João Daniel Dantas de Oliveira

Gödel's Slingshot Revisited: Does Russell's Theory of Descriptions Really Evade The Slingshot?

Natal, Brasil

2016

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Dissertação de Mestrado apresentada ao Programa de Pós-Graduação em Filosofia para obtenção do título de Mestre em Filosofia.

Universidade Federal do Rio Grande do Norte – UFRN

Centro de Ciências Humanas, Letras e Artes – CCHLA

Programa de Pós-Graduação em Filosofia – PPGFIL

Supervisor: Prof. Dr. João Marcos de Almeida

Natal, Brasil

2016

Universidade Federal do Rio Grande do Norte - UFRN
Sistema de Bibliotecas - SISBI
Catalogação de Publicação na Fonte. UFRN - Biblioteca Setorial do Centro de Ciências Humanas, Letras e Artes - CCHLA

Oliveira, João Daniel Dantas de.
Gödel's Slingshot Revisited: Does Russell's Theory of Descriptions Really Evade The Slingshot / João Daniel Dantas de Oliveira. - 2016.
76f.: il.

Dissertação (mestrado) - Universidade Federal do Rio Grande do Norte. Centro de Ciências Humanas, Letras e Artes. Programa de Pós-Graduação em Filosofia.
Orientador: Prof. Dr. João Marcos de Almeida.

1. Lógica - filosofia. 2. Pluralismo lógico. 3. Descrições definidas. I. Almeida, João Marcos de. II. Título.

RN/UF/BS-CCHLA CDU 16

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Trabalho aprovado. Natal, Brasil, 30 de Setembro de 2016:

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2016

To my parents Weydes R. D. de Araújo and Antônio Kydelmir D. de Oliveira

Acknowledgements

I would like to thank:

My advisor Prof. João Marcos for all the patience in carefully reading so many early drafts of this manuscript and making important comments and also the defense panel for their time and comments.

My other professors for the many lessons learned: Elaine Pimentel, Bruno Vaz, Maria da Paz, Frode Bjørdal and specially Prof. Daniel Durante and Prof. José Eduardo Moura who were in my qualification panel and whose comments helped shape this text.

My friends from the philosophy department for all the good talks, academic and otherwise: Everton Thiago Timboo, João Paulo Rodrigues, Sanderson Molick, Diego Wendell, Ricardo Pereira, Hudson dos Anjos Benevides, Patrick Terrematte, David Gomes, Carolina Blasio, Samir Gorsky, André Ângelo, and Thiago Nascimento.

Those in Bochum who so kindly took me in: Prof. Heinrich Wansing for the useful discussions, Daniel Skurt, Andrea Kruse, Mathieu Beirlaen and Jesse Heyninck for the valued companionship and my roommates Serena Alfarano and Clara Paglialonga for teaching me to cook risotto.

My family for all the support, specially my brother Joaquim Adelino Dantas de Oliveira for always maintaining a healthy competitive environment, my mother Weydes Régia Dantas de Araújo for always encouraging me to pursue my interests and my father Antônio Kydelmir Dantas de Oliveira for supporting my studies.

The friends: Clara Liberalino, Daniel Liberalino, Diego Cirilo and the folk of the Conjunto Urich Graff.

The Erickson family: Prof. Glenn W. Erickson for the wise words of advice and careful revisions of my English, Prof. Sandra S. F. Erickson for the kinds words of support, and Rebecca and Marília for making me feel at home. Finally, Evelyn, whom I can't thank enough, but I can try: for all the love and companionship and the help in writing this thesis.

Last, but not least, my dog Nick for never letting me sleep too much.

Left to the reader.

Resumo

A família de argumentos chamada "*Slingshot Arguments*" é uma família de argumentos subjacente à visão fregeana de que se sentenças tem referência, a sua referência é os seus valores de verdade. Usualmente visto como um espécie de argumento colapsante, o argumento consiste em demonstrar que, uma vez que você suponha que há alguns itens que são as referências das sentenças (como fatos ou situações, por exemplo), estes itens colapsam em apenas dois: O Verdadeiro e O Falso. Esta é uma dissertação sobre o slingshot que é denominado o slingshot de Gödel. Gödel argumentou que há uma conexão profunda entre estes argumentos e descrições definidas. Mais precisamente, de acordo com Gödel, adotando-se a interpretação de Russell de descrições definidas (que diverge da visão de Frege de que descrições definidas são termos singulares) é possível escapar do slingshot. Nós desafiamos a posição de Gödel de duas formas, primeiramente por apresentar um slingshot mesmo com uma interpretação russelliana de descrições definidas em segundo lugar por apresentar um slingshot mesmo se mudarmos de termos singulares para termos plurais à luz do recente desenvolvimento da chamada Lógica Plural. A dissertação está dividida em três capítulos. No primeiro capítulo apresentamos o debate entre Frege e Russell sobre descrições definidas, no segundo capítulo apresentamos a posição de Gödel e reconstruções de seu argumento e no terceiro capítulo demonstramos nosso próprio slingshot para a Lógica Plural. Através desses resultados pretendemos concluir que podemos recuperar slingshots mesmo com uma interpretação russelliana de descrições definidas ou em um contexto de Lógica Plural.

Palavras-chaves: Slingshot Arguments; Descrições Definidas; Lógica Plural.

Abstract

"Slingshot Arguments" are a family of arguments underlying the Fregean view that if sentences have reference at all, their references are their truth-values. Usually seen as a kind of collapsing argument, the slingshot consists in proving that, once you suppose that there are some items that are references of sentences (as facts or situations, for example), these items collapse into just two items: The True and The False. This dissertation treats of the slingshot dubbed "Gödel's slingshot". Gödel argued that there is a deep connection between these arguments and definite descriptions. More precisely, according to Gödel, if one adopts Russell's interpretation of definite descriptions (which clashes with Frege's view that definite descriptions are singular terms), it is possible to evade the slingshot. We challenge Gödel's view in two manners, first by presenting a slingshot even with a Russellian interpretation of definite descriptions and second by presenting a slingshot even when we change from singular terms to plural terms in the light of new developments of the so-called Plural Logic. The text is divided in three chapters, in the first, we present the discussion between Russell and Frege regarding definite descriptions, in the second, we present Gödel's position and reconstructions of Gödel's argument and in the third we prove our slingshot argument for Plural Logic. In light of these results we conclude that we can maintain the validity of slingshot arguments even within a Russellian interpretation of definite descriptions or in the context of Plural Logic.

Key-words: Slingshot Arguments; Definite Descriptions; Plural Logic.

Contents

Introduction

This dissertation will argue against Gödel's solution to slingshot arguments. It will also argue that a term-forming operator is not a necessary ingredient to such arguments. The so-called "slingshot arguments" are a family of arguments underlying the view that if sentences have reference at all, their references are their truth-values. In order to properly present the slingshot argument we will study Frege's, as well as Russell's theory of definite descriptions and Gödel's deliberation on the dispute between the two. Further more, we will analyse Plural Logic as a way to deal with a plural term-forming operator. We will argue that none of these attempts to get rid of the slingshot works.

The dissertation comprises three chapters. In Chapter One, "Frege and Russell On Descriptions", we present Frege's and Russell's theories of definite descriptions and compare both theories. In Chapter Two, "Gödel's Slingshot Revisited", we analyse Gödel's proposal of evading the slingshot by using Russell's theory of descriptions. By featuring a more technical approach to the subject, we show that this solution does not work and the slingshot comes back stronger. Finally, in Chapter Three, "Plural Logics and Slingshot Arguments", we explore another approach to a theory of descriptions that can also deal with a plural description in its formal language. Here we also show that the slingshot follows. In the Conclusion, we argue that definite description is not an essential ingredient to the slingshot, as is generally held in the literature. This is a problem for those who believe that logic is somehow connected to the real world and don't want there to be but two facts. We also comment on other possible solutions to the slingshot, indicating future work.

Main authors discussed are Frege [Frege, 1948], Russell [Russell, 1905] [Russell, 1919], Gödel [Gödel, 1944] and Oliver and Smiley [Oliver and Smiley, 2013a]. We also rely on Neale's [Neale, 1995] account of the slingshot and Shramko and Wansing's reconstruction of the slingshot [Shramko and Wansing, 2011], as well as Hatcher's formal system for dealing with definite descriptions [Hatcher, 1982].

1 Frege and Russell On Descriptions

> Principles and logic didn't give birth to reality. Reality came first, and the principles and logic followed.
>
> — Haruki Murakami, *1Q84*

1.1 Introduction

The present chapter deals with Frege's theory of descriptions, Russell's theory of descriptions and slingshot arguments. Frege's theory states that the reference of a definite description is a singular term and the reference of a sentence is its truth-value. For all purposes here we treat slingshot arguments as arguments which underlie the Fregean thesis that if sentences have references at all, such references are their truth-values.

Slingshots are usually seen as a kind of collapsing argument[1], since once you suppose that there are some items that are references of sentences (as facts or situations, for example), these items collapse into just two items, which one might call 'The True' and 'The False'. Donald Davidson, Alonzo Church, Willard Von Orman Quine and Kurt Gödel are usually counted among those receiving the title of slingshooter.

Davidson [Davidson, 1967] used the slingshot argument against a theory of facts that assumes that true sentences correspond to facts in the world. If true sentences refer to facts, then all true sentences have to refer to the same fact, which he later [Davidson, 1969] called *The Great Fact*. Church [Church, 1943] agreed with Frege about the reference of sentences, and used the slingshot against Carnap's earlier position, according to which the references of sentences are propositions. Quine's [Quine, 1953] argument concerning the collapse of modal distinction it is sometimes labelled as a slingshot as well[2]. Although the argument appearing in Gödel's paper *Russell's Mathematical Logic* [Gödel, 1944] was

1 See [Neale, 1995, p. 761].

2 In works by Jon Barwise and John Perry [Barwise and Perry, 1981, p. 398] and Stephen Neale [Neale, 1995, p. 772] they claim that Quine uses the slingshot, who himself refers to Church and Gödel as his "fellow slingshooters" [Quine, 2000, p. 426].

dubbed Gödel's slingshot, Gödel himself wasn't a slingshooter, so to speak, in the sense that it is not clear if Gödel endorsed or not the slingshot conclusion. On the one hand, Gödel sees a relation between definite descriptions and the slingshot, namely, according to Gödel, by adopting Frege's interpretation of definite descriptions (together with some minimal assumptions) it is necessary to draw the conclusion that equivalent sentences have the same referent. On the other hand, by adopting Russell's interpretation of definite descriptions it is possible to evade the previous conclusion, according to Gödel.

Slingshots are especially problematic for those who maintain a realistic view about logic, that is, the view that logic somehow describes the real world and that a sentence is true if it refers to some fact or situation in the world. Given that, in other words, the slingshot states that there are no two equivalent sentences that describe different facts of the reality. And it should be expected, from a realistic point of view, that there are distinct true sentences that refer to distinct facts in the world. For example, the sentences "snow is white" and "grass is green" should be regarded as equivalent in value, but distinct in reference. One is true if snow is actually white and the other if grass is actually green. Among those philosophers who adopt a realistic view about logic, one encounters the likes of Bertrand Russell and Ludwig Wittgenstein (in the *Tractatus Logico-Philosophicus*).

Several recipes were proposed as essential ingredients of the slingshot. One not so famous in the slingshot literature was proposed by J. Michael Dunn [Dunn, 1988]. Dunn had another goal when talking about slingshots, namely, with proving that certain higher-order non-classical logics do not enjoy any form of extensionality. To Dunn, extensionality is the principle according to which two properties, P and Q, have the same extension if and only if, every object that has the property P also has the property Q, and vice-versa. Yet, in the last section of his paper, Dunn asks himself if the argument he gives is a kind of (what he calls) 'FCGDQ argument' where the letters stand for Frege, Church, Gödel, Davidson and Quine, respectively. Since those are logicians usually associated to the use of the slingshot, we assume Dunn is referring to Slingshot arguments. Dunn gives a nice recipe for the argument, noticing that the argument makes use of: (i) Indiscernibility of Identicals, (ii) a certain notion of Replacement, and (iii) a term-forming operator. In this Chapter we are going to focus on one of these machineries, namely, we are going to focus on whether the use of a variable-binding term-forming operator is an essential ingredient for the slingshot argument to prove its conclusion. We believe this recipe is deeply connected to Gödel's version of the slingshot because Gödel employs Russell's interpretation of definite descriptions to eliminate the term-forming operators in the language and therefore to claim that the slingshot was evaded.

That said, our plan of attack, is as follows. First, we discuss the birth of the slingshot in Frege's idea of proper name in the section "Frege on Descriptions". Then, following Frege's idea of definite descriptions as singular terms, we introduce First-order Logic, its syntax and semantics, with a term-forming operator as primitive sign in the language in section "First-order Logic with Definite Descriptions". Finally, since Russell questions Frege's interpretation of definite descriptions as singular terms, in section "Russell on Descriptions" we sketch Russell's treatment of definite descriptions.

1.2 Frege on Descriptions

Given that Frege is held as the father of the slingshot, and since Gödel explicitly says that he is proposing a way of evading "Frege's puzzling conclusion", in this section we explore Frege's notion of proper names and how this lead him to conclude that the reference of a sentence is its truth-value. This section is based mostly on Frege's work [Frege, 1948] and Francis Jeffry Pelletier and Linsky's papers [Pelletier and Linsky, 2005] [Pelletier and Linsky, 2008]. Frege's notion of proper names states that the reference of a proper name is the unique object for which it stands, if there is such unique object. Among the sort of things Frege recognizes as proper names are definite descriptions (that is, phrases of the form 'the so-and-so') and sentences. A definite description, such as "the author of *Moby Dick*", is a proper name referring to a single object, namely, Herman Melville; and the sentence "the author of *Moby Dick* is Herman Melville" is a proper name referring to either of two single objects, The True or The False. At this point we shall examine the singular terms and the proper names, and discuss whether they are equivalent or not.

