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On Some Considerations of Mathematical Physics: May we Identify Clifford Algebra as a Common Algebraic Structure for Classical Diffusion and Schrödinger Equations?

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Abstract

We start from previous studies of G.N. Ord and A.S. Deakin showing that both the classical diffusion equation and Schrödinger equation of quantum mechanics have a common stump. Such result is obtained in rigorous terms since it is demonstrated that both diffusion and Schrödinger equations are manifestation of the same mathematical axiomatic set of the Clifford algebra. By using both such $A(S_i)$ and the $N_{i,\pm 1}$ algebra, it is evidenced, however, that possibly the two basic equations of the physics cannot be reconciled.

1. INTRODUCTION

It is well known that several tools enable us to derive the governing equations of classical dynamics from Lagrange equations and variational principles or directly from classical Newton's laws of motion. In all the cases we arrive to consider differential equations that are admitted to describe the dynamical system under consideration.

Very often, however, it is tacitly admitted that they express the universal deterministic behavior of our reality. Instead we outline that determinism is not spontaneously exhibited from such equations. On the contrary, determinism is forced to be exhibited from such equations imposing from the outside that they must satisfy Lipschitz conditions. They are a mathematical restriction that we impose from outside to the set of considered differential equations, and such imposed restriction guarantees the uniqueness of solutions when fixed the initial conditions.

However, there are some results that legitimate doubts on the validity of such conditions as universally admitted as it has been largely discussed by us elsewhere [1]. There are many cases in which Lipschitz conditions do not result compatible with the physical or the biological nature of the dynamics in consideration. The finding of Lipschitz violation in different systems regarding biological and physical dynamics, constitutes a promising acquisition in a current attempt to coherently and correctly describe the time dynamics of reality. The discovery of chaos contributed to better understanding the dynamics of evolution of systems as well as the interpretation and modeling of complex phenomena in physics and biology. The discovery of possible Lipschitz violation in a given set of differential equations cleared up on a new step in the framework of such chaotic studies. It follows from Lipschitz violation that a class of phenomena cannot be represented by deterministic chaos. In these cases the behavior of systems is governed by a new kind of dynamics that has been called discrete event dynamics. Here randomness appears as point events so that there is a sequence of random occurrences at fixed or random times but there are not additional components of uncertainty between these times [2-12]. In conclusion, the role of the mathematics is central in physics, and, in this particular case, it evidences in a substantial manner that are possible real conditions in which the determinism, generally assumed as an universal tenet regarding physical and biological dynamics, may be violated.

In an excellent paper M. Zak [13-17] gave a non-Lipschitz approach to quantum mechanics. He made an effort to reconcile quantum mechanics with Newton's laws corrected this time by the non Lipschitz formalism. In this paper we should pursue a step one. We would attempt to investigate if we may acknowledge an algebraic structure from which both classical diffusion equation and Schrödinger equation may be derived. It is well known that diffusion equation and Schrödinger equation evidence some profound conceptual differences in physics. We should be able to derive their common algebraic structure, if existing and the mathematical point in which such two basic and fundamental equations of physics bifurcate.

2. INTRODUCTORY MATHEMATICS

Let us fix the basic mathematical framework of our paper.

The Clifford algebra Cl_3 of $\Re^3$ is the real associative algebra generated by the set of abstract mathematical objects (e_1, e_2, e_3) satisfying the following two basic axioms

$$e_1^2 = e_2^2 = e_3^2 = 1 \tag{2.1}$$

and

$$e_1 e_2 = - e_2 e_1 ; \quad e_1 e_3 = - e_3 e_1 ; \quad e_2 e_3 = - e_3 e_2 \tag{2.2}$$

The (2.1) and the (2.2) represent the two basic mathematical axioms of Clifford algebra.

In this paper we will show that only such two mathematical axioms are required in order to derive both the diffusion and Schrödinger equations.

The employed algebra is 8-dimensional with the following basis

$$\begin{array}{ll} 1 & \text{the scalar} \\ e_1, e_2, e_3 & \text{vectors} \\ e_1 e_2, e_1 e_3, e_2 e_3 & \text{bivectors} \\ e_{123} = e_1 e_2 e_3 & \text{a volume element} \end{array} \tag{2.3}$$

An arbitrary element in Cl_3 is the sum of a scalar, a vector, a bivector, and a volume element, and it must be written in the following manner

$$q = \gamma + \mathbf{a} + \mathbf{b}\, e_{123} + \mu\ e_{123} \tag{2.4}$$

with $q \in Cl_3$; $\gamma, \mu \in \Re$,and $a, b \in \Re^3$.

As in particular we will discuss also in the following section, we are mainly concerned with the problem of a matrix representation of Cl_3. Therefore, let us denote the set of 2 x 2 matrices with complex numbers as entries by Mat (2, C). This set may be regarded also as a real algebra with scalar multiplication taken over the real numbers in $\Re$ also if the matrix entries are in the complex field C.

Let us remember that the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.5}$$

satisfy the multiplication rules

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I$$

and (2.6)

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\,\sigma_3$$
$$\sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\,\sigma_2$$
$$\sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\,\sigma_1$$

The (2.5) and the (2.6) generate the real algebra Mat(2,C).

The correspondence

$$\begin{array}{lcl} I & \leftrightarrow & 1 \\ \sigma_1,\sigma_2,\sigma_3 & \leftrightarrow & e_1, e_2, e_3 \\ \sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_1\sigma_3 & \leftrightarrow & e_1e_2, e_2e_3, e_1e_3 \end{array} \tag{2.7}$$

establish an isomorphism between real algebras. We have that

$$Cl_3 \cong \text{Mat}\ (2, C) \tag{2.8}$$

An essential difference between the Clifford algebra Cl_3 and Mat (2, C) remains in the fact that in Cl_3 we distinguish, by definition, a particular subspace, the vector space $\Re^3$, while no distinguished subspace is signed in the definition of Mat (2, C).

The theory of Clifford algebra includes basically the statement that each Clifford algebra is isomorphic to a matrix representation.

