



INFORMATION RECOVERY PROBLEMS

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Dedication: to Peter H. Hare for a quarter century of leadership, encouragement and support of "The Buffalo School of Logic", with affection and admiration.

ABSTRACT

An information recovery problem is the problem of constructing a proposition containing the information dropped in going from a given premise to a given conclusion that follows. The proposition(s) to be constructed can be required to satisfy other conditions as well, e.g. being independent of the conclusion, or being "informationally unconnected" with the conclusion, or some other condition dictated by the context. This paper discusses various types of such problems, it presents techniques and principles useful in solving them, and it develops algorithmic methods for certain classes of such problems. The results are then applied to classical number theory, in particular, to questions concerning possible refinements of the 1931 Gödel Axiom Set, e.g. whether any of its axioms can be analyzed into "informational atoms". Two propositions are "informationally unconnected" [with each other] if no informative (nontautological) consequence of one also follows from the other. A proposition is an "informational atom" if it is informative but no information can be dropped from it without rendering it uninformative (tautological). Presentation, employment, and investigation of these two new concepts are prominent features of this paper.

1. Introduction. In the broadest sense an information recovery problem arises when we have a given premise and a given conclusion that follows. In this situation, as a rule but not in every case, "information" has been "lost" or "dropped" in going from the premise to the conclusion. The problem is to "recover" the information, if any, that may have been dropped. Strictly speaking, the problem is to find another consequence [of the premise] whose conjunction with the given conclusion implies the premise in return.

In order to discuss situations of this sort we use the following terminology: *the premise P*; *the first (or given) conclusion C*; *the (or a) second conclusion S*. Second conclusions are also called *solutions* and where it is of interest to present several solutions the letter S will be indexed: S1, S2, S3, and so on.

1.1. For example, if the premise P is "Every number divides itself" and the given conclusion C is "Two divides two", then one solution S1 may be taken to be "Every number

other than two divides itself". A second solution S2 is the conditional proposition "If two divides two then every number divides itself". The solution S1 comes readily to mind; the solution S2 is less natural; but it contains less information; and, in fact, it "overlaps" less with the conclusion. Thus, for some purposes, S2 is preferable to S1. Once we have established the conditional proposition S2 as a solution we see that dropping information from its antecedent "Two divides two" produces further solutions. Take S3 to be "If two or three divides itself then every number divides itself". Take S4 to be "If two, three or four divides itself then every number divides itself". This produces an infinite sequence of ever more informative propositions whose limit, so to speak, is the proposition "If some number divides itself then every number divides itself", which we call S*.

The last solution S* is very close to the original premise "Every number divides itself", which, strictly speaking according to the definition of the problem, is also a solution. Notice that none of the propositions mentioned are logically equivalent and that they are all situated between two limiting cases; the premise itself and the conditional S2. The premise implies each and every solution; each and every solution implies the conditional S2. S2 is what we will call *the standard conditional* for this information recovery problem; *the standard conditional for a given information recovery problem* is the conditional of the given conclusion C with the given premise P, that is, the conditional whose antecedent is C and whose consequent is P.

The fact that the premise implies each and every solution is part of the definition of the problem. The fact that each and every solution implies the standard conditional can be inferred immediately from a form of "the conditional deduction theorem" [Corcoran 1985]: in order for a given proposition X to imply a given conditional (*If Y then Z*) it is sufficient for (*X and Y*), the conjunction of the given proposition with the antecedent, to imply the consequent Z. In the present context "the given proposition X" is an arbitrarily chosen solution, "the antecedent Y" is the first conclusion and "the consequent Z" is the premise. The condition involving the premise being implied by the conjunction of the arbitrarily chosen solution with the conclusion is simply the definition of being a solution.

As just mentioned, nothing has been said which rules out "trivial" solutions, i.e. solutions equivalent to the original premise. Thus we can take as another solution ST1 "Every number is divided by itself" and as yet another ST2 "Every number that does not divide itself divides itself". Ingenuity will produce many more propositions equivalent to the premise, that is, many more trivial solutions.

For several reasons including the need to rule out trivialities, it is convenient to consider qualifications or restrictions, and thus to define classes of qualified information recovery problems. But before we take up this task we need to describe the presuppositions that we are making in our examples. In this paper, we subscribe to standard classical predicate logic with identity and functions. It goes without saying that universes of discourse are presumed to be nonempty and that the universal quantifier is presumed to have existential import in the sense that, e.g. "Every number divides itself" implies "Some number divides itself". Of course, "Every number that exceeds itself is both odd and even" does not imply "Some number that exceeds itself is both odd and even"; the former is vacuously true and the latter is false.

In most cases, the universe of discourse can be taken to be the class of natural numbers. In these cases the individual variables range over that class as in Gödel 1931, which is also the source of the arithmetic axioms used below; and the word 'number' in the English sentences used corresponds to an individual variable. The expression 'Every number' is paraphrased 'For every number x'. When the universe of discourse is understood to be indicated by the word 'number',

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that word becomes redundant and we have 'For every x ', which is formalized using the universal quantifier. The examples are chosen so that they are easily formalizable. However, it is by no means necessary to formalize them in order to understand this paper. Cf. Corcoran 1993, xxxviii-xlii.

All arithmetic concepts are assumed to be primitive unless a formal definition has been indicated; and when a concept has been introduced by means of a formal definition each of its occurrences is presumed to be in terms of primitives. For example, the concepts "odd" and "even" are treated as primitives; therefore, "Three is odd" is not logically equivalent to "Three is non-even" and the proposition "Every number is either even or odd" is logically independent of "No number is both even and odd". However, since the first few number words are used for defined concepts, it will turn out that "One precedes two" is logically equivalent to "The successor of zero precedes its own successor".

In order to facilitate smooth and faithful translation of English into symbolic language it is convenient to assume that a symbol for the converse-relation operator is available. Thus, for example, the two propositions, "Zero precedes one" and "One is preceded by zero", though logically equivalent, are expressed symbolically by different sentences, as they are in English. One of the symbolic sentences involves the relation symbol ' $<$ ' for precedence (the less-than relation) without the converse-relation operator symbol while the other uses the precedence symbol with the converse-relation operator symbol affixed. The availability of the converse-relation operator symbol makes translation easier by lessening the need to rephrase before translating; as a result it also helps to keep translational processes separate from inferential processes since rephrasing often involves the inference of logical equivalents.

In order to distinguish the above class of recovery problems from other classes where the solution is required to meet special qualifications, those mentioned above are called *unrestricted* information recovery problems. It is worth noting that every unrestricted information recovery problem has at least one solution, the trivial solution. The standard conditional is of course also a solution to an unrestricted problem.

However, in the case where the given conclusion is tautological (or uninformative), all information in the premise has been dropped and thus every solution is automatically equivalent to the premise. In this case the standard conditional is also a trivial solution. In fact, in this case, there is "virtually" only one solution; there is only one solution "up to equivalence"; literally, every two solutions are equivalent to each other.

In the rest of this paper we follow the convention of counting solutions "up to equivalence", that is, counting equivalence classes of solutions. Since there are infinitely many propositions logically equivalent to any given proposition, literally, every unrestricted problem has infinitely many solutions. Thus, we say that a problem has only one solution, or only two solutions, or only finitely many solutions, as we will have occasion to do below, when we are only counting solutions separately if they are non-equivalent.