Frege opens his paper *Sense and Reference* [Frege, 1948] with a puzzle about identity. Frege, in his *Begriffsschrift*, had assumed that identity is a relation between names or signs of objects and "[w]hat is intended to be said by $a = b$ seems to be that the sign or names 'a' and 'b' designate the same thing" [Frege, 1948, p. 209]. That said, Frege invites us to contrast the two identity statements $a = a$ and $a = b$. The statements should have different cognitive values, as the first is known a priori (its knowledge doesn't require an empirical observation) and the second is known a posteriori (its knowledge depends on an empirical observation). As in some of Frege's examples, it is not necessary to do an empirical experiment to know that the morning star is identical to the morning star (since we expect that every object is identical to itself), but an experiment was needed in order to discover that the morning star is the same as the evening star. But, by assuming that identity is a relation between names, then $a = a$ and $a = b$ are not different, if $a = b$ is true.

A *proper name* is some expression that uniquely refers to a single object, or in Frege's words: "The designation of a single object can also consist of several words or other signs. For brevity, let every such designation be called a proper name" [Frege, 1948, p. 210]. According to Frege, there are two ingredients coupled to a proper name, namely, the *sense* and the *reference*[3] of the name.

> It is natural, now, to think of there being connected with a sign (name, combination of words, letter), besides that to which the sign refers, which may be called the referent of the sign, also what I would like to call the *sense* of the sign, wherein the mode of presentation contained. [Frege, 1948, p. 210]

In other words, first we have a name and connected to this name we have two components: The reference of this name, which is the object referred to by the name, and the sense of the name, which is the mode of presentation of the name. The next natural question, after talking about the sense and reference of proper names, is to ask whether there are proper names which have sense, but no reference. To that Frege would say yes, there are proper names with no reference, such as Odysseus or the English detective created by Sir Arthur Conan Doyle.

Do sentences have sense and reference as well? "What is the sense of declarative sentences" (to which Frege answers as being the thought) is a question that has inspired many philosophers to create formal theories of sense, but it is Frege's answer to what is

[3] We follow Max Black's translation of Frege's original terminology in [Frege, 1960]. The German nouns 'der Sinn' and 'die Bedeutung' are translated as 'sense' and 'reference' respectively. It is important to point out that in the literature some have chosen to translate Bedeutung as denotation or designation, or even some have considered this nouns as untranslatable (See [Pelletier and Linsky, 2005, p. 61, note 3]).

the reference of sentences that the slingshot discussion is more connected to (more than to the notion of sense).

> "We are therefore driven into accepting the *truth value* of a sentence as its referent. By the truth value of a sentence I understand the circumstance that it is true or false. There are no further truth values. For brevity I call the one the true, the other the false. Every declarative sentence concerned with the referents of its words is therefore to be regarded as a proper name, and its referent, if it exists, is either the true or the false." [Frege, 1948, p. 216]

Frege then raises the question "What else but the truth value could be found, that belongs quite generally to every sentence concerned with the referents of its components and remains unchanged by substitutions [...]?" [Frege, 1948, p. 217] to which some would answer situations or facts designated by the sentence[4]. Beyond that, Frege concludes: "If now the truth value of a sentence is its referent, then on the one hand all true sentences have the same referent and so, on the other hand, do all false sentences" [Frege, 1948, p. 217]

Going back to the puzzle about identity, given that, according to Frege, names do have sense and reference, and that the reference of a sentence is its truth-value, the reference of $a = a$ is the same as the reference $a = b$ (if $a = b$ is true), namely The True, although the sentences differ in sense.

> If now "$a = b$", then indeed the referent of "b" is the same as that of "a", and hence the truth value of "$a = b$" is the same as that of "$a = a$". In spite of this, the sense of "b" may differ from that of "a", and thereby the sense expressed in "$a = b$" differs from that of "$a = a$". [Frege, 1948, p. 230]

What would be latter called by Russell definite descriptions, Frege denominated as *compound proper names*, that is, a proper name that was constructed from a predicate together with a definite article[5].

---

[4] See [Barwise and Perry, 1981, p. 395].

[5] In Section "Russell on Descriptions" we elucidate more clearly what is Russell's account on definite descriptions.

> Adjective clauses also serve to construct compound proper names even if, unlike noun clauses, they are not sufficient by themselves for this purpose. These adjective clauses are to be regarded as equivalent to adjectives. Instead of "the square root of 4 which is smaller than 0," one can also say "the negative square root of 4." We have here the case of a compound proper name constructed from the predicate expression with the help of the singular definite article. This is at any rate permissible if the predicate applies to one and only one single object. [Frege, 1948, p. 223]

Regardless of how was constructed, a compound proper name (or a definite description) should be equated as a proper name, in Frege's view.

According to Pelletier and Linsky [Pelletier and Linsky, 2008], there are actually three Fregean theories of descriptions, which Pelletier and Linsky call Frege-Strawson theory, Frege-Carnap theory and Frege-Grundgesetze theory and each differ in how to interpret proper names that have no reference. The aim of Pelletier and Linsky's paper [Pelletier and Linsky, 2008] is to show that, even if there are actually three Fregean theories of descriptions, "each of these theories has some claim to be Fregean" [Pelletier and Linsky, 2008, p. 40], which is, each of these treat definite descriptions (and proper names) as singular terms[6].

> In all the theories suggested by Frege's words, he sought to make definite descriptions be *terms*, that is, be name-like in character. By this we mean that not only are they *syntactically* singular in nature, like proper names, but also that (as much as possible) they behave *semantically* like paradigm proper names in that they designate some item of reality, i.e., some object in the domain of discourse. [Pelletier and Linsky, 2005, p. 5]

According to Pelletier and Linsky, the theory presented in "Sense and Reference" is not a theory with a scientific purpose, but rather a theory about how ordinary language works. A theory where it is possible that a proper name has sense but no reference (by being an empty name) and a sentence containing an empty name could also be meaningful but without reference. Of course, since Frege takes the reference of a sentence as being its

[6] In the next section we properly define what are *terms* in a formal language.

truth-value, then in the case where one compound of the sentence is a empty name the sentence would be neither true nor false.

We have seen that treating definite descriptions as terms is an essential feature of Frege's ideas and some have believed that this could be an essential feature that led him to conclude that sentences work as proper names as well. Thus, in what follows, we will present First-order Logic with definite description as terms, as a term-forming operator.

1.3 First-order Logic with Definite Descriptions

> Every word she writes is a lie,
> including *and* and *the*.
>
> — Mary MacCarthy

In this section we introduce First-order Logic, its syntax and semantics, having a term-forming operator as primitive sign in the language. We hope the following technical introduction helps to clarify the discussion coming next. We follow mostly the first chapter of [Hatcher, 1982]. Our term-forming operator in our initial language is the description operator denoted by the symbol ι.

Syntax

The Language $\mathcal{L}_\iota$ of First-order Logic with the description operator consists in:

(1) A denumerable set VAR of variables,[7] $x_0, x_1, ...$

(2) A set CON of constants, $c_0, c_1, ...$

(3) Connectives $\neg, \supset, \leftrightarrow, \wedge, \vee$ and brackets (,)

(4) Quantifiers $\forall$ and $\exists$

(5) The description operator ι

(6) A set $PRED$ of predicate symbols $P_1^n, ..., P_m^n$ of arity n, where $n \geq 0$

(7) A set FUN of function symbols $f_1^n, ..., f_m^n$ of arity n, where $n \geq 0$

(8) A two place predicate $\doteq$ called Equality

Thanks to the description operator, the set of terms and the set of formulas are defined by mutual recursion. This is so because the definition of the description operator mentions

[7] Later in the Chapter Plural Logics and Slingshot Arguments we make a distinction between plural and singular variables, but here we don't need this difference yet.

the set of formulas and the definition of formulas mentions the set of terms.

(9) The set $TERM$ of terms is defined as follows[8]:

$$t ::= \quad x \;\Big|\; c \;\Big|\; f^n(t_1, ..., t_n) \;\Big|\; \iota x \varphi$$

where $x \in VAR$, $c \in CON$, $f^n \in FUN$, φ is a formula and $t_1, ..., t_n \in TERM$.

(10) The set $FORM$ of formulas is defined as follows:

$$\varphi ::= \quad P(t_1, ..., t_n) \;\Big|\; t_1 \doteq t_2 \;\Big|\; \neg\varphi \;\Big|\; \varphi \wedge \varphi \;\Big|\; \varphi \vee \varphi \;\Big|\; \varphi \supset \varphi \;\Big|\; \forall x\varphi \;\Big|\; \exists x\varphi \;\Big|\; \bot$$

where $P^n \in PRED$, $t_1, ..., tn \in TERM$ and $x \in VAR$.

With this syntax we are capable of formalizing sentences like (i) "there is a yellow cat", (ii) "every dog is an animal", (iii) "the author of Waverley is bald" and many others. For example, let $Y(x)$ stand for the predicate "x is yellow", $C(x)$ stand for "x is a cat" and we can formalize (i) by (i') $\exists x(Y(x) \wedge C(x))$. Similarly, let $D(x))$ stand for "x is a dog", $A(x)$ stand for "x is an animal", $W(x)$ stand for "x is author of Waverley", and $B(x)$ stand for "x is bald" and we can formalize (ii) and (iii) by (ii') $\forall x(D(x) \supset A(x))$ and (iii') $B(\iota x W(x))$ respectively. The question whether these sentences are true or false belongs to the semantics.

Semantics

Definition 1.0: (Interpretation Structure) An *interpretation structure* $\mathcal{I} = \langle \mathcal{D}, (.)^{\mathcal{I}} \rangle$ is such that:

$\mathcal{D}$ is the domain of discourse and $.^{\mathcal{I}}$ is an interpretation mapping such that each function symbol f_i^n is interpreted as an operation $f^{\mathcal{I}} : \mathcal{D}^n \to \mathcal{D}$ and each n-ary predicate P_i^n as $P^{\mathcal{I}} \subseteq \mathcal{D}^n$

Definition 1.1: (Assignment) The function v is a function from linguistic items to semantical values. Given an interpretation structure $\mathcal{I} = \langle \mathcal{D}, .^{\mathcal{I}} \rangle$, then $v : VAR \to \mathcal{D}$ is a function such that:

$\tilde{v}(x) = v(x)$ where $\tilde{v} : TERM \to \mathcal{D}$,

$\tilde{v}(c) = c^{\mathcal{I}}$ where $c^{\mathcal{I}} \in \mathcal{D}$.

For each n-place function symbol f:

$\tilde{v}(f(t_1, ..., t_n)) = f^{\mathcal{I}}(\tilde{v}(t_1), ..., \tilde{v}(t_n))$

[8] Throughout this thesis we are going to use the Backus-Naur formalism when describing our languages. The meta symbols ::= and | can be read as "is defined as" and "or", respectively.

$$\tilde{v}(\iota x\varphi) = v'(x) \text{ iff } \langle \mathcal{I}, v' \rangle \Vdash \varphi \text{ for a unique } v' = v[x := \iota\varphi x]$$

Definition of 1.2: (x-equivalence) Given a variable $x \in VAR$, two assignments v and v' are said to be *x-equivalent* if $v(z) = v'(z)$ for all $z \in VAR \setminus \{x\}$

Notation 1.0: We use $v[x := k]$ to denote an assignment v' which is x-equivalent to v and such that $v'(x) = k$.

Definition 1.3: Given an interpretation structure $\mathcal{I}$ and a valuation function v, we say that $\langle \mathcal{I}, v \rangle$ satisfies a formula φ (Notation:$\langle \mathcal{I}, v \rangle \Vdash \varphi$) if v gives the value 1 for φ.
$\langle \mathcal{I}, v \rangle \Vdash P(t_1, ..., t_n)$ iff $\langle (\tilde{v}(t_1), ..., \tilde{v}(t_n) \rangle \in P^{\mathcal{I}}$,
$\langle \mathcal{I}, v \rangle \Vdash \neg\varphi$ iff $\langle \mathcal{I}, v \rangle \nVdash \varphi$,
$\langle \mathcal{I}, v \rangle \Vdash \varphi \vee \psi$ iff $\langle \mathcal{I}, v \rangle \Vdash \varphi$ or $\langle \mathcal{I}, v \rangle \Vdash \psi$,
$\langle \mathcal{I}, v \rangle \Vdash \varphi \wedge \psi$ iff $\langle \mathcal{I}, v \rangle \Vdash \varphi$ and $\langle \mathcal{I}, v \rangle \Vdash \psi$,
$\langle \mathcal{I}, v \rangle \Vdash \varphi \to \psi$ iff $\langle \mathcal{I}, v \rangle \nVdash \varphi$ or $\langle \mathcal{I}, v \rangle \Vdash \psi$,
$\langle \mathcal{I}, v \rangle \Vdash \forall x\varphi$ iff $\langle \mathcal{I}, v[x := k] \rangle \Vdash \varphi$ for all $k \in \mathcal{D}$,
$\langle \mathcal{I}, v \rangle \Vdash \exists x\varphi$ iff $\langle \mathcal{I}, v[x := k] \rangle \Vdash \varphi$ for some $k \in \mathcal{D}$,

Definition 1.4: (Semantical Consequence) Given $\Gamma \cup \{\varphi\} \subseteq FORM$, we say that φ is semantical consequence from Γ (Notation: $\Gamma \models \varphi$) if for all valuations that satisfy Γ (that is $\langle \mathcal{I}, v \rangle \Vdash \Gamma$) also satisfies φ (that is $\langle \mathcal{I}, v \rangle \Vdash \varphi$).