Idempotents are members of the previous algebra with the particular well known property that, when squared, give themselves.
For Cl_2, we have two basic members (basic elements) e_i (i = 1,2) ($e_1e_2 = - e_2e_1$), and one such idempotent involves only one basic element, i.e.,

$$\psi_1 = \frac{1}{2}(1 + e_1), \qquad \psi_1\psi_1 = \psi_1 \tag{2.9}$$

If the idempotent is multiplied by the other basic element, e_2, other functions can be generated, as it follows:

$$\psi_2 = e_2\psi_1 = (\frac{1}{2} - \frac{1}{2}e_1)\, e_2; \quad \psi_3 = \psi_1 e_2 = (\frac{1}{2} + \frac{1}{2} e_1)\, e_2;$$

$$\psi_4 = e_2\psi_1 e_2 = \frac{1}{2} - \frac{1}{2} e_1. \tag{2.10}$$

In addition, we have also that

$$\psi_1 e_1 \psi_1 = 0$$

The important thing that we must retain for the following arguments is that the four functions ψ_i (i=1, 2, 3, 4) provide means to represent any member of the space. A general member q is given in terms of the basis members of the algebra in the following manner

$$q = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_1e_2 \tag{2.11}$$

and q may be represented by the series of terms of the idempotents

$$q= a_{11}\psi_1 + a_{21}\psi_2 + a_{12}\psi_3 + a_{22}\psi_4 \tag{2.12}$$

with

$$a_{11}+a_{22} = 2a_0 \qquad a_{12} + a_{21} = 2a_2 \tag{2.13}$$
$$a_{11} - a_{22} = 2a_1 \qquad a_{12} - a_{21} = 2a_3$$

On the other hand, calculating $\psi_1 q\psi_1$, $\psi_1 q\psi_2$, $\psi_3 q\psi_1$, $\psi_3 q\psi_2$, one finds that a matrix A may be defined

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{2.14}$$

with

$$a_{11} = a_0 + a_1, \quad a_{22} = a_0 - a_1, \quad a_{12} = a_2 + a_3, \quad a_{21} = a_2 - a_3$$

and

$$(1 \; e_2)\, A\, \psi_1 \begin{pmatrix} 1 \\ e_2 \end{pmatrix} = q = a_{11}\psi_1 + a_{21}\psi_2 + a_{12}\psi_3 + a_{22}\psi_4 \tag{2.15}$$

Thus, we may conclude that the (2.15) generates the general Clifford number q. When equating q with 1, e_1, e_2, e_1e_2, respectively, we obtain the final matrix expressions multiplied by the idempotent:

$$q = 1,\; A\psi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\psi_1; \quad q = e_2,\; A\psi_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\psi_1$$

$$q = e_1,\; A\psi_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\psi_1; \quad q = e_1e_2,\; A\psi_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\psi_1 \tag{2.16}$$

These are the usual basis matrices for Cl_2.

A set of basis matrices for Cl_3 may be obtained following the same previous procedure. In this case one obtains that

$$e_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.17}$$

With a proper choice of the idempotent, we may also arrive to express new sets of basis matrices at n = 4 , 8 , ...

They are expressed in the following manner

$$E_{0i} = I^1 \otimes e_i ; \qquad E_{i0} = e_i \otimes I^2 \tag{2.18}$$

The notation $\otimes$ denotes direct product of matrices, and I^i is the ith 2x2 unit matrix. Thus, in the case of n= 4 we have two distinct sets of basis matrices, $E_{0\,i}$ and $E_{i\,0}$, with

$$E_{0i}^2 = 1 ; \qquad E_{i0}^2 = 1 ; \qquad i = 1, 2, 3; \tag{2.19}$$

$$E_{0i}\, E_{0j} = i\, E_{0k} ; \qquad E_{i0}\, E_{j0} = i\, E_{k0} ; \qquad j = 1, 2, 3; \qquad i \neq j$$

and

$$E_{i0}\, E_{0j} = E_{0j}\, E_{i0} \tag{2.20}$$

with (i, j, k) cyclic permutation of (1, 2, 3).

Let us examine now the following result

$$(I^1 \otimes e_i)\,(e_j \otimes I^2) = E_{0i}\, E_{j0} = E_{ji} \tag{2.21}$$

We have that $E_{0i}\, E_{j0} = E_{ji}$ with i = 1, 2, 3 and j =1, 2, 3, with $E_{ji}^2 = 1$,

$E_{ij}\, E_{km} \neq E_{km}\, E_{ij}$, and $E_{ij}\, E_{km} = E_{pq}$ where p results from the cyclic permutation (i, k, p) of (1, 2, 3) and q results from the cyclic permutation (j, m, q) of (1, 2, 3).

In the case n = 4 we have two distinct basic matrices E_{0i} , E_{i0} and, in addition, basic sets of unities (E_{ij}, E_{ip}, E_{0m}) with (j, p, m) basic permutation of (1, 2, 3). Similarly, we may realize other basic sets of basis matrices using (E_{ji}, E_{pi}, E_{m0}).

Note the basic role explained from cyclic permutations of (1, 2, 3) and their strong connection with non commutativity of the chosen basic elements.

In the case of matrix representation at order n= 8, we have the possibility to introduce three sets of biquaternion basic unities. We will have E_{00i}, E_{0i0}, and E_{i00}, i = 1, 2, 3 and

$$E_{00i} = I^1 \otimes I^1 \otimes e_i ; \qquad E_{0\,i0} = I^2 \otimes e_i \otimes I^2 ; \qquad E_{i\,0\,0} = e_i \otimes I^3 \otimes I^3$$

and

$$(I^1 \otimes I^1 \otimes e_i)(I^2 \otimes e_i \otimes I^2)(e_i \otimes I^3 \otimes I^3) = e_i \otimes e_i \otimes e_i = E_{00i} E_{0i0} E_{i00} = E_{iii} \tag{2.22}$$

Still we will have that

$$E_{00i}\, E_{0i0} = E_{0i0}\, E_{i00}; \quad E_{00i}\, E_{i00} = E_{i00}\, E_{00i} ; \quad E_{0i0}\, E_{i00} = E_{i00}\, E_{0i0} \tag{2.23}$$

In the case n = 8 we have three distinct basic unities, and, in addition, we have other sets of basis matrices, as example (E_{ijk}, E_{iji}, E_{ijj}). Other cases are obviously possible.