It is easy to see that the case of the tautological given conclusion is the only case where the standard conditional is equivalent to the premise. In fact, if the standard conditional implies the premise then the given conclusion is tautological. In every other case, the standard conditional is a second, non-trivial, solution to an unrestricted information recovery problem. Normally, one might say, an unrestricted information recovery problem has at least two solutions, the two limiting cases mentioned above: "the" trivial solution which superimplies every non-trivial solution and "the" standard conditional which follows from every solution.

By 'superimplies' we mean of course "implies without being implied by", which amounts to "implies without being equivalent to". Thus a given premise [logically] implies a given conclusion if and only if the premise contains all of the information of the conclusion (regardless of whether the premise contains more or contains no more); it [logically] superimplies the conclusion if and only if it contains all of the information of the conclusion *and more*; it is [logically] equivalent to the conclusion if and only if it contains all of the information of the conclusion *and no more*.

2. Independent Information Recovery Problems. Perhaps the most convenient qualification that rules out trivial solutions is to require that the solution be independent of the given conclusion. To be more explicit, we say that one proposition is *independent* of another if the first neither implies nor is implied by the second. In other words, two propositions are said to be *independent* [of each other] if and only if neither one implies the other.

It turns out that this qualification yields an interesting class of information recovery problems, perhaps the most interesting class from a pedagogical point of view. Let us call these the *independent information recovery problems*: given a premise P that implies a given conclusion C to find another conclusion S also implied by P but not implying nor implied by the conclusion C; of course, it is necessary for the conjunction of the two conclusions C and S to imply the premise P.

Paradigm cases of the independent information recovery problem arise when the premise is equivalent to a conjunction of independent propositions one of which is the first conclusion; in such a case the other can be taken to be a solution.

2.1. For example, if the given premise P is "Two is even and prime" and the given conclusion C is "Two is even", then "Two is prime" is a solution S to the independent information recovery problem determined by the given premise and the given conclusion. The conditional proposition "If two is even then two is prime" is also a solution. Notice that this conditional is shorter than the standard conditional; nevertheless, the two are logically equivalent.

Even though the premise and its equivalents are not eligible to be solutions, to this independent information recovery problem we still have two non-equivalent solutions: the "natural" solution S1, "Two is prime"; the conditional solution S2, "If two is even then two is prime". It is obvious, of course, that the conclusion "Two is even" neither implies nor is implied by either S1 or S2. Moreover, it is easy to see that there are infinitely many more non-equivalent solutions: S3 "If two or three is even then two is prime", S4 "If two or three or four is even then two is prime"; and so on. In a sense, the limit of this sequence of solutions is the conditional S* "If some number is even then two is prime".

Thus restricting solutions to consequences independent of the given conclusion leaves interesting solutions, some of which may seem extraneous. This suggests that further restrictions may prove fruitful. Another, superficially different, class of independent information recovery problems worth consideration in its own right, turns out on further examination to fall under the above paradigm. I have in mind here the situation where the given conclusion is the [non-exclusive] disjunction of the premise with a proposition independent of the premise.

2.2. For example take as the premise P "Three is even" and as the conclusion C "Three is even or prime". Clearly the proposition that three is even also implies S1 "Three is even or non-prime", which is independent of the given conclusion. Moreover, the conjunction of "Three is even or non-prime" with the given conclusion implies the premise. In fact, it is clear that the premise "Three is even" is equivalent to the conjunction of the two conclusions. Hence this class of problems

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involving disjunctions turns out to be a subclass of the paradigm class mentioned above where the premise is equivalent to a conjunction of independent propositions one of which is the given conclusion.

The conjunctive connective "AND" is routinely used to add information, so that deletion of an "and-clause" often drops information. The information dropped in going from "Two is even and prime" to "Two is even" is contained in "Two is prime".

The [non-exclusive] disjunctive connective "OR" is routinely used to drop information. This application of disjunction has been seen already above. The information dropped in going from "Two is square" to "Two is even or square" is contained in "Two is non-even or square". In the process of revising conversation or non-fiction writing the need to drop information arises when one realizes that one had already asserted or was about to assert something that goes beyond the available evidence, something containing more information than desired. Here the facility of adjoining an "or-clause" at the end of a sentence is convenient. When the assertion that Abe is American is not warranted, the assertion that Abe is American or Canadian may be entirely in order.

Disjunction is often a sign of incomplete knowledge or incomplete memory. In the course of learning arithmetic a student may be confident in asserting that zero or one is not prime without being able to say which with confidence. The connection of disjunction to incomplete knowledge or to incomplete memory or, more generally, to an undecided state of mind is so strong that students often find disjunctive reasoning to be awkward when all relevant facts are known. This may be one of the reasons why it is more effective pedagogically in presenting an implication to also present the information dropped; instead of simply dropping information it is better also to identify the information dropped. Rather than presenting the student only with the fact that "Two is prime" implies "Two is even or prime" it is better at the same time to note that "Two is prime" also implies "Two is non-even or prime". For further discussion of the entangled interrelations among the logical, the pragmatic, the social and the ethical dimensions of communication, see Part I of Grice 1989.

Although disjunction is perhaps the most common or most prominent device for dropping information it is by no means the only one. Another prominent information-dropping device is the adjunction of a condition as in going from "Ben speaks French" to "If Ben is Canadian then Ben speaks French". The information dropped here is contained in the conditional "If Ben is not Canadian then Ben speaks French". This leads us to another class of independent information recovery problems where the given conclusion is a conditional whose consequent is the given premise and whose antecedent is a proposition independent of the premise. Here one natural solution is the conditional obtained from the conclusion by replacing the antecedent by its denial, or by a contradictory opposite of the antecedent.

2.3. For the premise P take "Zero is even". For the conclusion C take "If one is prime then zero is even". The most natural independent solution is the conditional S1 "If one is non-prime then zero is even".

In this case the standard conditional S2 is a double conditional, a conditional whose antecedent is itself a conditional: "*If* if one is prime then zero is even, *then* zero is even". Since S2 is the standard conditional it contains the dropped information. Moreover, as is easily seen, S2 is independent of the conclusion C and thus S2 is also a solution. However, S2 is not a second solution "up to equivalence"; S2 is equivalent to S1 even though they are both conditionals having

identical consequents and non-equivalent antecedents. This is one of many examples where counting solutions "up to equivalence" tends to mask some important differences.

To see that C is independent of S1 two transformations are needed: substituting "two" for "one" and "one" for "zero" shows that S1 does not imply C; interchanging "prime" with "even" shows that C does not imply S1. See Corcoran 1989, 27, 28 and Corcoran 1993, xx, xxxi, xxxii.

It is worth reminding ourselves that independence as used here is simply mutual non-implication, which of course entails self-consistency not only of each of the two propositions but also self-consistency of their two negations. If two propositions are independent, then the first does not imply the second and thus the first is consistent with the negation of the second. Since every two propositions that are consistent with each other are each self-consistent, it follows then that the first and the negation of the second are each self-consistent. Parallel reasoning shows that the second is consistent with the negation of the first and thus that the second and the negation of the first are each self-consistent. Putting these two argumentations together we have that independence entails self-consistency of the two propositions and of their two negations.