Proof System

$$\dfrac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix}}{\varphi \supset \psi} \supset I \qquad\qquad \dfrac{\varphi \wedge \psi}{\varphi} \wedge E$$

Where we adopt the notation that $[\varphi]$ means that the hypothesis φ is cancelled after used in the derivation.

$$\dfrac{\varphi \supset \psi \qquad \varphi}{\psi} \supset E \qquad\qquad \dfrac{\varphi \qquad \psi}{\varphi \wedge \psi} \wedge I$$

$$\dfrac{\begin{matrix}[\varphi] \\ \vdots \\ \bot\end{matrix}}{\neg\varphi} \neg I \qquad\qquad \dfrac{\neg\varphi \qquad \varphi}{\bot} \neg E$$

$$\dfrac{\varphi}{\varphi \vee \psi} \vee I \qquad\qquad \dfrac{\varphi \vee \psi \qquad \begin{matrix}[\varphi] \\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\psi] \\ \vdots \\ \chi\end{matrix}}{\chi} \vee E$$

$$\frac{t_1 \doteq t_2}{t_2 \doteq t_1}\ sym \qquad \frac{t_1 \doteq t_2 \qquad t_2 \doteq t_3}{t_1 \doteq t_3}\ trans$$

$$\frac{t_1 \doteq t_2 \qquad \varphi(t_1)}{\varphi(t_2)}\ sub$$

$$\frac{\varphi(x)}{\forall x \varphi}\ \forall I$$

$$\frac{\forall x \varphi(x)}{\varphi(t)}\ \forall E$$

Given that, in the rule $\forall I$ the variable x can not occur free in any hypothesis and in the rule $\forall E$ it is required that t is free for x.

$$\frac{\exists x \varphi(x) \qquad \begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix}}{\psi}\ \exists E$$

In the rule $\exists E$ it is require that x is not free in ψ.

$$\frac{\varphi(t)}{\exists x \varphi(x)}\ \exists I$$

$$\frac{\varphi(a)}{a \doteq \iota x(x \doteq a \land \varphi(x))}\ \iota - I$$

A proper explanation of the proof system would require to present the meta-theorems of soundness and completeness, proving that every derivation in the proof system is true and every true proposition in the semantics are provable using this proof system respectively. The theorem of soundness for a similar system as the one presented here can be found in [Hatcher, 1982, p. 56] and completeness in [Hatcher, 1982, p. 61].

1.4 Russell on Descriptions

Our goal in this section is to sketch Russell's treatment of definite descriptions in [Russell, 1905] and [Russell, 1919]. Russell questions Frege's interpretation of definite descriptions as singular terms. Russell holds that definite descriptions are meaningless without context and that they perform as complex formula constructed with quantifiers rather then as singular terms. To clarify the notion of context and what Russell means by a context elimination, we follow David Kaplan in [Kaplan, 1970] and, especially, [Gratzl, 2015], to see this as a result of eliminating definite descriptions.

Descriptions might be of two sorts: a definite description is a phrase of the form "the so-and-so" and an indefinite description is of the form "a so-and-so". Although it is sometimes claimed that the theory of definite descriptions was first born in [Russell, 1905], it was in [Russell, 1919, chapter 16] that Bertrand Russell pursued his doctrine of descriptions with more precision:

> In this chapter we shall consider the word *the* in the singular, and in the next chapter we shall consider the word *the* in the plural. It may be thought excessive to devote two chapters to one word, but to the philosophical mathematician it is a word of very great importance [...] I would give the doctrine of this word if I were 'dead from the waist down' and not merely in prison. [Russell, 1919, p. 94]

Indeed, there was the seed of the theory of definite descriptions in [Russell, 1905]. Russell starts *On Denoting* by giving examples of what he considers a "denoting phrase". In this class of phrases are those of the form: all men, some women, no rational animal and the president of France in 1653. They all share a certain form: *all*, *some*, *no* and *the*. As to how these phrases might denote, it is possible to distinguish three cases: (i) a phrase like "the president of France in 1653" is a denoting phrase, because it has the structure of a denoting phrase, but does not denote anything; (ii) a phrase like "the present coach of Alecrim" denotes a specific person; and finally (iii), a phrase might denote ambiguously like "a man" denotes at least one man, but not necessarily many. Among the denoting phrases, those containing *the* are "by far the most interesting and difficult of denoting phrases" [Russell, 1905, p.481].

Some philosophers, such as Scott Soames, think that the recognition of definite descriptions as an expression involving quantifiers, rather then singular terms, is a really good insight[9]. More than that, it marks the point where Russell disagrees with Frege. According to Russell, definite descriptions like the one in "the author of *Moby Dick* had a beard" must guarantee the uniqueness of the individual who wrote *Moby Dick*, that is, it

[9] See [Soames, 2010, p.24]

must guarantee that there is someone who wrote *Moby Dick* and no other individual is such that he also wrote *Moby Dick*. Therefore, in the example, there is an x who wrote *Moby Dick* and x had a beard, and if y wrote *Moby Dick* then y and x are the same, for every y. More generally, according to Russell's analysis of definite description, a formula such as $\psi(\iota x\varphi(x))$ is just a shorthand for a sentence like: $\exists x(\varphi(x) \wedge \forall y(\varphi(y) \supset x \doteq y) \wedge \psi(x))$[10].

Recall that Frege's theory distinguishes two components of a sentence: the sense and the reference. The denotation is the object to which the sentence refers, and the sense is the way the sentence refers to this object. Thus, a sentence like "Scott is the author of *Waverley*" establishes an identity between the object denoted by "Scott" and the object denoted by "the author of *Waverley*", but their names differ on sense. First, Russell disagrees that this is a simple statement about the identity between two terms:

> The meaning of such propositions cannot be stated without the notion of identity, although they are not simply statements that Scott is identical with another term, the author of *Waverley* [...]. The shortest statement of 'Scott is the author of *Waverley*' seems to be: 'Scott wrote *Waverley*; and it is always true of y that if y wrote *Waverley*, y is identical with Scott'. It is in this way that identity enters in 'Scott is the author of *Waverley*'; and it is owing to such uses that identity is worth affirming. [Russell, 1905, p. 492]

Secondly, Russell considers the example of sentences that have as a constituent a term without denotation as "the president of France in 1653". Frege's solution is to postulate a null class to which every empty term denotes. Russell does not want to adopt this strategy since, to him, this kind of sentence should not be consider as a sentence without reference, but plainly false.

Russell calls attention to the fact that "[c]onfusion of primary and secondary occurrences is a ready source of fallacies where descriptions are concerned"[11]. As an example of how this distinction may be useful, let us take a look at one puzzle that Russell presents in *On Denoting*. By the law of excluded middle, either a sentence or its negation is true, therefore either (i) "the president of France in 1653 had a beard" is true or (ii) "the president of France in 1653 did not had a beard" is true. Yet none seems to be the case, in

10 See [Russell, 1919, p.100]
11 [Russell, 1919, p.100]

as much as there was no president of France in 1653. Let $\varphi(x)$ stand for "x is president of France in 1653" and $\psi(x)$ "stand for x had a beard", the sentence (ii) is ambiguous between :

(ii') $\exists x(\varphi(x) \wedge \forall y(\varphi(y) \supset x \doteq y) \wedge \neg\psi(x))$ and

(ii") $\neg\exists x(\varphi(x) \wedge \forall y(\varphi(y) \supset x \doteq y) \wedge \psi(x))$

The first case, i.e, (ii'), is a case of primary occurrence, and in this case the sentence is false because there is no president of France in 1653. The second, (ii"), is a case of secondary occurrence and it is true. Thus, the correct instantiation of the law of excluded middle should be between (i) and (ii").

According to David Kaplan, what is meant to be Russell's contextual definition of definite description is a new phrase introduced to clarify the meaning of an old phrase previously in the language.

> [W]hat is a contextual definition? Ordinarily we think of definitions as being either stipulative or explicative (in an older terminology, nominal or real). That is, either a new expression is introduced and assigned a meaning of a phrase whose meaning is antecedently known, or else an old expression is given a more precise, or in some other way slightly adjusted, meaning in terms of some antecedently understood phrase. (...) But what meaning is given to it, or even to the full description? None! For the central thesis of Russell's theory is that this phrase *has* no meaning in isolation. [Kaplan, 1970, p. 282]

Thus, we assume that it is in accord with Kaplan that a definite description operator was already primitive in the initial language, and the new phrase introduced is an equivalent expression. Latter in the paper Kaplan add the following:

> I believe, somewhat more appropriate to Russell's contextual definition, is that of *abbreviation*. In an abbreviation a new expression is introduced to *stand for* an old phrase. (...) Abbreviation is purely a matter of syntax. [Kaplan, 1970, p. 282]

Russell's view on proper names was very peculiar, in fact, not even a ordinary name was to be considered to be a proper name. The closest thing to a proper name was demonstratives as "this" and "that", and all the rest were just descriptions in disguise.

> We may even go so far as to say that, in all such knowledge as can be expressed in words – with exception of 'this' and 'that' and a few other words of which the meaning varies on different occasions – no names, in the strict sense, occur, but what seems like names are really descriptions. [Russell, 1919, p. 100]

This chapter has explained Frege's and Russell's theories of description. While for Frege definite descriptions should be equated to terms, in a modern terminology, for Russell should be equated as a with a complex expression constructed with quantifier. This dispute has been prominent in the literature of definite descriptions. In the next chapter we will analyse Gödel's position on this matter.

2 Gödel's Slingshot Revisited

2.1 Introduction

In the last chapter we saw that Frege and Russell disagreed about the construal of definite descriptions and that the concept of truth-values was already in Frege's work. Gödel suggested to use Russell's theory of definite description in order to evade the slingshot argument. In this chapter we question if Gödel's solution to the slingshot really kills the argument or only makes the argument stronger.

We are going to challenge the idea that term-forming operators are an essential ingredient for the slingshot. If we look at Russell's elimination procedure as a theorem of description elimination — as [Gratzl, 2015] and [Grabmayer et al., 2011] do — then the result is just stating that whatever was provable in the initial language containing definite descriptions is also provable in the new language without this operation (as long as we rephrase it with a formula that serves the same purpose).

That said, in section Gödel On Russell's Mathematical Logic we explore Gödel's tribute to Russell and how Gödel thought that Russell's interpretation of definite descriptions could evade the slingshot. Yet since Gödel did not explicitly state his argument, there are several reconstructions and in Two Ways of Reconstructing Gödel's Slingshot we explore two of those reconstructions. We go back to Gödel's way of evading the slingshot by means of Russell's interpretation of descriptions in Gödel's Way of Evading The Slingshot and prove that this manoeuver does not work.

2.2 Gödel On Russell's Mathematical Logic

In this section we will explore Kurt Gödel's tribute to Russell in [Gödel, 1944] and the reasons why Gödel claims that Russell's theory of definite descriptions could avoid the conclusion of the slingshot. Gödel's argument was well discussed in [Neale, 1995] (and several other places) and, as we will see latter in Two Ways of Reconstructing Gödel's Slingshot, according to Neale, although the argument called little attention at the time, Gödel's Slingshot should be consider as philosophically important.

Gödel is primarily concerned with Russell's earlier work, rather than Russell's standpoint at the time. Particularly, he is concerned with Russell's work on the analysis of concepts and axioms of mathematical logic. Gödel seems to endorse Russell's realistic view about Logic that is well presented in his early writings. The view is that Logic is concerned with reality and the truth of a sentence should be established by the correlation with a fact in the real world. More than that, according to [Kovač and Świętorzecka, 2015, p. 125] Gödel's commentary aim to show a compatibility between Russell's theory of reference and Russell's realism towards logic. Gödel calls attention to a passage of Russell's *Introduction to Mathematical Philosophy* where he states that "logic is concerned with the real world just as truly as zoology, though with its more abstract and general features" [Russell, 1919, p. 95].

As an example of Russell's analysis of concepts in mathematical logic, Gödel turns his attention to Russell's treatment of definite descriptions. Before going into any detail, Gödel clarifies the terminology he is going to use. According to Gödel, the question arising about definite descriptions is: What do definite descriptions signify? Here Gödel is using the verb *to signify* (which he takes as corresponding to the German verb *bedeuten*)[1] as a translation of Frege's terminology, since, according to Gödel, Frege was the first to consider the previous question[2]. As for Russell's terminology, Russell uses *indication* to mean the relation between sentences and facts in the outer world and uses *denotation* to be the relation between names and objects, given that "[Russell] holds that the relation between a sentence and a fact is quite different from that of a name to the thing named" [Gödel, 1944, p. 122]. Thus, using Frege's terminology, more precisely, Gödel's translation of Frege's terminology:

> The apparently obvious answer that, e.g, 'the author of *Waverley*' signifies Walter Scott, leads to unexpected difficulties. For, if we admit the further apparently obvious axiom that the signification of a composite expression, containing constituents which have themselves a signification, depends only on the signification of these constituents (not on the manner in which this signification is expressed), then it follows that the sentence 'Scott is the author of *Waverley* signifies the same thing as 'Scott is Scott'; and this again leads

[1] See [Gödel, 1944, p. 122, note 5]

[2] As for our own terminology, we prefer to translate the noun *Bedeuntung*, in Frege's terminology, by reference, and *bedeuten* by *to refer*, as was already done before in many places.