Generally speaking, fixed the order n of the matrix representation of the set of basis matrices, we will have that

$$\Gamma_1 = \Lambda_n$$

$$\Gamma_{2m} = \Lambda_{n-m} \otimes e_2^{(n-m+1)} \otimes I^{(n-m+2)} \otimes \ldots \ldots \ldots \otimes I^n \tag{2.24}$$

$$\Gamma_{2m+1} = \Lambda_{n-m} \otimes e_3^{(n-m+1)} \otimes I^{(n-m+2)} \otimes \ldots \ldots \ldots \otimes I^n$$

$$\Gamma_{2n} = e_2 \otimes I^{(2)} \otimes \ldots \ldots \ldots \otimes I^n$$

with
$\Lambda_n = e_1^{(1)} \otimes e_1^{(2)} \otimes .....\otimes e_1^{(n)} = (e_1 \otimes I^{(1)} \otimes .....\otimes I^n)\cdot(........)\cdot(I^{(1)} \otimes I^{(2)} ...\otimes I^{(n)} \otimes e_1)$;
m = 1, ..., n - 1
according to the n-possible dispositions of e_1 and $I^1, I^2, ..., I^n$ in the distinct direct products.
Basis matrices are determined by the number of dispositions possible for e_i and $I^{(n)}$.
In this manner we have established that matrix representations of Cl_3 exist at different orders n = 2, 4, 8, ...
We may now give the explicit expressions of E_{0i}, E_{i0}, and E_{ij}.

$$E_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad E_{02} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}; \quad E_{03} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$E_{10} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \quad E_{20} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad E_{30} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$E_{11} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; E_{22} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \quad E_{33} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad (2.25)$$

$$E_{12} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}; \quad E_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}; \quad E_{21} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix};$$

$$E_{31} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}; \quad E_{23} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}; \quad E_{32} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

Let us return to our Cl_3 mathematical formulation and let us explore still some other important features of the idempotents.
In order to semplify our elaboration we remain to consider matrix representation of Cl_3 at order n=2.
Consider that, with regard, as example, to the basic element e_3 of Cl_3, we may identify an idempotent ψ, and we write that

$$e_3 \psi = \psi \quad \text{and} \quad \psi e_3 = \psi \qquad (2.26)$$

Still, we may identify an idempotent φ so that

$$e_3 \varphi = - \varphi \quad \text{and} \quad \varphi e_3 = - \varphi \qquad (2.27)$$

We may also generalize such definitions.
In fact, we may have that

$$e_3\,\psi_1 = \psi_1 \quad \text{and} \quad \psi_2 \quad e_3 = \psi_2 \tag{2.28}$$

and

$$e_3\,\varphi_1 = -\varphi_1 \quad \text{and} \quad \varphi_2\,e_3 = -\varphi_2 \tag{2.29}$$

Calculations show that in the case of the (2.25), we have that

$$\psi = \frac{1+e_3}{2} \tag{2.30}$$

In the case of the (2.26), we have

$$\varphi = \frac{1-e_3}{2} \tag{2.31}$$

In the case of the (2.27), we have

$$\psi_1 = \frac{1+e_3}{2} + \frac{e_1+ie_2}{2} \quad \text{and} \quad \psi_2 = \frac{1+e_3}{2} + \frac{e_1-ie_2}{2} \tag{2.32}$$

In the case of the (2.28), we have

$$\varphi_1 = \frac{1-e_3}{2} + \frac{e_1-ie_2}{2} \quad \text{and} \quad \varphi_2 = \frac{1-e_3}{2} + \frac{e_1+ie_2}{2} \tag{2.33}$$

Consider again the (2.26) that now we rewrite as it follows

$$(e_3 - 1)\,\psi = 0 \quad \text{and} \quad \psi(e_3 - 1) = 0 \tag{2.34}$$

As rigorously admitted from our elaboration of the previous pages, let us calculate now the (2.34) with regard to e_i, i = 1,2 . We obtain that

$$e_i\,(e_3 - 1)\,\psi = 0 \quad \text{and} \quad \psi\,(e_3 - 1)\,e_i = 0 \tag{2.35}$$

Thus, we may conclude, with regard to ψ that, as example, we have that

$$(i\,e_1 - e_2)\,\psi = 0 \quad \text{and} \quad \psi\,(i\,e_1 + e_2) = 0 \tag{2.36}$$

Considering the (26), we may rewrite that

$$(e_3 + 1)\,\varphi = 0 \quad \text{and} \quad \varphi\,(e_3 + 1) = 0 \tag{2.37}$$

We may calculate the (2.36) with regard to e_i, i = 1,2, obtaining

$$e_i\,(e_3 + 1)\,\varphi = 0 \quad \text{and} \quad \varphi\,(e_3 + 1)\,e_i = 0 \tag{2.38}$$

Thus we may conclude that with regard to φ we have that

$$(i\,e_1 + e_2)\,\varphi = 0 \quad \text{and} \quad \varphi\,(i\,e_1 - e_2) = 0 \tag{2.39}$$

In previous papers we have largely used Clifford algebra to realize a bare bone skeleton of quantum mechanics and therefore we will not discuss such basic features furterly here.[18-19]

3. THE DERIVATION OF DIFFUSION AND SCHRÖDINGER EQUATIONS

Let us attempt to show that we may derive Diffusion and Schrödinger equations using only the two basic algebraic axioms (2.1) and (2.2) that we introduced in the previous section without assuming quantum mechanics physical foundations.
In this elaboration we will utilize some results that were previously obtained by G.N. Ord and A.S. Deakin [20-22].

Consider the diffusion equation in 1+1 dimensions

$$\frac{\delta u}{\delta t} = D\frac{\delta^2 u}{\delta x^2} \tag{3.1}$$

According to A. Einstein in 1905 it goes with a microscopic model of Brownian motion with u(x,t) describing the ensemble average concentration of small particles undergoing such Brownian motion on a scale much less than the scale of observation.
Schrödinger equation is written as

$$\frac{\delta \psi}{\delta t} = iD\frac{\delta^2 \psi}{\delta x^2} \tag{3.2}$$

It is seen that apparently one may obtain the (3.2) from the (3.1) in a mathematical way considering the time coordinate to be imaginary in the previous diffusion equation (3.1). This was correctly indicated to be a mathematical extension, a formal analytic continuation (FAC) and this indicated that we may compare the solutions of the two equations also if the solutions of diffusion equations result qualitatively very different from those of free particle Schrödinger's equation.
G.N. Ord and A.S. Deakin [20-22] discussed in detail this problem. According to such authors, both diffusion and Schrödinger equation happen in the domain of classical statistical mechanics. The model they used is the standard lattice random walk model of Brownian motion: solutions of the diffusion equation appeared directly in a first order projection out of the considered space while solutions of Schrödinger's equation appeared directly in a second order projection. The conclusion was that two qualitatively different behaviors coexist in the same physical system since the two considered projections resulted to be orthogonal. Ord and Deakin's conclusion was that no FAC is required to be conceived in order to introduce Schrödinger equation from this classical model: both the equations delineate different views of the same system represented by an ensemble of Brownian particles. FAC indicated a link of the two equations, while, according to [20-22], in this model we have a "wave function" that is an observable property of the examined ensemble of real point particles.
As basic framework the authors [20-22] considered a space-time lattice with spacings δ and ε. Particles were considered to hop on lattice a distance $\pm\,\delta$ at each time step ε. The walks are symmetric and at each lattice site walks are equally likely to take either direction. Also we shall be interested in the statistics of the number of direction changes in trajectories on the lattice. G.N. Ord and Deakin considered that between lattice sites, each particle will be in one of two direction states, the right or the left movement, respectively with one of the two spin states. Ising spin variable was used with $\sigma = \pm 1$ in order to describe such two states. The direction state was considered to change with every collision, the direction change, while instead the spin was considered to change with every two collisions. In this manner, a particle starting off in state 1 (right moving and $\sigma = +1$) changes to state 2 having left moving and $\sigma = +1$ at the first collision, it will have state 3, right moving and $\sigma = -1$, at the second collision, while instead it will arrive at state 4 it with left moving and $\sigma = -1$ at the third collision, and back to state 1 at the fourth collision. States one and three will correspond both to right