However, as Leibniz emphasized in the *New Essays*, the self-consistency of both of two propositions does not entail that they are mutually consistent, that is, that they are consistent with each other. Moreover, it will become important in understanding the structure of classification of information recovery problems to realize that the concept of independence used here does not entail mutual consistency nor does it entail mutual consistency of the negations. It is perhaps overly concise to illustrate both points with one example; but the example deserves being considered anyway.

Incidentally, the first of these two points was made explicitly by Lewis and Langford 1932, 338; and it appears that they may have been trying to make the second point as well but what they actually accomplished was to make the first point twice. They say that if two propositions are independent "it may still happen" that the first implies the negation of the second or that the second implies the negation of the first. These two clauses are equivalent. What they probably meant to say was "the negation of the second implies the first", a very different condition equivalent to the inconsistency of the two negations. Since they miss the second possibility, they necessarily also miss the fact that both possibilities "may still happen" together, that is, they miss the possibility that two independent propositions may be self-consistent contradictory opposites.

2.4. For the given premise P take "Two is even and non-even". For the given conclusion C take "Two is even". The proposition "Two is non-even" may be taken as an independent solution S, since it neither implies nor is implied by "Two is even". Nevertheless, the conclusion is inconsistent with the solution; and the negation of the conclusion is inconsistent with the negation of the solution.

The following two examples illustrate separately the point that independence of two propositions does not entail their mutual consistency and the point that independence does not entail the mutual consistency of their negations.

2.5. Take "Every number is both even and non-even" as the given premise P and take "Every number is even" as the given conclusion C. Since the given conclusion neither implies nor is implied by "No number is even" and since the conjunction of the given conclusion with "No number is even" implies the given premise, the proposition S that no number is even may be taken as an independent solution even though it is inconsistent with the conclusion. However, the negations of the solution and of the conclusion are both true and hence consistent with each other.

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2.6. Now take "Some but not every number is even" as the given premise P and take "Some number is not even" as the given conclusion C. The proposition S "Some number is even" may be taken as an independent solution even though its negation is inconsistent with the negation of the given conclusion.

The above discussion of independence may have suggested to some readers that in order for two propositions to be independent of each other it is necessary and sufficient for their two respective negations to be independent of each other. This proposition, which is easily seen to be true, provides a criterion for independence which is handy in cases where it is obvious that the two negations are independent but not obvious, or not as obvious, that the propositions themselves are independent.

The class of all propositions divides into three mutually exclusive and jointly exhaustive subclasses: those that are tautological (non-informative), those that are [self-]contradictory ([self-]inconsistent) and those that are neither. The propositions in the last class are both consistent and informative (consistent-informative) and it is exclusively within this class that the independence relation holds. Thus the conclusion and the solutions of an independent information recovery problem are necessarily consistent-informative, even though the premise is sometimes contradictory.

Every unrestricted information recovery problem has at least one solution. In some cases they have only one solution, the trivial solution equivalent to the premise. In every other case there are at least two non-equivalent solutions. Below we will consider the question of whether there is an unrestricted problem with only two solutions, or only finitely many solutions, or whether every such problem has infinitely many non-equivalent solutions as was the case in the example considered above.

Not every independent information recovery problem has a solution; and even in cases that have solutions none of their solutions are trivial in the sense of being equivalent to the premise. If the conclusion is tautological, contradictory, or logically equivalent to the premise there are no solutions. In every other case the standard conditional is a solution. This follows from the fact that in every information recovery problem if the conclusion is consistent-informative and not equivalent to the premise then the standard conditional is independent of the conclusion. The question remains whether there is an independent problem with only one solution, or only a finite number of solutions, or whether every such problem has infinitely many non-equivalent solutions as was the case in every example considered above.

This paper is dedicated to information recovery problems and as such it approaches every valid one-premise argument with the aim of finding a proposition to add to the conclusion in order to restore the information that had been dropped. This can be succinctly characterized as the adding-of-dropped-information perspective. However, it is possible to do a kind of figure-ground shift and to approach a valid one-premise argument with the aim of transforming the premise into a proposition that no longer implies the conclusion, that is, into a proposition that drops the information retained in the conclusion. This can be characterized as the dropping-of-retained-information perspective. Just as the former perspective leads to variously qualified information *recovery* problems the latter perspective leads to variously qualified information *removal* problems.

It is not within the scope of this paper to deal with these problems. However, there is a widely known, and even more widely felt, point to be made in this connection, a point that complements and rounds out several points made above and that, at the same time, will prove useful below: adjoining an informative conclusion as a condition removes at least some of the information of the

conclusion from the premise. More formally, if the conclusion C is informative then the standard conditional (*If C then P*) no longer implies the conclusion C . To see this notice that the negation of any conditional implies the antecedent and that any proposition implied by a given proposition and by the negation of the given proposition is uninformative. Thus any conditional that implies its own antecedent has an uninformative antecedent. The application to information removal problems is evident: in order to transform the premise into a proposition that no longer implies the conclusion simply adjoin the conclusion as a condition forming the conditional of the conclusion with the premise, that is, forming the standard conditional. Before leaving this point it should be noticed that we have not identified the information dropped; all we have said is that the standard conditional does not imply the conclusion, that is, the standard conditional does not contain *all* of the information of the conclusion. It turns out that the standard conditional contains *none* of the information of the conclusion, or as we will say, the standard conditional is unconnected with the conclusion.

3. Unconnected Information Recovery Problems. An unrestricted problem can be described as follows: given a conclusion that follows from a given premise find a consequence of the premise that contains the information dropped in going from the premise to the conclusion. An independent problem requires in addition that the consequence to be found be logically independent of the conclusion. As we have seen above in example after example, in neither case is it required to find a consequence that contains *only* information dropped *without* containing any of the information not dropped, that is, without containing any of the information in the conclusion.

3.1. To be sure, the condition requiring a solution with no information in common with the conclusion is not precluded by the condition requiring a solution independent of the conclusion. For example, if the premise P is "Two is even and non-even" and the conclusion C is "Two is even" then the solution S "Two is non-even" is independent of the conclusion and it has no information in common with the conclusion. It is easy to see that every proposition implied both by the conclusion "Two is even" and by the solution "Two is non-even", for example "Two is even or non-even", is uninformative (or tautological). This example illustrates what we are seeking in this section.

It may seem to some readers that the condition of having a solution with no information common with the conclusion is automatically satisfied by a solution independent of the conclusion. But this is far from true. To see that independence does not entail this added requirement consider the following example.

3.2.1. For the premise P take "Four is even, composite, and square". For the conclusion C take "Four is even and composite" and for the solution S_1 take "Four is composite and square". It is clear that the solution neither implies nor is implied by the conclusion; the solution implies "Four is square", which is not implied by the conclusion, and the conclusion implies "Four is even", which is not implied by the solution. Thus the conclusion and the solution are independent even though they have "Four is composite" as a common consequence.

