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Generalized Quantifiers: Logic and Language

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1. From "natural" to "formal" quantification
2. Introducing generalized quantifiers
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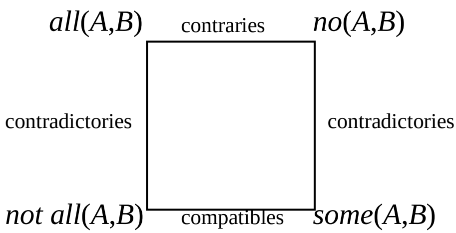
ABSTRACT. The Generalized Quantifiers Theory, I will argue, in the second half of last Century has led to an important rapprochement, relevant both in logic and in linguistics, between logical quantification theories and the semantic analysis of quantification in natural languages. In this paper I concisely illustrate the formal aspects and the theoretical implications of this rapprochement.

1. From "natural" to "formal" quantification

Aristotle, when invented the very idea of logic, was concerned with natural quantification. He focused on the meaning of some natural language determiners: *all, some, no, not all*, whose reciprocal logical relations are graphically expressed in the well-known square of opposition:



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The Aristotelian logic can be considered a segment of the logic of natural quantification, since it formally characterizes quantifiers expressible as Noun Phrases of a natural language.

When Frege invented the very idea of formal logic, he introduced two artificial quantifiers, as variable-binding operators, structurally very different from natural language quantifiers: $\forall$and $\exists$. It was the "divorce" between logic and natural language in the Twentieth Century. The following is the so-called square of duality relations between the two fregean quantifiers, showing their interdefinability by means of inner and outer negation:

$$\forall \equiv \neg\exists\neg \qquad\qquad \forall\neg \equiv \neg\exists$$

$$\neg\forall \equiv \exists\neg \qquad\qquad \exists \equiv \neg\forall\neg$$

The main reason of this divorce is that the quantifiers of first order logic are inadequate to deal with quantified sentences of natural languages in at least two respects: (1) The syntactic structure of quantified sentences in predicate calculus is totally different from the syntactic structure of natural language phrases expressing quantification (the "interface" problem); (2) There are quantifiers in natural language (expressed by some Noun Phrases) which simply cannot be represented in a logic which is restricted to the standard first-order quantifiers (the "expressiveness" problem).

Now, the Generalized Quantifier theory, based on the seminal works of Mostowski (1957) and Lindström (1966), has led to new insights into the nature of quantifiers, insights which permit logical syntax to correspond more closely to natural language syntax. Moreover, Generalized Quantifier Theory shows an increased expressive power than the standard first-order quantification.

2. Introducing Generalized Quantifiers

After having briefly introduced the Generalized Quantifiers (GQ, hereafter) theory, I'll proceed addressing the two mentioned problems.

Let E be a non-empty set, the universe, of a model M and Q_E any set of subsets of E ($Q_E \subseteq$ power(E)). Then Q_E is a type $\langle 1\rangle$ GQ whose meaning is given by:

$$Q_E(A) \Leftrightarrow A \in Q_E$$

The same quantifier can be written as variable-binding operator (where $M \models \phi$ is the usual satisfaction relation between a model and a formula, and $[[\phi(x)]]_M$ is the "extension" of $\phi(x)$ in M):

$$M \models Qx\phi(x) \text{ iff } [[\phi(x)]]_M \in Q_E$$

This notation suggests that standard quantifiers are conceivable as type $\langle 1\rangle$ quantifiers and, consequently, that you can provide a compositional account for first order formulas with quantifiers. It is well-known that standard universal and existential quantifiers do not have a denotation as such in the formulas in which they occur, and that, as a consequence, that it is not possible to compositionally interpret these formulas. But, as type $\langle 1\rangle$ quantifiers, standard quantifiers denote sets of sets:

$$\forall_E = \{\mathrm{E}\}$$
$$\exists_E = \{\mathrm{A} \subseteq \mathrm{E}: \mathrm{A} \neq \varnothing\}$$

Now, we can compositionally express the meaning of a formula such as $\forall x\phi(x)$ as follows:

$$M \models \forall x\phi(x) \Leftrightarrow [[\phi(x)]]_M \in [[\forall_E]] \Leftrightarrow [[\phi(x)]]_M = E$$

The denotation of $\forall x\phi(x)$ is constructed out by applying the denotation of the quantifier to the denotation of the open formula.