moving particles and states two and four will correspond to left moving. A particle starting in state 1 (2) and ending in the same direction state 3 (4) will result to have changed its spin from +1 to –1.Ising spin was not considered a new property added to the particle but a simple label helping to classify particle trajectories. Finally, Ord and Deakin considered a so called parity in order to account for differences between two states with identical directions.

The authors [20-22] indicated by $p_\mu(m\delta, s\varepsilon)\delta$ ($\mu = 1,2,3,4$) the probability that the particle, leaving the space-time point $(m\delta, s\varepsilon)$, is in state μ ($m=0, \pm 1,\ldots\ldots$; $s=0, \pm 1,\ldots\ldots$).

The master equation expressed by the following set of difference equations was written

$$
\begin{aligned}
p_1(m\delta,(s+1)\varepsilon) &= \frac{1}{2}\, p_1((m-1)\delta,s\varepsilon) + \frac{1}{2}\, p_4((m+1)\delta,s\varepsilon) \\
p_2(m\delta,(s+1)\varepsilon) &= \frac{1}{2}\, p_2((m+1)\delta,s\varepsilon) + \frac{1}{2}\, p_1((m-1)\delta,s\varepsilon) \\
p_3(m\delta,(s+1)\varepsilon) &= \frac{1}{2}\, p_3((m-1)\delta,s\varepsilon) + \frac{1}{2}\, p_2((m+1)\delta,s\varepsilon) \\
p_4(m\delta,(s+1)\varepsilon) &= \frac{1}{2}\, p_4((m+1)\delta,s\varepsilon) + \frac{1}{2}\, p_3((m-1)\delta,s\varepsilon)
\end{aligned}
\tag{3.3}
$$

If we multiply the first equation by δ, we have that the probability for a particle leaving the node $(m\delta, (s+1)\varepsilon)$ in state 1, is equal to the sum of two probabilities, the first representing the probability that a particle leaves node $((m-1)\delta, s\varepsilon)$ in state 1 and remains in this state when it leaves node $(m\delta, (s+1)\varepsilon)$ and the second representing probability that particle leaves the node $((m+1)\delta, s\varepsilon)$ in state 4 and changes to state 1 when it leaves the node $(m\delta, (s+1)\varepsilon)$. The remaining equations may be interpreted in a similar way. The probability p_μ is uniquely determined since the initial conditions are fixed. Fixed $s \geq 0$, we have also that

$$\sum_{\mu=1}^{4} \sum_{m=-\infty}^{+\infty} p_\mu(m\delta, s\varepsilon)\delta = 1$$

so that the probability for particle to be somewhere on the lattice at a given time is equal to one.

Assuming $q_i(s)$ ($i=1,2,3,4$) is the probability for a particle to be in the i-th state ($i=1,2,3,4$) at the s-th step on the lattice, one has that

$$(q_1(s+1),q_2(s+1),q_3(s+1), q_4(s+1))^T=T(1)(q_1(s),q_2(s),q_3(s),q_4(s))^T \tag{3.4}$$

where the transition matrix T(1) defines a Markov chain.

In this manner we have reached the central point of our paper.

Generally speaking, the most general Markov chain with four states may be defined by the following matrix

$$T(\alpha) = \frac{\alpha}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{3.5}$$

where, with $\alpha = 1$, T_{ij} is the probability of a transition from state j to state i in one step. Ord and Deakin just started with (3.5) in their elaboration. The problem that at this point we pose is the following:
have we a set of a purely mathematical axioms by which we may derive all the results that were previously obtained by these authors?
Let us start observing that $T(\alpha)$, as indicated in the (3.5), is a general member of the Clifford algebra that we introduced in the previous section. In detail we may write that

$$T(\alpha) = \frac{\alpha}{2}\left[1\otimes Z_1 + e_1 \otimes Z_2\right] \tag{3.6}$$

where

$$Z_1 = 1 + \frac{1}{2} e_1 - \frac{i}{2} e_2;\ Z_2 = \frac{1}{2} e_1 + \frac{i}{2} e_2 \tag{3.7}$$

In order to discuss other features of the Markov process, one considers a change of variables from $q_i(s)$ ($i = 1, 2, 3, 4$) to $\mu_i(s)$ ($i = 1, 2$) and $\xi_i(s)$ ($i = 1, 2$):

$$\mu_1 = q_1 + q_2 + q_3 + q_4\ ; \qquad \mu_2 = (q_1 + q_3) - (q_2 + q_4)$$
$$\xi_1 = q_1 - q_3 \qquad \xi_2 = q_2 - q_4 \tag{3.8}$$

where $\mu_1 \equiv \mu_1(s)$ is the sum of all the occupation probabilities, $\mu_2 \equiv \mu_2(s)$ is the difference of occupation probabilities by direction, $\xi_1 \equiv \xi_1(s)$ and $\xi_2 \equiv \xi_2(s)$ represent instead the differences of occupation of the 2 spin states for right and left moving particles, respectively. In matrix form it is obtained that

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \xi_1 \\ \xi_2 \end{pmatrix} = R \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \quad \text{with} \quad R = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \tag{3.9}$$

