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Three Ways of Being Non-Material

Abstract. This paper develops a probabilistic analysis of conditionals which hinges on a quantitative measure of evidential support. In order to spell out the interpretation of 'if' suggested, we will compare it with two more familiar interpretations, the suppositional interpretation and the strict interpretation, within a formal framework which rests on fairly uncontroversial assumptions. As it will emerge, each of the three interpretations considered exhibits specific logical features that deserve separate consideration.

Keywords: Conditionals, Probability, Evidential support, Connexivity, Suppositional.

1. Preliminaries

Although it is widely agreed that indicative conditionals as they are used in ordinary language do not behave as material conditionals, there is little agreement on the nature and the extent of such deviation. Different theories of conditionals tend to privilege different intuitions, and there is no obvious way to tell which of them is the correct theory. At least two non-material readings of 'if' deserve attention. One is the *suppositional interpretation*, according to which a conditional is acceptable when it is likely that its consequent holds on the supposition that its antecedent holds. The other is the *strict interpretation*, according to which a conditional is acceptable when its antecedent necessitates its consequent. This paper explores a third non-material reading of 'if' — the *evidential interpretation* — which rests on the idea that a conditional is acceptable when its antecedent supports its consequent, that is, when its antecedent provides a reason for accepting its consequent.

The first two interpretations have been widely discussed, and have prompted quite distinct formal accounts of conditionals. The suppositional interpretation has been articulated by Adams [1] and others by defining a suitable probabilistic semantics. The strict interpretation, which goes back to the Stoics, has been mainly treated in standard modal logic. Instead, the evidential interpretation is relatively underdeveloped. The idea of support

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is hard to capture at the formal level, and the same goes for its closest relatives, such as the notion of reason. This explains the heterogeneity and the multiplicity of the attempts that have been made to define a conditional with such a property.[1]

In what follows, we outline a precise and well defined version of the evidential interpretation which hinges on a quantitative measure of evidential support. In order to spell out in a perspicuous way the relations between the evidential interpretation so understood and the other two interpretations, we will adopt a unified formal framework that rests on fairly uncontroversial assumptions. Basically, all we need as a background theory is propostional logic and the probability calculus. This framework provides a good basis for comparing the three interpretations and elucidating their logic. As we shall see, some interesting principles that involve different kinds of non-material conditionals can be expressed and assessed in our language. Moreover, it allows for a probabilistic reduction of the strict interpretation, which is interesting in itself.[2]

The structure of the paper is as follows. Section 2 introduces a language that includes three symbols $\Rightarrow, \rhd, \prec$ that stand respectively for the suppositional conditional, the evidential conditional, and the strict conditional.[3] Section 3 defines validity along the lines suggested by Adams. Section 4 outlines a set of principles of conditional logic and states some important relations between them. Sections 5–7 explain, for each of the three symbols, what kind of considerations can justify its use, and how it behaves with respect to the principles outlined. Section 8 adds some general remarks on the relations between the three interpretations considered. The remaining part of the paper is a technical appendix which contains the proofs of all the facts set out in the previous sections.

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[1] Rott [33,34] develop an account of "difference-making" conditionals within the framework of belief revision theory. Spohn [35] oulines a ranking-theoretic account of conditionals along the same lines. The approach outlined in [14,15], perhaps the closest precedent of our analysis, employs the notion of evidential support in Bayesian epistemology. [5,39,41] provide further examples.

[2] One might wonder whether a similar comparison can be carried out by relying on a modal semantics, given that there are modal treatments of the suppositional interpretation, and that the strict interpretation is naturally understood in modal terms. The modal semantics provided in [10] shows that such discussion is possible, although it will not be pursued here. Raidl, Iacona and Crupi [32] provides a completeness result in the modal framework for the logic of the evidential conditional.

[3] The symbol $\Rightarrow$ is used exactly as in Adams. The symbol $\rhd$ is borrowed from [35]. The symbol $\prec$ is a tribute to the seminal work on strict implication in [28].

2. The Language L

Let **P** be a standard propositional language whose alphabet is constituted by a finite set of sentence letters $p, q, r...$, the connectives $\sim, \supset, \wedge, \vee$, and the brackets (,). We will call *propositional formulas* the formulas of **P**, and use the symbol $\vDash_{PL}$ to indicate logical consequence in **P**. Let **L** be a language with the following alphabet:

$p, q, r, ...$

$\sim, \supset, \wedge, \vee$

$\Box$

$\Rightarrow, \rhd, \multimapdotinv$

$(,)$

The formulas of **L** are defined by induction as follows:

Definition 1. 1 If $\alpha \in \mathbf{P}$, then $\alpha \in \mathbf{L}$;

2 if $\alpha \in \mathbf{P}$, then $\Box\alpha \in \mathbf{L}$;

3 if $\alpha \in \mathbf{P}$ and $\beta \in \mathbf{P}$, then $\alpha \Rightarrow \beta \in \mathbf{L}$;

4 if $\alpha \in \mathbf{P}$ and $\beta \in \mathbf{P}$, then $\alpha \rhd \beta \in \mathbf{L}$;

5 if $\alpha \in \mathbf{P}$ and $\beta \in \mathbf{P}$, then $\alpha \multimapdotinv \beta \in \mathbf{L}$;

6 if $\alpha \in \mathbf{L}$, then $\sim\alpha \in \mathbf{L}$.

Note that clause 2 rules out multiple occurrences of $\Box$ in the same formula. For example, $\Box(p \wedge \Box q)$ is not a formula of **L**. Similarly, clauses 3-5 rule out multiple occurrences of $\Rightarrow, \rhd, \multimapdotinv$ in the same formula. For example, $p \Rightarrow (p \Rightarrow q)$ is not a formula of **L**. Moreover, since clause 6 is the only clause that applies to formulas of **L**, $\sim$ is the only connective whose scope can include the scope of $\Box, \Diamond, \Rightarrow, \rhd, \multimapdotinv$.

The idea that underlies the semantics of **L**, which is in line with a tradition initiated by Adams, is to define a valuation function for sentences depending on the probability of their propositional constituents, that is, the constituents that are adequately formalized in a standard propositional language. The function V is defined as follows for any probability function P over **P**:

Definition 2. 1 For every $\alpha \in \mathbf{P}$, $V_P(\alpha) = P(\alpha)$;

2 $V_P(\Box\alpha) = \begin{cases} 1 \text{ if } P(\alpha) = 1 \\ 0 \text{ otherwise;} \end{cases}$

3 $V_P(\alpha \Rightarrow \beta) = \begin{cases} P(\beta|\alpha) & \text{if } P(\alpha) > 0 \\ 1 & \text{if } P(\alpha) = 0; \end{cases}$

4 $V_P(\alpha \rhd \beta) = \begin{cases} \frac{P(\beta|\alpha)-P(\beta)}{1-P(\beta)} & \text{if } P(\beta|\alpha) \geq P(\beta), P(\alpha) > 0 \text{ and } P(\beta) < 1, \\ 1 & \text{if } P(\alpha) = 0 \text{ or } P(\beta) = 1, \\ 0 & \text{otherwise;} \end{cases}$

5 $V_P(\alpha \dashv\!\!\!\!\:\exists\; \beta) = \begin{cases} 1 \;\; if\; P(\beta|\alpha) = 1\; or\; P(\alpha) = 0 \\ 0 \text{ otherwise;} \end{cases}$

6 $V_P(\sim\alpha) = 1 - V_P(\alpha)$.

One straightforward way to understand V is as a measure of assertibility: for any α, $V_P(\alpha)$ represents the degree of assertibility of α given P. From now on we will take this interpretation for granted. But we would like to emphasize that our formal treatment is consistent with other interpretations. P, in turn, may be understood in more than one way. A natural option is to take its values to represent epistemic probabilities. But our formal treatment is also compatible with a reading of P in terms of objective chance.

Clause 1 says that V_P assigns to the propositional formulas the same values as P. This means that the degree of assertibility of any propositional formula amounts to its probability.

Clause 2 says that a formula $\Box\alpha$ takes either 1, the maximal value, or 0, the minimal value, depending on whether or not $P(\alpha) = 1$.

Clause 3 says that the value that V_P assigns to $\alpha \Rightarrow \beta$ is the conditional probability of β given α, with the proviso that $V_P(\alpha \Rightarrow \beta) = 1$ if $P(\alpha) = 0$ (normally, $P(\beta|\alpha)$ would be undefined in that case). This is essentially Adam's idea that the degree of assertibility of a conditional is the conditional probability of its consequent given its antecedent.[4]

Clause 4 is the crucial one. The value that V_P assigns to $\alpha \rhd \beta$ is the degree of evidential support (if any) that α provides to β relative to P (with the proviso that $V_P(\alpha \rhd \beta) = 1$ if $P(\beta) = 1$ or $P(\alpha) = 0$, in which case the evidential support measure loses mathematical meaning). This conveys the assumption that the antecedent has to contribute positively to the credibility of the consequent, because the measure employed to quantify $V_P(\alpha \rhd \beta)$ gives a definite positive value if and only if $P(\beta|\alpha) > P(\beta)$. The latter condition is a straightforward characterization of positive probabilistic relevance, and also the standard qualitative definition of evidential support (or incremental confirmation) in a Bayesian framework. It is well known that several quantitative measures retain this fundamental idea, and here we will make

[4]See [2]. About the stipulation that $V_P(\alpha \Rightarrow \beta) = 1$ if $P(\alpha) = 0$, see [3, p. 150].

no attempt to justify our specific choice, which has been spelled out and defended elsewhere.[5] We will simply point out that there is a coherent sense in which the measure so defined characterizes positive evidential impact as the degree of partial entailment of β by α. In this sense, the degree of assertibility of $\alpha \triangleright \beta$ may be understood as the degree of partial entailment — if any — from α to β.[6]

Clause 5 says that the value that V_P assigns to $\alpha \multimap \beta$ is 1 when $P(\beta|\alpha) = 1$ or $P(\alpha) = 0$, otherwise it is 0, given that $P(\beta|\alpha) = 1$ or $P(\alpha) = 0$ if and only if there is no chance that α is true and β is false. As far as we can see, this is the best approximation in the language of probability to the modal condition that characterizes the strict conditional as traditionally understood, namely, that it is impossible that α is true and β is false. Obviously, as long as probability is understood epistemically, the same goes for the corresponding notion of possibility.

Clause 6 defines negation in the classical way, as it entails that the value of $\sim\alpha$ is 1 when the value of α is 0, and that the value of $\sim\alpha$ is 0 when the value of α is 1. In particular, when $V_P(\sim\Box\sim\alpha) = 1$, we get that $V_P(\Box\sim\alpha) = 0$, which means that $\sim\alpha$ is not necessary, hence that α is possible. This shows that $\Diamond\alpha$ can be defined in the usual way as $\sim\Box\sim\alpha$.

Note that clauses 3-5 entail that each of the three conditionals defined is fully assertible whenever $\alpha \vDash_{PL} \beta$. Suppose that $\alpha \vDash_{PL} \beta$. Then, if $P(\alpha) = 0$, $V_P(\alpha \Rightarrow \beta) = V_P(\alpha \triangleright \beta) = V_P(\alpha \multimap \beta) = 1$. If $P(\alpha) > 0$, then again $V_P(\alpha \Rightarrow \beta) = V_P(\alpha \triangleright \beta) = V_P(\alpha \multimap \beta) = 1$, because $P(\beta|\alpha) = 1$.

As emerges from the definitions just outlined, our threefold formal account of non-material conditionals inherits certain features of the probabilistic tradition which some reader may legitimately see as potential threats or limitations. It is then sensible to comment on them at this stage. The main point to emphasize is that these features do not depend on specific properties of $\triangleright$ or $\multimap$, and that the account itself does not introduce any new technical or conceptual difficulty.

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[5] The details are in [12, 13]. Related ideas are thoroughly discussed in [14, 15].

[6] An alternative definition, which would deserve careful consideration, is obtained by removing the second line, that $V_P(\alpha \triangleright \beta) = 1$ when $P(\alpha) = 0$ or $P(\beta) = 1$. The rationale would be that in these two cases α is incapable of making a difference to β. We owe this interesting suggestion to an anonymous reviewer.

First of all, there is nothing exceptional in the limited expressive power of **L**. In particular, setting aside compounds and embeddings of non-propositional formulas is a standard move in probabilistic approaches to non-material conditionals, if only for technical reasons. A more subtle limitation is perhaps the choice to work with a finite set of sentence letters. The motivation here is again instrumental: our symbolic apparatus is sufficient for many applications targeting reasoning and language, while also allowing for a smooth connection between necessity and probability. Overall, it is significantly richer than usual in the literature, and anyway comprehensive enough to express a large set of important principles that are relevant for our purposes.

So-called triviality results are another case in point. One might think that these results pose a crucial challenge to Adams' analysis of conditionals, what we call the suppositional interpretation, see [20]. However, no additional problem is raised by the evidential and the strict interpretation in this respect. In fact, the semantic values attached to formulas in corresponding clauses of Definition 2 are not even probabilities in the first place, so no straightforward extension of traditional triviality results would apply.

A third issue, which has been widely discussed in connection with the previous one, is whether conditionals have truth-conditions. Surely, the approach we take here has been popular among authors, such as Edgington, who firmly reject the idea of truth-conditions for non-material conditionals, see [16]. For readers with such inclinations, a key expected outcome of our work is to show that two further kinds of non-material conditional are tractable within their own favourite framework. As it happens, however, we do not think that our approach is incompatible with the idea that conditionals have truth conditions. Even if a plausible truth-conditional theory of non-material conditionals exists, it is still legitimate and potentially fruitful to spell out their assertibility conditions in probabilistic terms.

3. Validity

In order to define validity in **L** it is convenient to adopt a function U such that $U_P(\alpha) = 1 - V_P(\alpha)$ for any α and any P. As long as V is understood as a measure of assertibility, $U_P(\alpha)$ represents the lack of assertibility of α given P, what Adams calls the *uncertainty* of α relative to P. Following Adams, we will define validity in terms of U:

Definition 3. $\alpha_1, ...\alpha_n \vDash \beta$ if and only if, for any P, $U_P(\alpha_1)+...+U_P(\alpha_n) \geq U_P(\beta)$

In Adams' terminology, a valid argument is an argument in which the uncertainty of the conclusion cannot exceed the total uncertainty of the premises.

Definition 3 preserves the classical notion of logical consequence in at least two crucial respects. First, as Adams has shown, all classically valid arguments expressible in **P** remain valid. That is, if $\alpha_1, ..., \alpha_n, \beta \in \mathbf{P}$ and $\alpha_1, ...\alpha_n \vDash_{PL} \beta$, then $\alpha_1, ...\alpha_n \vDash \beta$. We will use the label PL whenever we rely on this fact, see [3, p. 38]. Second, the rule of *Substitution of Logical Equivalents* (SLE) is valid:

FACT 1. *If $\alpha \dashv\vDash_{PL} \beta$, α occurs in γ, and γ' is obtained from γ by replacing α with β, then $\gamma \dashv\vDash \gamma'$.*

As will be shown in the appendix, SLE plays an important role in the proof of many technically useful results. This rule entails two rules for conditionals that are sometimes treated separately. One is Left Logical Equivalence: if $\alpha \dashv\vDash_{PL} \beta$, then $\alpha > \gamma$ is equivalent to $\beta > \gamma$. The second is Right Logical Equivalence: if $\beta \dashv\vDash_{PL} \gamma$, then $\alpha > \beta$ is equivalent to $\alpha > \gamma$.

From Definitions 2 and 3 we also get two important results concerning the connection between $\Rightarrow, \triangleright, \dashv$, namely, that $\triangleright$ is stronger than $\Rightarrow$ and $\dashv$ is stronger than $\triangleright$.

FACT 2. $\alpha \triangleright \beta \vDash \alpha \Rightarrow \beta$ *but* $\alpha \Rightarrow \beta \nvDash \alpha \triangleright \beta$

FACT 3. $\alpha \dashv \beta \vDash \alpha \triangleright \beta$ *but* $\alpha \triangleright \beta \nvDash \alpha \dashv \beta$

This makes perfect sense. If α supports β, then it is reasonable to expect that β is credible enough given α, and if α necessitates β, then it is reasonable to expect that α supports β. In fact necessitation may be regarded as the strongest kind of support.

4. Principles of Conditional Logic

Before dealing with the symbols $\Rightarrow, \triangleright, \dashv$ one by one, it is useful to list thirty principles of conditional logic that we will discuss in connection with them, and spell out some important relations between these principles. In what follows, $>$ indicates a conditional without specifying its interpretation, $\top$ and $\bot$ stand for tautology and contradiction, $\Box$ and $\Diamond$ stand for 'necessarily' and 'possibly', and the long arrow $\Longrightarrow$ indicates valid inference.[7]

Superclassicality (S): If $\alpha \vDash_{PL} \beta$, then $\alpha > \beta$ must hold

[7]Some sources for our list are [8, 15, 22, 37, 38].

Material Implication (MI): $\alpha > \beta \Longrightarrow \alpha \supset \beta$

Detachment (DET): $\top > \alpha \Longrightarrow \alpha$

Modus Ponens (MP): $\alpha > \beta, \alpha \Longrightarrow \beta$

Conjunction of Consequents (CC): $\alpha > \beta, \alpha > \gamma \Longrightarrow \alpha > (\beta \wedge \gamma)$

Disjunction of Antecedents (DA): $\alpha > \gamma, \beta > \gamma \Longrightarrow (\alpha \vee \beta) > \gamma$

Necessary Consequent (NC): $\Box\alpha \Longrightarrow \beta > \alpha$

Impossible Antecedent (IA): $\Box{\sim}\alpha \Longrightarrow \alpha > \beta$

Cautious Monotonicity (CM): $\alpha > \beta, \alpha > \gamma \Longrightarrow (\alpha \wedge \beta) > \gamma$

Negation Rationality (NR): $\alpha > \gamma, {\sim}((\alpha \wedge {\sim}\beta) > \gamma) \Longrightarrow (\alpha \wedge \beta) > \gamma$

Rational Monotonicity (RM): $\alpha > \gamma, {\sim}(\alpha > {\sim}\beta) \Longrightarrow (\alpha \wedge \beta) > \gamma$

Right Weakening (RW): If $\beta \vDash_{PL} \gamma$, then $\alpha > \beta \Longrightarrow \alpha > \gamma$

Conversion (CON): $\alpha \Longrightarrow \top > \alpha$

Conjunctive Sufficiency (CS): $\alpha \wedge \beta \Longrightarrow \alpha > \beta$

Conditional Excluded Middle (CEM): ${\sim}(\alpha > \beta) \Longrightarrow \alpha > {\sim}\beta$

Limited Transitivity (LT): $\alpha > \beta, (\alpha \wedge \beta) > \gamma \Longrightarrow \alpha > \gamma$

Conditional Equivalence (CE): $\alpha > \beta, \beta > \alpha, \beta > \gamma \Longrightarrow \alpha > \gamma$

False Antecedent (FA): ${\sim}\alpha \Longrightarrow \alpha > \beta$

True Consequent (TC): $\beta \Longrightarrow \alpha > \beta$

Monotonicity (M): $\alpha > \gamma \Longrightarrow (\alpha \wedge \beta) > \gamma$

Transitivity (T): $\alpha > \beta, \beta > \gamma \Longrightarrow \alpha > \gamma$

Contraposition (C): $\alpha > \beta \Longrightarrow {\sim}\beta > {\sim}\alpha$

Conditional Proof (CP): If $\Gamma, \alpha \vDash_{PL} \beta$, then $\Gamma \Longrightarrow \alpha > \beta$

Empty Antecedent Strengthening (EAS): $\top > \alpha \Longrightarrow \beta > \alpha$

Prelinearity (PRE): ${\sim}(\alpha > \beta) \Longrightarrow \beta > \alpha$

Complementary Antecedent (CA): ${\sim}(\alpha > \beta) \Longrightarrow {\sim}\alpha > \beta$

Restricted Selectivity (RS): If $\beta \vDash_{PL} {\sim}\gamma$, then $\Diamond\alpha, \alpha > \beta \Longrightarrow {\sim}(\alpha > \gamma)$

Restricted Conditional Non-Contradiction (RCN): $\Diamond\alpha, \alpha > \beta \Longrightarrow {\sim}(\alpha > {\sim}\beta)$

Restricted Aristotle's Thesis (RAT): $\Diamond\alpha \Longrightarrow {\sim}(\alpha > {\sim}\alpha)$

Restricted Aristotle's Second Thesis (RAST): $\Diamond{\sim}\beta, \alpha > \beta \Longrightarrow {\sim}({\sim}\alpha > \beta)$

This list includes both principles that only involve conditional formulas, such as CM or T, and principles that also involve non-conditional formulas, with or without modal operators, such as NC or MI. Apart from the last four principles, which may be labelled *connexive principles*, the principles listed above hold for the material conditional.[8] This means that they hold if $>$ is replaced by $\supset$. As we shall see, the suppositional interpretation, the evidential interpretation, and the strict interpretation differ from the material interpretation — and from each other — with respect to these principles, because we get different results if we replace $>$ with $\Rightarrow$, $\triangleright$, or $\prec$.