There are several expressions used to indicate that one given proposition has information in common with a second given proposition, i.e. that there is at least one informative proposition implied at the same time by both of them. For example, we can say that they are connected or redundant or repetitive or that they overlap in information content. Moreover, there are several coextensive conditions each of which can be used as a formal definition of this relation, which is

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called *connectedness* in this paper. One definition that seems to bring out the intuitive import especially vividly is based on exhibiting common information content: in order for two propositions A and B to be *connected* [to each other] it is necessary and sufficient for there to be three propositions A*, B*, and C* such that C* is informative, A is equivalent to the conjunction (A* and C*) of the first with the third and B is equivalent to the conjunction (C* and B*) of the third with the second. For example, to use this definition to show that "One is neither composite nor prime" is connected with "One is neither prime nor even" it would not be sufficient simply to notice that they both imply "One is not prime"; we would have to notice that they are respectively equivalent to conjunctions having a common conjunct: roughly, "One is both non-composite and non-prime" and "One is both non-prime and non-even"; and we would have to note that the common conjunct "One is non-prime" is informative.

In keeping with the connotation of the word 'connected' we can say that two connected propositions A and B are *connected by* a proposition C that is informative and that they both imply. Thus, "Four is composite" connects "Four is even and composite" with "Four is composite and square". This shows that instead of defining the two-placed connectedness relation first, as we have done, we could have defined the three-place connectedness relation (as in "C connects A and B" or "C connects B to A") and then used that to define the two-placed relation: in order for one given proposition to be connected to a second given proposition it is necessary and sufficient for there to exist a third proposition that connects the first to the second.

By the way, since every two connected propositions have an informative common implication, no two tautologies are connected. In fact no tautology is connected to another proposition; not even itself. It is easy to forget that the word 'connected' has many senses other than the precise sense that we have stipulated for it here. Here connected means "connected by an informative proposition". When there is a danger of confusion or where a sentence may sound uncomfortably paradoxical, the adverb 'informationally' can be adjoined as redundant rhetoric. Thus we can say that no tautology is *informationally* connected to itself, that a given proposition is *informationally* connected to itself if and only if it is informative, and that every contradictory proposition is *informationally* connected to every informative proposition but to no uninformative one.

Thus, in the sense used here, since "Every odd number is odd" is tautological, it is not connected with "If three is an odd prime then three is odd", with "Three is an odd prime", with "Three is odd", or with any other proposition. Sharing concepts does not by itself mean being connected. Moreover, not sharing concepts does not by itself mean unconnected; "One is square but two isn't" and "Three divides six but not conversely" both imply "There are at least two numbers".

In order for two propositions to be unconnected (that is, informationally unconnected), it is necessary and sufficient for them to have no common informative consequence. This means that if two propositions are unconnected then every one of their common consequences, that is, every proposition implied by both separately, is tautological. It also means that if every proposition implied by each of two propositions is tautological then they are unconnected, that is, informationally unconnected. Thus we may take as our initial paradigm case of an unconnected pair a proposition and its denial, say "Two is even" and "Two is not even". More generally, we can take a pair of propositions one of which is equivalent to the negation of the other, in other words, a proposition and one of its contradictory opposites: "No number divides itself" paired with "Some number divides itself", "Every oblong number is even" paired with "Some oblong number is not even", "Two is either odd or square" with "Two is both non-odd and non-square".

Now, if two propositions are connected then adding information to either, or to both, transforms them into another pair of connected propositions. It of course does not matter whether the information added to both is the same; once we have a connection, adding information can only strengthen it. Likewise if two propositions are unconnected then dropping information from either or both results in another pair of unconnected propositions. This gives us the capacity to generate from a paradigmatic unconnected pair an infinite sequence of ever more intricate unconnected pairs.

Suppose for our original unconnected pair we have A1 "One is even" paired with B1 "One is not even". Starting with A1 and dropping information from each subsequent proposition we have A2 "One or two is even", A3 "One or two or three is even", and so on. Now starting with B1 and dropping information from each subsequent proposition using a different rule we have B2 "One is not both even and square", B3 "One is not even, square, and prime" which is equivalent to "It is not the case that one is even, square, and prime". For B4 we use four numerical properties: "It is not the case that one is even, square, prime, and perfect". For B5 we use five properties, and so on. As a result of the way these sequences are constructed, pairing any one of the As with any one of the Bs gives an unconnected pair. For example, pairing A2 with B3 we have that "One or two is even" is unconnected with "It is not the case that one is even, square, and prime". In a sense of course, there is a "connection" between the two; in fact, the proposition "One is even" could be said to be (in a sense) "contained" in both. But, the sense of 'connection' needed and the sense of 'contain' needed are not the senses being used in this paper. It is clear that the false proposition "One is even" is not implied by any true propositions and, in particular, that it is not implied by A2 "One or two is even" nor by B3 "It is not the case that one is even, square, and prime". The information of "One is even" is not *contained* in that of A2 or of B3 in the sense of 'contain' used in this paper.

Readers may have been led by these examples to notice that if two propositions are unconnected then their disjunction is tautological. The reason for this is that every proposition implies the disjunction of itself with an arbitrary proposition and, therefore, given two propositions, each implies their (common) disjunction. If two propositions are unconnected then none of their common implications are informative; and hence their disjunction is not informative.

The converse is also true, that is, if the disjunction of two given propositions is tautological then the two propositions are unconnected. The reason for this is that the disjunction of two propositions implies each and every proposition that is a common consequence of the two, or in other words, that is at one and the same time a consequence of each of them. Putting the result of this paragraph together with the result of the previous paragraph gives us an alternative definition of unconnectedness: in order for two propositions to be unconnected it is necessary and sufficient for their disjunction to be tautological.

The logical principle used in the previous two paragraphs has been called a *disjunctive deduction theorem*: in order for a given disjunction to imply a given proposition it is necessary and sufficient for each of the disjuncts to imply the given proposition. See Corcoran 1985. This means that the information contained in the disjunction is exactly the information common to the two disjuncts. Thus, to repeat a point made above, two propositions have information in common, that is, are connected if and only if their disjunction contains some information.

Now if two propositions are both false then their disjunction is false and hence nontautological. Thus, no two false propositions are unconnected. In other words, every two false propositions are connected and, in fact, are connected by their disjunction. We just saw that any

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two propositions that are connected are connected by their disjunction. Thus having a false disjunction is a sufficient condition for being connected. To see that it is not necessary, consider the two propositions "Four is even and composite" and "Four is composite and square". However, once the sufficient condition has been established as such, it is easy to arrive at a necessary and sufficient condition; in order for two propositions to be connected it is necessary and sufficient that their disjunction be in the same logical form as some false proposition. Of course, this result could have been gotten more directly in view of the fact that the property of being nontautological is coextensive with the property of having the same logical form as some false proposition. This coextensiveness has even been used to define the property of being nontautological, cf. Quine 1978, 47f and Corcoran 1979.

Notwithstanding the connotation of the words 'independent' and 'unconnected', neither relation entails the other. In fact the two relations are "orthogonal", that is, in any given case the fact that one holds or does not hold does not determine whether the other holds or not. This is proved by the following four pairs: "One is even" and "One is not even"; "Two is even" and "Two is prime"; "Three is three" and "Three is not three"; "Four precedes five" and "Five is preceded by four".