In our mathematical framework the (3.9) becomes the following equation involving general members of Clifford algebra

$$U = R\,P \tag{3.10}$$

where R is the Clifford general member

$$R = Z_3 \otimes Z_4 + Z_5 \otimes 1 \tag{3.11}$$

and

$$Z_3 = \frac{1}{2} + \frac{1}{2} e_1 + \frac{i}{2} e_2 + \frac{1}{2} e_3\ ; \qquad Z_4 = e_1 + e_3\ ;$$

$$Z_5 = -\frac{1}{2} + \frac{1}{2} e_1 - \frac{i}{2} e_2 + \frac{1}{2} e_3$$

P is the following Clifford member

$$P = Z_6 \otimes Z_7 + Z_8 \otimes Z_9 \tag{3.12}$$

with

$$Z_6 = \frac{1}{2} + \frac{1}{2} e_3\ ; \qquad Z_7 = \frac{q_1}{2} + \frac{q_1}{2} e_3 + \frac{q_2}{2} e_1 - \frac{i}{2} q_2 e_3\ ; \qquad Z_8 = \frac{1}{2} e_1 - \frac{i}{2} e_2\ ;$$

And

$Z_9 = \frac{q_3}{2} + \frac{q_3}{2}e_3 + \frac{q_4}{2}e_1 - \frac{iq_4}{2}e_2$

Finally, U is given in the following manner

$$U = Z_6 \otimes Z'_7 + Z_8 \otimes Z'_9 \qquad (3.13)$$

where

$Z'_7 = \frac{\mu_1}{2} + \frac{\mu_1}{2}e_3 + \frac{\mu_2}{2}e_1 - \frac{i\mu_2}{2}e_2$ and $Z'_9 = \frac{\xi_1}{2} + \frac{\xi_1}{2}e_3 + \frac{\xi_2}{2}e_1 - \frac{i\xi_2}{2}e_2$

At this stage a basic result may be evidenced from a theoretical as well as mathematical physics view points: the equations contained in Ord and Deakin's paper may be obtained in terms of Clifford algebra.

The interesting conclusion is only one. The results obtained in [20-22] are expression of the basic set of axioms of Clifford algebra.

In the calculations, Ord and Deakin normalized $\xi_i(s)$ defining

$$\xi'_i(s) = (\sqrt{2})^s \xi_i(s)$$

and introducing the (3.9) in the (3.5), they splitted the problem in two final equations that again are given in terms of Clifford members

$$\Xi(s+1) = V\,\Xi(s) \qquad (3.14)$$

with

$$V = \frac{1}{\sqrt{2}}(1 - ie_2)$$

and

$$\Xi(s) = \frac{\xi_1(s)}{2} + \frac{\xi_1(s)}{2}e_3 + \frac{\xi_2(s)}{2}e_1 - \frac{i\xi_2(s)}{2}e_2$$

and similar Clifford member for $\Xi(s+1)$.

The equations (3.3) characterize the full random walk of the system. It remains to show that the solutions of these equations can be approximated in terms of solutions of diffusion and of Schroödinger equations. To this purpose in [20-22] it was used the usual diffusive scaling of random walks for small δ

$$\frac{\delta^2}{2\varepsilon} = D + O(\delta) \quad \text{or} \quad \varepsilon = \frac{\delta^2}{2D} + O(\delta^3) \qquad (3.15)$$

where D is the constant of diffusion. Let us consider the following shift operators, $E_x^{\pm 1}$ and E_t, acting in the following manner

$$E_x^{\pm 1}\, p_i(m\delta, s\varepsilon) = p_i((m \pm 1)\delta, s\varepsilon)\ ;$$

$$E_t\, p_i(m\delta, s\varepsilon) = p_i(m\delta, (s+1)\varepsilon) \qquad (3.16)$$

The (3.3), the basic master equation of the paper, results to be still a basic equation expressed by Clifford members. In fact, we have that

$$P(m\delta, (s+1)\varepsilon) = L\, P(m\delta, s\varepsilon) \qquad (3.17)$$

with

$$L = \frac{1}{2}(1 \otimes T_1 + e_1 \otimes T_2)$$

and (3.18)

$$T_1 = \frac{(E_x + E_x^{-1})}{2} + \frac{E_x^{-1}}{2} e_1 - i\frac{E_x^{-1}}{2} e_2 + \left(\frac{E_x^{-1} - E_x}{2}\right) e_3 ;$$

$$T_2 = \frac{E_x}{2} e_1 + i\frac{E_x}{2} e_2$$

Instead P is given by the following Clifford member

$$P = Z_6 \otimes Q_7 + Z_8 \otimes Q_9$$

where

$$Q_7 = \frac{p_1}{2} + \frac{p_2}{2} e_1 - i\frac{p_2}{2} e_2 + \frac{p_1}{2} e_3 \quad \text{and} \quad Q_9 = \frac{p_3}{2} + \frac{p_4}{2} e_1 - i\frac{p_4}{2} e_2 + \frac{p_3}{2} e_3$$

Thus, in conclusion, the master equation, given in the (3.3), is expression of the basic axiom set of the Clifford algebra.

Ord and Deakin operated the following change of variables

$$z_1 = p_1+p_2+p_3+p_4 ; \qquad z_2 = (p_1+p_3)-(p_2+p_4) ; \tag{3.19}$$

$$\phi_1 = p_1-p_3 ; \qquad \phi_2 = p_2-p_4$$

and we may arrive to write the following and final Clifford equation

$$\chi_1 = N\chi \tag{3.20}$$

with

$$N = \frac{1}{2}\left[Z_6 \otimes N_1 + Z_{10} \otimes N_2\right]$$

$$N_1 = \frac{\left(E_x + E_x^{-1}\right)}{2} + \frac{\left(E_x + E_x^{-1}\right)}{2} e_3 + \frac{\left(E_x^{-1} - E_x\right)}{2} e_1 + i\frac{\left(E_x^{-1} - E_x\right)}{2} e_2 ;$$

$$Z_{10} = \frac{1}{2} - \frac{1}{2} e_3 ;$$

$$N_2 = \frac{\left(E_x^{-1} + E_x\right)}{2} + \frac{\left(E_x^{-1} - E_x\right)}{2} e_3 + \frac{\left(E_x^{-1} - E_x\right)}{2} e_1 - i\frac{\left(E_x + E_x^{-1}\right)}{2} e_2$$

and

$$\chi = Z_6 \otimes B_1 + Z_8 \otimes B_2$$

where

$$B_1 = \frac{z_1}{2} + \frac{z_1}{2} e_3 - \frac{i}{2} z_2 e_2 + \frac{i}{2} z_2 e_1 ; \quad B_2 = \frac{\phi_1}{2} + \frac{\phi_1}{2} e_3 + \frac{\phi_2}{2} e_1 - \frac{i}{2}\phi_2 e_2$$

By using such mathematical procedure, we have arrived now to explain in mathematical terms the reason by which both the diffusion and the Schrödinger equations appear within the domain of classical statistical mechanics and the reason by which diffusion equations appears directly in a first order projection of the considered space while Schrödinger equation appears in a second order projection. We have two equations that express two qualitatively different behaviors to coexist in the same physical system.