The principles listed above are related in various ways, so they cannot be accepted or rejected independently of each other. In particular, we will rely on the following facts, some of which are well known, which hold for any reading of $>$.

Fact 4. *If MI holds, then DET holds as well, given PL.*

Fact 5. *If MI holds, then MP holds as well, given PL.*

Fact 6. *If LT and S hold, then RW holds as well.*

Fact 7. *If M and CON hold, then TC holds as well.*

Fact 8. *If CM and CEM hold, then RM holds as well.*

Fact 9. *If T and S hold, then M holds as well.*[9]

Fact 10. *If C and RW hold, then M holds as well.*[10]

Fact 11. *If C and CC hold, then DA holds as well.*

Fact 12. *If S, CC, CE hold, then LT holds as well.*[11]

Fact 13. *If CP holds, then FA and TC hold as well, given PL.*

Fact 14. *If CON and EAS hold, then TC holds as well.*

Fact 15. *If NC holds, and either C holds or RW, S, CC hold, then IA holds as well.*

Fact 16. *If RS holds, then RCN holds as well.*

Fact 17. *If RCN and S hold, then RAT holds as well.*

Fact 18. *if C and RS hold, then RAST holds as well.*

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[8]On connexive principles in general see [31,40]. Some restricted versions of connexive principles, which require contingent antecedents, have been considered in [24,25,38], although, as far as we know, no similar restriction on consequents has been suggested so far.

[9]See [26, pp. 180–181].

[10]See [26, pp. 180–181].

[11]See [26, pp. 179]. Fact 12 also follows from a result given in [18, p. 54].

5. The Suppositional Conditional

Since the logic of the suppositional conditional is well known, we will simply recall some established results and add some details that matter for our purposes. The first seventeen principles in our list hold for $\Rightarrow$, see [3, chap. 7], [11]. Of these principles, we will prove only NC and IA, which are seldom discussed in the literature because they involve modal operators.

Fact 19. $\Box\alpha \vDash \beta \Rightarrow \alpha$ (*Necessary Consequent* ✓)

Fact 20. $\Box{\sim}\alpha \vDash \alpha \Rightarrow \beta$ (*Impossible Antecedent* ✓)

Now consider FA, TC, M, T, and C. These principles do no hold for $\Rightarrow$.

Fact 21. ${\sim}\alpha \nvDash \alpha \Rightarrow \beta$ (*False Antecedent* $\times$)

Fact 22. $\beta \nvDash \alpha \Rightarrow \beta$ (*True Consequent* $\times$)

Fact 23. $\alpha \Rightarrow \gamma \nvDash (\alpha \wedge \beta) \Rightarrow \gamma$ (*Monotonicity* $\times$)

Fact 24. $\alpha \Rightarrow \beta, \beta \Rightarrow \gamma \nvDash \alpha \Rightarrow \gamma$ (*Transitivity* $\times$)

Fact 25. $\alpha \Rightarrow \beta \nvDash {\sim}\beta \Rightarrow {\sim}\alpha$ (*Contraposition* $\times$)

According to Adams, Facts 21–25 speak in favour of the suppositional reading of 'if'. His point is that if the material reading of 'if' is adopted, some apparently invalid arguments that instantiate FA, TC, M, T, and C must be treated as valid, which is quite implausible, see [1, pp. 166–167].

Note that, while plain monotonicity fails for $\Rightarrow$, other weaker principles — CM, RM, and NR — license similar inferences under additional conditions.[12] Similarly, while plain transitivity fails for $\Rightarrow$, LT and CE remain valid.

Now let us consider CP and EAS. These two principles also fail.

Fact 26. *Not: if* $\Gamma, \alpha \vDash_{PL} \beta$, *then* $\Gamma \vDash \alpha \Rightarrow \beta$ (*Conditional Proof* $\times$)[13]

Fact 27. $\top \Rightarrow \alpha \nvDash \beta \Rightarrow \alpha$ (*Empty Antecedent Strengthening* $\times$)

PRE and CA are further principles that hold for $\supset$ but not for $\Rightarrow$.

Fact 28. ${\sim}(\alpha \Rightarrow \beta) \nvDash \beta \Rightarrow \alpha$ (*Prelinearity* $\times$)

Fact 29. ${\sim}(\alpha \Rightarrow \beta) \nvDash {\sim}\alpha \Rightarrow \beta$ (*Complementary Antecedent* $\times$)

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[12] For discussions of Rational Monotonicity and Negation Rationality, see [26, p. 197], [27], [4, p. 332].

[13] See [17, p. 176].

Facts 28 and 29 may be regarded as desirable results. It reasonable to expect that in some cases it is right to deny $\alpha > \beta$ even though it is wrong to assert $\beta > \alpha$, or $\sim\alpha > \beta$, see [17, p. 171].

Finally, RS, RCN, and RAT hold for $\Rightarrow$, while RAST does not.

FACT 30. *If* $\beta \vDash_{PL} \sim\gamma$, *then* $\Diamond\alpha, \alpha \Rightarrow \beta \vDash \sim(\alpha \Rightarrow \gamma)$ (*Restricted Selectivity* ✓)

FACT 31. $\Diamond\alpha, \alpha \Rightarrow \beta \vDash \sim(\alpha \Rightarrow \sim\beta)$ (*Restricted Conditional Non-Contradiction* ✓)

FACT 32. $\Diamond\alpha \vDash \sim(\alpha \Rightarrow \sim\alpha)$ (*Restricted Aristotle's Thesis* ✓)

FACT 33. $\Diamond\sim\beta, \alpha \Rightarrow \beta \nvDash \sim(\sim\alpha \Rightarrow \beta)$ (*Restricted Aristotle's Second Thesis* ×)

All things considered, $\Rightarrow$ is preferable to $\supset$ in some respects, in that it invalidates some principles that hold for $\supset$ but may be perceived as counterintuitive, such as FA or TC, while it retains other principles that hold for $\supset$ and are widely accepted as correct, such as S or MP. However, the behaviour of $\Rightarrow$ is not satisfactory in all respects, and this explains at least in part why the debate on conditionals has moved on after Adams. Here we will provide four observations, each of which points out a possible source of perplexity.

Observation 1: it is not obvious that CS should be preserved. Several critics have regarded this principle as an unsettling contamination of the truth-functional account in the logic of non-material conditionals, and we are inclined to agree with them, see [6], [4, pp. 239–240]. The sheer fact that α and β hold seems not enough to claim $\alpha > \beta$, unless some further connection holds between them. For example, there may be something wrong in the following conditional even if Susan is actually a red-haired doctor:

(1) If Susan is red-haired, then she is a doctor

Remarkably, Adams himself labels CS a "rather strange inference" and retreats on a Gricean escape to accommodate it. However, this is the same kind of move that is usually regarded as insufficient to relieve the truth-functional account from the counterintuitive effects of FA and TC.[14]

Observation 2: 'it is not obvious that CEM should be preserved. The intuitive status of this principle is notoriously controversial, see e.g. [9, 42].

[14] See [3, p. 157]. CS has been a matter of discussion within non-probabilistic variants of the suppositional interpretation. Most notably, Lewis [29] does not treat CS as an essential principle.

There seem to be cases in which it is correct to deny $\alpha > \beta$ while it is incorrect to assert $\alpha > \sim\beta$. For example, even if it may be right to deny (1), this does not make it right to assert (2):

(2) If Susan is red-haired, then she is not a doctor

Adams assumes that $\sim(\alpha > \beta)$ simply means $\alpha > \sim\beta$, and other theorists of conditionals agree on this assumption, see [1, p. 181]. But CEM can hardly be defended by appealing to meaning, given that the whole debate on conditionals stems precisely from the fact that it is not entirely clear what 'if' means.[15]

Observation 3: it is not entirely clear that C is to be rejected. Although many theorists of conditionals follow Adams and think that C must fail, others are apt to think that C can coherently be preserved. The alleged counterexamples to C, such as the following, have been widely discussed, and there is no obvious way to handle them.

(3) If John makes a mistake, it is not a big mistake

(4) If John does not make a big mistake, it is not a mistake

In particular, one thing that has been noted is that such counterexamples imply that their premise is naturally understood as a concessive conditional. (3) can be rephrased by using 'even if' instead of 'if'. This means that such counterexamples would loose their grip on any account of $>$ which rules out the concessive reading.[16]

Observation 4: the suppositional treatment of the connexive principles is not ideal. On the one hand, $\Rightarrow$ validates RS, RCN, and RAT, which are quite reasonable principles. On the other hand, however, it invalidates RAST, which is also reasonable to some extent. Consider the following conditionals:

(5) If it is cold, then it is not raining

(6) If it is not cold, then it is not raining

As long as one thinks that the assertibility of a conditional requires that its antecedent supports its consequent, one can hardly accept that both (5) and (6) hold. This is probably why many people would naturally refrain from asserting (5) and (6) together.

[15] As in the case of CS, CEM has been a matter of discussion within non-probabilistic variants of the suppositional interpretation. While Stalnaker [36] accepts it, Lewis [29] rejects it.

[16] See [30, p. 34], [4, pp. 32, 143–144]. A recent and forceful defense of contraposition for non-concessive conditionals is provided in [19].

As the next two sections show, the evidential interpretation and the strict interpretation provide different but equally motivated answers to the questions raised by these four observations. So they may be regarded as interesting alternatives to the suppositional interpretation.

6. The Evidential Conditional

As we have seen, much of the appeal of the suppositional interpretation lies in the fact that it invalidates some principles that hold for $\supset$ but may be perceived as counterintuitive, while it retains other principles that hold for $\supset$ and are widely accepted as correct. The evidential interpretation preserves this virtue, although it significantly differs from the suppositional interpretation in some crucial respects which are directly relevant to observations 1-4.

First of all, consider S, MI, DET, MP, CC, C, and DA. These principles hold for $\rhd$, and the same goes for C:

FACT 34. *If* $\alpha \vDash_{PL} \beta$, *then* $\alpha \rhd \beta$ (*Superclassicality* ✓)

FACT 35. $\alpha \rhd \beta \vDash \alpha \supset \beta$ (*Material Implication* ✓)

FACT 36. $\top \rhd \alpha \vDash \alpha$ (*Detachment* ✓)

FACT 37. $\alpha \rhd \beta, \alpha \vDash \beta$ (*Modus Ponens* ✓)

FACT 38. $\alpha \rhd \beta, \alpha \rhd \gamma \vDash \alpha \rhd (\beta \wedge \gamma)$ (*Conjunction of Consequents* ✓)

FACT 39. $\alpha \rhd \beta \vDash {\sim}\beta \rhd {\sim}\alpha$ (*Contraposition* ✓)

FACT 40. $\alpha \rhd \gamma, \beta \rhd \gamma \vDash (\alpha \vee \beta) \rhd \gamma$ (*Disjunction of Antecedents* ✓)

As shown in the appendix, Fact 39 is technically useful to connect Facts 38 and 40. C marks a key difference between $\rhd$ and $\Rightarrow$, and constitutes one of the most interesting features of $\rhd$. The reason is that $\rhd$, unlike $\Rightarrow$, rules out the concessive reading of conditionals. $\alpha \Rightarrow \beta$ can be highly assertible even though α is irrelevant to β, or is at odds with β: a high probability of β given α is enough. Instead, the assertibility of $\alpha \rhd \beta$ requires not only that β is highly probable if α is assumed, but that it is so at least in part *because* α is assumed. Since the alleged counterexamples to C typically involve the concessive reading of conditionals, as noted in observation 3, it makes sense that C holds for $\rhd$. For example, on the evidential interpretation (3) is hardly assertible, so the inference from (3) to (4) is no counterexample to C.

Now consider principles NC, IA, CM, and NR. These principles hold for $\rhd$.

FACT 41. $\Box\alpha \vDash \beta \rhd \alpha$ (*Necessary Consequent* ✓)

FACT 42. $\Box{\sim}\alpha \vDash \alpha \rhd \beta$ (*Impossible Antecedent* ✓)

FACT 43. $\alpha \rhd \beta, \alpha \rhd \gamma \vDash (\alpha \wedge \beta) \rhd \gamma$ (*Cautious Monotonicity* ✓)

FACT 44. $\alpha \rhd \gamma, {\sim}((\alpha \wedge {\sim}\beta) \rhd \gamma) \vDash (\alpha \wedge \beta) \rhd \gamma$ (*Negation Rationality* ✓)

From what has been said so far it turns out that the first ten principles of our list hold for $\rhd$. Since the same principles hold for $\Rightarrow$, this shows that there is a considerable overlap between $\rhd$ and $\Rightarrow$, and consequently between $\rhd$, $\Rightarrow$, and $\supset$.

Now we will focus on the principles that do not hold for $\rhd$, and thereby highlight some significant differences between $\rhd$ and $\Rightarrow$. We have already seen that $\rhd$, unlike $\Rightarrow$, validates C. Another difference is that $\rhd$, unlike $\Rightarrow$, violates RM.

FACT 45. $\alpha \rhd \gamma, {\sim}(\alpha \rhd {\sim}\beta) \nvDash (\alpha \wedge \beta) \rhd \gamma$ (*Rational Monotonicity* ×)

In the appendix, we prove this fact by means of an example. Suppose that we are interested in the blood type of a person named Sara. Let α be 'Sara's mother's blood type is A', let β be 'Sara's father's blood type is B', and let γ be 'Sara's blood type is A'. The following probability distribution arises from plausible background assumption and basic genetic theory:[17]

$$\begin{aligned}
&P(\alpha \wedge \beta \wedge \gamma) = 0{,}018\\
&P(\alpha \wedge \beta \wedge {\sim}\gamma) = 0{,}052\\
&P(\alpha \wedge {\sim}\beta \wedge \gamma) = 0{,}152\\
&P(\alpha \wedge {\sim}\beta \wedge {\sim}\gamma) = 0{,}078\\
&P({\sim}\alpha \wedge \beta \wedge \gamma) = 0{,}003\\
&P({\sim}\alpha \wedge \beta \wedge {\sim}\gamma) = 0{,}160\\
&P({\sim}\alpha \wedge {\sim}\beta \wedge \gamma) = 0{,}127\\
&P({\sim}\alpha \wedge {\sim}\beta \wedge {\sim}\gamma) = 0{,}409
\end{aligned}$$

In the case described, the uncertainty of 'If Sara's mother's blood is type A, then Sara's blood type is A' is moderate. The uncertainty of 'It is not the

[17] The underlying hypothetical distribution of blood phenotypes 0, A, B, AB is 40%, 30%, 23%, 7%. Assuming a Hardy-Weinberg model, the corresponding genotype distribution for AA, BB, 00, AB, A0, B0 is 4%, 3%, 40%, 7%, 26%, 20% respectively (figures rounded). All other figures are implied given random mating (another standard background condition) and a basic Mendelian model of inheritance, with alleles A and B dominant and 0 recessive.

case that, if Sara's mother's blood type is A, then Sara's father's blood type is not B' is null, because the antededent and the consequent of the negated conditional are statistically independent, so the negation of such evidential conditional is fully assertible. Instead, the uncertainty of 'If Sara's mother's blood type is A and Sara's father's blood type is B, then Sara's blood type is A' is maximal, because the probability of the consequent is not increased (in fact it is decreased) by the probability of the antecedent.

Fact 45 shows that there is at least one sense in which $\rhd$ is *less* monotonic than $\Rightarrow$. This is clear if one thinks that, in the case decribed above, the inference would go through if the suppositional interpretation were adopted. For $U_P(\alpha \Rightarrow \gamma) = 1 - V_P(\alpha \Rightarrow \gamma) = 1 - P(\gamma|\alpha) = 0,44$, and $U_P({\sim}(\alpha \Rightarrow {\sim}\beta)) = 1 - V_P({\sim}(\alpha \Rightarrow {\sim}\beta) = 1 - P(\beta|\alpha) = 1 - P(\beta) = 0,77$, so $U_P(\alpha \Rightarrow \gamma) + U_P({\sim}(\alpha \Rightarrow {\sim}\beta) = 1,21$, while $U_P((\alpha \wedge \beta) \Rightarrow \gamma) = 1 - V_P((\alpha \wedge \beta) \Rightarrow \gamma) = 1 - P(\gamma|\alpha \wedge \beta) = 0,75$. The main difference concern the second premise, whose uncertainty is null for $\rhd$ but quite high for $\Rightarrow$.

Two corollaries of Fact 45 are that M and RW do not hold for $\rhd$.

FACT 46. $\alpha \rhd \gamma \not\vDash (\alpha \wedge \beta) \rhd \gamma$ (*Monotonicity* $\times$)

FACT 47. *Not: if* $\beta \vDash_{PL} \gamma$, *then* $\alpha \rhd \beta \vDash \alpha \rhd \gamma$ (*Right Weakening* $\times$)

Fact 46 shows that $\rhd$ is exactly like $\Rightarrow$ as far as M is concerned. Instead, Fact 47 shows that $\rhd$ and $\Rightarrow$ differ with respect to RW, which is quite interesting. RW is one of the most entrenched and technically powerful rules of traditional logics for conditionals. Yet, as puzzling as it may seem at first sight, the failure of RW is a very natural outcome for evidential conditionals.[18] Note that, at least since the debate between Hempel and Carnap, it is clear that evidential support must fail the so-called "special consequence condition", see [7,21]. In fact if α and β are probabilistically independent propositional formulas, and $0 < P(\alpha), P(\beta) < 1$, then α provides evidential support to $\alpha \wedge \beta$ but not to β, that is, $P(\alpha \wedge \beta|\alpha) > P(\alpha \wedge \beta)$ while $P(\beta|\alpha) = P(\beta)$, in spite of the fact that $\alpha \wedge \beta \vDash_{PL} \beta$.

From the failure of M and RW we can also conclude that T, LT, and CE do not hold for $\rhd$.

FACT 48. $\alpha \rhd \beta, \beta \rhd \gamma \not\vDash \alpha \rhd \gamma$ (*Transitivity* $\times$)

FACT 49. $\alpha \rhd \beta, (\alpha \wedge \beta) \rhd \gamma \not\vDash \alpha \rhd \gamma$ (*Limited Transitivity* $\times$)

FACT 50. $\alpha \rhd \beta, \beta \rhd \alpha, \beta \rhd \gamma \not\vDash \alpha \rhd \gamma$ (*Conditional Equivalence* $\times$)

[18] Rott [34, p. 7], takes the failure of RW to be the hallmark of "difference-making conditionals". Crupi and Iacona [10] provides a more detailed discussion of RW.