> almost inevitably to the conclusion that all true sentences have the same signification (as well as all false ones). [Gödel, 1944, p. 122]

The apparently obvious axiom that Gödel refers to might capture the idea of a compositionality of signification, where the signification of an expression (which is composed by constituents) is determined by the signification of its constituents. As to the expression "almost inevitably", Gödel seems to have in mind a possible way to avoid this conclusion, as we will see later. Yet following the example of other philosophers who left details of important proofs in footnotes, Gödel hinted to three further details of the final proof in the following footnote:

> The only further assumptions one would need in order to obtain a rigorous proof would be (1) that '$\phi(a)$' and the proposition 'a is the object which has the property ϕ and is identical with a' mean the same thing and (2) that every proposition 'speaks about something', i.e., can be brought to the form $\phi(a)$. Furthermore one would have to use the fact for any two objects a, b, there exists a true proposition of the form $\phi(a, b)$ as, e.g., $a \neq b$ or $a = a \wedge b = b$.[3] [Gödel, 1944, p. 122, note 5]

The assumptions above include that sentences of the form $F(a)$ and $a \doteq \iota x(x = a \wedge F(x))$ have the same signification (or stand for the same fact); that any sentence can be rewritten into a predicate form; and that any two objects a and b are comparable, namely, either they are the same object or they are different. This last assumption can be translated in the language that we presented in Chapter One as the following axiom:

$$\forall x \forall y(x \doteq y \vee x \neq y)$$

According to Gödel, assuming the previous machinery leads "almost inevitably" to the conclusion all true sentences indicate the same object, and Frege had no problem with this conclusion.

> Frege actually drew this conclusion; and he meant it in almost a metaphysical sense [...]. 'The True' – according to Frege's view – is analyzed by us in

[3] Here we are using the notation introduced in section 1.3 instead of the notation in *Principia Mathematica* as in Gödel's original paper.

> different ways in different propositions, 'the True' being the name he uses for the common signification of all true propositions." [Gödel, 1944, p. 122]

Gödel considers Russell's theory of descriptions as an advance towards avoiding the slingshot conclusion. Therefore, it seems that it is possible to summarize Gödel's point in two "if... then" theses, namely, if someone wants to use Frege's interpretation of the signification of definite descriptions, then he (or she) must draw the same conclusion as Frege does that all true sentences have the same signification, namely, The True, as well as all false ones also have the same signification, The False. If someone wants to avoid this conclusion, then he (or she) must reject one of the previous machineries used, and that is where Russell's interpretation of definite descriptions is helpful.

> Therefore this view concerning sentences makes it necessary either to drop the above mentioned principle about the signification (i.e., in Russell's terminology the corresponding one about the denotation and indication) of composite expressions or to deny that a descriptive phrase denotes the object described. Russell did the latter by taking the viewpoint that a descriptive phrase denotes nothing at all but has meaning only in context. [Gödel, 1944, p. 123]

Gödel takes Russell's doctrine that definite descriptions are incomplete symbols and have no meaning without context as a way dropping one of the previous assumptions and, therefore, avoid the Slingshot conclusion. This lead Gödel to the following conclusion:

> As to the question in the logical sense, I cannot help but feeling that the problem raised by Frege's puzzling conclusion has only been evaded by Russell's theory of descriptions and that there is something behind it which is not yet completely understood. [Gödel, 1944, p. 123]

There are, however, a few things to notice before we go any further. In a recent work by [Kovač and Świętorzecka, 2015] the authors claim that Gödel is not completely satisfied with the use of Russell's interpretation of definite descriptions to block Frege's puzzling conclusion, and that would explain the use of the word "evaded" instead of "solved". After all, we think it seems like a very brutal solution to a problem. Borrowing an analogy made by Susan Haack in [Haack, 1978, p. 142], the so called solution by cutting of the nose to spite the face. The same applies to the slingshot. Even though every commentary on Gödel's slingshot assumes that chopping off descriptions from the language is the "obvious solution", it does not seems obvious to us at all, given that many arguments can still be validated (by taking another path) even if you drop some of the assumptions. Another important thing to notice is: Frege's puzzling conclusion is puzzling to whom? It is problematic for those who agreed with Russell that a sentence is true if it corresponds to a fact in the real world. In this case, the slingshot would show that all true sentences correspond to the same fact. Lastly, since Gödel did not formally detail his proof in the paper, many proposals were made for reconstructing in detail what Gödel's argument could have been. We will go back to this matter in section 2.3.

2.2.1 Digression: On Two Sorts of Slingshots

Before we proceed any further and analyse some ways of reconstructing Gödel's slingshot we would like to make a little digression to distinguish two sorts of slingshots. One, that we might call *Value-chain Slingshot* and another that we might call $Slingshot_2$ by lack of a better name. The first, where one starts with a true sentence and by making substitutions of terms with the same reference one arrives to another completely different sentence and the only thing the first and the last sentence have in common are their truth-value. This is similar to a word-chain game where you start from a word and by substituting the letters (as long as the steps don't go through a non-existent word) one arrives to a completely different word. The second is a slingshot where we extend the language with a connective that deals with the denotation of sentences (whether they are situations or facts or other candidates), and prove that every sentence with the same truth-value shares the same denotation.

We use Church's Slingshot as an example of what we call value-chain slingshot. There are two versions of what is called Church's Slingshot. The first appears in a review of Rudolf Carnap's *Introduction to Semantics* [Church, 1943] and the second version appears in Church's book *Introduction to Mathematical Logic* [Church, 1956]. Both arguments were well documented in several places, for example in Marco Ruffino's paper *Church's and Gödel's Slingshot arguments* [Ruffino, 2004] and in chapter two of Shramko and Wansing's *Truth and Falsehood* [Shramko and Wansing, 2011]. History tells us that the version of Church's Slingshot presented in [Church, 1943] was enough to convert Carnap into

Church's church given that in Carnap's next book (*Meaning and Necessity* published in 1947) the author adopted the view that the reference of a sentence is its truth-value. But since we only need one example to illustrate our category of value-chain slingshot, hopefully just its most popular version, namely, the second one, will do.

Definition 2.0: (Co-referentiality) We say that two terms $t_1, t_2 \in TERM$ are co-referential, according to a given assignment υ, if the assignment of t_1 is equal to the assignment of t_2, that is, $\upsilon(t_1) = \upsilon(t_2)$.

Definition 2.1: (Synonymy) We say that φ and ψ are synonymous if, for every unary sentential context χ, $\chi(\varphi) \vdash \chi(\psi)$ and $\chi(\psi) \vdash \chi(\varphi)$, where $\chi(\varphi)$ is the result of replacing the occurrences of ψ in $\chi(\psi)$ by φ.

According to Ruffino [Ruffino, 2004], the version of the argument appearing in [Church, 1956] makes use of the following principles:

(S) Synonymous sentences have the same reference.

(R) The reference of a complex expression does not change when one of its constituent expression is replaced by another co-referential term.

Now, given that the sentences 1 and 4 below are true, the informal version of Church's slingshot goes as follows.

(1) Sir Walter Scott is the author of *Waverley*.

(2) Sir Walter Scott is the man who wrote 29 *Waverley* novels altogether.

(3) 29 is the number, such that Sir Walter Scott is the man who wrote that many *Waverley* novels altogether.

(4) The number of counties in Utah is 29.

Sentence (2) is a result of replacing in one of the constituent expressions of (1) by a co-referential expression, namely, by replacing 'the author of *Waverley*' by 'the man who wrote 29 *Waverley* novels altogether'. Thus sentences (1) and (2) above must have the same reference, in view of principle (R). By a similar argument we can conclude that sentences (3) and (4) have the same reference, since sentence (4) is the result of replacing 'the number, such that Sir Walter Scott is the man who wrote that many *Waverley* novels altogether' by 'the number of counties in Utah'. Again, by using principle (R) we reach that (3) and (4) have the same reference. Note that if we assume that 'the number, such that Sir Walter Scott is the man who wrote that many *Waverley* novels altogether' is co-referential with 'the man who wrote 29 *Waverley* novels altogether', then what the argument shows is actually that 'Sir Walter Scott is the number 29'. Which is odd! Now, to justify that sentences (2) and (3) have the same reference, Church supposes that, even though if (2) and (3) are not synonymous they are "at least so nearly so as to ensure its having the same denotation" [Church, 1956, p. 25]. To which Church concludes:

> Now the two sentences "Sir Walter Scott is the author of *Waverley*" and "The number of counties in Utah is twenty-nine", though they have the same denotation according to the preceding line of reasoning, seem actually to have very little in common. The most striking thing that they do have in common is that both are true. Elaboration of examples of this kind leads us quickly to the conclusion, as at least plausible, that all true sentences have the same denotation. And parallel examples may be used in the same way to suggest that all false sentences have the same denotation (...). [Church, 1956, p. 25]

Church has endorsed very much Frege's conclusion, and together with Frege has used the argument to "postulate abstract objects called *truth-values*, one of them being *truth* and the other one *falsehood*. And we declare all true sentences to denote the truth-value truth, and all false sentences to denote the truth-value falsehood." [Church, 1956, p. 25]. Whether we call these two objects The True and The False or Truth and Falsehood is a matter of choice. If one wants, however, to postulate that the reference of a sentence is a situation, and if this one is convinced by Church's argument, then this person must also defend that there is only two situations. One positive situation and one negative situation.

Although this sort of slingshot is very effective for Carnap, others don't feel that way. For instance, Barwise and Perry argued that "[f]rom one perspective the first and the last steps are fine but the middle step is all wrong. From a second perspective the middle step is reasonably good but the first and last steps are completely unfounded." [Barwise and Perry, 1981, p. 396]. That is to say that, from the perspective that the contribution of the definite descriptions in (1) and (2) is just to fix the referent, namely, Walter Scott, as well as the contribution of (3) and (4), then the step from (2) to (3) is "all wrong" as they claim.

That having been said, since the value-chain slingshot can generate some concerns, we prefer to use a kind of slingshot more connected to Gödel's view that employs, to talk about the denotation of a sentence, a extended language with a connective that internalizes in the object language a way of talking about the denotations of sentences. Thus, in this new language the slingshot takes the form of a derivation according to which every sentence with the same truth-value denotes the same abstract *object* (whether it is a situation, a fact or another entity).

2.3 Two Ways of Reconstructing Gödel's Slingshot

Since Gödel only claims to have a proof, but does not actually construct carefully an argument for the Slingshot, several reconstructions of Gödel's Slingshot were made. In this section we shall compare two of these reconstructions, one by Stephen Neale in [Neale, 1995] and another by Yaroslav Shramko and Heinrich Wansing in [Shramko and Wansing, 2011], and show that they differ in a crucial point, namely the use of the compositionality. On the one hand, Neale's reconstruction has a problematic passage where he claims to use the compositionality principle but is not clear how he is using the principle or what is his definition of compositionality. This shortcoming was pointed out by Graham Oppy in [Oppy, 1997]. On the other hand, Shramko and Wansing's reconstruction does not seem to face the same problem (even though they say their reconstruction is based on Neale's) since they do not invoke compositionality in their proof but rather definite description rules and properties of equality.

In the paper [Gödel, 1944], Gödel covers a few concepts of Russell's earlier work on mathematical logic such as Russell's interpretation of definite descriptions. Gödel's point is that Russell's theory of definite descriptions could avoid the slingshot conclusion. Although the argument appears as an argument to show that there is a deep relation between theories of facts and the semantical content of a definite description, surprisingly, the argument called little attention at the time. According to Neale, Gödel's Slingshot is philosophically important for imposing a challenge (if we may call it a challenge) to theories of facts, namely, in order to live in a world where different true sentences may indicate different facts, then one must drop either the assumption of compositionality or the assumption that definite descriptions are singular terms[4]. Yet, in one of Neale's formalization of Gödel's Slingshot, it is not clear how he is using compositionality in the argument. Neale summarizes Gödel's claim:

> Gödel's claim boils down to this: if an expression of the form $\ulcorner(\iota x)\phi\urcorner$ were viewed as a genuine singular term standing for the unique object satisfying ϕ, then by invoking minimal logical principles in connection with formulae containing descriptions of the form $[(\iota x)(x = a \wedge Fx)]$ it would be possible to demonstrate that all true sentences must stand for the same fact. [Neale, 1995, p. 777]

According to Neale, Gödel hints to a proof that can be done by using the further assumptions that: (G1) sentences of the form $F(a)$ and $a = \iota x(x = a \wedge F(x))$ stand for the same fact; (G2) any sentence can be rewritten into a predicate form; and (G3) compositionality, i.e, the semantical principle that the value of a sentence is uniquely determined by the value of it constituents[5].

The idea is to prove that, considering three distinct true sentences, they all denote the same fact. The proof starts by supposing that the following sentences, (I), (II) and (III), are true. Each of them denotes a specific fact. Stephen Neale's reconstruction goes as follows:

(I) $F(a)$ denotes $f1$

(II) $a \neq b$ denotes $f2$

(III) $G(b)$ denotes $f3$

Using (G1) we see that the following sentence denotes the same fact as (I):

(IV) $a = \iota x(x = a \wedge F(x))$ denotes $f1$

Using a similar argument, by (G1) we have that (V) denotes the same fact as (II):

(V) $a = \iota x(x = a \wedge x \neq b)$ denotes $f2$.

The problem arises when Neale tries to justify that (IV) and (V) denote the same fact by using compositionality. According to Neale, since (IV) and (V) are sentences about the same object, a, then, by using compositionality we have that

4 See [Neale, 1995, p.776] .

5 Here we are using G1, G2 and G3 as labels for the three basic assumptions used in Gödel's claim as Neale did (see [Neale, 1995, p. 777-778]) in order to maintain continuity with Neale's paper.

(VI) $f1 \doteq f2$.

In Neale's words:

> If a definite description $\ulcorner(\iota x)\phi\urcorner$ stands for the unique thing satisfying ϕ, then the descriptions in (IV) and (V) both stand for the same thing, viz. a. So, by (G3), sentences (IV) and (V) stand for the same fact [...] [Neale, 1995, p. 778].

Then Neale goes on with the argument and states that from (III), by (G1), we have that (VII) also denotes $f3$.