The definition of type $\langle 1,1\rangle$ quantifiers is a natural extension of that of type $\langle 1\rangle$ quantifiers. Let E be the universe of a model M and Q_E any (second order) binary relation on subsets of E ($Q_E \subseteq$ power(E) $\times$ power(E)). Then Q_E is a type $\langle 1,1\rangle$ GQ whose meaning is given by:

$$Q_E(A, B) \Leftrightarrow \langle A, B\rangle \in Q_E$$

Now, we can address the two problems: For the interface problem, I shall try to show the advantages of the approach based on GQ theory in designing a syntactic structure of quantification that is very similar to the syntax of quantification in natural languages. Then, I'll show that there are some generalized quantifiers that can capture the meaning of some natural language determiners, meaning that seems otherwise intractable (if we limit ourselves to the standard quantification).

2. The “interface” problem

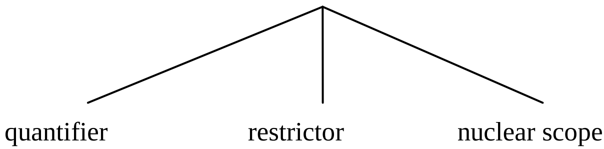
We presuppose a type-logical framework for the semantic component of the linguistic theory. The role of this component is to derive a semantic representation for a well-formed sentence of a natural language, starting from the meaning of the lexical units occurring in the sentence, according to its syntactic structure. Thus, the problem of the interface between logical syntax and natural language syntax turns into the problem of syntax/semantics interface, a mere linguistic “affair”, according to the following table:

syntax	linguistics
semantics	
logical relations	logic

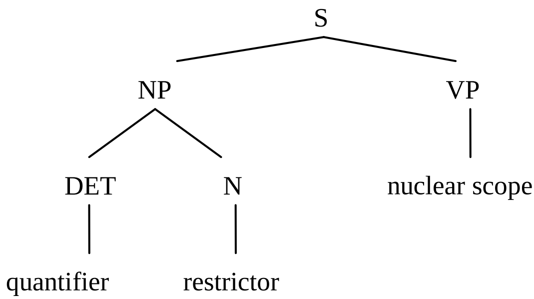
But without generalized quantifiers we have to insert logical representations during the process of meaning derivation. Consider the meaning derivation for the sentence “some boy runs”:

$$\dfrac{\dfrac{\dfrac{\mathit{some}}{(et)((et)t):\lambda PQ.\exists x(P(x)\wedge Q(x))} \qquad \dfrac{\mathit{boy}}{et:\mathit{boy}}}{(et)t:\lambda Q.\exists x(\mathit{boy}(x)\wedge Q(x))} \qquad \dfrac{\mathit{runs}}{et:\mathit{run}}}{t:\exists x(\mathit{boy}(x)\wedge \mathit{run}(x))}$$

As you can see, you are forced to conventionally stipulate that the meaning of “some” corresponds to a whole sentence of first order logic, with two “slot”, corresponding to the to variables bounded by the lambda operator. $\exists$ does not matches any words in the sentence. This is because $\exists$ is an unrestricted generalized quantifier of type $\langle 1\rangle$ and, simply, in natural languages there aren’t determiners directly matching unrestricted GQs of type $\langle 1\rangle$. The syntax of quantification in natural languages exhibits some typical, idiosyncratic, properties. In many natural languages, quantified sentences normally show the following “tripartite” structure:

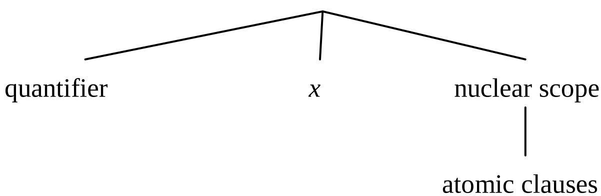


where the Quantifier usually is a pronoun or an adjective expressing a quantificational relation, the Restrictor is a predicate constraining the domain of quantification (the set of entities concerning the quantification), the Nuclear Scope expresses the set with which the restrictor is confronted in order to determining if the quantificational relation is satisfied (briefly, the scope of quantification). For instance, for a quantified noun phrase in subject position, we have something like the following syntactic structure:



It is worth noting that Restrictor and Nuclear Scope are clearly distinct since they belong to different syntactic categories and have different positions and role in the phrase structure of the sentence.