Other more interesting examples of independent pairs of connected propositions are easy to find. Consider the three propositions that Gödel 1931 took as axioms for arithmetic [Cohen-Nagel 1993, xli]: the Zero Axiom ZA "Zero is not the successor of any number"; the Successor Axiom SA "Every two distinct numbers have distinct successors"; the Mathematical Induction Axiom IA "Every property that belongs to zero and to the successor of every number to which it belongs [also] belongs to every number". It is easy to see that each pair of these is independent. The basic idea can be found in many places, e.g. Russell 1903, 125. To see that ZA has information in common with SA substitute "integer" for "number" and "square" for "successor". In fact, the same transformation shows that ZA is connected to IA and that SA is connected to IA. Zero is the square of itself. One is the square of itself and it is the square of minus one. And the property of being a square belongs to zero and to the square of every integer to which it belongs but it does not belong to every integer; it does not belong to the integer two. Given a disjunction of any two Gödel axioms, the above substitution transforms it into a false proposition of the same logical form. Thus, by the observation enunciated two paragraphs above, the two given axioms are shown to be connected. Cf. Cohen-Nagel 1993, xxxii and Corcoran 1989, 27, 31. This observation can be called *the principle of possibly false disjunction*: two propositions are connected if their disjunction is in the same logical form as a false proposition.

3.2.2. We have still not solved the unconnected problem whose premise P is "Four is even, composite, and square" and whose conclusion C is "Four is even and composite". We noted above that the independent solution S1 "Four is composite and square" is connected to the conclusion by means of "Four is composite". The natural next candidate to consider is the result of dropping this connecting proposition from S1 arriving at S2 "Four is square". However, although S2 is also independent, it is likewise also connected to the conclusion, by means of the nontautological disjunctive proposition "Four is either both even and composite or square", which can be seen to be informative by substituting "three" for "four", thus transforming it into a false proposition having the same logical form. Thus S2 still contains too much information, namely, the information of the disjunction "Four is either both even and composite or square". As we saw above, the information contained in an informative consequence of a given proposition can be dropped by adjoining the consequence as a condition, that is, by taking the conditional whose

antecedent is the consequence and whose consequent is the given proposition: "If one is even then one is both odd and even" does not imply "One is even" although its consequent does. Using this idea we arrive at S3 "If four is either both even and composite or square then four is square", a strange proposition indeed, which turns out to be logically equivalent to S4 "If four is both even and composite then four is square". Now, it is easy to see that S4 is unconnected with C "Four is both even and composite"; S4 is a result of dropping information from the negation of C, since C itself is the antecedent of S4.

3.3. For this example we have some choices of context among which the following two are natural options. First, we can take the universe to be the class of persons and then take each of the names used to denote a person. Second, we can retain the class of natural numbers and then take each of the names used to denote an "unidentified" or "unknown" number as in elementary algebra.

The given premise P is the conjunction "Abe is Ben and Ben is Carl". The given conclusion C is the identity "Carl is Abe". The conditional proposition S1 "If Carl is Abe, then Ben is Abe and Carl" is [logically] equivalent to the standard conditional. S2 "If Carl is Abe then Abe is Ben" and S3 "If Carl is Abe then Ben is Carl" are also solutions to the unconnected information recovery problem; but it is easy to see that these are also equivalent to S the standard conditional. Further attempts to find non-equivalent solutions prove futile.

There are several ways of seeing that C and S have no information in common. As was suggested above, one easy method is to notice that the disjunction (C or S) is implied by a proposition whose negation also implies it: C implies the disjunction and the negation of C implies every conditional which, like S, has C as antecedent. This means, as explained above, that every proposition implied by each of C and S is tautological.

Even though the two propositions C and S have no information in common and their conjunction exhausts the information in P, it is by no means the case that every piece of information contained in P is contained in one or in the other. For example "Ben is Carl" is contained in P but it is not contained in C "Carl is Abe" and it is not contained in S "If Carl is Abe then, Abe is Ben and Ben is Carl". To see the latter substitute "Zero", "One", and "Two" respectively for "Abe", "Ben" and "Carl". This exemplifies the *synergistic* effect of conjunction: every conjunction of independent propositions implies consequences which do not follow from either conjunct. This obvious point is crucial in understanding certain otherwise puzzling aspects of information containment. Its importance, to the best of my knowledge, was first emphasized by Lewis and Langford 1932, 357.

Let us conclude the discussion of unconnected information recovery problems by formally stating what has by now become obvious to many readers: each such problem has a solution, indeed, a solution unique up to equivalence; and there is a uniform method for constructing "the" solution given the premise and the conclusion: the solution, a function of the premise P and the conclusion C, is the standard conditional (*If C then P*). This result is based on a theorem, *the unique [up to equivalence] complement theorem*, that may be of interest beyond the context of information recovery problems and, for this reason, we describe it in more widely usable terminology.

Following established practice in set theory and in geometry we define one proposition A to be a *complement* of another proposition B with respect to a third proposition C if and only if the first two propositions A and B are unconnected and their conjunction (A and B) is logically equivalent to the third proposition C. The cognates and variants familiar from set theory and

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geometry are automatically presupposed. For example, we will say that A "One is even" and B "One is not even" are complements with respect to C "One is both even and non-even", that the two are complementary with respect to C, etc. *The unique complement theorem* is that every consequence of a given proposition has a unique complement with respect to the given proposition.

The unique complement theorem, as we have seen, gives us a necessary and sufficient condition for a given proposition to be a solution for a given *unconnected* information recovery problem. It is possible to extend this result to a useful necessary and sufficient condition for solutions to *unrestricted* information recovery problems. There are two ideas that bring about the extending. First, every solution to a given unrestricted problem implies the standard conditional and, therefore, is equivalent to a conjunction one of whose conjuncts is the standard conditional. Second, every such solution, other than the standard conditional, is connected to the conclusion and, therefore, is equivalent to a conjunction one of whose conjuncts is a consequence of the conclusion. Thus we are led to the hypothesis that in order for a given proposition to be a solution to a given unrestricted information recovery problem it is necessary and sufficient for the given proposition to be logically equivalent to the conjunction of a consequence of the conclusion with the standard conditional. This hypothesis is easily proved given the results already developed. Thus we have that every solution S to a given unrestricted problem with premise P and conclusion C is equivalent to a conjunction (C1 *and* (If C then P)), where C1 is a consequence of C.

4. Conclusion. I would like to conclude with brief treatments of some topics that complement what has been done above and that will indicate the vistas opened by the kind of deliberations inspired by information recovery problems.

4.1. Gödel's Axiom Set. Using ideas originating with Peano, Gödel 1931 axiomatized arithmetic taking as primitive concepts "number", "zero", and "successor". By number is meant a natural number, either zero or the result of repeated addition of one to zero, a so-called inductive number. By successor is meant the function which produces when applied to an arbitrary number, the immediately succeeding number. Thus "The successor of a given number is the sum of the given number with one" is a true proposition of arithmetic which, however, involves the concept of addition (not among the Gödel primitives). In formalizations, zero is denoted by the usual symbol '0' and the successor function is denoted by the small letter 's'. In this paper we treat the first few number-concepts as defined: "one" is "the successor of zero", "two" is "the successor of one", "three" is "the successor of two", and so on as far as necessary. Thus the sequence of natural numbers is: 0, s0, ss0, sss0,...