(VII) $b = \iota x(x = b \wedge G(x))$ denotes $f3$

By (G1), we also have that (VIII) denotes the same fact as (II):

(VIII) $b = \iota x(x = b \wedge x \neq a)$ denotes $f2$

By a similar argument as before, since (VII) and (VIII) are sentences about the same object, by using compositionality one would conclude that:

(IX) $f3 \doteq f2$, leading to the conclusion that $f1 \doteq f2 \doteq f3$. However, the use of compositionality in the justification of the steps (VI) and (IX) seems odd. This problem was noticed by Graham Oppy in his paper in response to Neale's, where Oppy remarks that:

> Consider, for example, those friends of facts who start from metaphysics, and who insist that facts are simply truth-makers. Such theorists might well say that 'Fa' and '$a = (\iota x)(x = a \wedge Fx)$' are made true by the same fact – namely, the instantiation of the property F by the object a. However, such theorists will also insist that '$a = (\iota x)(x = a \wedge x \neq b)$' is made true by a different fact – namely, the failure of the instantiation of the relation of identity between a and b. [Oppy, 1997, p. 128]

That is, if we take a and b as standing to regular proper names as *George Harrison* and *Paul McCartney*, for instance, and the predicate $F(x)$ standing for 'x is a musician', the "friends of facts" (as Oppy states) would agree that the sentence 'George Harrison is a musician' is made true by the same fact as the sentence 'George Harrison is the x such that x is George Harrison and is a musician'. However, the sentence 'George Harrison is the x such that x is George Harrison and is different from Paul MacCartney' is made true by a different fact.

Shramko and Wansing's reconstruction of Gödel's Slingshot does not face the same problem, although they claim to be essentially the same as Neale's reconstruction. Their assumptions are similar to Neale's, but they do not try to use forcefully compositionality as a rule of inference in the argument. The only inference rules that are considered are the description operator rules and the identity rules. The reconstruction starts by extending our initial language to include the binary connective $\equiv$ where $\varphi \equiv \psi \in FORM$ stands for "φ and ψ denote the same situation". First of all, let us define what is a theory and what is a conservative extension.

let $\mathcal{L}_{\equiv}$ be $\mathcal{L}_0 \bigcup \{\equiv\}$, that is, an extension of our previous language with the situation connective, such that, we add the following rules to our set of the situation connective:

$$\frac{\varphi \equiv \chi \qquad \chi \equiv \psi}{\varphi \equiv \psi}\ trans \equiv$$

$$\frac{\varphi \equiv \psi}{\psi \equiv \varphi}\ symm \equiv$$

$$\frac{P(a)}{P(a) \equiv a \doteq \iota x(x \doteq a \wedge P(x))}\ (G1)$$

We decided to maintain (G1) as the label for the above rule because it is essentially the same principle (G1) advocated by Gödel and previously discussed here. Note also that, it is expected that the slingshot have two sides, namely, one side that states that all true sentences have the same denotation and another side that states that all false sentences have the same denotation. Thus we have the following slingshots in our new language:

(i) $F(a), G(b) \vdash F(a) \equiv G(b)$,

(ii) $\neg F(a), \neg G(b) \vdash \neg F(a) \equiv \neg G(b)$,

Let Δ be the premisses of the first slingshot argument, that is, $F(a), G(b)$. Due to the width of the of the trees of slingshots, we choose to discuss the strategy of proving the argument by dividing in different sub-proofs. The first idea is that, by an instantiation of the axiom $\forall x \forall y(x \doteq y \vee x \neq y)$, namely, $a \doteq b \vee a \neq b$ by a proof by cases, from the assumption that $a \doteq b$ we prove $F(a) \equiv G(b)$, as well from the assumption that $a \neq b$ we prove $F(a) \equiv G(b)$, then we conclude $F(a) \equiv G(b)$. First let us prove the derivation with the assumption that $a \neq b$, in this proof we need to prove that $F(a) \equiv a \neq b$ and

$a \neq b \equiv G(b)$, then, by transitivity of the situation connective we get $F(a) \equiv G(b)$. Looking first for the "left side" of the proof, to prove that $F(a) \equiv a \neq b$, we need to prove $a \neq b \equiv a \doteq \iota x(x \doteq a \wedge x \neq b)$ and $a \doteq \iota x(x \doteq a \wedge x \neq b) \equiv F(a)$, which we divided into derivations $\mathcal{D}_0$ and $\mathcal{D}_1$.

$$\dfrac{\dfrac{\begin{matrix}[a \neq b]\\ \mathcal{D}_0\\ a \neq b \equiv a \doteq \iota x(x = a \wedge x \neq b)\end{matrix} \qquad \begin{matrix}[F(a)]\\ \mathcal{D}_1\\ a \doteq \iota x(x = a \wedge x \neq b) \equiv F(a)\end{matrix}}{F(a) \equiv a \neq b}\, trans \equiv \qquad \begin{matrix}\vdots\\ a \neq b \equiv G(b)\end{matrix}}{F(a) \equiv G(b)}\, trans \equiv$$

The proof $\mathcal{D}_0$ is just an instantiation of the $(G1)$ rule.

$$\dfrac{a \neq b}{a \neq b \equiv a \doteq \iota x(x = a \wedge x \neq b)}\,(G1)$$

Now, the proof $\mathcal{D}_1$ is a more complicated proof that uses our assumption that $a \neq b$ and the premiss $F(a)$ together with rules for the iota operator and identity rules.

$$\mathcal{D}_1$$

$$\dfrac{\dfrac{\dfrac{\dfrac{a \neq b}{a \doteq \iota x(x \doteq a \wedge x \neq b)}\,\iota - intro}{\iota x(x \doteq a \wedge x \neq b) \doteq a}\,\doteq -sym \qquad \dfrac{F(a)}{a \doteq \iota x(x \doteq a \wedge F(x))}\,\iota - intro}{\iota x(x \doteq a \wedge x \neq b) \doteq \iota x(x \doteq a \wedge F(x))}\,\doteq trans \qquad \dfrac{F(a)}{F(a) \equiv a \doteq \iota x(x \doteq a \wedge F(x))}\,(G1)}{a \doteq \iota x(x \doteq a \wedge x \neq b) \equiv F(a)}\, sub$$

In a similar way, now looking at the "right side" of the proof, to prove that $a \neq b \equiv G(b)$ we need to prove that $a \neq b \equiv b \doteq \iota x(x = b \wedge a \neq x)$ and $b \doteq \iota x(x = b \wedge a \neq x) \equiv G(b)$, which we divided into the derivations $\mathcal{D}_2$ and $\mathcal{D}_3$.

$$\dfrac{\begin{matrix}\vdots\\ F(a) \equiv a \neq b\end{matrix} \qquad \dfrac{\begin{matrix}[a \neq b]\\ \mathcal{D}_2\\ a \neq b \equiv b \doteq \iota x(x = b \wedge a \neq x)\end{matrix} \qquad \begin{matrix}[G(b)]\\ \mathcal{D}_3\\ b \doteq \iota x(x = b \wedge a \neq x) \equiv G(b)\end{matrix}}{a \neq b \equiv G(b)}\, trans \equiv}{F(a) \equiv G(b)}\, trans \equiv$$

Like in $\mathcal{D}_0$, $\mathcal{D}_2$ is just an instantiation of the $(G1)$ rule.

$$\dfrac{a \neq b}{a \neq b \equiv b \doteq \iota x(x = b \wedge a \neq x)}\,(G1)$$

The proof $\mathcal{D}_3$ uses the premisses $a \neq b$ and $G(b)$ to prove that $b \doteq \iota x(x \doteq b \wedge a \neq x) \equiv G(b)$.

$$\mathcal{D}_3$$

$$\dfrac{\dfrac{\dfrac{\dfrac{a \neq b}{b \doteq \iota x(x \doteq b \wedge a \neq x)}\,\iota-intro}{\iota x(x \doteq b \wedge a \neq x) \doteq b}\,\doteq-sym \qquad \dfrac{G(b)}{b \doteq \iota x(x \doteq b \wedge G(x))}\,\iota-intro}{\iota x(x \doteq b \wedge a \neq x) \doteq \iota x(x \doteq b \wedge G(x))}\,\doteq trans \qquad \dfrac{G(b)}{G(b) \equiv b \doteq \iota x(x \doteq b \wedge G(x)}\,(G1)}{b \doteq \iota x(x \doteq b \wedge a \neq x) \equiv G(b)}\,sub$$

Of course, this side already strikes as odd, given that states that, for two distinct objects, for example take a standing for $Nick$ and b for *Buddha*, the situation described by the assertion 'Nick is a dog' is the same situation described by the assertion 'Buddha is a god'. For the other part of this proof by cases, we take a look at the case with the assumption that $a \doteq b$ and from this we can prove that $F(a) \equiv G(b)$.

$$\dfrac{\dfrac{\begin{matrix}[a \doteq b]\\ \mathcal{D}_4\\ a \doteq b \equiv a \doteq \iota x(x \doteq a \wedge x \doteq b)\end{matrix} \qquad \begin{matrix}[F(a)]\\ \mathcal{D}_5\\ a \doteq \iota x(x \doteq a \wedge x \doteq b) \equiv F(a)\end{matrix}}{F(a) \equiv a \doteq b}\,trans \equiv \qquad \begin{matrix}\vdots\\ a \doteq b \equiv G(b)\end{matrix}}{F(a) \equiv G(b)}\,trans \equiv$$

The proof $\mathcal{D}_4$ is just an use of $(G1)$.

$$\dfrac{a \doteq b}{a \doteq b \equiv a \doteq \iota x(x \doteq a \wedge x \doteq b)}\,(G1)$$

In $\mathcal{D}_5$ we prove from premisses $a \doteq b$ and $F(a)$, using the latter twice, that $a \doteq \iota x(x \doteq a \wedge x \doteq b) \equiv F(a)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{a \doteq b}{a \doteq \iota x(x \doteq a \wedge x \doteq b)}\,\iota-intro}{\iota x(x \doteq a \wedge x \doteq b) \doteq a}\,\doteq-sym \qquad \dfrac{F(a)}{a \doteq \iota x(x \doteq a \wedge F(x))}\,\iota-intro}{\iota x(x \doteq a \wedge x \doteq b) \doteq \iota x(x \doteq a \wedge F(x))}\,\doteq trans \qquad \dfrac{F(a)}{F(a) \equiv a \doteq \iota x(x \doteq a \wedge F(x)}\,(G1)}{a \doteq \iota x(x \doteq a \wedge x \doteq b) \equiv F(a)}\,sub$$

$$\dfrac{\begin{matrix}\vdots\\ F(a) \equiv a \doteq b\end{matrix} \qquad \dfrac{\begin{matrix}[a \doteq b]\\ \mathcal{D}_6\\ a \doteq b \equiv b \doteq \iota x(x \doteq b \wedge a \doteq x)\end{matrix} \qquad \begin{matrix}[G(b)]\\ \mathcal{D}_7\\ b \doteq \iota x(x \doteq b \wedge a \doteq x) \equiv G(b)\end{matrix}}{a \doteq b \equiv G(b)}\,trans \equiv}{F(a) \equiv G(b)}\,trans \equiv$$

$\mathcal{D}_6$

$$\dfrac{a \doteq b}{a \doteq b \equiv b \doteq \iota x(x = b \wedge a \doteq x)}\,(G1)$$

$\mathcal{D}_7$

$$\dfrac{\dfrac{\dfrac{\dfrac{a \doteq b}{b \doteq \iota x(x \doteq b \wedge a \doteq x)}\;\iota-intro}{\iota x(x \doteq b \wedge a \doteq x) \doteq b}\;\doteq -sym \qquad \dfrac{G(b)}{b \doteq \iota x(x \doteq b \wedge G(x))}\;\iota-intro}{\iota x(x \doteq b \wedge a \doteq x) \doteq \iota x(x \doteq b \wedge G(x))}\;\doteq trans \qquad \dfrac{G(b)}{G(b) \equiv b \doteq \iota x(x \doteq b \wedge G(x)}\;(G1)}{b \doteq \iota x(x \doteq b \wedge a \doteq x) \equiv G(b)}\;sub$$

Note that from the above derivations we have proved the following:

$$\dfrac{a \doteq b \vee a \neq b \qquad \begin{matrix}[a \doteq b]\\ \vdots \\ F(a) \equiv G(b)\end{matrix} \qquad \begin{matrix}[a \neq b]\\ \vdots \\ F(a) \equiv G(b)\end{matrix}}{F(a) \equiv G(b)}\;\vee E$$

The reader might identify slingshot (i) as regarding truth about "positive facts", in this sense, slingshots on the following kind, as in (ii), would be regarded as slingshots about "negative facts".

