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The emergence of some of the nonlogical paradoxes of the theory of sets, 1903–1908. (French, Spanish summaries)
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A "paradox" is an argumentation that appears to deduce a conclusion believed to be false from premises believed to be true. An "inconsistency proof for a theory" is an argumentation that actually deduces a negation of a theorem of the theory from premises which are all theorems of the theory. An "indirect proof of the negation of a hypothesis" is an argumentation that actually deduces a conclusion known to be false from the hypothesis alone or, more commonly, from the hypothesis augmented by a set of premises known to be true. A "direct proof of a hypothesis" is an argumentation that actually deduces the hypothesis itself from premises known to be true. Since 'appears', 'believes' and 'knows' all make elliptical reference to a "participant", it is clear that 'paradox', 'indirect proof' and 'direct proof' are all participant-relative. In normal mathematical writing the "participant" is presumed to be "the community of mathematicians" or some more or less well-defined subcommunity and, therefore, omission of explicit reference to the participant is often warranted. However, in historical, critical, or philosophical writing focused on emerging branches of mathematics such omission often invites confusion. One and the same argumentation has been a paradox for one mathematician, an inconsistency proof for another, and an indirect proof to a third. One and the same argumentation-text can appear to one mathematician to express an indirect proof while appearing to another mathematician to express a direct proof. Of the above four sorts of argumentation only the paradox invites "solution" or "resolution", and ordinarily this is to be accomplished either by discovering a logical fallacy in the "reasoning" of the argumentation or by discovering that the conclusion is not really false or by discovering that one of the premises is not really true. Resolution of a paradox by a participant amounts to reclassifying a formerly paradoxical argumentation—either as a "fallacy", as a direct proof of its conclusion, as an indirect proof of the negation of one of its premises, as an inconsistency proof, or as something else depending on the participant's state of knowledge or belief. This illustrates why an argumentation which is a paradox to a given mathematician at a given time may well not be a paradox to the same mathematician at a later time.

The present article considers several set-theoretic argumentations that appeared in the period 1903–1908. The year 1903 saw the publication of B. Russell's *Principles of mathematics,* [Cambridge Univ. Press, Cambridge, 1903; Jbuch **34**, 62]. The year 1908 saw the publication of Russell's article on type theory as well as Ernst Zermelo's two watershed articles on the axiom of choice and the foundations of set theory. The argumentations discussed concern "the largest cardinal", "the largest ordinal", the well-ordering principle, "the well-ordering of the continuum", definability of ordinals and definability of reals. The article shows that these argumentations were variously classified by various mathematicians and that the surrounding atmosphere was one of confusion and misunderstanding—partly as a result of failure to make or to heed distinctions similar to those made above. The article implies that historians have made the situation worse by not observing or not analysing the nature of the confusion.

Russell's role in discovering and disseminating paradoxes, both logical and nonlogical, is shown to be rather more extensive than previously thought. Jacques Hadamard emerges as one of the few mathematicians of the time who saw the difficulties in set theory in historical perspective, i.e., as natural consequences of an attempt to broaden the scope of mathematics. Henri Poincaré's inclination to classify all paradoxes as inconsistency proofs is regarded as a result of his aim to discredit Cantor's theory of sets and Russell's logicism.

This well-written and well-documented article exemplifies the fact that clarification of history can be achieved through articulation of distinctions that had not been articulated (or were not being heeded) at the time. The article presupposes extensive knowledge of the history of mathematics, of mathematics itself (especially set theory) and of philosophy. It is therefore not to be recommended for casual reading.

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