This structural diversification completely vanishes in the syntax of quantified first order formulas. Neither a Restrictor nor a Nuclear Scope can be isolated. Conversely, they are indistinguishable, insofar as they are generally expressed by atomic clauses within the scope of the quantifier:



Now, thanks to Lindström generalization of Mostowski theory of generalized quantifiers, we can restore a closer correspondence between logical syntax and natural language syntax. Generalized quantifiers allow us to keep the linguistic job free from intrusion of logical representations, as you can easily see in the following meaning derivation of the sentence "some boy runs", in which the determiner "some" perfectly matches the meaning of the generalized quantifier $some(A\ ,B) = A \cap B \neq \varnothing$:

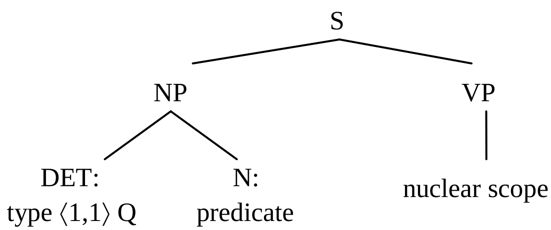
$$\cfrac{\cfrac{\cfrac{some}{(et)((et)t) : \lambda PQ.some(P,Q)} \quad \cfrac{boy}{et : boy}}{(et)t : \lambda Q.some(boy,Q)} \qquad \cfrac{runs}{et : run}}{t : some(boy,run)}$$

As already mentioned, the type $\langle 1,1 \rangle$ GQ $some(A\ ,B)$ matches the meaning of the word "some", while the type $\langle 1 \rangle$ GQ $some(A\ ,B)$ matches the meaning of the noun phrase "some boy", and the meaning of the whole sentence is straightforwardly rewritten into a set-theoretical representation to be interpreted in a model:

$$\cfrac{\cfrac{some(boy,run)}{\langle boy,run \rangle \in \{A,B \subseteq E : A \cap B \neq \varnothing\}}}{boy \cap run \neq \varnothing}$$

Hence, adopting the generalized quantifiers, the one-to-one correspondence between syntax and semantics is re-established (the so-called syntax/semantics isomorphism), at least for the meaning of quantified noun phrases.

Indeed, the following tree illustrates a quite general pattern of the semantic structure of nuclear sentences, in many natural languages:



Noun phrases can be uniformly interpreted as type $\langle 1 \rangle$ quantifiers, while determiners can be uniformly interpreted as type $\langle 1, 1 \rangle$ quantifiers. Moreover, proper names can be treat as type $\langle 1 \rangle$ quantifiers as well. This result is obtained performing the so-called "type-raising" of the meaning of proper names, from the semantic type of *e* (entities) to the semantic type (*et*)*t* (functions mapping sets onto truth-values). The following derivation is then licensed, for a sentence like "John drinks":

$$\cfrac{\cfrac{\cfrac{\cfrac{john}{\lambda P.\,john(P)} \qquad \cfrac{drinks}{drink}}{john(drink)}}{drink \in \{A \subseteq E : john \in A\}}}{john \in drink}$$

1. The expressive power of GQ

Addressing the expressiveness problem, it is easy to show the expressive power of GQ theory by examples. Let me start presenting the generalized quantifiers corresponding to the four determiners of the classical square of opposition:

$$\begin{array}{lcl}
all(A, B) & \Leftrightarrow & A \subseteq B \\
no(A, B) & \Leftrightarrow & A \cap B = \varnothing \\
some(A, B) & \Leftrightarrow & A \cap B \neq \varnothing \\
not\ all\ (A, B) & \Leftrightarrow & A - B \neq \varnothing
\end{array}$$

Just to give an idea of the expressive power of GQ theory in representing the meaning of natural language quantifiers, some definite and indefinite determiners are translated as generalized quantifiers:

$$\begin{array}{lcl}
the(A, B) & \Leftrightarrow & A \subseteq B \wedge |\mathrm{A}| = 1 \\
the\ ten(A, B) & \Leftrightarrow & |A| = 10 \wedge A \subseteq B \\
John's(A_{pl}, B) & \Leftrightarrow & A \cap \{b\text{: owner}(j,b)\} \subseteq B \\
 & & \wedge\, |\, A \cap \{b\text{: owner}(j,b)\}| > 1 \\
John's(A_{sg}, B) & \Leftrightarrow & A \cap \{b\text{: owner}(j,b)\} \subseteq B \\
 & & \wedge\, |\, A \cap \{b\text{: owner}(j,b)\}| = 1 \\
every\ but\ John\ (A, B) & \Leftrightarrow & A - B = \{john\}
\end{array}$$

n(A, B)	$\Leftrightarrow$	$A \cap B = n$
at most n(A, B)	$\Leftrightarrow$	$A \cap B \leq n$
at least n(A, B)	$\Leftrightarrow$	$A \cap B \geq n$
n out of m(A, B)	$\Leftrightarrow$	$m \times \mid A \cap B \mid = n \times \lvert A \rvert$
not one in ten(A, B)	$\Leftrightarrow$	$10 \times \lvert A \cap B \rvert < \lvert A \rvert$
all but n(A, B)	$\Leftrightarrow$	$\lvert A - B \rvert = n$
all but a tenth(A, B)	$\Leftrightarrow$	$10 \times \lvert A - B \rvert < \lvert A \rvert$
all but finitely many(A, B)	$\Leftrightarrow$	$\lvert A - B \rvert$ is finite
most(A, B)	$\Leftrightarrow$	$\lvert A \cap B \rvert > \lvert A - B \rvert$