(ii) $\neg F(a), \neg G(b) \vdash \neg F(a) \equiv \neg G(b)$

This slingshot follows a similar strategy as the previous one where from the assumption that $a \doteq b$ we prove that $\neg F(a) \equiv \neg G(b)$ and from the assumption that $a \neq b$ we also prove $\neg F(a) \equiv \neg G(b)$. Again, let Δ be the set of premisses of the slingshot, namely, $\neg F(a)$ and $\neg G(b)$. Let us begin by the proof with the assumption that $a \neq b$.

$$\dfrac{\dfrac{\begin{matrix}[a \neq b]\\ \mathcal{D}_0 \\ a \neq b \equiv a \doteq \iota x(x = a \wedge x \neq b)\end{matrix} \qquad \begin{matrix}[F(a)]\\ \mathcal{D}_1 \\ a \doteq \iota x(x = a \wedge x \neq b) \equiv \neg F(a)\end{matrix}}{\neg F(a) \equiv a \neq b}\;trans \equiv \qquad \begin{matrix}\vdots \\ a \neq b \equiv \neg G(b)\end{matrix}}{\neg F(a) \equiv \neg G(b)}\;trans \equiv$$

The derivation $\mathcal{D}_0$ is just an instantiation of the $(G1)$ rule.

$$\dfrac{a \neq b}{a \neq b \equiv a \doteq \iota x(x \doteq a \wedge x \neq b)}\;(G1)$$

In the derivation $\mathcal{D}_1$ we use our assumption $a \neq b$ and our premiss $F(a)$ together with rules for the iota operator and identity rules.

$$\dfrac{\dfrac{\dfrac{\dfrac{a \neq b}{a \doteq \iota x(x \doteq a \wedge x \neq b)}\;\iota-intro}{\iota x(x \doteq a \wedge x \neq b) \doteq a}\;\doteq -sym \qquad \dfrac{\neg F(a)}{a \doteq \iota x(x \doteq a \wedge \neg F(x))}\;\iota-intro}{\iota x(x \doteq a \wedge x \neq b) \doteq \iota x(x \doteq a \wedge \neg F(x))}\;\doteq trans \qquad \dfrac{\neg F(a)}{\neg F(a) \equiv a \doteq \iota x(x \doteq a \wedge \neg F(x))}\;(G1)}{a \doteq \iota x(x \doteq a \wedge x \neq b) \equiv \neg F(a)}\;sub$$

Now we have to prove that $a \neq b \equiv \neg G(b)$, and to prove that we need to prove that $a \neq b \equiv b \doteq \iota x(x \doteq b \wedge a \neq x)$ and $b \doteq \iota x(x \doteq b \wedge a \neq x) \equiv \neg G(b)$, which we divided into the derivations $\mathcal{D}_2$ and $\mathcal{D}_3$.

$$\dfrac{\begin{matrix}\vdots \\ \neg F(a) \equiv a \neq b\end{matrix} \qquad \dfrac{\begin{matrix}[a \neq b] \\ \mathcal{D}_2 \\ a \neq b \equiv b \doteq \iota x(x \doteq b \wedge a \neq x)\end{matrix} \qquad \begin{matrix}[G(b)] \\ \mathcal{D}_3 \\ b \doteq \iota x(x \doteq b \wedge a \neq x) \equiv \neg G(b)\end{matrix}}{a \neq b \equiv \neg G(b)}\, trans \equiv}{\neg F(a) \equiv \neg G(b)}\, trans \equiv$$

Like in $\mathcal{D}_0$, the derivation $\mathcal{D}_2$ is just an instantiation of the $(G1)$ rule.

$$\dfrac{a \neq b}{a \neq b \equiv b \doteq \iota x(x = b \wedge a \neq x)}\,(G1)$$

The proof $\mathcal{D}_3$ uses our assumption $a \neq b$ and the premiss $\neg G(b)$ to prove that $b \doteq \iota x(x \doteq b \wedge a \neq x) \equiv G(b)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{a \neq b}{b \doteq \iota x(x \doteq b \wedge a \neq x)}\,\iota - intro}{\iota x(x \doteq b \wedge a \neq x) \doteq b}\,\doteq -sym \qquad \dfrac{\neg G(b)}{b \doteq \iota x(x \doteq b \wedge \neg G(x))}\,\iota - intro}{\iota x(x \doteq b \wedge a \neq x) \doteq \iota x(x \doteq b \wedge \neg G(x))}\,\doteq trans \qquad \dfrac{\neg G(b)}{\neg G(b) \equiv b \doteq \iota x(x \doteq b \wedge \neg G(x))}\,(G1)}{b \doteq \iota x(x \doteq b \wedge a \neq x) \equiv \neg G(b)}\, sub$$

This was the first case of our prove by cases, now we prove from the assumption $a \doteq b$ and the slingshot premisses $\neg F(a)$ and $\neg G(b)$, $\neg F(a) \equiv \neg G(b)$.

$$\dfrac{\dfrac{\begin{matrix}[a \doteq b] \\ \mathcal{D}_4 \\ a \doteq b \equiv a \doteq \iota x(x \doteq a \wedge x \doteq b)\end{matrix} \qquad \begin{matrix}[\neg F(a)] \\ \mathcal{D}_5 \\ a \doteq \iota x(x \doteq a \wedge x \doteq b) \equiv \neg F(a)\end{matrix}}{\neg F(a) \equiv a \doteq b}\, trans \equiv \qquad \begin{matrix}\vdots \\ a \doteq b \equiv \neg G(b)\end{matrix}}{\neg F(a) \equiv \neg G(b)}\, trans \equiv$$

The derivation $\mathcal{D}_4$ is an use of the (G1) rule.

$$\dfrac{a \doteq b}{a \doteq b \equiv a \doteq \iota x(x \doteq a \wedge x \doteq b)}\,(G1)$$

$$\mathcal{D}_5$$

$$\dfrac{\dfrac{\dfrac{\dfrac{a \doteq b}{a \doteq \iota x(x \doteq a \wedge x \doteq b)}\,\iota - intro}{\iota x(x \doteq a \wedge x \doteq b) \doteq a}\,\doteq -sym \qquad \dfrac{\neg F(a)}{a \doteq \iota x(x \doteq a \wedge \neg F(x))}\,\iota - intro}{\iota x(x \doteq a \wedge x \doteq b) \doteq \iota x(x \doteq a \wedge \neg F(x))}\,\doteq trans \qquad \dfrac{\neg F(a)}{\neg F(a) \equiv a \doteq \iota x(x \doteq a \wedge \neg F(x)}\,(G1)}{a \doteq \iota x(x \doteq a \wedge x \doteq b)) \equiv \neg F(a)}\, sub$$

Now we have proved $\neg F(a) \equiv a \doteq b$ and we need to prove that $a \doteq b \equiv \neg G(b)$.

$$\dfrac{\begin{matrix}\vdots\\ \neg F(a) \equiv a \doteq b\end{matrix} \qquad \dfrac{\begin{matrix}[a \doteq b]\\ \mathcal{D}_6\\ a \doteq b \equiv b \doteq \iota x(x \doteq b \wedge a \doteq x)\end{matrix} \qquad \begin{matrix}[\neg G(b)]\\ \mathcal{D}_7\\ b \doteq \iota x(x \doteq b \wedge a \doteq x) \equiv \neg G(b)\end{matrix}}{a \doteq b \equiv \neg G(b)}\ trans \equiv}{\neg F(a) \equiv \neg G(b)}\ trans \equiv$$

$$\mathcal{D}_6$$

$$\dfrac{a \doteq b}{a \doteq b \equiv b \doteq \iota x(x = b \wedge a \doteq x)}\ (G1)$$

$$\mathcal{D}_7$$

$$\dfrac{\dfrac{\dfrac{\dfrac{a \doteq b}{b \doteq \iota x(x \doteq b \wedge a \doteq x)}\ \iota - intro}{\iota x(x \doteq b \wedge a \doteq x) \doteq b}\ \doteq -sym \qquad \dfrac{\neg G(b)}{b \doteq \iota x(x \doteq b \wedge \neg G(x))}\ \iota - intro}{\iota x(x \doteq b \wedge a \doteq x) \doteq \iota x(x \doteq b \wedge \neg G(x))}\ \doteq trans \qquad \dfrac{\neg G(b)}{\neg G(b) \equiv b \doteq \iota x(x \doteq b \wedge \neg G(x))}\ (G1)}{b \doteq \iota x(x \doteq b \wedge a \doteq x) \equiv \neg G(b)}\ sub$$

By using our previous derivations, what we have proved in the slingshot (ii) is the following.

$$\dfrac{a \doteq b \vee a \neq b \qquad \begin{matrix}[a \doteq b]\\ \vdots\\ \neg F(a) \equiv \neg G(b)\end{matrix} \qquad \begin{matrix}[a \neq b]\\ \vdots\\ \neg F(a) \equiv \neg G(b)\end{matrix}}{\neg F(a) \equiv \neg G(b)}\ \vee E$$

In this section we used an extension to our previous language with a situation connective to formalize the slingshots. In order to give a full account of the slingshot, however, one would need to prove this extension is a conservative extension, that is, is an extension of the previous language and it is not adding any new theorems in the proof system. Also, a good account of what is the semantics for the situation connective is needed. In the work of Wójcicki [Wójcicki, 1986] a semantics for the situation connective is provided. Together with that we should be able to prove soundness and completness theorems for the proof system in order to fully understand the slingshot. We leave, however, this issue to a future work.

2.4 Gödel's Way of Evading The Slingshot

It is the Russellian view about descriptions, namely, that definite descriptions are not proper names but are actually meaningless without context, that Gödel would later use to avoid Frege's argument to the conclusion that all true sentences denote The True and all false ones denote The False.

Here we decided to look for a more formal view and use Norbert Gratzl work [Gratzl, 2015] and [Grabmayer et al., 2011] to see this as a result in a formal language. According to these works, see Gratzl's theorem in [Gratzl, 2015, p. 15], if there is a proof of some formula that contains definite descriptions (as a singular term), then there is a proof of an equivalent formula which has no definite descriptions in it, by using Russell's interpretation. For the purpose of our work, we only need a particular case of this theorem, that is, if a slingshot is one proof made with definite descriptions as singular terms, then there is a slingshot with Russell's interpretation of definite descriptions. According to Gratzl [Gratzl, 2015, p. 2], it is useful to make a distinction between the thesis that definite descriptions are *incomplete symbols* and the thesis that definite descriptions *have no meaning in isolation*, being the first a thesis about the syntax of those expressions and the second about the semantics. In order to produce the slingshot with Russellian definite descriptions we consider the following rule using Russell's interpretation of definite descriptions:

$$\frac{\psi(\iota x\varphi(x))}{\exists x\forall y((\varphi(y) \supset x \doteq y) \wedge \psi(x))}\ \iota - R$$

Note that using this rule of introduction of the definite description operator by means of a Russellian interpretation, we have the following as an instantiation of the previous rule.

$$\frac{\varphi(a)}{\exists x\forall y((\varphi(y) \supset x \doteq y) \wedge a \doteq x)}\ \iota - R$$

We also consider a rule similar to (G1), but now using Russell's interpretation of definite descriptions.

$$\frac{\varphi(a)}{\varphi(a) \equiv \exists x\forall y((\varphi(y) \supset x \doteq y) \wedge a \doteq x)}\ (G1^*)$$

With this mechanism we can prove the slingshot in the following manner without the description operator:

$$\dfrac{\begin{matrix}[a \neq b] & [F(a)] & [F(a)] & [a \neq b] & [G(b)] & [G(b)] \\ \mathcal{D}_0 & \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 & \mathcal{D}_4 & \mathcal{D}_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & F(a) \equiv a \neq b & & & a \neq b \equiv G(b) & \end{matrix}}{F(a) \equiv G(b)}\ trans \equiv$$

By using the above Russellian interpretation we can get the following proofs:

$\mathcal{D}_0$

$$\dfrac{\dfrac{a \neq b}{\exists x \forall y((y \neq b \leftrightarrow x \doteq y)a \doteq x)}\,\iota - R \qquad \dfrac{[\forall y((y \neq b \leftrightarrow u \doteq y) \wedge a \doteq u)]}{((a \neq b \leftrightarrow u \doteq a) \wedge a \doteq u)}\,\forall E}{((a \neq b \leftrightarrow u \doteq a) \wedge a \doteq u)}\,\exists E$$

$\mathcal{D}_1$

$$\dfrac{\dfrac{F(a)}{\exists x \forall y((F(y) \leftrightarrow x \doteq y)a \doteq x)}\,\iota - R \qquad \dfrac{[\forall y((F(y) \leftrightarrow k \doteq y) \wedge a \doteq k)]}{((F(a) \leftrightarrow a \doteq k) \wedge a \doteq k)}\,\forall E}{((F(a) \leftrightarrow a \doteq k) \wedge a \doteq k)}\,\exists E$$

$\mathcal{D}_2$

$$\dfrac{\dfrac{F(a)}{F(a) \equiv \exists x \forall y((F(y) \leftrightarrow x \doteq y) \wedge a \doteq x)}\,(G1^*) \qquad \dfrac{[F(a) \equiv \forall y((F(y) \leftrightarrow k \doteq y) \wedge a \doteq k)]}{F(a) \equiv ((F(a) \leftrightarrow a \doteq k) \wedge a \doteq k)}\,\forall E}{F(a) \equiv ((F(a) \leftrightarrow a \doteq k) \wedge a \doteq k)}\,\exists E$$

From $\mathcal{D}_0$ and $\mathcal{D}_1$ we get $((a \neq b \leftrightarrow u \doteq a) \wedge a \doteq u) \leftrightarrow ((F(a) \leftrightarrow a \doteq k) \wedge a \doteq k)$ and from $\mathcal{D}_2$ we get $F(a) \equiv ((F(a) \leftrightarrow k \doteq a) \wedge a \doteq k)$. By using those proofs we can get the following:

$\mathcal{D}_3$

$$\dfrac{\dfrac{\dfrac{((a \neq b \leftrightarrow u \doteq a) \wedge a \doteq u) \leftrightarrow ((F(a) \leftrightarrow a \doteq k) \wedge a \doteq k) \qquad F(a) \equiv ((F(a) \leftrightarrow k \doteq a) \wedge a \doteq k)}{F(a) \equiv ((a \neq b \leftrightarrow u \doteq a) \wedge a \doteq u)}\,\textit{synonymy}}{F(a) \equiv \forall y((y \neq b \supset u \doteq y) \wedge a \doteq u)}\,\forall I}{F(a) \equiv \exists x \forall y((y \neq b \supset x \doteq y) \wedge a \doteq x)}\,\exists I$$

It seems, in the derivation $\mathcal{D}_4$, we need a rule similar to the definition of synonymy discussed in Digression: On Two Sorts of Slingshots since we are not using the substitutional rule anymore.