Another important difference between $\Rightarrow$ and $\triangleright$ concerns CS and CEM. These two principles do not hold for $\triangleright$.

FACT 51. $\alpha \wedge \beta \nvDash \alpha \triangleright \beta$ (*Conjunction Sufficiency* $\times$)

FACT 52. $\sim(\alpha \triangleright \beta) \nvDash \alpha \triangleright \sim\beta$ (*Conditional Excluded Middle* $\times$)

As it emerges from Facts 51 and 52, the evidential interpretation differs from the suppositional interpretation in that it provides opposite answers to the questions raised in observations 1 and 2. First, according to the evidential interpretation there are cases in which $\alpha > \beta$ can reasonably be denied even though α and β hold. For example, if one denies (1), one does so because 'Susan is red-haired' provides no support for 'Susan is a doctor', independently of Susan's actual hair colour or profession. Second, according to the evidential interpretation there are cases in which it is correct to deny $\alpha > \beta$ while it is incorrect to assert $\alpha > \sim\beta$. For example, it is perfectly consistent to deny both (1) and (2), for in both cases the antecedent does not support the consequent.

As far as FA, TC, CP, PRE, and CA are concerned, $\triangleright$ behaves exactly like the $\Rightarrow$, and unlike $\supset$.

FACT 53. $\sim\alpha \nvDash \alpha \triangleright \beta$ (*False Antecedent* $\times$)

FACT 54. $\beta \nvDash \alpha \triangleright \beta$ (*True Consequent* $\times$)

FACT 55. *Not: if* $\Gamma, \alpha \vDash_{PL} \beta$, *then* $\Gamma \vDash \alpha \triangleright \beta$ (*Conditional Proof* $\times$)

FACT 56. $\sim(\alpha \triangleright \beta) \nvDash \beta \triangleright \alpha$ (*Prelinearity* $\times$)

FACT 57. $\sim(\alpha \triangleright \beta) \nvDash \sim\alpha \triangleright \beta$ (*Complementary Antecedent* $\times$)

Two further differences between $\Rightarrow$ and $\triangleright$ concern EAS and CON. While $\Rightarrow$ validates the former but not the latter, $\triangleright$ validates the latter but not the former.

FACT 58. $\top \triangleright \alpha \vDash \beta \triangleright \alpha$ (*Empty Antecedent Strengthening* $\checkmark$)

FACT 59. $\alpha \nvDash \top \triangleright \alpha$ (*Conversion* $\times$)

Finally, consider the connexive principles. All these principles hold for $\triangleright$.

FACT 60. *If* $\beta \vDash_{PL} \sim\gamma$, *then* $\Diamond\alpha, \alpha \triangleright \beta \vDash \sim(\alpha \triangleright \gamma)$ (*Restricted Selectivity* $\checkmark$)

FACT 61. $\Diamond\alpha$, $\alpha \triangleright \beta \vDash \sim(\alpha \triangleright \sim\beta)$ (*Restricted Conditional Non-Contradiction* $\checkmark$)

FACT 62. $\Diamond\alpha \vDash \sim(\alpha \triangleright \sim\alpha)$ (*Restricted Aristotle's Thesis* $\checkmark$)

Fact 63. $\Diamond{\sim}\beta, \alpha \rhd \beta \vDash {\sim}({\sim}\alpha \rhd \beta)$ (*Restricted Aristotle's Second Thesis* ✓)

Facts 60–63 show that the evidential interpretation provides a coherent treatment of the connexive principles, including RAST.

From what has been said so far it turns out that $\rhd$ agrees with $\supset$ and $\Rightarrow$ on several plausible principles. This is a distinctive feature of the evidential interpretation as we understand it, which distinguishes it from similar accounts of conditionals that have been provided so far. In particular, our analysis crucially differs from Douven's, which is perhaps its closest relative. The logic generated by Douven's approach is rather weak, as it fails five of the first ten principles of our list, that is, MP, CC, DA, CM, and NR.[19]

As we have seen, $\rhd$ has a neatly distinctive logic and differs from $\Rightarrow$ with respect to each of the four issues raised in observations 1-4: the account outlined invalidates CS and CEM, validates C, and provides a uniform treatment of the connexive principles. So, the evidential interpretation may be regarded as a coherent alternative to the suppositional interpretation.

7. The Strict Conditional

The strict interpretation is a different alternative to the suppositional interpretation. On the strict reading of $>$, to assert $\alpha > \beta$ is to assert that there is no chance that α holds but β does not hold. As noted in section 2, 'chance' may plausibly understood in its objective reading, as opposed to epistemic or subjective probability. But in any case our account of $\dashv$ does not depend on this distinction. As we will see, Definition 2 implies that $\dashv$ preserves the logical profile of the strict conditional as traditionally understood, and warrants the equivalence between $\alpha \dashv \beta$ and $\Box(\alpha \supset \beta)$. This is why we take the label 'strict' to be appropriate, even though in a probabilistic framework.[20]

The strict interpretation agrees with the evidential interpretation in at least two important respects. First, it preserves all the classical principles preserved by the evidential interpretation. Second, it offers the same kind of responses to the questions raised in observations 1-4, including the treatment of the connexive principles. However, as we will see, $\dashv$ differs from $\rhd$ in other crucial respects.

Let us start with S, MI, DET, MP, CC, C, and DA. These principles hold for $\dashv$, and the same goes for C:

[19] See [15, Theorem 5.2.1, p. 130]. A more thorough discussion of Douven is provided in [11].

[20] Iacona [23] outlines some general arguments for the strict interpretation.

FACT 64. *If* $\alpha \vDash_{PL} \beta$, *then* $\alpha \dashv\!\!\!\!\!\!\to\!\! \beta$ (*Superclassicality* ✓)

FACT 65. $\alpha \rightarrow\!\!\!\!\!\!\!\dashv \beta \vDash \alpha \supset \beta$ (*Material Implication* ✓)

FACT 66. $\top \triangleright \alpha \vDash \alpha$ (*Detachment* ✓)

FACT 67. $\alpha \triangleright \beta, \alpha \vDash \beta$ (*Modus Ponens* ✓)

FACT 68. $\alpha \rightarrow\!\!\!\!\!\!\!\dashv \beta, \alpha \rightarrow\!\!\!\!\!\!\!\dashv \gamma \vDash \alpha \rightarrow\!\!\!\!\!\!\!\dashv (\beta \wedge \gamma)$ (*Conjunction of Consequents* ✓)

FACT 69. $\alpha \rightarrow\!\!\!\!\!\!\!\dashv \beta \vDash {\sim}\beta \rightarrow\!\!\!\!\!\!\!\dashv {\sim}\alpha$ (*Contraposition* ✓)

FACT 70. $\alpha \rightarrow\!\!\!\!\!\!\!\dashv \gamma, \beta \rightarrow\!\!\!\!\!\!\!\dashv \gamma \vDash (\alpha \vee \beta) \rightarrow\!\!\!\!\!\!\!\dashv \gamma$ (*Disjunction of Antecedents* ✓)

NC and IA also hold for $\rightarrow\!\!\!\!\!\!\!\dashv$.

FACT 71. $\Box\alpha \vDash \beta \rightarrow\!\!\!\!\!\!\!\dashv \alpha$ (*Necessary Consequent* ✓)

FACT 72. $\Box{\sim}\alpha \vDash \alpha \triangleright \beta$ (*Impossible Antecedent* ✓)

The crucial difference between $\triangleright$ and $\rightarrow\!\!\!\!\!\!\!\dashv$ is that $\rightarrow\!\!\!\!\!\!\!\dashv$ is fully transitive and monotonic, in that T and M hold for $\rightarrow\!\!\!\!\!\!\!\dashv$.

FACT 73. $\alpha \rightarrow\!\!\!\!\!\!\!\dashv \beta, \beta \rightarrow\!\!\!\!\!\!\!\dashv \gamma \vDash \alpha \rightarrow\!\!\!\!\!\!\!\dashv \gamma$ (*Transitivity* ✓)

FACT 74. $\alpha \rightarrow\!\!\!\!\!\!\!\dashv \gamma \vDash (\alpha \wedge \beta) \rightarrow\!\!\!\!\!\!\!\dashv \gamma$ (*Monotonicity* ✓)

Although T and M may not accord with the evidential interpretation, they make sense on the strict interpretation. If α necessitates β, and β necessitates γ, then clearly α necessitates γ. Similarly, if α necessitates γ, then clearly $\alpha \wedge \beta$ necessitates γ. In the literature on conditionals, there has been plenty of discussion about the alleged counterexamples to T and M, and there is no widespread agreement about them. The strict interpretation may be combined with some accounts of these cases that explain away the apparent violation of T and M.[21]

From Facts 73 and 74 we obtain some important corollaries: CE, LT, RW, and RM hold for $\rightarrow\!\!\!\!\!\!\!\dashv$, while they do not hold for $\triangleright$.

FACT 75. $\alpha \rightarrow\!\!\!\!\!\!\!\dashv \beta, \beta \rightarrow\!\!\!\!\!\!\!\dashv \alpha, \beta \rightarrow\!\!\!\!\!\!\!\dashv \gamma \vDash \alpha \rightarrow\!\!\!\!\!\!\!\dashv \gamma$ (*Conditional Equivalence* ✓)

FACT 76. $\alpha \rightarrow\!\!\!\!\!\!\!\dashv \beta, (\alpha \wedge \beta) \rightarrow\!\!\!\!\!\!\!\dashv \gamma \vDash \alpha \rightarrow\!\!\!\!\!\!\!\dashv \gamma$ (*Limited Transitivity* ✓)

FACT 77. *If* $\beta \vDash_{PL} \gamma$, *then* $\alpha \rightarrow\!\!\!\!\!\!\!\dashv \beta \vDash \alpha \rightarrow\!\!\!\!\!\!\!\dashv \gamma$ (*Right Weakening* ✓)

FACT 78. $\alpha \rightarrow\!\!\!\!\!\!\!\dashv \gamma, {\sim}(\alpha \rightarrow\!\!\!\!\!\!\!\dashv {\sim}\beta) \vDash (\alpha \wedge \beta) \rightarrow\!\!\!\!\!\!\!\dashv \gamma$ (*Rational Monotonicity* ✓)

[21] Iacona [23] outlines such an account.

The divergence between $\triangleright$ and $\strictif$ emerges clearly if we focus on RM. As we have seen, in the case of Sara the evidential interpretation makes the argument invalid because it implies that the second premise is certain: 'It is not the case that, if Sara's mother's blood type is A, then Sara's father's blood type is not B'. In fact this is the key difference between $\triangleright$ and $\Rightarrow$. The strict interpretation, instead, makes the argument valid because it raises the uncertainty of the first premise. More precisely, we have that $V_P(\alpha \strictif \gamma) = 0$, $V_P({\sim}(\alpha \strictif {\sim}\beta)) = 1$, and $V_P((\alpha \wedge \beta) \strictif \gamma) = 0$, so $U_P(\alpha \strictif \gamma) + U_P({\sim}(\alpha \strictif {\sim}\beta)) = U_P((\alpha \wedge \beta) \strictif \gamma) = 1$. So the strict interpretation agrees with the evidential interpretation on the certainty of the second premise, but it preserves RM because it poses higher constraints on the assertibility of the first premise.

Obviously, since M holds for $\strictif$, the same goes for CM, NR, and EAS, which are weaker. In this respect, $\strictif$ agrees with $\triangleright$.

Fact 79. $\alpha \strictif \beta, \alpha \strictif \gamma \vDash (\alpha \wedge \beta) \strictif \gamma$ (*Cautious Monotonicity* ✓)

Fact 80. $\alpha \strictif \gamma, {\sim}((\alpha \wedge {\sim}\beta) \strictif \gamma) \vDash (\alpha \wedge \beta) \strictif \gamma$ (*Negation Rationality* ✓)

Fact 81. $\top \strictif \alpha \vDash \beta \strictif \alpha$ (*Empty Antecedent Strengthening* ✓)

The agreement between $\strictif$ and $\triangleright$ also concerns FA, TC, CON, CS, CEM, CP, PRE, and CA, which do not hold for $\strictif$.

Fact 82. ${\sim}\alpha \nvDash \alpha \strictif \beta$ (*False Antecedent* $\times$)

Fact 83. $\beta \nvDash \alpha \strictif \beta$ (*True Consequent* $\times$)

Fact 84. $\alpha \nvDash \top \strictif \alpha$ (*Conversion* $\times$)

Fact 85. $\alpha \wedge \beta \nvDash \alpha \strictif \beta$ (*Conjunctive Sufficiency* $\times$)

Fact 86. ${\sim}(\alpha \strictif \beta) \nvDash \alpha \strictif {\sim}\beta$ (*Conditional Excluded Middle* $\times$)

Fact 87. *Not: if* $\Gamma, \alpha \vDash_{PL} \beta$, *then* $\Gamma \vDash \alpha \strictif \beta$ (*Conditional Proof* $\times$)

Fact 88. ${\sim}(\alpha \strictif \beta) \nvDash \beta \strictif \alpha$ (*Prelinearity* $\times$)

Fact 89. ${\sim}(\alpha \strictif \beta) \nvDash {\sim}\alpha \strictif \beta$ (*Complementary Antecedent* $\times$)

These principles do not hold for $\strictif$ for the same reason for which they do not hold for $\triangleright$, namely, that the connection between antecedent and consequent implied by $\strictif$ would not be preserved if they were valid.

As far as the connexive principles are concerned, $\strictif$ behaves exactly like $\triangleright$.

Fact 90. *If* $\beta \vDash_{PL} {\sim}\gamma$, *then* $\Diamond\alpha, \alpha \strictif \beta \vDash {\sim}(\alpha \strictif \gamma)$ (*Restricted Selectivity* ✓)

FACT 91. $\Diamond\alpha, \alpha \dashv\!\!\!\!\!\!\rightarrow \beta \vDash \sim(\alpha \dashv\!\!\!\!\!\!\rightarrow \sim\beta)$ (*Restricted Conditional Non-Contradiction* ✓)

FACT 92. $\Diamond\alpha \vDash \sim(\alpha \dashv\!\!\!\!\!\!\rightarrow \sim\alpha)$ (*Restricted Aristotle's Thesis* ✓)

FACT 93. $\Diamond\sim\beta, \alpha \dashv\!\!\!\!\!\!\rightarrow \beta \vDash \sim(\sim\alpha \dashv\!\!\!\!\!\!\rightarrow \beta)$ (*Restricted Aristotle's Second Thesis* ✓)

From Facts 90–93 it turns out that the strict interpretation provides a coherent treatment of the connexive principles, just like the evidential interpretation. The table below summarizes what has been said so far and provides an overall picture of our results. The difference between $\dashv\!\!\!\!\!\!\rightarrow$ and $\rhd$ lies in the fact that $\dashv\!\!\!\!\!\!\rightarrow$ validates five principles that do not hold for $\rhd$, namely, RM, RW, LT, CE, M, and T. By contrast, there is no principle that is validated by $\rhd$ but not by $\dashv\!\!\!\!\!\!\rightarrow$.

As a matter of fact, $\dashv\!\!\!\!\!\!\rightarrow$ is exactly as strong as the necessitation of $\supset$, as the following equivalence holds.

FACT 94. $\alpha \dashv\!\!\!\!\!\!\rightarrow \beta \dashv\vDash \Box(\alpha \supset \beta)$

This shows that, insofar as conditionals are adequately formalized as strict conditionals, we can express their logical properties in **L** instead of employing a standard modal language. Of course, modal logic works perfectly well, and many people take possible worlds to be entirely acceptable theoretical entities, or at least no more problematic than probabilities. However, one might be apt to believe that probabilities are theoretically kosher in some sense in which possible worlds are not, or prefer probabilities for purely instrumental reasons. If you belong to the second category, then here there is something for you. You can have the logic of the strict conditional, but without possible worlds.

8. Final Remarks

In the foregoing sections we have spelled out three non-material interpretations of 'if then' — the suppositional interpretation, the evidential interpretation, and the strict interpretation — by elucidating some characteristic properties of the symbols $\Rightarrow$, $\rhd$, $\dashv\!\!\!\!\!\!\rightarrow$. This last section provides some final remarks about the relations between these symbols.

As Facts 2 and 3 show, our three symbols can be ordered in terms of increasing strength as follows: $\Rightarrow, \rhd, \dashv\!\!\!\!\!\!\rightarrow$. It is important to note, however, that this does not mean that the logic of each of these three symbols is an extension of the logic of the symbol that precedes it, in the sense that it preserves all the principles that hold for the symbol that precedes it. Although it may

		$\supset$	$\Rightarrow$	$\triangleright$	$-\!\!3$
S	Superclassicality	✓	✓	✓	✓
MI	Material Implication	✓	✓	✓	✓
DET	Detachment	✓	✓	✓	✓
MP	Modus Ponens	✓	✓	✓	✓
CC	Conjunction of Consequents	✓	✓	✓	✓
DA	Disjunction of Antecedents	✓	✓	✓	✓
NC	Necessary Consequent	✓	✓	✓	✓
IA	Impossible Antecedent	✓	✓	✓	✓
CM	Cautious Monotonicity	✓	✓	✓	✓
NR	Negation Rationality	✓	✓	✓	✓
RM	Rational Monotonicity	✓	✓	×	✓
RW	Right Weakening	✓	✓	×	✓
CON	Conversion	✓	✓	×	×
CS	Conjunctive Sufficiency	✓	✓	×	×
CEM	Conditional Excluded Middle	✓	✓	×	×
LT	Limited Transitivity	✓	✓	×	✓
CE	Conditional Equivalence	✓	✓	×	✓
FA	False Antecedent	✓	×	×	×
TC	True Consequent	✓	×	×	×
M	Monotonicity	✓	×	×	✓
T	Transitivity	✓	×	×	✓
C	Contraposition	✓	×	✓	✓
CP	Conditional Proof	✓	×	×	×
EA	Empty Antecedent Strengthening	✓	×	✓	✓
PRE	Prelinearity	✓	×	×	×
CA	Complementary Antecedent	✓	×	×	×
RS	Restricted Selectivity	×	✓	✓	✓
RC	Restricted Conditional Non-Contradiction	×	✓	✓	✓
RAT	Restricted Aristotle's Thesis	×	✓	✓	✓
RAST	Restricted Aristotle's Second Thesis	×	×	✓	✓

be reasonable to conjecture that the logic of $-\!\!3$ is an extension of the logic of $\triangleright$, it is certainly not the case that the logic of $\triangleright$ is an extension of the logic of $\Rightarrow$, that is, $\Rightarrow$ and $\triangleright$ have different logics, neither of which is an extension of the other. We take this to be a major implication of our results.

In fact, the distinction between the suppositional interpretation and the evidential interpretation deserves careful consideration. As the facts stated in section 6 show—the most important results of this paper—these two interpretations may be regarded as two alternative and complementary ways to depart from the material interpretation *and* abandon full monotonicity. Consider M. As Fact 10 shows, the rejection of M forces the failure of at least one among RW and C: the logic of $\Rightarrow$ retains the former, while the logic of $\rhd$ retains the latter, and each option finds a coherent theoretical motivation in the corresponding reading of 'if'. A very similar pattern arises from the rejection of TC, which is a distinctive "paradox" of the material conditional, because this rejection imposes a choice between CON and EAS, as is shown by Fact 14.