The reason of such two different projections resides in the intrinsic mathematical features of Clifford algebra. As said in the previous section, a significant feature of such algebra is that it admits primitive idempotents that here we may consider to be given in the following manner

$$\psi_i = \frac{1+e_i}{2} \quad , \qquad i = 1, 2, 3$$

and whose important properties were outlined in the (2.26) – (2.39) of the previous section. We also outlined in other papers their importance in order to explain the actual origin of quantum mechanics [23-26]. Here we obtain that they have a decisive role in determining the two projections that give origin to diffusion and Schrödinger equations respectively. We have a basic mathematical set that from one hand determines both the two equations and at the same time is able to differentiate between diffusion and Schrödinger equations on the basis of its intrinsic mathematical features. Consider as example the idempotent

$$\psi_3 = \frac{1+e_3}{2}$$

One has that

$$e_3\psi_3 = \psi_3 \; ; \; \psi_3 \, e_3 = \psi_3 \; ; (e_3 - 1)\, \psi_3 = 0 \quad \text{and} \quad \psi_3\, (e_3 - 1) = 0$$

In the same manner one has that

$$e_j(e_3 - 1)\, \psi_3 = 0 \quad \text{and} \quad \psi_3\, (e_3 - 1)e_j = 0 \; ; \qquad j = 1, 2.$$

Such idempotents have a decisive role in characterizing Clifford algebra since some their products, as seen in the previous trivial example, give zero. At this stage one may note that all the Clifford members that we calculated for N and χ of the (3.20) are just expressed always in terms of the idempotents that we have here indicated. Owing to the intrinsic mathematical features of such idempotents in the (2.20) one obtains that the general Clifford member N_1 will act only on the basic Clifford element B_1 of χ, depending only from z_1 and z_2, while the other Clifford member N_2 will act only on the Clifford member B_2 of χ, depending this time only from the completely different variables. As a result of such basic mathematical features, using N_1 we will arrive to establish only the usual diffusion equation, while, using N_2, we will arrive only to Schrödinger's equation, both formulated in the framework of the same model, and, mathematically speaking, on the basis of the same basic axiom set defining the employed algebra.
In a previous paper we demonstrated two theorems in Clifford algebra showing the manner in which they arise giving algebraic solution to the well known quantum measurement problem in quantum mechanics. They arose showing that during a measurement we have a transition from the the standard Clifford A(S_i) algebra to the Dihedral algebra $N_{i,\pm 1}$ where new commutation rules appear and linking in some manner the present formulation [27]. In addition, using idempotents, interpreted, according to von Neumann, as logical statements, we also arrived to support the logical origin of quantum mechanics [23-26]. An intrinsic cognition principle appears in our formulation as this problem is investigated in detail in [28-37].
Here, we have given a clear Clifford algebraic explanation on the common mathematical structure that determines such two equations but at the same time we have given large evidence about the mathematical origin of their formal and conceptual differentiation in physics.

We may now conclude moving the calculations in the direction of the continuum limit as previously was performed in ref. [20-22]. One considers

$$E_x\ \gamma_i(M\delta,s\varepsilon) = \gamma_i(M\delta+\delta,s\varepsilon)$$

expanding $\gamma_i(M\delta+\delta,s\varepsilon)$ in a power series of δ with

$$E_x = 1 + L + \frac{1}{2}L^2 + O\left(\delta^3\right) \tag{3.23}$$

where $L = \delta\frac{\delta}{\delta x}$, one arrives to

$$L\ \gamma_i(M\delta,s\varepsilon) = L\ \gamma_i\ (x,t) \tag{3.24}$$

with the domain definition of γ_i extended to all the (x, t) and γ_i (x, t) considered to be continuously differentiable.

Since the (3.23), one has that

$$E_x^{-1} = 1 - L + \frac{L^2}{2} + O\left(\delta^3\right) \tag{3.25}$$

and for N_2 one has that

$$N_2 = \frac{1}{\sqrt{2}}\left[\frac{2+L^2}{2} - Le_1 - i\frac{\left(2+L^2\right)}{2}e_2 - Le_3\right] + O(\delta^3) \tag{3.26}$$

that may be written in the following manner

$$N_2 = (1+\frac{1}{2}L^2)\frac{1}{\sqrt{2}}(1-ie_2) - \frac{1}{\sqrt{2}}L(e_1+e_3) + O(\delta^3) \tag{3.27}$$

One may call

$V=\frac{1}{\sqrt{2}}(1-ie_2)$ as in the (3.14)

and

$$B=-\frac{1}{\sqrt{2}}(e_1+e_3) \tag{3.28}$$

and thus obtains that

$$N_2 = V + BL + \frac{1}{2}L^2V + O(\delta^3) \tag{3.29}$$

Owing to the intrinsic features of Clifford algebra, one has that

$$V^2 = \sqrt{2}V - 1 = -ie_2\text{; } BV=-e_3\text{; } VB=-e_1\text{; } VB+BV = \sqrt{2}\,B\text{; } B^2 = 1 \tag{3.30}$$

Therefore, it is obtained that

$$N_2^2 = (1+L^2)\,V^2 + L\,(VB+BV) + L^2\,B^2 + O\left(\delta^3\right) =$$
$$\sqrt{2}\,L^2\,B + \sqrt{2}\,L\,B - ie_2 \tag{3.31}$$

In the same manner one has that

$$N_2^4 = -1 - 2ie_2\,L^2 + O\left(\delta^3\right) \tag{3.32}$$

since it is that

$$V(-ie_2)=(-ie_2)V=V-\sqrt{2}\text{; } \quad B\,(-ie_2) = -(-ie_2)B = \frac{\sqrt{2}}{2}(e_1-e_3) \tag{3.33}$$