$\mathcal{D}_4$

$$\dfrac{a \neq b}{a \neq b \equiv \exists x \forall y((x \neq y \supset y \doteq x) \wedge x \doteq y)}\,(G1^*)$$

Note that by $(G1^*)$ we have that $a \neq b \equiv \exists x \forall y((x \neq y \supset y \doteq x) \wedge x \doteq y)$, and finally we can prove that $F(a) \equiv a \neq b$. This is for the left side of the slingshot.

$\mathcal{D}_5$

$$\dfrac{F(a) \equiv \exists x\forall y((x \neq y \supset y \doteq x) \wedge x \doteq y) \qquad \exists x\forall y((x \neq y \supset y \doteq x) \wedge x \doteq y) \equiv a \neq b}{F(a) \equiv a \neq b} \equiv trans$$

We proceed in a similar manner in our second part of the slingshot.

$\mathcal{D}_6$

$$\dfrac{\dfrac{a \neq b}{\exists x\forall y((y \neq b \leftrightarrow x \doteq y)a \doteq x)}\ \iota - R \qquad \dfrac{[\forall y((y \neq b \leftrightarrow u \doteq y) \wedge a \doteq u)]}{((a \neq b \leftrightarrow u \doteq a) \wedge a \doteq u)}\ \forall E}{((a \neq b \leftrightarrow u \doteq a) \wedge a \doteq u)}\ \exists E$$

$\mathcal{D}_7$

$$\dfrac{\dfrac{G(b)}{\exists x\forall y((G(y) \leftrightarrow x \doteq y) \wedge b \doteq x)}\ \iota - R \qquad \dfrac{[\forall y((G(y) \leftrightarrow k \doteq y) \wedge b \doteq k)]}{((G(b) \leftrightarrow b \doteq k) \wedge b \doteq k)}\ \forall E}{((G(b) \leftrightarrow b \doteq k) \wedge b \doteq k)}\ \exists E$$

$\mathcal{D}_8$

$$\dfrac{\dfrac{G(b)}{G(b) \equiv \exists x\forall y((G(y) \leftrightarrow x \doteq y) \wedge a \doteq x)}\ (G1^{*}) \qquad \dfrac{[G(b) \equiv \forall y((G(y) \leftrightarrow k \doteq y) \wedge a \doteq k)]}{G(b) \equiv ((G(b) \leftrightarrow b \doteq k) \wedge b \doteq k)}\ \forall E}{G(b) \equiv ((G(b) \leftrightarrow b \doteq k) \wedge b \doteq k)}\ \exists E$$

$\mathcal{D}_9$

$$\dfrac{\dfrac{\dfrac{((a \neq b \leftrightarrow u \doteq b) \wedge b \doteq u) \leftrightarrow ((G(b) \leftrightarrow b \doteq k) \wedge b \doteq k) \qquad G(b) \equiv (G(b) \leftrightarrow b \doteq k) \wedge b \doteq k)}{G(b) \equiv ((a \neq b \leftrightarrow u \doteq b) \wedge b \doteq u)}\ synonymy}{G(b) \equiv \forall y((y \neq a \leftrightarrow u \doteq y) \wedge b \doteq u)}\ \forall I}{G(b) \equiv \exists x\forall y((y \neq a \supset x \doteq y) \wedge b \doteq x)}\ \exists I$$

$\mathcal{D}_{10}$

$$\dfrac{a \neq b}{a \neq b \equiv \exists x\forall y((x \neq y \supset y \doteq x) \wedge b \doteq x)}\ (G1^{*})$$

$\mathcal{D}_{11}$

$$\dfrac{a \neq b \equiv \exists x\forall y((x \neq y \supset y \doteq x) \wedge x \doteq y) \qquad \exists x\forall y((x \neq y \supset y \doteq x) \wedge x \doteq y) \equiv G(b)}{a \neq b \equiv G(b)} \equiv trans$$

Note that now that we proved $F(a) \equiv a \neq b$ and $a \neq b \equiv G(b)$ it is easy to conclude, by the transitivity of the situation connective, that $F(a) \equiv G(b)$.

In this chapter we have challenged Gödel's solution to the slingshot by means of using Russell's theory of descriptions. Gödel thought this was a good solution because he thought that by eliminating descriptions of the language one could also evade the slingshot. We presented a reconstruction of the argument for a language with definite description, based on Shramko and Wansing's work, and we proved that the slingshot still follows even without the definite description. In the next chapter we explore another approach to descriptions as terms developed in Plural Logic. We also prove that the slingshot follows even if we change the notion of terms to plural terms.

3 Plural Logics and Slingshot Arguments

3.1 Introduction

In last chapter we have explored Gödel's slingshot and its connection with definite descriptions. In this chapter we explore the idea of extending the treatment of singular terms to plural terms (and from definite descriptions to plural descriptions) and propose our own slingshot argument with regard to plural logics. There is an extensive literature about plural logics or how to deal with pluralities in a formal language and this chapter is based mostly on the work of Alex Oliver and Timothy Smiley in [Oliver and Smiley, 2013a]. The idea of proposing such a puzzling conclusion comes from a passage where, although the authors agree with Frege (and against Russell) that definite descriptions are a kind of proper name, they do not agree with Frege (as neither does Russell) in considering sentences to be proper names as well. Thus, the importance of our slingshot would be to force Oliver and Smiley to put sentences, as well as definite descriptions, "under the same umbrella" of proper names. We argue that a Gödelian solution is not available for them, since they reject completely the idea of Russell's interpretation. But this is not equivalent to claim that there is no context elimination of plural descriptions. We point out that, even if there was a elimination procedure to plural descriptions, that would be not a solution to the slingshot, if we were right in the last chapter regarding the definite descriptions elimination.

That said, the strategy of attack will be as follows. In section "Motivations and Fundamental Concepts of Plural Logic", we look at some examples presented by Oliver and Smiley as motivations to consider plural denotation and we will give our own example by using Euclid's construction of a equilateral triangle.

In the section "Full Plural Logic" we present the system of Full Plural Logic that appears in [Oliver and Smiley, 2013a], but with changing some regarding the notation. We point out some choices they make in their system, like the choice of dealing with non-denoting terms, as one of the choices that are in the root of Oliver and Smiley's rejection of Russell's interpretation of definite descriptions. We choose to maintain an axiomatic system as was presented in [Oliver and Smiley, 2013a, p. 233-244] unlike what we did in the previous chapters where we presented the proof system as in natural deduction. In the section "A Slingshot for Full Plural Logic" we propose our own slingshot using a similar strategy as presented in section "2.3" and discuss whether this is an undesirable for Oliver and Smiley's logic or not.

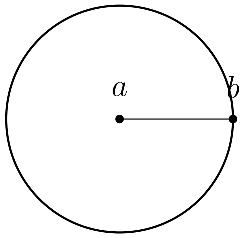
3.2 Motivations and Fundamental Concepts of Plural Logic

Classical Logic can be stretched in many directions. The creation of a non-classical logic can be motivated by some aspect of Natural Language that Classical Logic cannot properly deal with or by a formal problem within its own Language. The motivations for creating a Plural Logic arise from aspects of Natural Language that Classical Logic, or according to Alex Oliver and Timothy Smiley's terminology a "singular approach", fails to translate. Certain sentences of a natural language do not have a translation in Classical Logic that captures their full meaning. For instance, the Geach-Kaplan sentence "Some critics admire only one another" was proved by Boolos in [Boolos, 1984] [1] to have no first-order translation. This chapter will focus on the Plural Logic developed in [Oliver and Smiley, 2013a]. We examine whether if the slingshot argument can also be recovered in the context of a Plural Logic if the concept of terms is changed to a slightly different notion such as plural terms. We are going to use mainly the book *Plural Logic* because we think that proving a slingshot argument for Oliver and Smiley's system of Plural Logic might be an undesirable result for them.

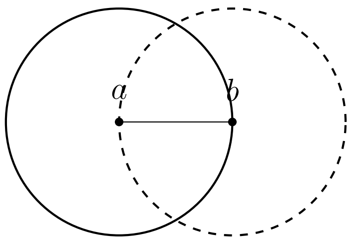
The main thesis of the *Plural Logic* book is that there is a Plural Phenomena in natural language. These Phenomena occur because in natural language there is *plural denotation*, a phenomenon occurring when a term denotes several things at the same time. This is a recurring theme throughout the book. We begin by giving our own example of plural denotation. Consider Euclid's construction of an equilateral triangle. In the first step we fix two points, here we will name them a and b, and by Euclid's Axiom that 'between two points there is always a line', we can draw a line between a and b.

[1] See [Linnebo, 2014] for an overview how to prove it.

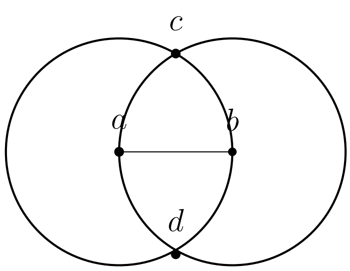
Now, in step two we are commanded to rotate the line *ab* to construct the circle with center *a* and radius *ab*.



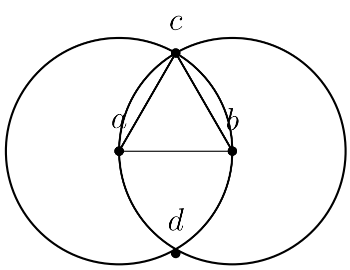
Again, use the line *ab* to construct a circle in the other direction with center *b* and radius *ab*.



Here is where the things get tricky! In step four we are told to fix a point in the intersection of the circles. Note, however, that there are two intersections in the circle, namely *c* and *d* [2].

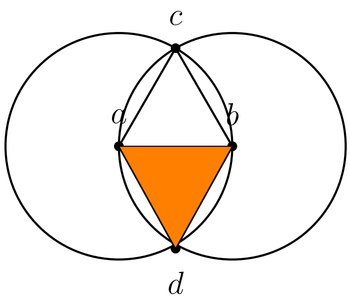


Again, by using the axiom that 'between two points there is always a line' we can draw a line between *a* and *c* (and *b* and *c*)



[2] It remains to be prove that there are intersections between the two circles at all, but since we are in a two dimensional Flat World this should be provable.

Note that there are two triangles constructed by following the same rules, that is, the triangle *abc* and the triangle *abd*.



At the end, the triangles constructed by following the rules are actually two. More than that, if two people are told to construct a triangle by following the previous rule, they might construct each one a different triangle. They might talk with each other about their triangles and never find out that they have different triangles in their hands.

Oliver and Smiley took a different path and started the conversation about plural phenomena by using a different example. In the first chapter they invite us to consider the following example:

(1) Whitehead and Russell were logicians,

(2) Whitehead and Russell wrote *Principia Mathematica*. [3]

There are many different ways of answering the question "What are the roles of the list 'Whitehead and Russell' in (1) and (2)?". One possible answer is that (1) stands as a shorthand for "Whitehead was a logician and Russell was a logician". The authors claim that this reduction cannot be done for the sentence (2), while, according to Oliver and Smiley, some philosophers as Michael Dummett has defended that the sentence (2) can in fact be reduced to 'Russell contributed to writing *Principia Mathematica* and Whitehead contributed to writing *Principia Mathematica* and no one else did'. The authors devote a great time arguing against views such as of Dummett's who defended that the creation of a Plural Logic is unnecessary, i.e, that it is possible to deal with any plural sentence within Classical Logic[4]. Another possible answer, according to Oliver and Smiley, is Russell's proposal. They claim that Russell proposed in *Principles of Mathematics* to analyse the list 'Whitehead and Russell' in (2) as standing for a class of objects.

[3] This is illustrated in [Oliver and Smiley, 2013a, p. 1].

[4] Here we are not going to go in many details about the discussion but most of Oliver and Smiley's arguments are in [Oliver and Smiley, 2013a, p. 9-10]

Oliver and Smiley think that both these answers are wrong. They defend that the list ‘Whitehead and Russell’ is a case of plural denotation, in which a term may denote several things at once. So in both (1) and (2) the list denotes the two men Whitehead and Russell. Once we accept the existence of plural denotation, we can classify and distinguish terms in our language. In distinction from a plural term, which may denote several things at once, a singular term cannot: it denotes a single thing. Thus, is also expected to extend the notion of denotation so it could deal with terms as plural definite descriptions (such as ‘The men who wrote *Plural Logic*’) and plural proper names such as ‘The Grimms’ (standing for Jacob and Wilhelm).

When it comes to predicates, we face a distributive-collective distinction. A predicate F is collective if it is analytic that F is true of some things iff it is true of all of these things jointly. For example: Oliver and Smiley wrote *Plural Logic*; it is not true that each of them wrote the book separately, they did it jointly. A predicate is distributive if it is true for each of the things that applies separately. For example: Nick and Rex are dogs; it is true about each of them that they are dogs.

About the question “why study Plural Logic?”, the authors answer by saying that the reason to study Plural Logic is because there is a plural phenomena appear both in natural languages and in formal languages. One of the examples in formal languages that the authors use is “the square root of 4”. The definite description “the square root of 4” should denote only one object, but oddly enough (or not) there are two numbers that squared give you 4. So what is the expression $\sqrt{4} = \pm 2$ informing? That there is such a number ‘plus-or-minus two’? Oliver and Smiley defend that this is an example of a plural description. In the beginning of this chapter we gave our own example of a plural description appearing in a formal language with “the triangles constructed by following the previous rules”.

The authors compare their strategy for studying a logic of plurals with the problem of empty singular terms. Take for instance the sentence, “The president of France in 1653 is bearded”. Since the term ‘The president of France in 1653’ is empty, i.e, it has no denotation, how can we decide whether the sentence is true or false? On the one hand there is Frege’s solution, which is to postulate a null class that all empty terms denote. On the other hand, there is Russell’s solution which is to create a way of analysing these sentences and deciding that they are false, which lead him to create his Theory of

Descriptions. But there is yet another way (hand), which is the approach of Free Logics that allows us to directly represent empty terms and deal with them by changing the underlying formal logic. Similarly, while some logicians want to reduce plural terms to lists of singular terms, Oliver and Smiley chose to change the formal logic to a Plural Logic that can deal with the Plural Phenomena.