More generally, each of the three interpretations considered has interesting logical implications, and finds some support in the ordinary use of 'if'. These three interpretations may be regarded either as three distinct meanings that speakers attach to 'if', or as three ways of explicating a single indeterminate meaning by replacing it with a precise and well defined counterpart. The second option leaves open the question of whether there is a unique correct analysis of conditionals. Some theorists of conditionals work under the assumption that there is such an analysis, while others are inclined to think that different accounts of conditionals may be equally correct. We believe that the contents presented here are to a large extent neutral with respect to this divide. If there is a unique correct analysis of conditionals, the results presented in the foregoing sections may shed some light on such analysis. On the other hand, if different formal accounts of conditionals are equally correct, the distinction between $\Rightarrow$, $\rhd$, and $\dashv$ suggests one definite way to carve the space of the possible options.

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Appendix

In what follows we will adopt three methodological conventions. First, we will use the letters α, β, γ to refer to propositional formulas, without specifying that they belong to $\mathbf{P}$. Second, we will use the letter P to refer to an arbitrary probability function. Third, since Definition 3 says that $\alpha_1, ...\alpha_n \vDash \beta$ if and only if $U_P(\alpha_1) + ... + U_P(\alpha_n) \geq U_P(\beta)$, we will take for granted that in order to prove that $\alpha_1, ...\alpha_n \vDash \beta$ it suffices to show that $U_P(\beta) = 0$, or that $U_P(\alpha_i) = 1$ for at least some α_i in $\alpha_1, ...\alpha_n$.

FACT 1: If $\alpha \dashv\vDash_{PL} \beta$, α occurs in γ, and γ' is obtained from γ by replacing α with β, then $\gamma \dashv\vDash \gamma'$.

PROOF. The proof is by induction on the complexity of γ, assuming that $\alpha \dashv\vDash_{PL} \beta$, that α occurs in γ, that γ' is obtained from γ by replacing α with β, and that P is any probability function. The basis of the induction is the case in which $\gamma \in \mathbf{P}$. In this case $\gamma \dashv\vDash_{PL} \gamma'$, therefore $\gamma \dashv\vDash \gamma'$. In the inductive step we assume that the result to be proved holds for any formula of complexity less than or equal to n, and that γ is a formula of complexity $n + 1$. The possible cases are five.
Case 1: γ has the form $\Box\delta$. In this case $\gamma' = \Box\delta'$. Since $\delta \in \mathbf{P}$, $\delta \dashv\vDash_{PL} \delta'$. So, $P(\delta) = P(\delta')$. By clause 2 of Definition 2, it follows that $V_P(\gamma) = V_P(\gamma')$, hence that $U_P(\gamma) = U_P(\gamma')$. Therefore, $\gamma \dashv\vDash \gamma'$.
Case 2: γ has the form $\delta \Rightarrow \phi$. In this case $\gamma' = \delta' \Rightarrow \phi$ or $\gamma' = \delta \Rightarrow \phi'$. Suppose that $\gamma' = \delta' \Rightarrow \phi$. Since $\delta \in \mathbf{P}$, $\delta \dashv\vDash_{PL} \delta'$. So, $P(\delta) = P(\delta')$, and $P(\phi \wedge \delta) = P(\phi \wedge \delta')$. It follows that $P(\phi|\delta) = P(\phi|\delta')$ whenever $P(\delta) > 0$. By clause 3 of Definition 2 this entails that $V_P(\gamma) = V_P(\gamma')$, hence that $U_P(\gamma) = U_P(\gamma')$. The reasoning is similar if $\gamma' = \delta \Rightarrow \phi'$. Therefore $\gamma \dashv\vDash \gamma'$.
Case 3: γ has the form $\delta \rhd \phi$. This case is like case 2 but relies on clause 4 of Definition 2.
Case 4: γ has the form $\delta \dashv\!\!\!\exists\, \phi$. This case is like 2 but relies on clause 5 of Definition 2.
Case 5: γ has the form $\sim\delta$. In this case $\gamma' = \sim\delta'$. Since δ has complexity n, by the inductive hypothesis $\delta \dashv\vDash \delta'$, so $V_P(\delta) = V_P(\delta')$. By clause 6,

$V_P(\sim\delta) = 1 - V_P(\delta)$ and $V_P(\sim\delta') = 1 - V_P(\delta')$, so $V_P(\sim\delta) = V_P(\sim\delta')$. It follows that $U_P(\sim\delta) = U_P(\sim\delta')$, hence that $\gamma \mathrel{=}\!\mathrel{|}\!\mathrel{\models} \gamma'$. ∎

FACT 2: $\alpha \triangleright \beta \vDash \alpha \Rightarrow \beta$ but $\alpha \Rightarrow \beta \nvDash \alpha \triangleright \beta$

PROOF. In order to prove that $\alpha \triangleright \beta \vDash \alpha \Rightarrow \beta$, three cases must be considered.
Case 1: $P(\alpha) = 0$ or $P(\beta) = 1$. In this case $V_P(\alpha \Rightarrow \beta) = 1$, so $U_P(\alpha \Rightarrow \beta) = 0$. Therefore, $U_P(\alpha \triangleright \beta) \geq U_P(\alpha \Rightarrow \beta)$.
Case 2: $P(\alpha) > 1$, $P(\beta) < 1$, and $P(\beta|\alpha) < P(\beta)$. In this case $V_P(\alpha \triangleright \beta) = 0$, hence $U_P(\alpha \triangleright \beta) = 1$. Therefore, $U_P(\alpha \triangleright \beta) \geq U_P(\alpha \Rightarrow \beta)$.
Case 3: $P(\alpha) > 1$, $P(\beta) < 1$, and $P(\beta|\alpha) \geq P(\beta)$. In this case we have that

$$
\begin{aligned}
&0 \leq P(\beta)(P(\alpha) - P(\alpha \wedge \beta)) \\
&P(\alpha \wedge \beta) - P(\alpha \wedge \beta) \leq P(\alpha)P(\beta) - P(\beta)P(\alpha \wedge \beta) \\
&P(\alpha \wedge \beta) - P(\alpha)P(\beta) \leq P(\alpha \wedge \beta) - P(\beta)P(\alpha \wedge \beta) \\
&\frac{P(\alpha \wedge \beta)}{P(\alpha)} - P(\beta) \leq \frac{P(\alpha \wedge \beta) - P(\beta)P(\alpha \wedge \beta)}{P(\alpha)} \\
&P(\beta|\alpha) - P(\beta) \leq \frac{P(\alpha \wedge \beta)(1 - P(\beta))}{P(\alpha)} \\
&\frac{P(\beta|\alpha) - P(\beta)}{1 - P(\beta)} \leq \frac{P(\alpha \wedge \beta)}{P(\alpha)}
\end{aligned}
$$

This means that $V_P(\alpha \triangleright \beta) \leq V_P(\alpha \Rightarrow \beta)$, so that $U_P(\alpha \triangleright \beta) \geq U_P(\alpha \Rightarrow \beta)$.
To prove that $\alpha \Rightarrow \beta \nvDash \alpha \triangleright \beta$ it suffices to note that it can happen that $0 < P(\beta|\alpha) \leq P(\beta)$. In this case $V_P(\alpha \Rightarrow \beta) > V_P(\alpha \triangleright \beta)$, so $U_P(\alpha \Rightarrow \beta) < U_P(\alpha \triangleright \beta)$. ∎

FACT 3: $\alpha \dashv\!\!\!\exists\, \beta \vDash \alpha \triangleright \beta$ but $\alpha \triangleright \beta \nvDash \alpha \dashv\!\!\!\exists\, \beta$

PROOF. In order to prove that $\alpha \dashv\!\!\!\exists\, \beta \vDash \alpha \triangleright \beta$, three cases must be considered.
Case 1: $P(\alpha) = 0$ or $P(\beta) = 1$. In this case $V_P(\alpha \triangleright \beta) = 1$, so $U_P(\alpha \triangleright \beta) = 0$. Therefore, $U_P(\alpha \dashv\!\!\!\exists\, \beta) \geq U_P(\alpha \triangleright \beta)$.
Case 2: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) < 1$. In this case $V_P(\alpha \dashv\!\!\!\exists\, \beta) = 0$, so $U_P(\alpha \dashv\!\!\!\exists\, \beta) = 1$. Therefore, $U_P(\alpha \dashv\!\!\!\exists\, \beta) \geq U_P(\alpha \triangleright \beta)$.
Case 3: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) = 1$. In this case $P(\beta|\alpha) - P(\beta) = 1 - P(\beta)$, so $V_P(\alpha \triangleright \beta) = 1$ and $U_P(\alpha \triangleright \beta) = 0$. Therefore, $U_P(\alpha \dashv\!\!\!\exists\, \beta) \geq U_P(\alpha \triangleright \beta)$.
To prove that $\alpha \triangleright \beta \nvDash \alpha \dashv\!\!\!\exists\, \beta$ it suffices to note that it can happen that $P(\beta) < P(\beta|\alpha) < 1$. In this case $V_P(\alpha \triangleright \beta) > V_P(\alpha \dashv\!\!\!\exists\, \beta)$, so $U_P(\alpha \triangleright \beta) < U_P(\alpha \dashv\!\!\!\exists\, \beta)$. ∎

FACT 4: If MI holds, then DET holds as well, given PL.

Proof.

1	$\top > \alpha$	A
2	$\top \supset \alpha$	1 MI
3	$\top$	PL
4	α	2, 3 PL

■

Fact 5: If MI holds, then MP holds as well, given PL.

Proof.

1	$\alpha > \beta$	A
2	α	A
3	$\alpha \supset \beta$	1 MI
4	β	2, 3 PL

■

Fact 6: If LT and S hold, then RW holds as well.

Proof.

1	$\alpha > \beta$	A
2	$(\alpha \wedge \beta) > \gamma$	S [assuming that $\beta \vDash_{PL} \gamma$]
3	$\alpha > \gamma$	1, 2 LT

■

Fact 7: If M and CON hold, then TC holds as well.

Proof.

1	β	A
2	$\top > \beta$	1 CON
3	$(\top \wedge \alpha) > \beta$	2 M
4	$\alpha > \beta$	3 SLE

■

Fact 8: If CM and CEM hold, then RM holds as well.

Proof.

1	$\alpha > \gamma$	A
2	$\sim(\alpha > \sim\beta)$	A
3	$\alpha > \beta$	2 CEM
4	$(\alpha \wedge \beta) > \gamma$	1, 3 CM

■

Fact 9: If T and S hold, then M holds as well.

Proof.

1	$\alpha > \beta$	A
2	$(\alpha \wedge \gamma) > \alpha$	S
3	$(\alpha \wedge \gamma) > \beta$	1, 2 T

■

Fact 10: If C and RW hold, then M holds as well.

Proof.

1 $\alpha > \gamma$	A
2 $\sim\gamma > \sim\alpha$	1 C
3 $\sim\gamma > (\sim\alpha \vee \sim\beta)$	2 RW
4 $\sim(\sim\alpha \vee \sim\beta) > \sim\sim\gamma$	3 C
5 $(\alpha \wedge \beta) > \gamma$	4 SLE

■

Fact 11: If C and CC hold, then DA holds as well.

Proof.

1 $\alpha > \gamma$	A
2 $\beta > \gamma$	A
3 $\sim\gamma > \sim\alpha$	1 C
4 $\sim\gamma > \sim\beta$	2 C
5 $\sim\gamma > (\sim\alpha \wedge \sim\beta)$	$3, 4$ CC
6 $\sim(\sim\alpha \wedge \sim\beta) > \sim\sim\gamma$	5 C
7 $(\alpha \vee \beta) > \gamma$	6 SLE

■

Fact 12: If S, CC, CE hold, then LT holds as well.

Proof.

1 $\alpha > \beta$	A
2 $(\alpha \wedge \beta) > \gamma$	A
3 $\alpha > \alpha$	S
4 $(\alpha \wedge \beta) > \alpha$	S
5 $\alpha > (\alpha \wedge \beta)$	$1, 3$ CC
6 $\alpha > \gamma$	$2, 4, 5$ CE

■

Fact 13: If CP holds, then FA and TC hold as well, given PL.

Proof. Assume that CP holds. Since $\beta, \alpha \vDash_{PL} \beta$, we get that $\beta \Longrightarrow \alpha > \beta$. So TC holds. Similarly, since $\alpha, \sim\alpha \vDash_{PL} \beta$, we get that $\sim\alpha \Longrightarrow \alpha > \beta$. So FA holds. ■

Fact 14: If CON and EAS hold, then TC holds as well.

Proof.

1	β	A
2	$\top > \beta$	1 CON
3	$\alpha > \beta$	2 EAS

■

FACT 15: If NC holds, and either C holds or RW, S, CC hold, then IA holds as well.

PROOF.

1	$\Box{\sim}\alpha$	A
2	${\sim}\beta > {\sim}\alpha$	1 NC
3	${\sim}\,{\sim}\alpha > {\sim}\,{\sim}\,\beta$	2 C
4	$\alpha > \beta$	3 SLE

1	$\Box{\sim}\alpha$	A
2	$\alpha > {\sim}\alpha$	1 NC
3	$\alpha > ({\sim}\alpha \vee \beta)$	2 RW
4	$\alpha > \alpha$	S
5	$\alpha > (\alpha \vee \beta)$	4 RW
6	$\alpha > ({\sim}\alpha \vee \beta) \wedge (\alpha \vee \beta)$	$3, 5$ CC
7	$\alpha > \beta$	7 SLE

■

FACT 16: If RS holds, then RCN holds as well.

PROOF.

1	$\Diamond\alpha$	A
2	$\alpha > \beta$	A
3	${\sim}(\alpha > {\sim}\beta)$	$1, 2$ RS [because $\beta \vDash_{PL} {\sim}\,{\sim}\,\beta$]

■

FACT 17: If RCN and S hold, then RAT holds as well.

PROOF.

1	$\Diamond\alpha$	A
2	$\alpha > \alpha$	S
3	${\sim}(\alpha > {\sim}\alpha)$	$1, 2$ RCN

■

FACT 18: if C and RS hold, then RAST holds as well.

PROOF. First note that, if C and RS hold, then from $\Diamond{\sim}\beta$ and $\alpha > \beta$ we obtain ${\sim}({\sim}\beta > \alpha)$:

1	$\Diamond{\sim}\beta$	A
2	$\alpha > \beta$	A
3	${\sim}\beta > {\sim}\alpha$	2 C
4	${\sim}({\sim}\beta > \alpha)$	$1, 3$ RS [because ${\sim}\alpha \vDash_{PL} {\sim}\alpha$]

Second, note that if $\Gamma, \alpha \vDash \beta$, then $\Gamma, {\sim}\beta \vDash {\sim}\alpha$. This can be seen as follows. Either ${\sim}\beta \in \Gamma$ or ${\sim}\beta \notin \Gamma$. If ${\sim}\beta \in \Gamma$, that is, if $\Gamma = \{{\sim}\beta, \gamma_1, ...\gamma_n\}$, then $\Gamma, \alpha \vDash \beta$ and $\Gamma, \alpha \vDash {\sim}\beta$, which entails that $U_P({\sim}\beta)+U_P(\gamma_1)+...+U_P(\gamma_n)+U_P(\alpha) \geq 1$. It follows that $U_P({\sim}\beta)+U_P(\gamma_1)+...+U_P(\gamma_n)+U_P(\alpha)-U_P(\alpha) \geq 1-U_P(\alpha)$, hence that $U_P({\sim}\beta)+U_P(\gamma_1)+...+U_P(\gamma_n) \geq U_P({\sim}\alpha)$. If ${\sim}\beta \notin \Gamma$, and $\Gamma = \{\gamma_1, ...\gamma_n\}$, it suffices to think that $\gamma_1, ...\gamma_n, \alpha \vDash \beta$ if and only if $U_P(\gamma_1)+...+U_P(\gamma_n)+U_P(\alpha) \geq U_P(\beta)$, if and only if $U_P(\gamma_1)+...+U_P(\gamma_n)+(1-U_P(\beta)) \geq 1-U_P(\alpha)$, if and only if $U_P(\gamma_1)+...+U_P(\gamma_n)+U_P({\sim}\beta) \geq U_P({\sim}\alpha)$.

Given that if $\Gamma, \alpha \vDash \beta$, then $\Gamma, {\sim}\beta \vDash {\sim}\alpha$, from 4 above we obtain ${\sim}({\sim}\alpha > \beta)$, because ${\sim}\alpha > \beta \vDash {\sim}\beta > \alpha$ by C and SLE. ■

Fact 19: $\Box\alpha \vDash \beta \Rightarrow \alpha$

Proof. Let $\alpha, \beta \in \mathbf{P}$ and let P be any probability function. Two cases must be considered.
Case 1: $P(\alpha) = 1$. In this case $V_P(\beta \Rightarrow \alpha) = 1$ no matter whether $P(\beta) = 0$ or $P(\beta) > 0$, so $U_P(\beta \Rightarrow \alpha) = 0$. Therefore, $U_P(\Box\alpha) \geq U_P(\beta \Rightarrow \alpha)$.
Case 2: $P(\alpha) < 1$. In this case $V_P(\Box\alpha) = 0$, so $U_P(\Box\alpha) = 1$. Therefore, $U_P(\Box\alpha) \geq U_P(\beta \Rightarrow \alpha)$. ■

Fact 20: $\Box{\sim}\alpha \vDash \alpha \Rightarrow \beta$

Proof. Since RW, S, and CC hold for $\Rightarrow$, from facts 15 and 19 we get that $\Box{\sim}\alpha \vDash \alpha \Rightarrow \beta$. ■

Fact 21: ${\sim}\alpha \nvDash \alpha \Rightarrow \beta$

Proof. Suppose that $0 < P(\beta) < 1$, and posit $\alpha = {\sim}\beta$. Then, $U_P({\sim}\alpha) = 1 - V_P({\sim}\alpha) = 1 - P({\sim}\alpha) = 1 - P(\beta) < 1$, but $U_P(\alpha \Rightarrow \beta) = 1 - V_P(\alpha \Rightarrow \beta) = 1 - P(\beta|\alpha) = 1 - P(\beta|{\sim}\beta) = 1$. Therefore, $U_P({\sim}\alpha) < U_P(\alpha \Rightarrow \beta)$. ■

Fact 22: $\beta \nvDash \alpha \Rightarrow \beta$

Proof. Suppose that $0 < P(\alpha) < 1$, and posit $\beta = {\sim}\alpha$. Then, $U_P(\beta) = 1 - V_P(\beta) = 1 - P(\beta) = 1 - P({\sim}\alpha) < 1$, but $U_P(\alpha \Rightarrow \beta) = 1 - V_P(\alpha \Rightarrow \beta) = 1 - P(\beta|\alpha) = 1 - P({\sim}\alpha|\alpha) = 1$. Therefore, $U_P(\beta) < U_P(\alpha \Rightarrow \beta)$. ■

Fact 23: $\alpha \Rightarrow \gamma \nvDash (\alpha \wedge \beta) \Rightarrow \gamma$

Proof. Suppose that $\alpha \Rightarrow \gamma \vDash (\alpha \wedge \beta) \Rightarrow \gamma$. Since CON holds for $\Rightarrow$, by Fact 7 we get that $\beta \vDash \alpha \Rightarrow \beta$, contrary to Fact 22. ■

Fact 24: $\alpha \Rightarrow \beta, \beta \Rightarrow \gamma \nvDash \alpha \Rightarrow \gamma$

Proof. Suppose that $\alpha \Rightarrow \beta, \beta \Rightarrow \gamma \vDash \alpha \Rightarrow \gamma$. Since S holds for $\Rightarrow$, by Fact 9 we get that $\alpha \Rightarrow \gamma \vDash (\alpha \wedge \beta) \Rightarrow \gamma$, contrary to Fact 23. ■

FACT 25: $\alpha \Rightarrow \beta \not\models \sim\beta \Rightarrow \sim\alpha$

PROOF. Suppose that $\alpha \Rightarrow \beta \models \sim\beta \Rightarrow \sim\alpha$. Since RW holds for $\Rightarrow$, by Fact 10 we get that $\alpha \Rightarrow \gamma \models (\alpha \wedge \beta) \Rightarrow \gamma$, contrary to Fact 23. ■

FACT 26: Not: if $\Gamma, \alpha \models_{PL} \beta$, then $\Gamma \models \alpha \Rightarrow \beta$.