Since, as mentioned above, the class of natural numbers is the universe of discourse, the variable 'n' expresses the concept of number. No "constant" symbol is needed; no constant symbol is useful; it would just clutter the formalization thereby defeating some of its purposes. Of course, a single variable is not sufficient and thus other letters are also used. These are so-called "dummy variables" that are simply alternative notational devices expressing the same concept that 'n' expresses, viz. "number". As long as the arrangement of the variables is the same, one and the same proposition is expressed by any number of sentences using different variables. The arithmetic law that every number is exceeded by a prime number (so that there is no end to the sequence of primes) is expressed by each of the following sentences.

Every number n is exceeded by a prime number p.

Every number x is exceeded by a prime number y .

For every u , u is exceeded by some v where v is prime.

For every [number] m there exists [a number] n such that m is exceeded by n and n is prime.

The bracketed material in the last sentence is routinely dropped.

As mentioned above, the following three propositions form the Gödel Axiom Set GAX: the Zero Axiom ZA "Zero is not the successor of any number", the Successor Axiom SA "Every two distinct numbers have distinct successors", the Mathematical Induction Axiom IA "Every property that belongs to zero and that belongs to the successor of every number to which it belongs [also] belongs to every number".

4.1.1. The Zero Axiom ZA "Zero is not the successor of any number" is logically equivalent to "Every number is such that its successor is not zero", which in turn may be paraphrased for formalization as "Every number n is such that its successor sn is not zero". As usual we define a given number to be a *successor* if and only if it is *the* successor of some number. Using this defined concept, ZA is logically equivalent to "No [number which is a] successor is zero" which we take to be our given premise P.

Now we define, again as usual, one given number is a *successor of* a second given number if and only if the first is in the sequence of numbers obtained by repeatedly applying the successor function to the second. Below we will see a way of formalizing this that was discovered, apparently independently, by Frege and by Dedekind. It is clear that P implies "No successor of zero is zero", which we take as our conclusion C.

The proposition "No successor of a non-zero number is zero" or "No successor of a number other than zero is zero" can be taken as a solution S1 to the independent information recovery problem determined by P and C. However, S1 is connected to C by "If zero isn't its own successor then it isn't the successor of its own successor".

It may occur to readers that the Zero Axiom is curiously "redundant" or "wasteful of information" in view of the fact that every successor is a successor of zero: why say that no successor is zero when "the same point" is made by saying less, viz. that no successor of zero is zero? Such considerations may lead to recognition of another solution S2 "If some successor is zero then some successor of zero is zero", which turns out to be equivalent to the standard conditional.

Incidentally, the converse of the "Every successor is a successor of zero" is tautological and therefore the proposition itself is equivalent to the equivalence "In order for a number to be a successor it is necessary and sufficient for it to be a successor of zero", which is called *the successor [equivalence] theorem*.

The idea of Frege and Dedekind is this: in order for one given number to be a successor of a second given number it is necessary and sufficient for every property belonging to the successor of the second and also belonging to the successor of any number to which it belongs to belong to the first given number. Frege 1879 and Dedekind 1888.

4.1.2. The Successor Axiom SA "Every two distinct numbers have distinct successors" is equivalent to "Given any two numbers, if they are unequal, their respective successors are unequal" or "Given any two unequal numbers the successor of the first is unequal to the successor of the second". In arithmetic, but in almost no other context, the words 'equal' and

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'unequal' mean "is" and "isn't" in the sense of identity and inidentity respectively. Tarski 1994, 56f has a fine discussion contrasting the use of 'equals' in arithmetic with its use in geometry.

The functional property "one-one" as in "The identity function is one-one" can be defined in familiar ways so that SA is (or is equivalent to) "The successor function is one-one". We can also define a predicate 'the successor function is one-one on' as in "the successor is one-one on zero": in order for *the successor function to be one-one on* a given number it is necessary and sufficient for the successor of the given number to be unequal to the successor of any other number. Thus the Successor Axiom is equivalent to the proposition that the successor function is one-one on every number, which would be formalized as an ordinary universal ('For every n') from which we immediately deduce the instances: "The successor [function] is one-one on zero", "The successor is one-one on one", "The successor is one-one on two", and so on.

Take SA as the premise P and for the conclusion C take "The successor [function] is one-one on every number whose successor is zero", which is implied both by SA and by ZA. There are many interesting points to be made about this proposition C, but let us first consider "the" natural solution S1 "The successor [function] is one-one on every number whose successor isn't zero", which, though immediately deducible from SA, doesn't follow from ZA at all. To see the latter point, substitute "two" for "zero", "square" for "successor" and "integer" for "number".

Notice that C is informative: substitute "square" for "successor", "integer" for "number" and "one" for "zero". Now since the information in C is common to ZA and SA, there is a redundancy in the Gödel Axiom Set. Since S1 contains the information dropped going from SA to C, we can replace SA by its consequence S1 forming a new axiom set which is equivalent to GAX but which does not have *this* redundancy. Let us call this new axiom ZSA *the Zero-Successor Axiom* "The successor is one-one on every number whose successor isn't zero". The new axiom set, say GAX1, contains ZA, ZSA and IA. This may be thought of as one typical kind of application for information recovery problems; eliminating redundancy in implicationally independent axiom sets.

4.1.3. The word 'inductive' has two important uses in the foundations of arithmetic. It expresses the numerical (first-order) property of being zero or being "generated" from zero by repeated application of the successor function. In this sense, every [natural] number is inductive. It also expresses the qualitative (or second-order) property that belongs to a given property if and only if the given property belongs to zero and to the successor of every number to which it belongs. The last sentence can be taken as a definition of the qualitative property of being inductive. In this sense, the Mathematical Induction Axiom is or is equivalent to the proposition "Every inductive property belongs to every number". Cf Russell 1919, 21, 27, and Russell 1903/37, 123ff.

Using the Frege-Dedekind idea mentioned above we can define the first-order sense of 'inductive' as follows: in order for a given number to be *inductive* it is necessary and sufficient for that number to have every property that belongs to zero and to the successor of every number to which it belongs. Using this definition we can say that IA is equivalent to the proposition that every number is inductive. This is not quite obvious.

By the way, the proposition that zero is inductive is uninformative as is the proposition that the successor of every inductive number is inductive. But the proposition that every number is inductive, which is deducible from the two mentioned uninformative propositions using IA, is informative. This is actually not an uncommon situation: it is tautological that a certain property is inductive but it is informative that the very same property is [numerically] universal, that is, that it belongs to every number. It is perhaps amusing that the two definitions of 'inductive' have created a situation wherein the following proposition is tautological: the property of being inductive is

inductive, in other words, the second-order property "inductive" belongs to the first-order property "inductive".

Let us take the Mathematical Induction Axiom IA for our premise P. For our conclusion C take the proposition "If the successor of zero isn't zero and the successor of the successor of a number whose successor isn't zero isn't zero [either] then zero is not the successor of any number". Thus, C is logically equivalent to the proposition that *if* the property of having a non-zero [immediate] successor is inductive *then* it is universal. Thus C follows not only from IA but also from ZA.