We can now suggest two reasonable hypotheses. Given that the number of type $\langle 1, 1 \rangle$ GQs, not difficult to guess, is huge (only for a domain with two individual is $2^{16} = 65536$), the set of determiners of natural languages is a proper subset of the set of GQs, and the set of first-order definable determiners (determiners definable by means of a formula of first order logic) is, in turn, a proper subset of that of natural language determiners. We do not provide a formal proof of these hypotheses, but some considerations may suffice to show their plausibility.

That not all GQs are natural language determiners is a consequence of the formulation of some (hypothetical) semantic universals. A natural language determiner Q of type $\langle 1, 1 \rangle$ denotes a "quantirelation" if it is characterized by the property of "isomorphism" (ISOM). A type $\langle 1, 1 \rangle$ satisfies ISOM iff for any pair of universes E and E′ and any sets $A, B \subseteq E$ and $A', B' \subseteq E'$:

$$\text{if } |A - B| = |A' - B'| \text{ and } |A \cap B| = |A' \cap B'| \text{ then } Q_E(A, B) \Leftrightarrow Q_{E'}(A, B)$$

ISOM lexical determiners have one of the following properties (supposing $Q(A, B)$ is the quantifier denoted):

- Extension (EXT): only $|A \cap B|$ is relevant - a typical example is *some*$(A, B) \Leftrightarrow |A \cap B| = \varnothing$;
- co-Extension (co-EXT): only $|A - B|$ is relevant - a typical example is *all*$(A, B) \Leftrightarrow |A - B| = \varnothing$;
- proportionality (PRO): both $|A \cap B|$ and $|A - B|$ are relevant - a typical example is *most*$(A, B) \Leftrightarrow |A \cap B| > |A - B|$.

This implies that all natural language determiners have the Conservativity (CONS) property: $|B - A|$ does not matter. We can add that Extension means that $|E - (A \cup B)|$ does not matter. Considering that also not-ISOM determiners like the definite articles or bare plurals are EXT, and that the so-called "cardinal determiners" belong to one of the mentioned categories, we can call the class of CONS quantifiers fallen into the union of EXT, co-EXT and PRO the "natural quantifiers". The class of natural quantifiers is closed under Boolean operations. If these general properties of determiners are to be taken as universally valid, the number of generalized quantifiers that can represent the denotation of natural language determiners is significantly restricted, i.e., the set of natural quantifiers is a proper subset of the set of GQs.

That not all natural language determiners are first-order definable is a result of some proofs in literature. Let $Q(A, B)$ be a type $\langle 1, 1\rangle$ quantifier and ϕ a first-order sentence whose non-logical constants are just the two unary predicates A and B. $Q(A, B)$ is first-order definable iff ϕ is true in every model M for which $Q([[A]]_M, [[B]]_M)$ is true. For example, *at least two*(A, B) is definable by $\exists x \exists y(x \neq y \wedge A(x) \wedge B(x) \wedge A(y) \wedge B(y))$ and *at least n*(A, B) is defined analogously. Hence, they are first-order definable.

In general, the so-called "intersective" GQ (for which only $|A \cap B|$ matters) and the so-called "co-intersective" GQ (for which only $|A - B|$ matters) are first-order definable (for finite domains), while the "proportional" GQ (for which both $|A \cap B|$ and $|A - B|$ matter) are not first-order definable. Barwise and Cooper (1981) proved that proportional natural language determiners like *most*(A, B) are not first-order definable and, successively, stronger results has been found.

Summarizing, GQ theory, with its set-theoretical framework, represents a formal tool particularly suitable for characterizing natural language quantification in a more "naturalistic" fashion. I suggest that, in some sense, this theory marks a switch from a normativist to a descriptivist approach to quantification. Moreover, we have seen how the adoption of the set-theory machinery allows us to obtain a more flexible codification of the meaning of natural quantifiers.

As we have seen, the two main features of GQ theory, that are:

- the restored correspondence between logical and natural syntax of quantification, and
- the increased expressive power of the theory of quantification,

imply the possibility to get some insight into the structure of natural language quantification and to formulate generalizations that may represent as many hypotheses about universal features of the semantics of quantificational phrases of natural languages.

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