PROOF. Suppose that CP holds for $\Rightarrow$. Then by Fact 13 we get that FA and TC also hold for $\Rightarrow$, contrary to Facts 21 and 22. ■

FACT 27: $\top \Rightarrow \alpha \not\models \beta \Rightarrow \alpha$

PROOF. Suppose that $\top \Rightarrow \alpha \models \beta \Rightarrow \alpha$. Since CON holds for $\Rightarrow$, by Fact 14 we get that $\beta \models \alpha \Rightarrow \beta$, contrary to Fact 22. ■

FACT 28: $\sim(\alpha \Rightarrow \beta) \not\models \beta \Rightarrow \alpha$

PROOF. Suppose that $0 < P(\alpha) < 1$, and posit $\beta = \sim\alpha$. Then $U_P(\sim(\alpha \Rightarrow \beta)) = 1 - V_P(\sim(\alpha \Rightarrow \beta)) = 1-(1-V_P(\alpha \Rightarrow \beta)) = V_P(\alpha \Rightarrow \beta) = P(\sim\alpha|\alpha) = 0$, whereas $U_P(\beta \Rightarrow \alpha) = 1 - V_P(\beta \Rightarrow \alpha) = 1 - P(\alpha|\beta) = 1 - P(\alpha|\sim\alpha) = 1$. Therefore, $U_P(\sim(\alpha \Rightarrow \beta)) < U_P(\beta \Rightarrow \alpha)$. ■

FACT 29: $\sim(\alpha \Rightarrow \beta) \not\models \sim\alpha \Rightarrow \beta$

PROOF. Suppose that $P(\alpha) > 0$ and $P(\sim\alpha \wedge \sim\gamma) > 0$, and posit $\beta = \sim\alpha \wedge \gamma$. Then $U_P(\sim(\alpha \Rightarrow \beta)) = 1 - V_P(\sim(\alpha \Rightarrow \beta)) = 1 - (1 - V_P(\alpha \Rightarrow \beta)) = V_P(\alpha \Rightarrow \beta) = P(\beta|\alpha) = P(\sim\alpha \wedge \gamma|\alpha) = 0$, but $U_P(\sim\alpha \Rightarrow \beta) = 1 - V_P(\sim\alpha \Rightarrow \beta) = 1 - P(\sim\alpha \wedge \gamma|\sim\alpha) = 1 - P(\gamma|\sim\alpha) > 0$. Therefore, $U_P(\sim(\alpha \Rightarrow \beta)) < U_P(\sim\alpha \Rightarrow \beta)$. ■

FACT 30: If $\beta \models_{PL} \sim\gamma$, then $\Diamond\alpha, \alpha \Rightarrow \beta \models \sim(\alpha \Rightarrow \gamma)$

PROOF. Assume that $\beta \models_{PL} \sim\gamma$. Two cases must be considered.
Case 1: $P(\alpha) = 0$. In this case $U_P(\Diamond\alpha) = 1 - V_P(\Diamond\alpha) = 1 - V_P(\sim\Box\sim\alpha) = 1 - (1 - V_P(\Box\sim\alpha) = V_P(\Box\sim\alpha) = 1$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \Rightarrow \beta) \geq U_P(\sim(\alpha \Rightarrow \gamma))$.
Case 2: $P(\alpha) > 0$. In this case, since $\beta \models_{PL} \sim\gamma$, we have that $P(\beta|\alpha) + P(\gamma|\alpha) \leq 1$, so that $1 - P(\beta|\alpha) \geq P(\gamma|\alpha)$. Given that $U_P(\alpha \Rightarrow \beta) = 1 - V_P(\alpha \Rightarrow \beta) = 1 - P(\beta|\alpha)$, and that $U_P(\sim(\alpha \Rightarrow \gamma)) = 1 - V_P(\sim(\alpha \Rightarrow \gamma)) = 1-(1-V_p(\alpha \Rightarrow \gamma)) = 1-(1-P(\gamma|\alpha)) = P(\gamma|\alpha)$, we get that $U_P(\alpha \Rightarrow \beta) \geq U_P(\sim(\alpha \Rightarrow \gamma))$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \Rightarrow \beta) \geq U_P(\sim(\alpha \Rightarrow \gamma))$.
■

FACT 31: $\Diamond\alpha, \alpha \Rightarrow \beta \models \sim(\alpha \Rightarrow \sim\beta)$

PROOF. From Facts 16 and 30. ■

FACT 32: $\Diamond\alpha \vDash \sim(\alpha \Rightarrow \sim\alpha)$

PROOF. From Facts 17 and 31. ■

FACT 33: $\Diamond\sim\beta, \alpha \Rightarrow \beta \nvDash \sim(\sim\alpha \Rightarrow \beta)$

PROOF. Suppose that $P(\beta) > 0$ and $P(\sim\beta) > 0$, and posit $\alpha = \beta \wedge \sim\beta$. Then $U_P(\Diamond\sim\beta) + U_P(\alpha \Rightarrow \beta) = U_P(\Diamond\sim\beta) + U_P((\beta \wedge \sim\beta) \Rightarrow \beta) = 1 - V_P(\Diamond\sim\beta) + 1 - V_P((\beta \wedge \sim\beta) \Rightarrow \beta)) = 1 - V_P(\sim\Box\beta) + 1 - V_P(\bot \Rightarrow \beta) = 1 - (1 - V_P(\Box\beta)) + 1 - V_P(\bot \Rightarrow \beta)) = 1 - (1 - 0) + (1 - 1) = 0$, and $U_P(\sim(\sim\alpha \Rightarrow \beta)) = 1 - V_P(\sim(\sim\alpha \Rightarrow \beta)) = 1 - (1 - V_P(\sim\alpha \Rightarrow \beta)) = 1 - (1 - V_P(\sim(\beta \wedge \sim\beta) \Rightarrow \beta)) = 1 - (1 - V_P(\sim\bot \Rightarrow \beta)) = 1 - (1 - V_P(\top \Rightarrow \beta)) = 1 - (1 - P(\beta|\top)) = P(\beta|\top) = P(\beta) > 0$. Therefore, $U_P(\Diamond\sim\beta) + U_P(\alpha \Rightarrow \beta) < U_P(\sim(\sim\alpha \Rightarrow \beta))$. ■

FACT 34: If $\alpha \vDash_{PL} \beta$, then $\vDash \alpha \triangleright \beta$

PROOF. Assume that $\alpha \vDash_{PL} \beta$. Two cases must be considered.
Case 1: $P(\beta) = 1$. In this case, $V_P(\alpha \triangleright \beta) = 1$, so $U_P(\alpha \triangleright \beta) = 0$.
Case 2: $P(\beta) < 1$. In this case, if $P(\alpha) = 0$, then again $V_P(\alpha \triangleright \beta) = 1$, so $U_P(\alpha \triangleright \beta) = 0$. If $P(\alpha) > 0$, we have that $P(\beta|\alpha) = 1$ because $\alpha \vDash_{PL} \beta$. It follows that $P(\beta|\alpha) - P(\beta) = 1 - P(\beta)$, so that $V_P(\alpha \triangleright \beta) = 1$ and $U_P(\alpha \triangleright \beta) = 0$. ■

FACT 35: $\alpha \triangleright \beta \vDash \alpha \supset \beta$

PROOF. Three cases must be considered.
Case 1: $P(\alpha) = 0$ or $P(\beta) = 1$. In this case $V_P(\alpha \supset \beta) = 1$, so $U_P(\alpha \supset \beta) = 0$. Therefore, $U_P(\alpha \triangleright \beta) \geq U_P(\alpha \supset \beta)$.
Case 2: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) > P(\beta)$. In this case we have that

$$
\begin{aligned}
&P(\sim\beta|\alpha)P(\alpha)P(\sim\beta) \leq P(\sim\beta|\alpha)\\
&P(\beta|\alpha) + P(\sim\beta|\alpha)P(\alpha)P(\sim\beta) \leq P(\beta|\alpha) + P(\sim\beta|\alpha)\\
&P(\beta|\alpha) + P(\sim\beta|\alpha)P(\alpha)P(\sim\beta) \leq P(\beta) + P(\sim\beta)\\
&P(\beta|\alpha) - P(\beta) \leq P(\sim\beta) - P(\sim\beta|\alpha)P(\alpha)P(\sim\beta)\\
&\frac{P(\beta|\alpha) - P(\beta)}{P(\sim\beta)} \leq 1 - P(\sim\beta|\alpha)P(\alpha)\\
&\frac{P(\beta|\alpha) - P(\beta)}{1 - P(\beta)} \leq 1 - P(\alpha \wedge \sim\beta)\\
&\frac{P(\beta|\alpha) - P(\beta)}{1 - P(\beta)} \leq P(\sim(\alpha \wedge \sim\beta))
\end{aligned}
$$

This means that $V_P(\alpha \triangleright \beta) \leq V_P(\alpha \supset \beta)$. Therefore, $U_P(\alpha \triangleright \beta) \geq U_P(\alpha \supset \beta)$.

Case 3: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) \leq P(\beta)$. In this case $V_P(\alpha \triangleright \beta) = 0$, so $U_P(\alpha \triangleright \beta) = 1$. Therefore, $U_P(\alpha \triangleright \beta) \geq U_P(\alpha \supset \beta)$. ∎

FACT 36: $\top \triangleright \alpha \vDash \alpha$

PROOF. From Facts 4 and 35. ∎

FACT 37: $\alpha \triangleright \beta, \alpha \vDash \beta$

PROOF. From Facts 5 and 35. ∎

FACT 38: $\alpha \triangleright \beta, \alpha \triangleright \gamma \vDash \alpha \triangleright (\beta \wedge \gamma)$

PROOF. First, note that if $P(\alpha) = 0$, then $U_P(\alpha \triangleright (\beta \wedge \gamma)) = 1 - V_P(\alpha \triangleright (\beta \wedge \gamma)) = 1 - 1 = 0$, so $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P(\alpha \triangleright (\beta \wedge \gamma))$. Second, note that if $P(\beta) = 1$, then $V_P(\alpha \triangleright \gamma) = V_P(\alpha \triangleright (\beta \wedge \gamma))$, so $U_P(\alpha \triangleright \gamma) = U_P(\alpha \triangleright (\beta \wedge \gamma))$. Therefore, $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P(\alpha \triangleright (\beta \wedge \gamma))$. The same conclusion follows if $P(\gamma) = 1$. Third, note that if $P(\beta \wedge \gamma) = 1$, then $P(\beta) = 1$ and $P(\gamma) = 1$, so $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P(\alpha \triangleright (\beta \wedge \gamma))$ for the reasons just explained. Now let us reason under the assumption that $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\gamma) < 1$. Three cases must be considered.

Case 1: $P(\beta|\alpha) \leq P(\beta)$ or $P(\gamma|\alpha) \leq P(\gamma)$. In this case $V_P(\alpha \triangleright \beta) = 0$ or $V_P(\alpha \triangleright \gamma) = 0$, and consequently $U_P(\alpha \triangleright \beta) = 1$ or $U_P(\alpha \triangleright \gamma) = 1$. Therefore, $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P(\alpha \triangleright (\beta \wedge \gamma))$.

Case 2: $P(\beta|\alpha) > P(\beta)$, $P(\gamma|\alpha) > P(\gamma)$, and $P(\beta \wedge \gamma|\alpha) > P(\beta \wedge \gamma)$. We know that $P(\beta \wedge \sim\gamma)P(\sim\gamma)P(\sim\beta|\alpha) + P(\sim\beta \wedge \gamma)P(\sim\beta)P(\sim\gamma|\alpha) + P(\sim\gamma)P(\sim\beta)P(\sim\beta \wedge \sim\gamma|\alpha) \geq 0$. Therefore, $P(\beta \wedge \sim\gamma)(P(\beta \wedge \sim\gamma) + P(\sim\beta \wedge \sim\gamma))(P(\sim\beta \wedge \gamma|\alpha) + P(\sim\beta \wedge \sim\gamma|\alpha)) + P(\sim\beta \wedge \gamma)(P(\sim\beta \wedge \gamma) + P(\sim\beta \wedge \sim\gamma)(P(\beta \wedge \sim\gamma|\alpha) + P(\sim\beta \wedge \sim\gamma|\alpha) + (P(\beta \wedge \sim\gamma) + P(\sim\beta \wedge \sim\gamma))(P(\sim\beta \wedge \gamma) + P(\sim\beta \wedge \sim\gamma))P(\sim\beta \wedge \sim\gamma|\alpha) \geq 0$. From this, by means of purely algebraic steps, we get what follows.[22]

[22]The steps needed to get from the last formula to the next are conveniently stated if we let $a = P(\beta \wedge \sim\gamma)$, $b = P(\beta \wedge \sim\gamma|\alpha)$, $c = P(\sim\beta \wedge \gamma)$, $d = P(\sim\beta \wedge \gamma|\alpha)$, $e = P(\sim\beta \wedge \sim\gamma)$,

$$\frac{P(\sim\beta \wedge \gamma|\alpha) + P(\sim\beta \wedge \sim\gamma|\alpha)}{P(\sim\beta \wedge \gamma) + P(\sim\beta \wedge \sim\gamma)} + \frac{P(\beta \wedge \sim\gamma|\alpha) + P(\sim\beta \wedge \sim\gamma|\alpha)}{P(\beta \wedge \sim\gamma) + P(\sim\beta \wedge \sim\gamma)}$$
$$\geq \frac{P(\beta \wedge \sim\gamma|\alpha) + P(\sim\beta \wedge \gamma|\alpha) + P(\sim\beta \wedge \sim\gamma|\alpha)}{P(\beta \wedge \sim\gamma) + P(\sim\beta \wedge \gamma) + P(\sim\beta \wedge \sim\gamma)}$$
$$\frac{P(\sim\beta|\alpha)}{P(\sim\beta)} + \frac{P(\sim\gamma|\alpha)}{P(\sim\gamma)} \geq \frac{P(\sim\beta \vee \sim\gamma|\alpha)}{P(\sim\beta \vee \sim\gamma)}$$
$$\frac{P(\sim\beta|\alpha)}{P(\sim\beta)} + \frac{P(\sim\gamma|\alpha)}{P(\sim\gamma)} \geq \frac{P(\sim(\beta \wedge \gamma)|\alpha)}{P(\sim(\beta \wedge \gamma))}$$
$$\frac{1 - P(\beta|\alpha)}{P(\sim\beta)} + \frac{1 - P(\gamma|\alpha)}{P(\sim\gamma)} \geq \frac{1 - P(\beta \wedge \gamma|\alpha)}{P(\sim(\beta \wedge \gamma))}$$
$$\frac{P(\sim\beta) - P(\beta|\alpha) + 1 - P(\sim\beta)}{P(\sim\beta)} + \frac{P(\sim\gamma) - P(\gamma|\alpha) + 1 - P(\sim\gamma)}{P(\sim\gamma)}$$
$$\geq \frac{P(\sim(\beta \wedge \gamma)) - P(\beta \wedge \gamma|\alpha) + 1 - P(\sim(\beta \wedge \gamma))}{P(\sim(\beta \wedge \gamma))}$$
$$\frac{P(\sim\beta) - P(\beta|\alpha) + P(\beta)}{P(\sim\beta)} + \frac{P(\sim\gamma) - P(\gamma|\alpha) + P(\gamma)}{P(\sim\gamma)}$$
$$\geq \frac{P(\sim(\beta \wedge \gamma)) - P(\beta \wedge \gamma|\alpha) + P(\beta \wedge \gamma)}{P(\sim(\beta \wedge \gamma))}$$
$$1 - \frac{P(\beta|\alpha) - P(\beta)}{P(\sim\beta)} + 1 - \frac{P(\gamma|\alpha) - P(\gamma)}{P(\sim\gamma)} \geq 1 - \frac{P(\beta \wedge \gamma|\alpha) - P(\beta \wedge \gamma)}{P(\sim(\beta \wedge \gamma))}$$

This means that $1 - V_P(\alpha \triangleright \beta) + 1 - V_P(\alpha \triangleright \gamma) \geq 1 - V_P(\alpha \triangleright (\beta \wedge \gamma))$, so that $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P(\alpha \triangleright (\beta \wedge \gamma))$.

$f = P(\sim\beta \wedge \sim\gamma|\alpha)$. That is,

$$a(a+e)(d+f) + c(c+e)(b+f) + (a+e)(c+e)f \geq 0$$
$$a(a+e)d + a(a+e)f + c(c+e)b + c(c+e)f + a(c+e)f + e(c+e)f \geq 0$$
$$a^2d + aed + a^2f + acf + aef + aef + cef + e^2f + c^2b + ceb + c^2f + cef \geq 0$$
$$a^2d + acd + aed + aed + ced + e^2d + a^2f + acf + aef + aef + cef + e^2f + acb + c^2b$$
$$+ceb + aeb + ceb + e^2b$$
$$+acf + c^2f + cef + aef + cef + e^2f \geq acb + ceb + aeb + e^2b + acd + ced + aed + e^2d$$
$$+acf + cef + aef + e^2f$$
$$(a+e)(a+c+e)(d+f) + (c+e)(a+c+e)(b+f) \geq (c+e)(a+e)(b+d+f)$$
$$\frac{d+f}{c+e} + \frac{b+f}{a+e} \geq \frac{b+d+f}{a+c+e}$$

Case 3: $P(\beta|\alpha) > P(\beta)$, $P(\gamma|\alpha) > P(\gamma)$, and $P(\beta \wedge \gamma|\alpha) \leq P(\beta \wedge \gamma)$. In this case we have that

$$\frac{P(\beta \wedge \gamma|\alpha) - P(\beta \wedge \gamma)}{P(\sim(\beta \wedge \gamma))} \leq 0$$

From this and the last line of the reasoning set out in case 2 we obtain that

$$1 - \frac{P(\beta|\alpha) - P(\beta)}{P(\sim\beta)} + 1 - \frac{P(\gamma|\alpha) - P(\gamma)}{P(\sim\gamma)} \geq 1$$

This is to say that $1 - V_P(\alpha \triangleright \beta) + 1 - V_P(\alpha \triangleright \gamma) \geq 1$, hence that $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq 1$. Therefore, $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P(\alpha \triangleright (\beta \wedge \gamma))$. ■

Fact 39: $\alpha \triangleright \beta \vDash \sim\beta \triangleright \sim\alpha$

Proof. Three cases must be considered.
Case 1: $P(\alpha) = 0$ or $P(\beta) = 1$. In this case $P(\sim\alpha) = 1$ or $P(\sim\beta) = 0$, so $V_P(\sim\beta \triangleright \sim\alpha) = 1$. It follows that $U_P(\sim\beta \triangleright \sim\alpha) = 0$, hence that $U_P(\alpha \triangleright \beta) \geq U_P(\sim\beta \triangleright \sim\alpha)$.
Case 2: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) \leq P(\beta)$. In this case $V_P(\alpha \triangleright \beta) = 0$, so $U_P(\alpha \triangleright \beta) = 1$. Therefore, $U_P(\alpha \triangleright \beta) \geq U_P(\sim\beta \triangleright \sim\alpha)$.
Case 3: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) > P(\beta)$. In this case, by the probability calculus we have that $P(\sim\alpha|\sim\beta) > P(\sim\alpha)$, so that

$$V_P(\alpha \triangleright \beta) = \frac{P(\beta|\alpha) - P(\beta)}{1 - P(\beta)} = \frac{P(\sim\beta) - P(\sim\beta|\alpha)}{P(\sim\beta)} = 1 - \frac{P(\sim\beta|\alpha)}{P(\sim\beta)}$$
$$V_P(\sim\beta \triangleright \sim\alpha) = \frac{P(\sim\alpha|\sim\beta) - P(\sim\alpha)}{1 - P(\sim\alpha)} = \frac{P(\alpha) - P(\alpha|\sim\beta)}{P(\alpha)} = 1 - \frac{P(\alpha|\sim\beta)}{P(\alpha)}$$

