elaborations on this matter [11].

To conclude such section we have to explain in detail still the reason because logic and language, and thus the cognitive performance, as delineated by a Clifford algebraic formulation, explain a so fundamental role in quantum mechanics. In particular, Orlov [9]asks the reason because logic and language play such a decisive role in quantum mechanics, while in classical mechanics they play only an auxiliary one. He gives the following qualitative explanation on which we wholly agree (we add the terms as *cognition* and *cognitive performance* to enforce our viewpoint). Though undescribed Nature certainly exists, scientific knowledge and *cognition* of Nature exists only in the form of logically organized descriptions. When these descriptions become too precise at some level of accuracy, the fundamental features of logic and language and thus *of cognition* acquire the same importance as the features of what is being described. At this level, we no more may separate the features of matter per se from the features of the logic and language, and thus of the *cognitive performance*, that we use to describe it. To explain better, quantum mechanics can be understood if we realize that at the quantum level we encounter a kind of reality in which the logic, the language, and thus the cognitive performance become parts of Nature. Therefore, this understanding of quantum mechanics may be useful to elaborate and to develop a theory of mind.

We advance here the basic explanation that a cognitive function acquires the same importance in quantum mechanics as the features of what is being described because quantum mechanics takes in consideration the passage from ontic potentiality, that is to say of basic, irreducible randomness, to actualisation. A passage that instead is missing in standard , classical physics.

2. A Derivation of Heisenberg Uncertainty Principle by using a Clifford algebraic framework.

T.F. Jordan published a book on Quantum mechanics in simple matrix form [3] in which this author derived previously Heisenberg's uncertainty relation using matrices.

Consider the following two algebraic members of Clifford algebra

$$k \equiv k(e_1, e_2, e_3) = \alpha e_1 + \beta e_2 + \gamma e_3 ;$$

$$L \equiv L(e_1, e_2, e_3) = a e_1 + b e_2 + c e_3 ; \qquad (2.1)$$

with

$(\alpha, \beta, \gamma) \in \Re$ and $(a, b, c) \in \Re$

We have that

$$k^2 = \alpha^2 + \beta^2 + \gamma^2 \geq 0 \quad \text{and} \quad L^2 = a^2 + b^2 + c^2 \geq 0 \qquad (2.2)$$

According to our rough quantum scheme, they mean that k and L may assume respectively the numerical values

$$\pm\sqrt{\alpha^2+\beta^2+\gamma^2} \quad \text{and} \quad \pm\sqrt{a^2+b^2+c^2} \tag{2.3}$$

Let us consider the following Clifford member

$$U=(k+wL)(K+w^*L)=k^2+ww^*L^2+q(Lk+kL)+ip(Lk-kL) \tag{2.4}$$

where

$$(w=q+ip)\in C$$

It is

$$kL+Lk=2\alpha a+2\beta b+2\gamma c \tag{2.5}$$

and

$$Lk-kL=(2b\gamma-2c\beta)ie_1+(2c\alpha-2a\gamma)ie_2+(2a\beta-2b\alpha)ie_3 \tag{2.6}$$

Consider

$q=0$ and $p=1$

We have the Clifford algebraic element

$$U-k^2-L^2=Se_1+Re_2+Te_3 \tag{2.7}$$

where

$$S=2c\beta-2b\gamma \quad ; \quad R=2a\gamma-2c\alpha \quad ; \quad T=2b\alpha-2a\beta \tag{2.8}$$

Let us calculate the mean value of such algebraic element according to the formulation given in the previous section. We have that

$$<U>=<k^2>+<L^2>+<Se_1+Re_2+Te_3> \tag{2.9}$$

Let us consider three real numbers (m,n,r) such that

$$x+y=-\frac{m}{n} \quad \text{and} \quad xy=\frac{r}{n} \quad \text{with} \quad x=<k^2> \quad \text{and} \quad y=<L^2>$$

we have that

$$x+y=sxy \qquad \text{with} \quad s=-m/r$$

Consequently, we have that

$$<U>=s<k^2><L^2>+<Se_1+Re_2+Te_3> \tag{2.10}$$

Note that we have

$$<k^2><L^2>=(\alpha^2+\beta^2+\gamma^2)(a^2+b^2+c^2)=\omega+\lambda \tag{2.11}$$

where

$$\omega=b^2\alpha^2+c^2\alpha^2+a^2\beta^2+c^2\beta^2+a^2\gamma^2+b^2\gamma^2 \tag{2.12}$$