Now let the letter 'Z' indicate the property whose numerical universality is "asserted" in ZA, the property of having a non-zero [immediate] successor. Then, C is equivalent to the proposition that if Z is inductive then Z is numerically universal. This suggests the natural solution S1 "Every property other than Z is such that if it is inductive then it is universal".

Once we have seen this, it becomes obvious that the Gödel Axiom Set is "infinitely redundant". There are infinitely many universal propositions that follow either from ZA alone, or from SA alone, and from the two together. Each such proposition, say T, "asserts" the universality of a property Q; in other words T is logically equivalent to the proposition that Q belongs to every number. Now T and IA both and, thus, also the disjunction (T or IA) implies the proposition that if Q is inductive then Q is universal. By pursuing this line of reasoning we may arrive at the result that it would be futile to try to rid GAX of redundancy by the method employed above, the method of finding a common consequence and then doing an information recovery problem to find a suitably weakened replacement axiom.

One of the main purposes of this subsection has been to show that even though the Gödel Axiom Set is pairwise independent it is nevertheless redundant in the sense that every pair of its axioms is connected. This point will have less force if there is another widely used concept of independence which entails unconnectedness; so far the reader has not been dissuaded from thinking "if the right notion of independence were used then independence would entail unconnectedness". This doubt deserves a response.

Indeed, another common notion of independence is the following: a set of propositions is *systematically independent* if and only if no member of the set is implied by the rest. See Church 1956, 328; Lewis and Langford 1932, 337; Russell 1903, 124; Wilder 1952, 29f. With this notion of independence the situation is the same: it does not preclude connectedness. This conclusion is proved by the same example; GAX is systematically independent and pairwise connected.

Systematic independence amounts to the condition that every set obtained from the given set by negating one axiom is consistent. A much stronger concept called *complete independence* results if it is required to be able to negate *any* number of members of the given set without getting an inconsistency. A set of propositions is said to be *completely independent* if and only if every set obtained from the given set by negating any number or none of the members is consistent. Complete independence does indeed preclude some of the disadvantages noted above in section 2.3. Moreover, GAX is *not* completely independent; it turns out that IA is inconsistent with the negations of the other two axioms taken together. However, the joint failure of complete independence and unconnectedness in this case hardly undercuts the above point. In fact, complete independence not only does not entail unconnectedness it entails connectedness. It is easy to see that every completely independent set is pairwise connected. Being completely independent requires that the negation of any one member be consistent with the negation of any other member, a condition entailing pairwise connectedness. For more on complete

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independence, see Wilder 1952, 31 who credits this concept to the American Postulate Theorist E.H. Moore.

No relevant concept of independence known to me entails unconnectedness; one of the most important concepts of independence actually precludes it. The only logician known to me who notices that the usual notions of independence are not sufficient to preclude overlapping axioms is Eaton 1931, 365 who says of axiom sets, "(...) in the ideal case no part of one [axiom] should follow from the others". Before leaving this point, notice that in a connected axiom set one could find oneself with a false consequence that could not be eliminated by dropping one axiom (or even by dropping all but one).

4.2. Unconnected Axiom Sets. The purpose of this subsection is to produce from the Gödel Axiom Set an equivalent but unconnected set and then to present a method for producing from an arbitrary independent set an equivalent unconnected set. But first there are some definitional matters.

In this subsection the word 'set' refers to a set of propositions having at least two members; the null set and the singleton sets are degenerate cases in regard to axiom sets; in order to avoid tedious checking of trivialities we exclude them from consideration. Above we used 'unconnectedness' for a first-order binary (or two-placed) relation defined on a class of propositions. Here we extend the usage so that 'unconnected' indicates a property of sets of propositions, a second-order property. As above, a set is said to be *pairwise independent* if any two [distinct] members are independent of each other; and a set is said to be [*systematically*] *independent* if no member of the set is implied by the rest of the set. It is obvious that every systematically independent set is pairwise independent, but not conversely. An analogous situation is found in regard to unconnectedness. A set of propositions is defined to be *pairwise unconnected* if no two of its members are connected and it is [*systematically*] *unconnected* if no member has an informative consequence which also follows from the rest. [Notice that this suggests that we say that one set of propositions is *unconnected* with another set if no consequence of one follows from the other; it makes no difference whether the sets are finite or infinite and here we include the null set and the singletons.] Let us consider an independent set of two propositions: A1, A2. Previous results make it obvious that the following equivalent set is unconnected: A1, (*If A1 then A2*). In the case of a set of three propositions A1, A2, A3 there are basically two choices for constructing an equivalent and pairwise unconnected set: the straight conditional method: A1, (*If A1 then A2*), (*If A2 then A3*); and the conjunctive conditional method A1, (*If A1 then A2*), (*If (A1 and A2) then A3*). Only the latter can be proved to be systematically unconnected by ascertaining that each member is unconnected with the rest.

The same idea holds for a set of four propositions: A1, A2, A3, A4. Take the first proposition itself, then take the conditional of the first with the second, then take the conditional of the conjunction of the first two (as antecedent) with the third (as consequent), and finally the fourth of the new propositions is the conditional whose antecedent is the conjunction of the first three from the old set and whose consequent is the fourth member of the old set.

A1 (*If A1 then A2*) (*If (A1 and A2) then A3*) (*If (A1 and A2) and A3 then A4*)

In general an independent set of n propositions A1, A2,..., An is transformed into an unconnected set B1, B2,..., Bn by taking B1 to be A1, B2 to be (*If A1 then A2*), and with j exceeding 1, taking Bj to be the conditional whose antecedent is the conjunction of the first $j-1$ of

the A's and whose consequent is A_j . This algorithmic procedure can be proved to generalize to countably infinite independent sets of propositions.

The following are unconnected sets equivalent to GAX.

GAX1: ZA (If ZA then SA) (If (ZA and SA) then ZA)

GAX2: SA (If SA then IA) (If (SA and IA) then IA)

GAX3: IA (If IA then ZA) (If (IA and ZA) then SA)

There are of course three more sets generated by ordering the propositions "backward": IA , SA , ZA .

In the strict sense, the problem of removing the redundancy from a finite axiom set has been solved. It is ironic that the methods used in the solution undermine any satisfaction that we might have taken because they reveal that the problem of subdividing the information of an axiom set into ultimate unconnected "atoms" is far from solved; any informative superimplication of an axiom gives rise to the dividing of that axiom into two unconnected parts. In fact, as a corollary to the unique complement theorem we have *the reduction theorem*: every proposition having a superimplication reduces to an unconnected pair of informative propositions. By 'reduces to' we mean "is equivalent to the conjunction of".

4.3. Atoms and Saturations. Normally when propositions are discussed by logicians the class of *all* propositions is not being discussed. As a rule, a logician limits the discussion to the class of all propositions concerning a fixed universe of discourse *and* involving only certain "primitive" concepts specified in advance. Thus when it is said, for example that every proposition is either implied by GAX or contradicted by GAX what is meant is *not*, for example, that the Gödel arithmetic axiom set either implies "Cancer is curable" or implies "Cancer is not curable", but rather that every arithmetic proposition [concerning the universe of natural numbers and involving no concepts besides "number", "zero" and "successor"] is either implied by GAX or contradicted by GAX. The latter assertion is true, of course, being a consequence of the so-called categoricity of GAX. See Corcoran 1980.