But

$$1 - \frac{P(\sim\beta|\alpha)}{P(\sim\beta)} = 1 - \frac{P(\alpha|\sim\beta)}{P(\alpha)}$$

Therefore, $V_P(\alpha \triangleright \beta) = V_P(\sim\beta \triangleright \sim\alpha)$, and consequently $U_P(\alpha \triangleright \beta) \geq U_P(\sim\beta \triangleright \sim\alpha)$. ■

Fact 40 $\alpha \triangleright \gamma, \beta \triangleright \gamma \vDash (\alpha \vee \beta) \triangleright \gamma$

Proof. From Facts 11 and 39. ■

Fact 41: $\Box\alpha \vDash \beta \triangleright \alpha$

Proof. Two cases must be considered.
Case 1: $P(\alpha) < 1$. In this case $V_P(\Box\alpha) = 0$, so $U_P(\Box\alpha) = 1$. Therefore, $U_P(\Box\alpha) \geq U_P(\beta \triangleright \alpha)$.
Case 2: $P(\alpha) = 1$. In this case $V_P(\beta \triangleright \alpha) = 1$, so $U_P(\beta \triangleright \alpha) = 0$. Therefore, $U_P(\Box\alpha) \geq U_P(\beta \triangleright \alpha)$. ■

Fact 42: $\Box{\sim}\alpha \vDash \alpha \triangleright \beta$

Proof. From Facts 15 and 41. ■

Fact 43: $\alpha \triangleright \beta, \alpha \triangleright \gamma \vDash (\alpha \wedge \beta) \triangleright \gamma$

Proof. First, note that if $P(\alpha \wedge \beta) = 0$ or $P(\gamma) = 1$, then $V_P((\alpha \wedge \beta) \triangleright \gamma) = 1$, so $U_P((\alpha \wedge \beta) \triangleright \gamma) = 0$. Therefore, $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P((\alpha \wedge \beta) \triangleright \gamma)$. Second, note that if $P(\beta) = 1$ or $P(\alpha \wedge {\sim}\beta) = 0$, then $V_P(\alpha \triangleright \gamma) = V_P((\alpha \wedge \beta) \triangleright \gamma)$, so $U_P(\alpha \triangleright \gamma) = U_P((\alpha \wedge \beta) \triangleright \gamma)$. Therefore, $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P((\alpha \wedge \beta) \triangleright \gamma)$. Now let us reason under the assumption that $P(\beta) < 1$, $P(\gamma) < 1$, $P(\alpha \wedge \beta) > 0$, $P(\alpha \wedge {\sim}\beta) > 0$. Three cases are possible.

Case 1: $P(\beta|\alpha) \leq P(\beta)$ or $P(\gamma|\alpha) \leq P(\gamma)$. In this case $V_P(\alpha \triangleright \beta) = 0$ or $V_P(\alpha \triangleright \gamma) = 0$, which means that $U_P(\alpha \triangleright \beta) = 1$ or $U_P(\alpha \triangleright \gamma) = 1$. It follows that $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P((\alpha \wedge \beta) \triangleright \gamma)$.

Case 2: $P(\beta|\alpha) > P(\beta)$ and $P(\gamma|\alpha) > P(\gamma)$, but $P(\gamma|\alpha \wedge \beta) \leq P(\gamma)$. In this case $P(\alpha \wedge \beta|\gamma) \leq P(\alpha \wedge \beta) \leq P(\alpha \wedge \beta|{\sim}\gamma)$, and we have that

$$
\begin{aligned}
&P(\alpha \wedge {\sim}\beta|{\sim}\gamma) + P(\alpha|{\sim}\beta) \geq P(\alpha|{\sim}\beta)P({\sim}\beta)\\
&P(\alpha \wedge {\sim}\beta|{\sim}\gamma) + P(\alpha|{\sim}\beta) \geq P(\alpha \wedge {\sim}\beta)\\
&P(\alpha \wedge \beta|{\sim}\gamma) + P(\alpha \wedge {\sim}\beta|{\sim}\gamma) + P(\alpha|{\sim}\beta) \geq P(\alpha \wedge \beta) + P(\alpha \wedge {\sim}\beta)\\
&\frac{P(\alpha|{\sim}\beta)}{P(\alpha \wedge \beta) + P(\alpha \wedge {\sim}\beta)} + \frac{P(\alpha \wedge \beta|{\sim}\gamma) + P(\alpha \wedge {\sim}\beta|{\sim}\gamma)}{P(\alpha \wedge \beta) + P(\alpha \wedge {\sim}\beta)} \geq 1\\
&\frac{P(\alpha|{\sim}\beta)}{P(\alpha)} + \frac{P(\alpha|{\sim}\gamma)}{P(\alpha)} \geq 1\\
&\frac{P({\sim}\beta|\alpha)}{P({\sim}\beta)} + \frac{P({\sim}\gamma|\alpha)}{P({\sim}\gamma)} \geq 1\\
&\frac{1 - P(\beta|\alpha)}{P({\sim}\beta)} + \frac{1 - P(\gamma|\alpha)}{P({\sim}\gamma)} \geq 1\\
&\frac{P({\sim}\beta) + P(\beta) - P(\beta|\alpha)}{P({\sim}\beta)} + \frac{P({\sim}\gamma) + P(\gamma) - P(\gamma|\alpha)}{P({\sim}\gamma)} \geq 1\\
&1 + \frac{P(\beta) - P(\beta|\alpha)}{P({\sim}\beta)} + 1 + \frac{P(\gamma) - P(\gamma|\alpha)}{P({\sim}\gamma)} \geq 1\\
&1 - \frac{P(\beta|\alpha) - P(\beta)}{1 - P(\beta)} + 1 - \frac{P(\gamma|\alpha) - P(\gamma)}{1 - P(\gamma)} \geq 1
\end{aligned}
$$

This means that $1 - V_P(\alpha \triangleright \beta) + 1 - V_P(\alpha \triangleright \gamma) \geq 1$, hence that $U_P(\alpha \triangleright \beta) + U_P(\alpha \triangleright \gamma) \geq U_P((\alpha \wedge \beta) \triangleright \gamma)$.

Case 3: $P(\beta|\alpha) > P(\beta)$, $P(\gamma|\alpha) > P(\gamma)$, and $P(\gamma|\alpha\wedge\beta) > P(\gamma)$. In this case $P(\alpha\wedge\beta|\gamma) > P(\alpha\wedge\beta) > P(\alpha\wedge\beta|{\sim}\gamma)$, and we have

$$P(\alpha\wedge\beta) > P(\alpha\wedge\beta|{\sim}\gamma)$$

$$1 > \frac{P(\alpha\wedge\beta|{\sim}\gamma)}{P(\alpha\wedge\beta)}$$

$$\frac{P(\alpha\wedge{\sim}\beta|{\sim}\gamma)}{P(\alpha\wedge{\sim}\beta)} + 1 > \frac{P(\alpha\wedge\beta|{\sim}\gamma)}{P(\alpha\wedge\beta)}$$

$$\frac{P(\alpha\wedge{\sim}\beta)}{P({\sim}\beta)P(\alpha\wedge{\sim}\beta)} + \frac{P(\alpha\wedge{\sim}\beta|{\sim}\gamma)}{P(\alpha\wedge{\sim}\beta)} > \frac{P(\alpha\wedge\beta|{\sim}\gamma)}{P(\alpha\wedge\beta)}$$

$$\frac{P(\alpha|{\sim}\beta)}{P(\alpha\wedge{\sim}\beta)} + \frac{P(\alpha\wedge{\sim}\beta|{\sim}\gamma)}{P(\alpha\wedge{\sim}\beta)} > \frac{P(\alpha\wedge\beta|{\sim}\gamma)}{P(\alpha\wedge\beta)}$$

$$P(\alpha|{\sim}\beta)P(\alpha\wedge\beta) + P(\alpha\wedge{\sim}\beta|{\sim}\gamma)P(\alpha\wedge\beta) > P(\alpha\wedge\beta|{\sim}\gamma)P(\alpha\wedge{\sim}\beta)$$

$$P(\alpha|{\sim}\beta)P(\alpha\wedge\beta) + P(\alpha\wedge\beta|{\sim}\gamma)P(\alpha\wedge\beta) + P(\alpha\wedge{\sim}\beta|{\sim}\gamma)P(\alpha\wedge\beta)$$
$$> P(\alpha\wedge\beta|{\sim}\gamma)P(\alpha\wedge\beta) + P(\alpha\wedge\beta|{\sim}\gamma)P(\alpha\wedge{\sim}\beta)$$

$$P(\alpha|{\sim}\beta)P(\alpha\wedge\beta) + P(\alpha\wedge\beta)(P(\alpha\wedge\beta|{\sim}\gamma) + P(\alpha\wedge{\sim}\beta|{\sim}\gamma))$$
$$> P(\alpha\wedge\beta|{\sim}\gamma)(P(\alpha\wedge\beta) + P(\alpha\wedge{\sim}\beta))$$

$$\frac{P(\alpha|{\sim}\beta)}{P(\alpha\wedge\beta)+P(\alpha\wedge{\sim}\beta)} + \frac{P(\alpha\wedge\beta|{\sim}\gamma)+P(\alpha\wedge{\sim}\beta|{\sim}\gamma)}{P(\alpha\wedge\beta)+P(\alpha\wedge{\sim}\beta)} > \frac{P(\alpha\wedge\beta|{\sim}\gamma)}{P(\alpha\wedge\beta)}$$

$$\frac{P(\alpha|{\sim}\beta)}{P(\alpha)} + \frac{P(\alpha|{\sim}\gamma)}{P(\alpha)} > \frac{P(\alpha\wedge\beta|{\sim}\gamma)}{P(\alpha\wedge\beta)}$$

$$\frac{P({\sim}\beta|\alpha)}{P({\sim}\beta)} + \frac{P({\sim}\gamma|\alpha)}{P({\sim}\gamma)} > \frac{P({\sim}\gamma|\alpha\wedge\beta)}{P({\sim}\gamma)}$$

$$\frac{1-P(\beta|\alpha)}{P({\sim}\beta)} + \frac{1-P(\gamma|\alpha)}{P({\sim}\gamma)} > \frac{1-P(\gamma|\alpha\wedge\beta)}{P({\sim}\gamma)}$$

$$\frac{P({\sim}\beta)+P(\beta)-P(\beta|\alpha)}{P({\sim}\beta)} + \frac{P({\sim}\gamma)+P(\gamma)-P(\gamma|\alpha)}{P({\sim}\gamma)}$$
$$> \frac{P({\sim}\gamma)+P(\gamma)-P(\gamma|\alpha\wedge\beta)}{P({\sim}\gamma)}$$

$$1 + \frac{P(\beta)-P(\beta|\alpha)}{P({\sim}\beta)} + 1 + \frac{P(\gamma)-P(\gamma|\alpha)}{P({\sim}\gamma)} > 1 + \frac{P(\gamma)-P(\gamma|\alpha\wedge\beta)}{P({\sim}\gamma)}$$

$$1 - \frac{P(\beta|\alpha)-P(\beta)}{1-P(\beta)} + 1 - \frac{P(\gamma|\alpha)-P(\gamma)}{1-P(\gamma)} > 1 - \frac{P(\gamma|\alpha\wedge\beta)-P(\gamma)}{1-P(\gamma)}$$

This means that $1-V_P(\alpha\triangleright\beta)+1-V_P(\alpha\triangleright\gamma) > 1-V_P((\alpha\wedge\beta)\triangleright\gamma)$. Therefore, $U_P(\alpha\triangleright\beta) + U_P(\alpha\triangleright\gamma) \geq U_P((\alpha\wedge\beta)\triangleright\gamma$. ∎

FACT 44: $\alpha \rhd \gamma, \sim((\alpha \wedge \sim\beta) \rhd \gamma) \vDash (\alpha \wedge \beta) \rhd \gamma$

PROOF. First, note that if $P(\alpha \wedge \beta) = 0$, then $V_P((\alpha \wedge \beta) \rhd \gamma) = 1$, hence $U_P((\alpha \wedge \beta) \rhd \gamma) = 0$. Therefore, $U_P(\alpha \rhd \gamma) + U_P(\sim((\alpha \wedge \sim\beta) \rhd \gamma)) \geq U_P((\alpha \wedge \beta) \rhd \gamma)$. Second, note that if $P(\alpha \wedge \sim\beta) = 0$, then $U_P(\sim((\alpha \wedge \sim\beta) \rhd \gamma)) = 1 - V_P(\sim((\alpha \wedge \sim\beta) \rhd \gamma)) = 1 - (1 - (V_P((\alpha \wedge \sim\beta) \rhd \gamma))) = 1 - (1 - 1) = 1$. Therefore, $U_P(\alpha \rhd \gamma) + U_P(\sim((\alpha \wedge \sim\beta) \rhd \gamma)) \geq U_P((\alpha \wedge \beta) \rhd \gamma)$. Now let us reason under the assumption that $P(\alpha \wedge \beta) > 0$ and $P(\alpha \wedge \sim\beta) > 0$. Three cases are possible.
Case 1: $P(\gamma) = 1$. In this case $V_P((\alpha \wedge \beta) \rhd \gamma) = 1$, so $U_P((\alpha \wedge \beta) \rhd \gamma) = 0$. Therefore, $U_P(\alpha \rhd \gamma) + U_P(\sim((\alpha \wedge \sim\beta) \rhd \gamma)) \geq U_P((\alpha \wedge \beta) \rhd \gamma)$.
Case 2: $P(\gamma) < 1$ and $P(\gamma|\alpha) \leq P(\gamma)$. In this case $V_P(\alpha \rhd \gamma) = 0$, so $U_P(\alpha \rhd \gamma) = 1$. Therefore, $U_P(\alpha \rhd \gamma) + U_P(\sim((\alpha \wedge \sim\beta) \rhd \gamma)) \geq U_P((\alpha \wedge \beta) \rhd \gamma)$.
Case 3: $P(\gamma) < 1$ and $P(\gamma|\alpha) > P(\gamma)$. In this case we can assume that $P(\gamma|\alpha \wedge \beta) \geq P(\gamma|\alpha) \geq P(\gamma|\alpha \wedge \sim\beta)$ with no loss of generality, and we have:

$$
\begin{aligned}
&P(\gamma|\alpha \wedge \beta) \geq P(\gamma|\alpha) \\
&P(\gamma|\alpha \wedge \beta) - P(\gamma) \geq P(\gamma|\alpha) - P(\gamma) \\
&\frac{P(\gamma|\alpha \wedge \beta) - P(\gamma)}{P(\sim\gamma)} \geq \frac{P(\gamma|\alpha) - P(\gamma)}{P(\sim\gamma)} \\
&1 - \frac{P(\gamma|\alpha \wedge \beta) - P(\gamma)}{P(\sim\gamma)} \leq 1 - \frac{P(\gamma|\alpha) - P(\gamma)}{P(\sim\gamma)} \\
&1 - \frac{P(\gamma|\alpha \wedge \beta) - P(\gamma)}{1 - P(\gamma)} \leq 1 - \frac{P(\gamma|\alpha) - P(\gamma)}{1 - P(\gamma)} \\
&1 - V_P((\alpha \wedge \beta) \rhd \gamma) \leq 1 - V_P(\alpha \rhd \gamma) \\
&V_P(\sim((\alpha \wedge \beta) \rhd \gamma)) \leq V_P(\sim(\alpha \rhd \gamma)) \\
&1 - V_P(\sim((\alpha \wedge \beta) \rhd \gamma)) \geq 1 - V_P(\sim(\alpha \rhd \gamma)) \\
&U_P(\sim((\alpha \wedge \beta) \rhd \gamma)) \geq U_P(\sim(\alpha \rhd \gamma)) \\
&U_P(\sim((\alpha \wedge \beta) \rhd \gamma)) + U_P(\sim((\alpha \wedge \sim\beta) \rhd \gamma)) \geq U_P(\sim(\alpha \rhd \gamma)) \\
&1 + U_P(\sim((\alpha \wedge \beta) \rhd \gamma)) + U_P(\sim((\alpha \wedge \sim\beta) \rhd \gamma)) \geq 1 + U_P(\sim(\alpha \rhd \gamma)) \\
&1 - U_P(\sim(\alpha \rhd \gamma)) + U_P(\sim((\alpha \wedge \sim\beta) \rhd \gamma)) \geq 1 - U_P(\sim((\alpha \wedge \beta) \rhd \gamma)) \\
&1 - (1 - V_P(\sim(\alpha \rhd \gamma))) + U_P(\sim((\alpha \wedge \sim\beta) \rhd \gamma)) \geq 1 - (1 - V_P(\sim((\alpha \wedge \beta) \rhd \gamma))) \\
&1 - V_P(\sim\sim(\alpha \rhd \gamma)) + U_P(\sim((\alpha \wedge \sim\beta) \rhd \gamma)) \geq 1 - V_P(\sim\sim((\alpha \wedge \beta) \rhd \gamma)) \\
&1 - V_P(\alpha \rhd \gamma) + U_P(\sim((\alpha \wedge \sim\beta) \rhd \gamma)) \geq 1 - V_P((\alpha \wedge \beta) \rhd \gamma)
\end{aligned}
$$

This means that $U_P(\alpha \rhd \gamma) + U_P(\sim((\alpha \wedge \sim\beta) \rhd \gamma)) \geq U_P((\alpha \wedge \beta) \rhd \gamma)$. ■

FACT 45: $\alpha \rhd \gamma, \sim(\alpha \rhd \sim\beta) \nvDash (\alpha \wedge \beta) \rhd \gamma$

PROOF. Let us assume the distribution of probability listed in section 6. Then $P(\alpha) = P(\gamma) = 0,30$, and $P(\beta) = 0,23$. Moreover, $P(\gamma|\alpha) = 0,56$,

$P(\gamma|\alpha \wedge \beta) = 0,25$, and $P(\beta|\alpha) = P(\beta) = 0,23$. In this case, we have

$$\begin{aligned} U_P(\alpha \triangleright \gamma) &= 1 - V_P(\alpha \triangleright \gamma) = 1 - \frac{P(\gamma|\alpha) - P(\gamma)}{1 - P(\gamma)} \\ &= 1 - \frac{0,56 - 0,30}{1 - 0,30} = 1 - 0,38 = 0,62 \\ U_P(\sim(\alpha \triangleright \sim\beta)) &= 1 - V_P(\sim(\alpha \triangleright \sim\beta)) \\ &= 1 - (1 - V_P(\alpha \triangleright \sim\beta)) = 1 - (1 - 0) = 0 \\ U_P((\alpha \wedge \beta) \triangleright \gamma) &= 1 - V_P((\alpha \wedge \beta) \triangleright \gamma) = 1 - 0 = 1 \end{aligned}$$

Therefore, $U_P(\alpha \triangleright \gamma) + U_P(\sim(\alpha \triangleright \sim\beta) < U_P((\alpha \wedge \beta) \triangleright \gamma)$. ■