Finally, one obtains that

$$N_2^{\,8} = 1 + 4ie_2 L^2 + O\left(\delta^3\right) \tag{3.34}$$

In conclusion, remembering that in the (3.20) we have $\chi_1 = E_t\,\chi$ with E_t giving

$$E_t^{\,8} = 1 + 8\,\varepsilon\,\frac{\delta}{\delta t} + O(\varepsilon^2) \tag{3.35}$$

and χ given in the following manner

$$\chi = \frac{\phi_1}{2} + \frac{\phi_2}{2}e_1 - \frac{i\phi_2}{2}e_2 + \frac{\phi_1}{2}e_3 \tag{3.36}$$

we have that

$$(1+4ie_2L^2)\left(\frac{\phi_1}{2} + \frac{\phi_2}{2}e_1 - \frac{i\phi_2}{2}e_2 + \frac{\phi_1}{2}e_3\right) = \left(\frac{\phi_1}{2} + 2L^2\phi_2\right) +$$

$$+\left(\frac{\phi_2}{2} - 2L^2\phi_1\right)e_1 + \left(\frac{-i\phi_2}{2} + 2iL^2\phi_1\right)e_2 + \left(\frac{\phi_1}{2} + 2L^2\phi_2\right)e_3 \tag{3.37}$$

and calculating

$$E_t^{\,8}\chi = \left(1+8\,\varepsilon\frac{\delta}{\delta t}\right)\left(\frac{\phi_1}{2} + \frac{\phi_2}{2}e_1 - \frac{i\phi_2}{2}e_2 + \frac{\phi_1}{2}e_3\right) \tag{3.38}$$

and equaling the (3.37) with the (3.38) as it is on the basis of the (3.20) and of the (3.3), one obtains that such two Clifford members coincide if and only if it happens that

$$\frac{\delta^2\phi_1}{\delta x^2} = -\frac{2\varepsilon}{\delta^2}\frac{\delta\phi_2}{\delta t} \qquad \text{and} \qquad \frac{\delta^2\phi_2}{\delta x^2} = \frac{2\varepsilon}{\delta^2}\frac{\delta\phi_1}{\delta t} \tag{3.39}$$

Since

$$\frac{\delta^2}{2\varepsilon} = D + O(\delta)$$

we have that

$$\frac{\delta\phi_1}{\delta t} = D\frac{\delta^2\phi_2}{\delta x^2} \qquad \text{and} \qquad \frac{\delta\phi_2}{\delta t} = -D\frac{\delta^2\phi_1}{\delta x^2} \tag{3.40}$$

as they were also derived in ref.[20-22].

Again we may outline here that all the present matter seems to run about the important presence of new basic commutation rules that are involved in algebraic transition from the A(S_i) to $N_{i,\pm 1}$.

One may also introduce the following functions

$$\psi_+(x,t) = \phi_2(x,t) + i\phi_1(x,t) \quad \text{and} \quad \psi_-(x,t) = \phi_2(x,t) - i\phi_1(x,t)$$

and thus obtaining the final forms of Schrödinger's equations

$$i\frac{\delta\psi_+(x,t)}{\delta t} = -D\frac{\delta^2\psi_+(x,t)}{\delta x^2} \ \text{and}\ i\frac{\delta\psi_-(x,t)}{\delta t} = D\frac{\delta^2\psi_-(x,t)}{\delta x^2} \tag{3.41}$$

simultaneously admitted from Clifford algebra in complex and complex conjugate forms.

At the same time, using this time the general Clifford member N_1, given in the (3.20), following the same Clifford algebraic procedure one arrives to write the well-known diffusion equation

$$\frac{\delta z(x,t)}{\delta t} = D\frac{\delta^2 z(x,t)}{\delta x^2} \tag{3.42}$$

The important conclusion of such long derivation is thus clear: both diffusion and free Schrödinger equations are Clifford members. In detail, they are emanations of the same basic axiomatic set of the Clifford algebra, the A(S_i), in particular the (2.1) and the (2.2) that we enounced in the previous section, and, in case, the associated $N_{i,\pm 1}$ Dihedral algebra. By using both such algebras, it seems, however, that both such equations, also arising from the same axiomatic set, possibly they cannot be reconciled.

REFERENCES

[1] A more general discussion on Determinism as basic rule in time dynamics including also some features of the present paper, was held by us in
E. Conte, A. Federici, A. Khrennikov, J.P Zbilut, Is determinism the basic rule in dynamics of biological matter? Conference Proceedings on Quantum Theory: Reconsideration of Foundations-2, Sweden,1-7 June 2003, Vaxjo University press, (2003), 639-675

[2] The relevance of singularities in physiological systems was prospected for the first time by J.P. Zbilut in collaboration with A. Hubler and C.L. Webber at the Conference Fluctuations and Order: the New Synthesis (Conference sponsored by the Center for Nonlinear Studies, Loas Alamos National Laboratories and the Santa Fe Institute, Santa Fe, New Mexico, Sept. 9-12, 1993) by a contribution entitled Nondeterministic Equations of Motion as a Result of Noise Perturbation. The paper was subsequently published by J.P. Zbilut, A. Hubler, C.L.Webber, Physiological Singularities Modeled by Nondeterministic Equations of Motion and the Effect of Noise, in M. Millonas edition, Fluctuations and Order: The New Synthesis, Springer Verlag, New York, 397-417, 1996. Other papers may be quoted here: J.P Zbilut., M Zak., C.L. Webber, Nondeterministic Chaos Approach to Neural Intelligence. Intelligent Engineering Systems Through Artificial Neural Networks, 4 (1994), 819-824, ASME Press, New York,

[3] J.P. Zbilut, M. Zak, C.L. Webber, Physiological Singularities in Respiratory and Cardiac Dynamics, Chaos Solitons and Fractals, **5** (1995), 1509-1516.

[4] J.P. Zbilut, M. Zak, E.R. Meyers, A Terminal Dynamics Model of the Heartbeat, Biological Cybernetics, 75 (1996), 277-280.

[5] M. Zak, J.P. Zbilut, E.R. Meyers, From Instability to Intelligence Complexity and Predictability in Nonlinear Dynamics, Lecture Notes in Physics: New Series m49, Springer Verlag, Berlin Heidelberg, New York, 1997.

[6] J.P. Zbilut, J-M Zaldivar-Comenges, F. Strozzi, Recurrence Quantification Based-Lyapunov Exponents for Monitoring Divergence in Experimental Data, Physics Letters A297 (2002), 173-181.