Likewise, when we say here that every contradiction, for example "One is not one", "Two is even and not even", etc., implies every proposition and therefore contains all information, what is meant by 'proposition' is "arithmetic proposition [etc.]" and what is meant by 'information' is "information contained in arithmetic propositions and sets of arithmetic propositions [etc.]". In this section unless explicitly indicated otherwise we are talking exclusively about arithmetic propositions based on the concepts "number", "zero", and "successor" as in GAX.

As mentioned above, the satisfaction that might have been taken in the construction of a [systematically] unconnected axiom set for arithmetic was undermined by the thought that, since each axiom can be reduced to two unconnected parts, perhaps, it may be impossible to arrive at an equivalent axiom set containing only irreducible unconnected propositions. For convenience let G be a single proposition equivalent to GAX. In accord with the above comment on the categoricity of GAX, the proposition G is *saturated* in the sense that it contains as much information as possible without being inconsistent. More literally, since G implies or contradicts every proposition, every conjunction of G with another proposition is either equivalent to G or [self-]contradictory (and thus contains all information). G is maximally informative without being contradictory. A consistent proposition such as G which is implied by no proposition other than a

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contradiction or one of its own equivalents is called a *saturation*. Every two saturations are either equivalent or mutually inconsistent.

We want to consider whether it is possible to divide G up into unconnected parts which can not be further subdivided. More generally, we want to consider whether propositions, all or some, can be "atomized". Let us say that a proposition is *atomic* if it is informative but no information can be dropped from it without thereby transforming it into a tautology. More formally, a proposition is said to be [informationally] *atomic* if and only if it is informative but all of its superimplications (non-equivalent implications) are uninformative. The noun *atom* means "informationally atomic proposition".

By the way, the expression 'atomic proposition' usually means "structurally atomic proposition", loosely, a proposition that is not "composed" of other propositions. In the class of arithmetic propositions discussed here the only structurally atomic propositions are the identities: "Zero is zero", "Zero is one", "Zero is two", etc. Among the arithmetic structural atoms every one is either tautologous or false and each of those that are false implies the disjunction of itself with another false structural atom. Thus no structural atom is informationally atomic. It is a lucky accident, so to speak, that we do not have to use the principle of possibly false disjunction here. Of course, this result presupposes that we are speaking about arithmetic propositions. If we consider a different "propositional domain" the issue would have to be reconsidered. For example consider the situation in which 'proposition' refers exclusively to propositions expressible using only 'Abe', 'Ben', 'is', and 'isn't' where the first two words refer to, say, Abraham Lincoln and Benjamin Franklin and where the last two words are taken in the senses of identity and distinctness, as above. In this propositional domain, we have four propositions up to equivalence: "Abe is Abe", "Abe is Ben", "Abe isn't Ben" and "Abe isn't Abe". Here the consistent informative proposition "Abe is Ben" is both structurally and informationally atomic. Confusing the concept of structural atom with that of informational atom may be involved in the fallacy of thinking that, e.g. "Abe is wise" is unconnected with "Ben is kind" in the domain of propositions expressible in normal English [etc.]. Of course, in the domain of propositions limited to those expressible using only 'Abe', 'Ben', 'wise', 'kind', and 'is' suitably interpreted with 'is' expressing predication, "Abe is wise" is unconnected with "Ben is kind" and there are no tautologies and no contradictions. This shows that the concept of connectedness presupposes a background domain as do many other logical concepts. The concept of a limited universe of discourse, found by logicians beginning with Aristotle to be essential in the analysis of axiomatic sciences, plays an important role in logic itself. Weyl 1927/49, 7, 18, 24; Lewis and Langford 1932, 330; Mates 1965, 173.

Limitations of space do not permit full discussion of the relevant results, but a summary of them will help to put the above deliberations in perspective. An atom is an informative proposition from which no information can be dropped without rendering it uninformative. As yet we have seen no example of an arithmetic atom. A given atom is implied by every proposition that it is connected to; every two non-equivalent atoms are unconnected; and an atom is implied by every proposition whose negation does not imply it [and, equivalently, it is implied by the negation of every proposition that does *not* imply it].

A saturation is a consistent proposition to which no information can be added without rendering it inconsistent. The proposition G equivalent to the Gödel Axiom Set is an example. In fact it is easy to see that there are infinitely many saturations in the domain of arithmetic propositions expressible in the language of Gödel 1931. See Weaver 1970 and Corcoran 1980. A given saturation implies every proposition that it is consistent with; every two non-equivalent

saturations are inconsistent; and a saturation implies every proposition whose negation it does not imply [and, equivalently, a saturation implies the negation of every proposition that it does not imply].

In every propositional domain containing the negation of each proposition it contains, the negation of a given saturation is an atom and the negation of a given atom is a saturation. Saturations are to atoms much as contradictions are to tautologies. At any rate, in order for a proposition to be atomic it is necessary and sufficient for it to be equivalent to the negation of a saturation. This result enables us to identify the negation of G as an atom and it enables us to identify infinitely many atoms. This in turn gives rise to the following results about the Gödel Axiom Set.

The Gödel Axiom Set GAX implies infinitely many atoms. It is, therefore, not equivalent to any finite set of propositions whose members can not be further analyzed into two or more unconnected "parts". Moreover, the Gödel Axiom Set is not implied by any set of atomic propositions [expressible in the language of Gödel 1931]. Therefore, it is not equivalent to *any* set of propositions not further analyzable into unconnected parts, whether that set be finite or infinite. Nevertheless we should not lose sight of the fact that Eaton's ideal of a set of axioms wherein "no part of one should follow from the others" *is* realizable, however uninteresting and inconsequential this may be in individual cases.

Other conclusions concern the interrelations among the Gödel axioms. The Gödel Axiom Set is infinitely redundant. In fact, there are infinitely many atoms each of which is implied by each of the three axioms, thus, each of which connects each axiom to each of the other two. It is, hopeless, therefore to attempt to remove the redundancy on an atom-by-atom basis.

Moreover, each of the three Gödel axioms implies infinitely many atoms not implied by the other two and, therefore, there is no way to replace even one of them by a finite set of propositions that can not be further analyzed into unconnected parts. Some of the results of this section are due in part to George Weaver.

In the domain of arithmetic propositions just considered, every two non-equivalent atoms are consistent with each other, every two non-equivalent saturations are connected, and no atom is a saturation. Indeed, in any domain an atom is as weak as it can be without being uninformative and saturation is as strong as it can be without being inconsistent. It may occur to the imaginative reader to wonder whether there could be a "degenerate" domain containing both tautologies and contradictions and containing both atoms and saturations, but in which the class of atoms is coextensive with the class of saturations. This is indeed the case with one of the propositional domains mentioned above which contained only four propositions up to equivalence; its only propositions weaker than "Abe is Ben" are tautologies and its only propositions stronger than "Abe is Ben" are contradictions. Thus "Abe is Ben" is both an atom and a saturation. The same holds for "Abe isn't Ben". There are other strange features of this degenerate case: two atoms are inconsistent with each other and two saturations are unconnected. In typical non-degenerate domains, such as the arithmetic domain discussed above, every two atoms are consistent and every two saturations are connected.

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