FACT 46: $\alpha \triangleright \gamma \nvDash (\alpha \wedge \beta) \triangleright \gamma$

PROOF. Suppose that $\alpha \triangleright \gamma \vDash (\alpha \wedge \beta) \triangleright \gamma$. Then, $\alpha \triangleright \gamma, \sim(\alpha \triangleright \sim\beta) \vDash (\alpha \wedge \beta) \triangleright \gamma$, contrary to Fact 45. ■

FACT 47: Not: if $\beta \vDash_{PL} \gamma$, then $\alpha \triangleright \beta \vDash \alpha \triangleright \gamma$

PROOF. Suppose that, if $\beta \vDash_{PL} \gamma$, then $\alpha \triangleright \beta \vDash \alpha \triangleright \gamma$. Then, by Facts 10 and 39 we get that $\alpha \triangleright \gamma \vDash (\alpha \wedge \beta) \triangleright \gamma$, contrary to Fact 46. ■

FACT 48: $\alpha \triangleright \beta, \beta \triangleright \gamma \nvDash \alpha \triangleright \gamma$

PROOF. Suppose that $\alpha \triangleright \beta, \beta \triangleright \gamma \vDash \alpha \triangleright \gamma$. Then, by Facts 9 and 34 we get that $\alpha \triangleright \beta \vDash (\alpha \wedge \beta) \triangleright \gamma$, contrary to Fact 46. ■

FACT 49: $\alpha \triangleright \beta, (\alpha \wedge \beta) \triangleright \gamma \nvDash \alpha \triangleright \gamma$

PROOF. Suppose that $\alpha \triangleright \beta, (\alpha \wedge \beta) \triangleright \gamma \vDash \alpha \triangleright \gamma$. Then, by Facts 6 and 34 we get that, if $\beta \vDash_{PL} \gamma$, then $\alpha \triangleright \beta \vDash \alpha \triangleright \gamma$, contrary to Fact 47. ■

FACT 50: $\alpha \triangleright \beta, \beta \triangleright \alpha, \beta \triangleright \gamma \nvDash \alpha \triangleright \gamma$

PROOF. Suppose that $\alpha \triangleright \beta, \beta \triangleright \alpha, \beta \triangleright \gamma \vDash \alpha \triangleright \gamma$. Then, by Facts 12, 34, and 38 we get that $\alpha \triangleright \beta, (\alpha \wedge \beta) \triangleright \gamma \vDash \alpha \triangleright \gamma$, contrary to Fact 49. ■

FACT 51: $\alpha \wedge \beta \nvDash \alpha \triangleright \beta$

PROOF. Suppose that $P(\alpha \wedge \beta) > 0$, and that α and β are probabilistically independent, so that $P(\alpha \wedge \beta) = P(\alpha)P(\beta)$. Then $U_P(\alpha \wedge \beta) = 1 - V_P(\alpha \wedge \beta) = 1 - P(\alpha \wedge \beta) < 1$. However, $U_P(\alpha \triangleright \beta) = 1 - V_P(\alpha \triangleright \beta) = 1 - 0 = 1$, because $P(\beta|\alpha) = P(\beta)$. Therefore, $U_P(\alpha \wedge \beta) < U_P(\alpha \triangleright \beta)$. ■

FACT 52: $\sim(\alpha \triangleright \beta) \nvDash \alpha \triangleright \sim\beta$

PROOF. Suppose that $\sim(\alpha \triangleright \beta) \vDash \alpha \triangleright \sim\beta$. Then, by Facts 8 and 43 we get that $\alpha \triangleright \gamma, \sim(\alpha \triangleright \sim\beta) \vDash (\alpha \wedge \beta) \triangleright \gamma$, contrary to Fact 45. ■

FACT 53: $\sim\alpha \nvDash \alpha \rhd \beta$

PROOF. Suppose that $0 < P(\beta) < 1$, and posit $\alpha = \sim\beta$. Then $U_P(\sim\alpha) = 1 - V_P(\sim\alpha) = 1 - P(\sim\alpha) = 1 - P(\beta) < 1$. But $U_P(\alpha \rhd \beta) = 1 - V_P(\alpha \rhd \beta) = 1 - 0 = 1$, because $P(\beta|\alpha) = P(\beta|\sim\beta) = 0$. Therefore, $U_P(\sim\alpha) < U_P(\alpha \rhd \beta)$. ■

FACT 54: $\beta \nvDash \alpha \rhd \beta$

PROOF. Suppose that $0 < P(\alpha) < 1$, and posit $\beta = \sim\alpha$. Then $U_P(\beta) = 1 - V_P(\beta) = 1 - P(\sim\alpha) < 1$. But $U_P(\alpha \rhd \beta) = 1 - V_P(\alpha \rhd \beta) = 1 - 0 = 1$, because $P(\beta|\alpha) = P(\sim\alpha|\alpha) = 0$. Therefore, $U_P(\beta) < U_P(\alpha \rhd \beta)$. ■

FACT 55: Not: if $\Gamma, \alpha \vDash_{PL} \beta$, then $\Gamma \vDash \alpha \rhd \beta$

PROOF. Like that of Fact 26. ■

FACT 56: $\sim(\alpha \rhd \beta) \nvDash \beta \rhd \alpha$

PROOF. Suppose that $0 < P(\alpha) < 1$, and posit $\beta = \sim\alpha$. Then $U_P(\sim(\alpha \rhd \beta)) = 1 - V_P(\sim(\alpha \rhd \beta)) = 1 - (1 - V_P(\alpha \rhd \beta)) = 1 - (1 - 0) = 0$, because $P(\beta|\alpha) = P(\sim\alpha|\alpha) = 0$. But $U_P(\beta \rhd \alpha) = 1 - V_P(\beta \rhd \alpha) = 1 - 0 = 1$, because $P(\alpha|\beta) = P(\alpha|\sim\alpha) = 0$. Therefore, $U_P(\sim(\alpha \rhd \beta)) < U_P(\beta \rhd \alpha)$. ■

FACT 57: $\sim(\alpha \rhd \beta) \nvDash \sim\alpha \rhd \beta$

PROOF. Suppose that $P(\alpha) > 0$ and that $P(\sim\alpha \wedge \sim\gamma) > 0$, and posit $\beta = \sim\alpha \wedge \gamma$. Then $U_P(\sim(\alpha \rhd \beta)) = 1 - V_P(\sim(\alpha \rhd \beta)) = 1 - (1 - V_P(\alpha \rhd \beta)) = 1 - (1 - 0) = 0$, because $P(\beta|\alpha) = P(\sim\alpha \wedge \gamma|\alpha) = 0$. But $U_P(\sim\alpha \rhd \beta) = 1 - V_P(\sim\alpha \rhd \beta) > 0$, because we have

$$P(\beta|\sim\alpha) = P(\sim\alpha \wedge \gamma|\sim\alpha) = \frac{P(\sim\alpha \wedge \gamma)}{P(\sim\alpha)} = \frac{P(\sim\alpha \wedge \gamma)}{P(\sim\alpha \wedge \gamma) + P(\sim\alpha \wedge \sim\gamma)} < 1$$

Therefore, $U_P(\sim(\alpha \rhd \beta)) < U_P(\sim\alpha \rhd \beta)$. ■

FACT 58: $\top \rhd \alpha \vDash \beta \rhd \alpha$

PROOF. Let $\alpha, \beta \in \mathbf{P}$, and consider any probability function P. Two cases are possible.
Case 1: $P(\alpha) < 1$. In this case $P(\alpha|\top) = P(\alpha)$, and $U_P(\top \rhd \alpha) = 1 - V_P(\top \rhd \alpha) = 1 - 0 = 1$. Therefore, $U_P(\top \rhd \alpha) \geq U_P(\beta \rhd \alpha)$.
Case 2: $P(\alpha) = 1$. In this case, $U_P(\top \rhd \alpha) = 1 - 1 = 0$, and $U_P(\beta \rhd \alpha) = 1 - V_P(\beta \rhd \alpha) = 1 - 1 = 0$. Therefore, $U_P(\top \rhd \alpha) \geq U_P(\beta \rhd \alpha)$. ■

Fact 59: $\alpha \not\models \top \rhd \alpha$

Proof. Suppose that $\alpha \models \top \rhd \alpha$. Then, by Facts 14 and 58 we get that $\beta \models \alpha \rhd \beta$, contrary to Fact 54. ■

Fact 60: If $\beta \models_{PL} \sim\gamma$, then $\Diamond\alpha, \alpha \rhd \beta \models \sim(\alpha \rhd \gamma)$

Proof. Assume that $\beta \models_{PL} \sim\gamma$. First, note that if $P(\alpha) = 0$, then $V_P(\Diamond\alpha) = 0$, so $U_P(\Diamond\alpha) = 1$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \rhd \beta) \geq U_P(\sim(\alpha \rhd \gamma))$. Second, note that if $P(\beta) = 1$, then $P(\sim\gamma) = 1$, because $\beta \models_{PL} \sim\gamma$, so $P(\gamma) = 0$ and $V_P(\alpha \rhd \gamma) = 0$, which means that $V_P(\sim(\alpha \rhd \gamma)) = 1$, hence that $U_P(\sim(\alpha \rhd \gamma)) = 0$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \rhd \beta) \geq U_P(\sim(\alpha \rhd \gamma))$. Third, note that if $P(\gamma) = 1$, then $P(\sim\beta) = 1$, because $\beta \models_{PL} \sim\gamma$, so $P(\beta) = 0$. It follows that $V_P(\alpha \rhd \beta) = 0$, hence that $U_P(\alpha \rhd \beta) = 1$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \rhd \beta) \geq U_P(\sim(\alpha \rhd \gamma))$. Now let us reason under the assumption that $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\gamma) < 1$. Three cases are possible.

Case 1: $P(\beta|\alpha) \leq P(\beta)$. In this case $V_P(\alpha \rhd \beta) = 0$, so $U_P(\alpha \rhd \beta) = 1$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \rhd \beta) \geq U_P(\sim(\alpha \rhd \gamma))$.

Case 2: $P(\gamma|\alpha) \leq P(\gamma)$. In this case $V_P(\alpha \rhd \gamma) = 0$, so $V_P(\sim(\alpha \rhd \gamma)) = 1$ and $U_P(\sim(\alpha \rhd \gamma)) = 0$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \rhd \beta) \geq U_P(\sim(\alpha \rhd \gamma))$.

Case 3: $P(\beta|\alpha) > P(\beta)$ and $P(\gamma|\alpha) > P(\gamma)$. In this case, since $\beta \models_{PL} \sim\gamma$, we have that $1 \geq P(\beta \vee \gamma|\alpha) = P(\beta|\alpha) + P(\gamma|\alpha)$. Moreover, $P(\beta|\alpha) \geq V_P(\alpha \rhd \beta)$, because

$$\begin{aligned}
&P(\beta)(1 - P(\beta|\alpha)) \geq 0 \\
&P(\beta) - P(\beta)P(\beta|\alpha) \geq 0 \\
&P(\beta) + (-P(\beta))P(\beta|\alpha) \geq 0 \\
&P(\beta) + (-(1 - P(\sim\beta)))P(\beta|\alpha) \geq 0 \\
&P(\beta) + (P(\sim\beta) - 1)P(\beta|\alpha) \geq 0 \\
&P(\beta|\alpha)P(\sim\beta) - P(\beta|\alpha) + P(\beta) \geq 0 \\
&P(\beta|\alpha)P(\sim\beta) \geq P(\beta|\alpha) - P(\beta) \\
&P(\beta|\alpha) \geq \frac{P(\beta|\alpha) - P(\beta)}{P(\sim\beta)}
\end{aligned}$$

For the same reasons, $P(\gamma|\alpha) \geq V_P(\alpha \rhd \gamma)$. Thus, given that $1 \geq P(\beta \vee \gamma|\alpha) = P(\beta|\alpha) + P(\gamma|\alpha)$, we have that

$$
\begin{aligned}
&1 \geq V_P(\alpha \triangleright \beta) + V_P(\alpha \triangleright \gamma)\\
&1 - V_P(\alpha \triangleright \beta) \geq V_P(\alpha \triangleright \gamma)\\
&1 - V_P(\alpha \triangleright \beta) \geq 1 - (1 - V_P(\alpha \triangleright \gamma))\\
&1 - V_P(\alpha \triangleright \beta) \geq 1 - (V_P(\sim(\alpha \triangleright \gamma))\\
&U_P(\alpha \triangleright \beta) \geq U_P(\sim(\alpha \triangleright \gamma))
\end{aligned}
$$

This entails that $U_P(\Diamond\alpha) + U_P(\alpha \triangleright \beta) \geq U_P(\sim(\alpha \triangleright \gamma))$. ■

FACT 61: $\Diamond\alpha, \alpha \triangleright \beta \vDash \sim(\alpha \triangleright \sim\beta)$

PROOF. From Facts 16 and 60. ■

FACT 62: $\Diamond\alpha \vDash \sim(\alpha \triangleright \sim\alpha)$

PROOF. From Facts 17, 34, and 61. ■

FACT 63: $\Diamond\sim\beta, \alpha \triangleright \beta \vDash \sim(\sim\alpha \triangleright \beta)$

PROOF. From Facts 18, 39, and 60. ■

FACT 64: If $\alpha \vDash_{PL} \beta$, then $\vDash \alpha \dashv\!\!\!\! \ni \beta$

PROOF. Assume that $\alpha \vDash_{PL} \beta$. Then $V_P(\alpha \dashv\!\!\!\! \ni \beta) = 1$ no matter whether $P(\alpha) > 0$ or $P(\alpha) = 0$, so $U_P(\alpha \dashv\!\!\!\! \ni \beta) = 1 - V_P(\alpha \dashv\!\!\!\! \ni \beta) = 1 - 1 = 0$. ■

FACT 65: $\alpha \dashv\!\!\!\! \ni \beta \vDash \alpha \supset \beta$

PROOF. Two cases must be considered.
Case 1: $P(\beta|\alpha) = 1$. In this case $V_P(\alpha \supset \beta) = 1$, because we have that

$$
\begin{aligned}
&P(\sim\beta|\alpha)P(\alpha) \leq P(\sim\beta|\alpha)\\
&P(\beta|\alpha) + P(\sim\beta|\alpha)P(\alpha) \leq P(\beta|\alpha) + P(\sim\beta|\alpha)\\
&P(\beta|\alpha) + P(\sim\beta|\alpha)P(\alpha) \leq 1\\
&P(\beta|\alpha) \leq 1 - P(\sim\beta|\alpha)P(\alpha)\\
&P(\beta|\alpha) \leq 1 - P(\alpha \wedge \sim\beta)\\
&P(\beta|\alpha) \leq 1 - P(\sim(\alpha \supset \beta))\\
&P(\beta|\alpha) \leq P(\alpha \supset \beta)
\end{aligned}
$$

This entails that $U_P(\alpha \supset \beta) = 0$. Therefore, $U_P(\alpha \dashv\!\!\!\! \ni \beta) \geq U_P(\alpha \supset \beta)$.
Case 2: $P(\beta|\alpha) < 1$. In this case $V_P(\alpha \dashv\!\!\!\! \ni \beta) = 0$, so $U_P(\alpha \dashv\!\!\!\! \ni \beta) = 1$. Therefore, $U_P(\alpha \dashv\!\!\!\! \ni \beta) \geq U_P(\alpha \supset \beta)$. ■

FACT 66: $\top \triangleright \alpha \vDash \alpha$

PROOF. From Facts 4 and 65. ■

FACT 67: $\alpha \rhd \beta, \alpha \vDash \beta$

PROOF. From Facts 5 and 65. ■

FACT 68: $\alpha \multimap \beta, \alpha \multimap \gamma \vDash \alpha \multimap (\beta \wedge \gamma)$

PROOF. First, note that if $P(\alpha) = 0$, then $V_P(\alpha \multimap (\beta \wedge \gamma)) = 1$, so $U_P(\alpha \multimap (\beta \wedge \gamma)) = 0$. Therefore, $U_P(\alpha \multimap \beta) + U_P(\alpha \multimap \gamma) \geq U_P(\alpha \multimap (\beta \wedge \gamma))$. Now we will reason under the assumption that $P(\alpha) > 0$. Two cases must be considered.
Case 1: $P(\beta|\alpha) < 1$ or $P(\gamma|\alpha) < 1$. In this case $V_P(\alpha \multimap \beta) = 0$ or $V_P(\alpha \multimap \gamma) = 0$, which means that $U_P(\alpha \multimap \beta) = 1$ or $U_P(\alpha \multimap \gamma) = 1$. Therefore, $U_P(\alpha \multimap \beta) + U_P(\alpha \multimap \gamma) \geq U_P(\alpha \multimap (\beta \wedge \gamma))$.
Case 2: $P(\beta|\alpha) = 1$ and $P(\gamma|\alpha) = 1$. In this case $P(\alpha \wedge \beta) = P(\alpha)$. Since $P(\alpha) = P(\alpha \wedge \beta) + P(\alpha \wedge \sim\beta)$, then $P(\alpha \wedge \sim\beta) = 0$. Moreover, since $P(\alpha \wedge \sim\beta) = P(\gamma \wedge \sim\beta \wedge \alpha) + P(\sim\gamma \wedge \sim\beta \wedge \alpha)$, it follows that $P(\gamma \wedge \sim\beta \wedge \alpha) = P(\sim\gamma \wedge \sim\beta \wedge \alpha) = 0$. A similar reasoning leads from the premise that $P(\alpha \wedge \gamma) = P(\alpha)$ to the conclusion that $P(\beta \wedge \sim\gamma \wedge \alpha) = P(\sim\beta \wedge \sim\gamma \wedge \alpha) = 0$. Consequently, $P(\sim(\beta \wedge \gamma) \wedge \alpha) = P((\sim\beta \vee \sim\gamma) \wedge \alpha)) = P(\sim\beta \wedge \sim\gamma \wedge \alpha) + P(\beta \wedge \sim\gamma \wedge \alpha) + P(\sim\beta \wedge \gamma \wedge \alpha) = 0$, so that $P(\alpha) = P((\beta \wedge \gamma) \wedge \alpha)) + P(\sim(\beta \wedge \gamma) \wedge \alpha) = P((\beta \wedge \gamma) \wedge \alpha)$, which implies that $P(\beta \wedge \gamma|\alpha) = 1$. Thus, $V_P(\alpha \multimap (\beta \wedge \gamma)) = 1$, hence $U_P(\alpha \multimap (\beta \wedge \gamma)) = 0$. Therefore, $U_P(\alpha \multimap \beta) + U_P(\alpha \multimap \gamma) \geq U_P(\alpha \multimap (\beta \wedge \gamma))$. ■

FACT 69: $\alpha \multimap \beta \vDash \sim\beta \multimap \sim\alpha$

PROOF. Three cases must be considered.
Case 1: $P(\alpha) = 0$ or $P(\beta) = 1$. In this case $P(\sim\alpha) = 1$ or $P(\sim\beta) = 0$, so $V_P(\sim\beta \multimap \sim\alpha) = 1$ and $U_P(\sim\beta \multimap \sim\alpha) = 0$. Therefore, $U_P(\alpha \multimap \beta) \geq U_P(\sim\beta \multimap \sim\alpha)$.
Case 2: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) < 1$. In this case $V_P(\alpha \multimap \beta) = 0$, so $U_P(\alpha \multimap \beta) = 1$. Therefore, $U_P(\alpha \multimap \beta) \geq U_P(\sim\beta \multimap \sim\alpha)$.
Case 3: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) = 1$. In this case $P(\alpha \wedge \beta) = P(\alpha)$. Since $P(\alpha) = P(\alpha \wedge \beta) + P(\alpha \wedge \sim\beta)$, we get that $P(\alpha \wedge \sim\beta) = 0$, hence that $P(\alpha|\sim\beta) = 0$. It follows that $P(\sim\alpha|\sim\beta) = 1$, so that $V_P(\sim\beta \multimap \sim\alpha) = 1$, which means that $U_P(\sim\beta \multimap \sim\alpha) = 0$. Therefore, $U_P(\alpha \multimap \beta) \geq U_P(\sim\beta \multimap \sim\alpha)$. ■