[7] J.P. Zbilut, D.D Dixon, M. Zak, Detecting Singularities of Piecewise Deterministic (terminal) Dynamics in Experimental Data, Physics Letters A304 (2002), 95-101.
[8] J.P. Zbilut, N. Thomasson, C.L. Webber, Recurrence quantification analysis as a tool for nonlinear exploration of nonstationarity cardiac signals, Medical Engineering and Physics, 9 (2001), 1-8.
[9] E. Conte, A. Federici, J.P. Zbilut, On a simple case of possible non-deterministic chaotic behavior in compartment theory of biological observables, Chaos, Solitons and Fractals, 22 (2004), 277-284.
[10] E. Conte, A. Vena, A. Federici, R. Giuliani, J.P. Zbilut, A brief note on a possible detection of physiological singularities in respiratory dynamics by recurrence quantification analysis, Chaos, Solitons and Fractals, 21 (4) (2004), 869-877.
[11] A. Vena, E. Conte, G. Perchiazzi, A. Federici, R. Giuliani, A. Federici, J.P. Zbilut, Detection of physiological singularities in respiratory dynamics analised by recurrence quantification analysis, Chaos, Solitons and Fractals, 22 (4) (2004), 857-866.
[12] E. Conte, GP. Pierri, A. Federici, L. Mendolicchio, J.P. Zbilut, A model of biological neuron with terminal chaos and quantum like features, Chaos, Solitons and Fractals, 30 (2006), 774-780.
[13] M. Zak, The Problem of Irreversibility in Newtonian Dynamics, Int. Journal of Theoretical Physics, 31 (2) (1992), 333-342.
[14] M. Zak, Terminal Chaos for Information Processing in Neurodynamics, Biological Cybernetics, 64 (1991), 343-351.
[15] M. Zak, Non-Lipschitzian Dynamics for Neural Net Modelling, Applied Math. Letters, 2 (1) (1989), 69-74.
[16] M. Zak, Dynamical Simulation of Probabilities, Chaos, Solitons and Fractals, 5 (1997), 793-804.
[17] M. Zak, Non-Lipschitz Approach to Quantum Mechanics, Chaos, Solitons and Fractals, 9 (1998), 1183-1198.
[18] E. Conte, Advances in application of quantum mechanics in neuroscience and psychology: a Clifford algebraic approach, Nova Science Publishers, June 2012-10-07
[19] E. Conte, An investigation on the basic conceptual foundations of quantum mechanics by using Clifford algebra, Adv. Studies Theor. Phys, 5 (11) (2011), 485-544.
[20] G.N. Ord, A.S. Deakin, Continuum Limits and Scrödinger Equation, GNOASD 10, (1996)
[21] G.N. Ord, Physics Letters, A173 (1993), 343-346.
[22] G. N. Ord, Fractal Space-Time and the Statistical Mechanics of Random Walks, Chaos, Solitons and Fractals, 7 (5) (1996), 111-123.
[23] E. Conte On the logical origins of quantum mechanics demonstrated by using Clifford algebra, Neuroquantology, 9 (2) (2011), 123-175.

[24] E. Conte, O. Todarello, A. Federici, F. Vitiello, M. Lopane, A. Khrennikov, A Preliminary Evidence of Quantum Like Behavior in Measurements of Mental States, Intern. Conference on Quantum Theory: Reconsideration of Foundations, Proceedings of Vaxio June 1-6, 2003.

[25] E. Conte, A.Y. Khrennikov, O. Todarello, A. Federici, L. Mendolicchio, J.P Zbilut, Mental states follow quantum mechanics during perception and cognition of ambiguous figures, Journal of Open Systems and Information Dynamics, 16 (2009), 1-17.

[26] E. Conte, On the Logical Origins of Quantum Mechanics Demonstrated By Using Clifford Algebra: A Proof that Quantum Interference Arises in a Clifford Algebraic Formulation of Quantum Mechanics. Electronic Journal of Theoretical Physics, 8 (2011), 109-126.

[27] E. Conte, A Reformulation of von Neumann's Postulates on Quantum Measurement by Using Two Theorems in Clifford Algebra. International Journal of Theoretical Physics, 49 (2010), 587-614.
E. Conte A proof of von Neumann's postulate in quantum mechanics. Quantum Theory: Reconsideration of Foundations-5 A.I.P. Conference Proceedings 1232, (2010), 201-205.

[28] E. Conte, Testing quantum consciousness, Neuroquantology, 6 (2) (2008), 126-139.

[29] E. Conte What Is the Reason to use Clifford Algebra in Quantum Cognition: "It From Qubit": On the Possibility that the Amino Acids can Discern Between two Quantum Spin States. Neuroquantology, 10 (3) (2012), 561-565.

[30] E. Conte, O. Todarello, A. Federici, N. Santacroce, V. Laterza, A. Khrennikov, May We Verify Non-Existing Dispersion Free Ensembles By Application of Quantum Mechanics in Experiments at Perceptive and Cognitive Level?. Neuroquantology, 10 (2012), 14-19.

[31] E. Conte, On the Possibility that we Think in a Quantum Probabilistic Manner. Neuroquantology, 8 (2010), 3-47.

[32] E. Conte, A Brief Note on Time Evolution of Quantum Wave Function and of Quantum Probabilities During Perception and Cognition of Human Subjects. Neuroquantology, 7 (2009), 435-448.

[33] E. Conte, N. Santacroce, A. Federici, A Possible Quantum Model of Consciousness Interfaced with a Non-Lipschitz Chaotic Dynamics of Neural Activity (Part I). Journal of Consciousness Exploration & Research, 3 (2012), 905-921.

[34] E. Conte, N. Santacroce, A. Federici, A Possible Quantum Model of Consciousness Interfaced with a Non-Lipschitz Chaotic Dynamics of Neural Activity (Part II). Journal of Consciousness Exploration & Research, 3 (2012), 922-936.

[35] E. Conte, O. Todarello, V. Laterza, A. Khrennikov, L. Mendolicchio, A. Federici, A Preliminary Experimental Verification of Violation of Bell inequality in a Quantum Model of Jung Theory of Personality Formulated with Clifford Algebra. Journal of Consciousness Exploration & Research, 1 (2010), 831-849.
E. Conte, O. Todarello, A.Y. Khrennikov, A. Federici, J.P. Zbilut, A preliminary experimental verification on the possibility of Bell inequality violation in mental states, Neuroquatology, 6 (3) (2008) 214-221.
[36] E. Conte, A. Khrennikov, O. Todarello, A. Federici, J.P. Zbilut, On the Existence of Quantum Wave Function and Quantum Interference Effects in Mental States: An Experimental Confirmation During Perception and Cognition in Humans. Neuroquantology, 7 (2009), 204-212.
[37] E. Conte, O. Todarello, A. Federici, F. Vitiello, M. Lopane, A. Khrennikov, J.P. Zbilut, Some remarks on an experiment suggesting quantum-like behavior of cognitive entities and formulation of an abstract quantum mechanical formalism to describe cognitive entity and its dynamics. Chaos, Solitons and Fractals, 31 (2007), 1076-1088.

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