and

$$\lambda=a^2\alpha^2+b^2\beta^2+c^2\gamma^2 \tag{2.13}$$

Of course , according to our Clifford algebra, we have that

$$(Se_1+Re_2+Te_3)\,(Se_1+Re_2+Te_3)=S^2+R^2+T^2 \tag{2.14}$$

and this is to say that

$$-\sqrt{S^2+R^2+T^2}\le<Se_1+Re_2+Te_3>\le\sqrt{S^2+R^2+T^2} \tag{2.15}$$

In conclusion, the mean value $<Se_1+Re_2+Te_3>$ is a number enclosed between

$$\pm 2\sqrt{\omega-2bc\beta\gamma-2ac\alpha\gamma-2ab\alpha\beta}\,. \tag{2.16}$$

Therefore, always we have that

$$<k^2><L^2>\ \ge\ \left|\frac{1}{2}<kL-Lk>\right|^2. \tag{2.17}$$

Consider now the two following Clifford members

$$A=k+<A> \qquad \text{and} \qquad B=L+<B> \tag{2.18}$$

where k and L have been given in (2.1).

It is

$$kL-Lk=AB-BA \tag{2.19}$$

Inserting the (2.18) and the (2.19) in the (2.17), we obtain that

$$< (A-< A >)^2 >< (B-< B >)^2 > \geq \left| \frac{1}{2} < AB - BA > \right|^2 \quad (2.20)$$

that is the standard expression of Heisenberg uncertainty relation in quantum mechanics.
Note the salient feature that it has obtained only by algebraic means without recovering any concrete physics. Recently, we also obtained proof of quantum interference by using only Clifford algebraic means [1]. In this manner, a result of these papers is that the main quantum phenomena of indeterminism and interference may be explained on a purely logical basis by extending classical mathematical logic as Orlov made. In this extension, statements were represented by hermitean matrices, and we outline here by an algebraic Clifford structure. In this framework it is required that they transform according to unitary transformations, and thus they undergo unitary transformations of truth values, in accord with Orlov's postulated principle of relativity of truth values.

3. Conclusion

In his papers [9] Orlov outlines that two properties of classical mathematical logic are potentially quantum and become crucial at the micro-level of accuracy.
They are :
(a) The truth values of classical logic are naturally quantized.
(b) There exists a hidden, unformalized symmetry in classical logic — namely, any logical tautology remains the same tautology, regardless of how we change the meaning of the truth values of its constituent statements. The only requirement is that every newly redefined "no" be the negation of a correspondingly redefined "yes."
These are the two Orlov's fixed points.
He immediately gets a quantum-like logical system when he formalizes these properties using the assumptions that symmetries are linear, continuous and non Abelian [9].
The known features of quantum measurements are consistent with this logical picture. In fact, we know that in quantum mechanics, when our measurements become too precise, and this is to say that we reach a quantum scale of accuracy, we cannot exclude the influence of the measuring processes on the measured system. As result, we lose the possibility of unconditionally defining truth [9]. The definition of truth now depends on how we observe the physical system, on our choice of apparatus [9]. This logical relativism does not exist in classical mechanics. Here statements are precise logically also if not physically. As a result, all truth operators of logical statements about observables commute and truth values never require to be redefined (transformed). Not being transformed, classical truth values do not depend on the choice of apparatus. In quantum mechanics, truths of logical statements about dynamic variables become dynamic variables themselves because they depend on parameters of symmetry transformations that redefine truth values. If the truths of statements become dynamic variables, one may ask about the actual nature of such statements. Orlov indicates that the abstract sets of all possible languages and statements describing physical observables exist objectively, as do sets of the conditional truths of those statements, whereas the choice of a language and thus of the quantum representation and questions to be answered, as well as the formulation of statements describing the results of measurements, belong to the human subject.
Orlov poses also the other question whether we can verbally express statements when their truths are quantum operators. The answer is that since statements themselves are not operators, they can be (and always really are) expressed verbally in an ordinary way. That is to say, p and q are non commuting observables does not limit our formulations of any statements about exact numerical values of both non commuting p and q. In fact, when describing any observable informally, we always use its own representation. The truths of such statements, however, cannot be expressed explicitly and simultaneously for both p and q. If, for example, it is true that $p = p_0$, then nothing precise can be said about the truth of the statement $q = q_0$. Still, we can describe numerically the conditional truth of this statement, under the condition $p = p_0$. Finally, logical statements are not distinguished from their truth, so truth operators are simultaneously operators of statements themselves.

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