FACT 70: $\alpha \multimap \gamma, \beta \multimap \gamma \vDash (\alpha \vee \beta) \multimap \gamma$

PROOF. From Facts 11 and 69. ■

FACT 71: $\Box\alpha \vDash \beta \multimap \alpha$

PROOF. Two cases must be considered.
Case 1: $P(\alpha) < 1$. In this case $V_P(\Box\alpha) = 0$, hence $U_P(\Box\alpha) = 1$. Therefore, $U_P(\Box\alpha) \geq U_P(\alpha \dashv\!\!\!\!\dashv \beta)$.
Case 2: $P(\alpha) = 1$. In this case, if $P(\beta) = 0$, then $V_P(\beta \dashv\!\!\!\!\dashv \alpha) = 1$, and if $P(\beta) > 0$, then $P(\alpha|\beta) = 1$, given that $P(\alpha \wedge \beta) = P(\beta)$, so again $V_P(\beta \dashv\!\!\!\!\dashv \alpha) = 1$. It follows that $U_P(\beta \dashv\!\!\!\!\dashv \alpha) = 0$, hence that $U_P(\Box\alpha) \geq U_P(\alpha \dashv\!\!\!\!\dashv \beta)$. ■

FACT 72: $\Box{\sim}\alpha \vDash \alpha \rhd \beta$

PROOF. From Facts 15 and 71. ■

FACT 73: $\alpha \dashv\!\!\!\!\dashv \beta, \beta \dashv\!\!\!\!\dashv \gamma \vDash \alpha \dashv\!\!\!\!\dashv \gamma$

PROOF. First, note that if $P(\alpha) = 0$, then $V_P(\alpha \dashv\!\!\!\!\dashv \gamma) = 1$, hence $U_P(\alpha \dashv\!\!\!\!\dashv \gamma) = 0$. Therefore, $U_P(\alpha \dashv\!\!\!\!\dashv \beta) + U_P(\beta \dashv\!\!\!\!\dashv \gamma) \geq U_P(\alpha \dashv\!\!\!\!\dashv \gamma)$. Second, note that the same holds if $P(\alpha) > 0$ but $P(\beta) = 0$, because $V_P(\alpha \dashv\!\!\!\!\dashv \beta) = 0$, hence $U_P(\alpha \dashv\!\!\!\!\dashv \beta) = 1$. Now let us reason under the assumption that $P(\alpha) > 0$ and $P(\beta) > 0$. Two cases are possible.
Case 1: $P(\beta|\alpha) < 1$ or $P(\gamma|\beta) < 1$. In this case $V_P(\alpha \dashv\!\!\!\!\dashv \beta) = 0$ or $V_P(\beta \dashv\!\!\!\!\dashv \gamma) = 0$, hence $U_P(\alpha \dashv\!\!\!\!\dashv \beta) = 1$ or $U_P(\beta \dashv\!\!\!\!\dashv \gamma) = 1$. Therefore, $U_P(\alpha \dashv\!\!\!\!\dashv \beta) + U_P(\beta \dashv\!\!\!\!\dashv \gamma) \geq U_P(\alpha \dashv\!\!\!\!\dashv \gamma)$.
Case 2: $P(\beta|\alpha) = 1$ and $P(\gamma|\beta) = 1$. In this case we have that $P(\alpha \wedge \beta) = P(\alpha)$. Since $P(\alpha) = P(\alpha \wedge \beta) + P(\alpha \wedge {\sim}\beta)$, it follows that $P(\alpha \wedge {\sim}\beta) = 0$. Moreover, since $P(\alpha \wedge {\sim}\beta) = P(\gamma \wedge {\sim}\beta \wedge \alpha) + P({\sim}\gamma \wedge {\sim}\beta \wedge \alpha)$, it follows that $P({\sim}\gamma \wedge {\sim}\beta \wedge \alpha) = 0$. A similar reasoning leads from the premise that $P(\beta \wedge \gamma) = P(\beta)$ to the conclusion that $P(\beta \wedge {\sim}\gamma \wedge \alpha) = 0$. Thus we have that $P({\sim}\gamma \wedge \alpha) = P({\sim}\gamma \wedge \beta \wedge \alpha) + P({\sim}\gamma \wedge {\sim}\beta \wedge \alpha) = 0$. So $P(\alpha) = P(\alpha \wedge \gamma) + P(\alpha \wedge {\sim}\gamma) = P(\alpha \wedge \gamma)$, which entails that $P(\gamma|\alpha) = 1$. It follows that $V_P(\alpha \dashv\!\!\!\!\dashv \gamma) = 1$, so that $U_P(\alpha \dashv\!\!\!\!\dashv \gamma) = 0$. Therefore, $U_P(\alpha \dashv\!\!\!\!\dashv \beta) + U_P(\beta \dashv\!\!\!\!\dashv \gamma) \geq U_P(\alpha \dashv\!\!\!\!\dashv \gamma)$. ■

FACT 74: $\alpha \dashv\!\!\!\!\dashv \gamma \vDash (\alpha \wedge \beta) \dashv\!\!\!\!\dashv \gamma$

PROOF. From Facts 9, 64, and 73. ■

FACT 75: $\alpha \dashv\!\!\!\!\dashv \beta, \beta \dashv\!\!\!\!\dashv \alpha, \beta \dashv\!\!\!\!\dashv \gamma \vDash \alpha \dashv\!\!\!\!\dashv \gamma$

PROOF. Directly from Fact 73. ■

FACT 76: $\alpha \dashv\!\!\!\!\dashv \beta, (\alpha \wedge \beta) \dashv\!\!\!\!\dashv \gamma \vDash \alpha \dashv\!\!\!\!\dashv \gamma$

PROOF. From Facts 12, 64, 68, and 75. ■

FACT 77: If $\beta \vDash_{PL} \gamma$, then $\alpha \dashv\!\!\!\!\dashv \beta \vDash \alpha \dashv\!\!\!\!\dashv \gamma$

PROOF. From Facts 6, 64, and 76. ■

FACT 78: $\alpha \mathrel{-\!\!\!\ni} \gamma, \sim(\alpha \mathrel{-\!\!\!\ni} \sim\beta) \vDash (\alpha \wedge \beta) \mathrel{-\!\!\!\ni} \gamma$

PROOF. Directly from Fact 74. ■

FACT 79: $\alpha \mathrel{-\!\!\!\ni} \beta, \alpha \mathrel{-\!\!\!\ni} \gamma \vDash (\alpha \wedge \beta) \mathrel{-\!\!\!\ni} \gamma$

PROOF. Directly from Fact 74. ■

FACT 80: $\alpha \mathrel{-\!\!\!\ni} \gamma, \sim((\alpha \wedge \sim\beta) \mathrel{-\!\!\!\ni} \gamma) \vDash (\alpha \wedge \beta) \mathrel{-\!\!\!\ni} \gamma$

PROOF. Directly from Fact 74. ■

FACT 81: $\top \mathrel{-\!\!\!\ni} \alpha \vDash \beta \mathrel{-\!\!\!\ni} \alpha$

PROOF. Directly from Fact 74, given that β is equivalent to $\top \wedge \beta$. ■

FACT 82: $\sim\alpha \nvDash \alpha \mathrel{-\!\!\!\ni} \beta$

PROOF. Suppose that $0 < P(\beta) < 1$, and posit $\alpha = \sim\beta$. Then $U_P(\sim\alpha) = 1 - V_P(\sim\alpha) = 1 - P(\sim\alpha) < 1$. But $U_P(\alpha \mathrel{-\!\!\!\ni} \beta) = 1 - V_P(\alpha \mathrel{-\!\!\!\ni} \beta) = 1$, because $P(\beta|\alpha) = P(\beta|\sim\beta) = 0$. Therefore, $U_P(\sim\alpha) < U_P(\alpha \mathrel{-\!\!\!\ni} \beta)$. ■

FACT 83: $\beta \nvDash \alpha \mathrel{-\!\!\!\ni} \beta$

PROOF. Suppose that $0 < P(\alpha) < 1$, and posit $\beta = \sim\alpha$. Then $U_P(\beta) = 1 - V_P(\beta) = 1 - P(\sim\alpha) < 1$. But $U_P(\alpha \mathrel{-\!\!\!\ni} \beta) = 1 - V_P(\alpha \mathrel{-\!\!\!\ni} \beta) = 1$, because $P(\beta|\alpha) = P(\sim\alpha|\alpha) = 0$. Therefore, $U_P(\beta) < U_P(\alpha \mathrel{-\!\!\!\ni} \beta)$. ■

FACT 84: $\alpha \nvDash \top \mathrel{-\!\!\!\ni} \alpha$

PROOF. From Facts 14, 81, and 83. ■

FACT 85: $\alpha \wedge \beta \nvDash \alpha \mathrel{-\!\!\!\ni} \beta$

PROOF. Suppose that $P(\alpha \wedge \beta) > 0$ and $P(\alpha \wedge \sim\beta) > 0$. Then $U_P(\alpha \wedge \beta) = 1 - V_P(\alpha \wedge \beta) = 1 - P(\alpha \wedge \beta) < 1$, because $P(\alpha \wedge \beta) > 0$. But $U_P(\alpha \mathrel{-\!\!\!\ni} \beta) = 1 - V_P(\alpha \mathrel{-\!\!\!\ni} \beta) = 1$, because $P(\alpha \wedge \sim\beta) > 0$, thus $P(\alpha \wedge \beta) < P(\alpha)$ and $P(\beta|\alpha) < 1$. Therefore, $U_P(\alpha \wedge \beta) < U_P(\alpha \mathrel{-\!\!\!\ni} \beta)$. ■

FACT 86: $\sim(\alpha \mathrel{-\!\!\!\ni} \beta) \nvDash \alpha \mathrel{-\!\!\!\ni} \sim\beta$

PROOF. Suppose that $P(\alpha \wedge \beta) > 0$ and $P(\alpha \wedge \sim\beta) > 0$. Then $U_P(\sim(\alpha \mathrel{-\!\!\!\ni} \beta)) = 1 - V_P(\sim(\alpha \mathrel{-\!\!\!\ni} \beta)) = 1 - (1 - V_P(\alpha \mathrel{-\!\!\!\ni} \beta)) = 1 - (1 - 0) = 0$, because $P(\alpha \wedge \sim\beta) > 0$, so $P(\alpha \wedge \beta) < P(\alpha)$ and consequently $P(\beta|\alpha) < 1$. But $U_P(\alpha \mathrel{-\!\!\!\ni} \sim\beta) = 1 - V_P(\alpha \mathrel{-\!\!\!\ni} \sim\beta) = 1 - 0 = 1$, because $P(\alpha \wedge \beta) > 0$, so $P(\alpha \wedge \sim\beta) < P(\alpha)$ and consequently $P(\sim\beta|\alpha) < 1$. Therefore, $U_P(\sim(\alpha \mathrel{-\!\!\!\ni} \beta)) < U_P(\alpha \mathrel{-\!\!\!\ni} \sim\beta)$. ■

FACT 87: Not: if $\Gamma, \alpha \vDash_{PL} \beta$, then $\Gamma \vDash \alpha \mathrel{-\!\!\!\ni} \beta$

Proof. Like that of Facts 26 and 55. ■

Fact 88: $\sim(\alpha \multimapinv \beta) \nvDash \beta \multimapinv \alpha$

Proof. Suppose that $0 < P(\alpha) < 1$, and posit $\beta = \sim\alpha$. Then $U_P(\sim(\alpha \multimapinv \beta)) = 1 - V_P(\sim(\alpha \multimapinv \beta)) = 1 - (1 - V_P(\alpha \multimapinv \beta)) = 1 - (1 - 0) = 0$, because $P(\beta|\alpha) = P(\sim\alpha|\alpha) = 0$. But $U_P(\beta \multimapinv \alpha) = 1 - V_P(\beta \multimapinv \alpha) = 1 - 0 = 1$, because $P(\alpha|\beta) = P(\alpha|\sim\alpha) = 0$. Therefore, $U_P(\sim(\alpha \multimapinv \beta)) < U_P(\beta \multimapinv \alpha)$. ■

Fact 89: $\sim(\alpha \multimapinv \beta) \nvDash \sim\alpha \multimapinv \beta$

Proof. Suppose that $P(\alpha) > 0$ and $P(\sim\alpha \wedge \sim\gamma) > 0$, and posit $\beta = \sim\alpha \wedge \gamma$. Then $U_P(\sim(\alpha \multimapinv \beta)) = 1 - V_P(\sim(\alpha \multimapinv \beta)) = 1 - (1 - V_P(\alpha \multimapinv \beta)) = 1 - (1 - 0) = 0$, because $P(\beta|\alpha) = P(\sim\alpha \wedge \gamma|\alpha) = 0$. But $U_P(\sim\alpha \multimapinv \beta) = 1 - V_P(\sim\alpha \multimapinv \beta) = 1 - 0 = 1$, because $P(\beta|\sim\alpha) = P(\sim\alpha \wedge \gamma|\sim\alpha) = P(\gamma|\sim\alpha)$, and

$$P(\gamma|\sim\alpha) = \frac{P(\sim\alpha \wedge \gamma)}{P(\sim\alpha)} = \frac{P(\sim\alpha \wedge \gamma)}{P(\sim\alpha \wedge \gamma) + P(\sim\alpha \wedge \sim\gamma)} < 1$$

Therefore, $U_P(\sim(\alpha \multimapinv \beta)) < U_P(\sim\alpha \multimapinv \beta)$. ■

Fact 90: If $\beta \vDash_{PL} \sim\gamma$, then $\Diamond\alpha, \alpha \multimapinv \beta \vDash \sim(\alpha \multimapinv \gamma)$

Proof. Two cases must be considered.
Case 1: $P(\alpha) = 0$. In this case $V_P(\Diamond\alpha) = 0$, which entails that $U_P(\Diamond\alpha) = 1$. Therefore, $U_P(\Diamond\alpha) + U_P(\alpha \multimapinv \beta) \geq U_P(\sim(\alpha \multimapinv \gamma))$.
Case 2: $P(\alpha) > 0$. In this case, since $\beta \vDash_{PL} \sim\gamma$, we have that $1 \geq P(\beta \vee \gamma|\alpha) = P(\beta|\alpha) + P(\gamma|\alpha)$. It follows that either $V_P(\alpha \multimapinv \beta) = 1$ and $V_P(\alpha \multimapinv \gamma) = 0$, with the consequence that $U_P(\alpha \multimapinv \beta) = 1 - V_P(\alpha \multimapinv \beta) = 0$ and $U_P(\sim(\alpha \multimapinv \gamma)) = 1 - V_P(\sim(\alpha \multimapinv \gamma)) = 1 - (1 - V_P(\alpha \multimapinv \gamma)) = 0$, or $V_P(\alpha \multimapinv \beta) = 0$ and $V_P(\alpha \multimapinv \gamma) = 1$, with the consequence that $U_P(\alpha \multimapinv \beta) = 1 - V_P(\alpha \multimapinv \beta) = 1$ and $U_P(\sim(\alpha \multimapinv \gamma)) = 1 - V_P(\sim(\alpha \multimapinv \gamma)) = 1 - (1 - V_P(\alpha \multimapinv \gamma)) = 1$. In both cases, $U_P(\Diamond\alpha) + U_P(\alpha \multimapinv \beta) \geq U_P(\sim(\alpha \multimapinv \gamma))$. ■

Fact 91: $\Diamond\alpha, \alpha \multimapinv \beta \vDash \sim(\alpha \multimapinv \sim\beta)$

Proof. From Facts 16 and 90. ■

Fact 92: $\Diamond\alpha \vDash \sim(\alpha \multimapinv \sim\alpha)$

Proof. From Facts 17, 64, and 91. ■

Fact 93: $\Diamond\sim\beta, \alpha \multimapinv \beta \vDash \sim(\sim\alpha \multimapinv \beta)$

Proof. From Facts 18, 69, and 90. ■

Fact 94: $\alpha \rhd \beta \dashv\vDash \Box(\alpha \supset \beta)$

Proof. In order to prove that $\alpha \rhd \beta \vDash \Box(\alpha \supset \beta)$, three cases must be considered.
Case 1: $P(\alpha) = 0$ or $P(\beta) = 1$. In this case $P(\alpha \supset \beta) = 1$, so $V_P(\Box(\alpha \supset \beta)) = 1$ and $U_P(\Box(\alpha \supset \beta)) = 0$. Therefore, $U_P(\alpha \rhd \beta) \geq U_P(\Box(\alpha \supset \beta))$.
Case 2: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) = 1$. Since $P(\beta|\alpha) \leq P(\alpha \supset \beta)$, as has been shown in the proof of Fact 65, in this case $P(\alpha \supset \beta) = 1$, hence $V_P(\Box(\alpha \supset \beta)) = 1$ and $U_P(\Box(\alpha \supset \beta)) = 0$. Therefore, $U_P(\alpha \rhd \beta) \geq U_P(\Box(\alpha \supset \beta)$.
Case 3: $P(\alpha) > 0$, $P(\beta) < 1$, and $P(\beta|\alpha) < 1$, In this case $V_P(\alpha \rhd \beta) = 0$ and $U_P(\alpha \rhd \beta) = 1$, so $U_P(\alpha \rhd \beta) \geq U_P(\Box(\alpha \supset \beta)$.
In order to prove that $\Box(\alpha \supset \beta) \vDash \alpha \rhd \beta$, it suffices to note what follows. In cases 1 and 2, $V_P(\alpha \rhd \beta) = 1$ and $U_P(\alpha \rhd \beta) = 0$. In case 3, $P(\alpha \supset \beta) < 1$, because $P(\beta|\alpha) < 1$ and we have that

$$
\begin{aligned}
&P(\sim\beta|\alpha)\frac{P(\alpha)}{P(\alpha)} = P(\sim\beta|\alpha)\\
&P(\beta|\alpha) + P(\sim\beta|\alpha)\frac{P(\alpha)}{P(\alpha)} = P(\beta|\alpha) + P(\sim\beta|\alpha)\\
&P(\beta|\alpha) + P(\sim\beta|\alpha)\frac{P(\alpha)}{P(\alpha)} = 1\\
&P(\beta|\alpha) = 1 - \frac{P(\sim\beta|\alpha)P(\alpha)}{P(\alpha)}\\
&P(\beta|\alpha) = 1 - \frac{P(\alpha \wedge \sim\beta)}{P(\alpha)}\\
&P(\beta|\alpha)P(\alpha) = P(\alpha) - P(\alpha \wedge \sim\beta)\\
&P(\beta|\alpha)P(\alpha) + P(\sim\alpha) = P(\alpha) + P(\sim\alpha) - P(\alpha \wedge \sim\beta)\\
&P(\beta|\alpha)P(\alpha) + P(\sim\alpha) = 1 - P(\alpha \wedge \sim\beta)\\
&P(\beta|\alpha)P(\alpha) + P(\sim\alpha) = P(\sim\alpha \vee \beta)\\
&P(\beta|\alpha)P(\alpha) + P(\sim\alpha) = P(\alpha \supset \beta)
\end{aligned}
$$

It follows that $V_P(\Box(\alpha \supset \beta)) = 0$, hence that $U_P(\Box(\alpha \supset \beta)) = 1$. Therefore, $U_P(\Box(\alpha \supset \beta)) \geq U_P(\alpha \rhd \beta)$. ■

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