As one of the main representatives of what Oliver and Smiley classify as the conservative logician, Frege "would have nothing to do with plural denotation" [Oliver and Smiley, 2013a, p. 16]. Indeed, Frege reduces the plural subject to a singular approach and treats differently the distributive and the collective predications. In the case of distributive predication, Frege reduces the sentence with plural predication to a sentence in which its constituents are singular terms. For example, he would reduce a sentence such as 'Oliver and Smiley are philosophers' to 'Oliver is a philosopher' and 'Smiley is a philosopher'. In the case of collective predication Frege treats as a singular term that denotes an aggregate of objects. In an example like 'Oliver and Smiley wrote Plural Logic' Frege would defend that 'Oliver and Smiley' denotes a compound object from which something is being predicated. Similarly Frege would defend that for plural descriptions as 'The Romans' in 'The Romans conquered Gaul', the phrase 'The Romans' denotes the whole of the Roman people. In any case, plural denotation is not necessary and we can live in a singular world.

Russell's *The Principle of Mathematics* has classes as a fundamental idea to his theory. According to the authors, in Russell's book plural talk and class talk coincide [5], then, in fact Russell's fundamental doctrine that the subject of a proposition may be plural puts him as a pioneer on this kind of pluralism. Russell treats plural descriptions as 'The children of Israel' as standing for many things, i.e, for a class of terms, therefore he deals with plural denotation applying to plural descriptions and not just to simple common nouns. Unfortunately Russell abandoned his doctrine because he was convinced that plurality was inadmissible as it added a objectionable ambiguity to the language.

Oliver and Smiley give a good account of the most common critiques against the development of Plural Logics and give their own responses to those criticisms in the first four chapter of their book. Regarding fundamental concepts of Plural Logic, the authors divided the concepts between Plural terms, Plural Denotation, and Lists (as terms). In

[5] See [Oliver and Smiley, 2013a, p.25].

the formal languages that they present we have the following varieties of terms, which includes *Singular Terms*, *Plural Terms* and *Empty terms*. Each of these three classes is distinguished by the number of things the corresponding terms can denote[6]: A Singular term denotes only one thing and is basically the construction of the set of terms that we did in the first chapter of this dissertation, a Plural term can denote several things[7] and a Empty term denotes nothing.

Concerning singular terms, the authors share a view with Frege about proper names: "In one respect Frege is our friend here: our notion of singular term is very much in the spirit of his liberal notion of 'proper name' (but, unlike him, we do not include sentences under the same umbrella)." [Oliver and Smiley, 2013a, p. 74]

Recall that Frege's liberal notion of proper names states that the reference of a proper name is the unique object for which it stands. A definite description as "The author of *Moby Dick*" is a proper name that refers to a definite object. The name "Alex Oliver" is a proper name that refers to an object. In a similar way, according to Frege, a sentence is a proper name that refers to one of the objects The True or The False. As we saw in previous chapters, in the literature the argument that proves Frege's idea that the reference of sentences are their truth-values was labelled slingshot. In this case a slingshot in a Plural Logic would appear as an argument to force Oliver and Smiley to put sentences "under the same umbrella" as proper names.

Concerning Empty terms it will help to, aside from *Plural Logic*, take a look at [Oliver and Smiley, 2013b] where Oliver and Smiley discuss the distinguished empty term Zilch. The problem arises when traditionally 'nothing' (or 'nobody') is exclusively treated as a quantifier as in 'there is no such-and-such'. Oliver and Smiley diverge from this point of view in the sense that they defend that treating 'nothing' as a quantifier is not the only way, as it is possible to treat 'nothing' as a term. Thus they introduce the empty term Zilch (denoted by O) that nothing refers to: "O is empty – it does not denote anything, whether existent or subsistent, real or imaginary, concrete or abstract, possible or impossible. It denotes Zilch." [Oliver and Smiley, 2013b, p. 602]

This proposal faced a challenge in [Casati and Fujikawa, 2015], where Casati and Fujikawa treat Zilch as a singular term that denotes null objects. Nevertheless, although

[6] See [Oliver and Smiley, 2013a, p.74].
[7] Note that this does not implies that a plural term necessarily denotes several things.

this is a fruitful debate, for our purpose here we will focus on the difference between Plural identity (written as $=$) and Weak identity (written as $\doteqdot$). This difference concerns the empty terms where $a = b$ is false when a and b are empty terms and $a \doteqdot b$ is true when a and b are empty terms[8].

The theory of Descriptions presented in the book Plural Logic covers Singular and Plural Descriptions. In Full Plural Logic we will have four term-forming operators: Singular description and singular exhaustive operator, plural descriptions and a plural exhaustive operator. Is important to point out that exhaustive descriptions suit distributive predication while plural descriptions suits collective predication[9].

3.3 Full Plural Logic

When constructing the systems Oliver and Smiley make it clear that they do not want to propose Plural Logic as just another formalism, but rather as a project that reviews the foundations of Logic itself. For instance even the meta-theory should be Plural and should avoid any talk about sets since Set theory is a way of singular talk. Here we are going to take a different path from the one the authors propose. In their book, first they introduce the Singular Logic, then the system of Mid-Plural Logic is an extension of the first one, and finally the system of Full Plural Logic which is the main system in the book. Here we are going to focus only ins the system of Full Plural Logic.

Syntax

The Language $\mathcal{L}_{PL}$ of Full Plural Logic consists in:

(1) A set VAR of singular variables, $x_0, x_1, ...$

(2) A set $PVAR$ of plural variables, $\mathbf{x}_0, \mathbf{x}_1, ...$

(3) A set CON of constants, $c_0, c_1, ...$

(4) Connectives $\neg, \supset, \leftrightarrow, \wedge, \vee$ and brackets (,)

(5) Quantifiers $\forall$ and $\exists$

(6) The description operator ι and the exhaustive description operator :

(7) A two place distinguished predicate called Inclusion $\preccurlyeq$

(8) A set $PRED$ of Predicates $P^n_1, ..., P^n_m$ of arity n, where $n \geq 0$

8 See [Oliver and Smiley, 2013b, p.602].
9 See [Oliver and Smiley, 2013a, p.122].

(9) A set *FUN* of Functions $f_1^n, ..., f_m^n$ of arity n, where $n \geq 0$

(10) The set *TERM* of terms is defined as follows:

$$t ::= \quad x \;\Big|\; \mathbf{x} \;\Big|\; c \;\Big|\; f(t_1, ..., t_n) \;\Big|\; \iota x\varphi \;\Big|\; \iota\mathbf{x}\varphi \;\Big|\; x : \varphi \;\Big|\; \mathbf{x} : \varphi$$

where $x \in VAR$, $\mathbf{x} \in PVAR$, $c \in CON$, $f \in FUN$ and φ is a formula.

(11) The set *FORM* of formulas is defined as follows:

$$\varphi ::= \quad P(t_1, ..., t_n) \;\Big|\; \neg\varphi \;\Big|\; \varphi \wedge \varphi \;\Big|\; \varphi \vee \varphi \;\Big|\; \varphi \supset \varphi \;\Big|\; \forall x\varphi \;\Big|\; \exists x\varphi \;\Big|\; \forall\mathbf{x}\varphi \;\Big|\; \exists\mathbf{x}\varphi$$

where $t_1, ..., t_n \in TERM$, $x \in VAR$, $\mathbf{x} \in PVAR$ and $P \in PRED$.

(12) Defined Expressions:

Plural identity:	$a = b =_{df} a \preccurlyeq b \wedge b \preccurlyeq a$
Proper inclusion:	$a \prec b =_{df} a \preccurlyeq b \wedge \neg(b \preccurlyeq a)$
Inequality:	$a \neq b =_{df} \neg(a = b)$
Existence:	$E!a =_{df} \exists x\; x \preccurlyeq a$ where $x \in VAR$ and x is not free in a
Singular Existence:	$S!a =_{df} \exists x\; x = a$
Singularity:	$Sa =_{df} \forall x(x \preccurlyeq a \supset x = a)$
Strict plurality:	$E!!a =_{df} \exists x\; x \prec a$
Weak plural Identity:	$a \doteqdot b =_{df} a = b \vee (\neg E!a \wedge \neg E!b)$
Zilch:	$O =_{df} x : x \neq x$

To avoid confusion between notation, we are going to use the symbol '$\doteqdot$' to denote the weak identity. This was necessary since in the book they use the symbol '$\equiv$' to denote the weak identity and it is the same symbol that we use for the situation connective throughout the dissertation.

Axioms

Now we are going to introduce the axiomatization of the Full Plural Logic system as presented in [Oliver and Smiley, 2013a, p. 241]. Later we are going to use this axiomatization to prove a Lemma and a Fact in Full Plural Logic that will help us to prove the slingshot for Full Plural Logic.

1. A where A is tautologous

2. $\forall x(A \supset B) \supset (\forall x A \supset \forall x B)$

3. $A \supset \forall x A$ where x is not free in A

4. $\forall x A(x) \rightarrow (S!a \supset A(a))$ where $A(a)$ has a free a wherever $A(x)$ has a free x

5. $\forall x\ x = x$

6. $a = b \supset (A(a) \leftrightarrow A(b))$ where $A(b)$ has free b at any place where $A(a)$ has a free a

7. $(\neg E!a \wedge \neg E!b) \supset (A(a) \leftrightarrow A(b))$ where $A(b)$ has free b at any place where $A(a)$ has a free a

8. $a = b \supset E!a \wedge E!b$

9. $\forall y(y = \iota x A(x) \leftrightarrow \forall x(A(x) \leftrightarrow x = y))$ where y does not occur in $\iota x A$

10. $\forall y(y \preccurlyeq x : A(x) \leftrightarrow A(y))$ where $A(y)$ has free y wherever $A(x)$ has free x

11. $a \preccurlyeq b \leftrightarrow E!a \wedge \forall x(x \preccurlyeq a \supset x \preccurlyeq b)$ where x is not free in a or b

12. Sx

13. $\forall \mathbf{x}(A \supset B) \supset (\forall \mathbf{x} A \supset \forall \mathbf{x} B)$

14. $A \supset \forall \mathbf{x} A$ where $\mathbf{x}$ is not free in A

15. $\forall \mathbf{x} A(\mathbf{x}) \supset (\exists \mathbf{y}\ \mathbf{y} = a \supset A(a))$ where $A(a)$ has a free a wherever $A(\mathbf{x})$ has a free $\mathbf{x}$ and $\mathbf{y}$ is the first plural variable not free in a

16. $\forall \mathbf{x}\ \mathbf{x} = \mathbf{x}$

17. $(\neg \exists \mathbf{x}\ \mathbf{x} = a \wedge \neg \exists \mathbf{y}\ \mathbf{y} = b) \supset (A(a) \leftrightarrow A(b)))$ where $\mathbf{x}$ and $\mathbf{y}$ are the first plural variables not free in a and b, where where $A(b)$ has free b at any place where $A(a)$ has a free a

18. $a = b \supset \exists \mathbf{x}\ \mathbf{x} = a \wedge \exists \mathbf{y}\ \mathbf{y} = b$ with $\mathbf{x}$ and $\mathbf{y}$ as in axiom 17

19. $\forall \mathbf{y}(\mathbf{y} = \iota \mathbf{x} A \leftrightarrow \forall \mathbf{x}(A \leftrightarrow \mathbf{x} = \mathbf{y}))$ where $\mathbf{y}$ does not occur in $\iota \mathbf{x} A$

Semantics

Definition 3.0: A interpretation structure $\mathcal{I} = \langle \mathcal{D}, (.)^{\mathcal{I}} \rangle$ such that:

$\mathcal{D}$ is the (possible empty) domain of discourse and $.^{\mathcal{I}}$ is an interpretation function such that each function symbol f_i^n is interpreted as an operation $f^{\mathcal{I}} : \mathcal{D}^n \to \mathcal{D}$ and each n-ary predicate P_i^n as $P^{\mathcal{I}} : \mathcal{D}^n \to \{0, 1\}$.

Definition 3.1: The partial function v is a function from linguistic items to semantical values. Given an $\mathcal{I}$, v is a function $v : VAR \to \mathcal{D}$ such that: $v(x) \in \mathcal{D}$ or $v(x) = O$, and $\tilde{v}$ is the homomorphic extension of v such that $\tilde{v} : VAR \to TERM$

$\tilde{v}(x) = v(x)$,

$\tilde{v}(c) = c^{\mathcal{I}}$ where $c^{\mathcal{I}} \in \mathcal{D}$ or $c^{\mathcal{I}} = O$.

For each n-place function sign f:

$\tilde{v}(f(t_1, ..., t_n)) = f^{\mathcal{I}}(\tilde{v}(t_1), ..., \tilde{v}(t_n))$

Definition 3.2: Given an interpretation structure $\mathcal{I}$ and a valuation function v, we say that $\langle \mathcal{I}, v \rangle$ satisfies a formula φ (Notation:$\langle \mathcal{I}, v \rangle \Vdash \varphi$) if v gives the value 1 for φ.

$\langle \mathcal{I}, v \rangle \Vdash P(t_1, ..., t_n)$ iff $\langle (\tilde{v}(t_1), ..., \tilde{v}(t_n)) \rangle \in P^{\mathcal{I}}$,

$\langle \mathcal{I}, v \rangle \Vdash \neg\varphi$ iff $\langle \mathcal{I}, v \rangle \nVdash \varphi$,

$\langle \mathcal{I}, v \rangle \Vdash \varphi \vee \psi$ iff $\langle \mathcal{I}, v \rangle \Vdash \varphi$ or $\langle \mathcal{I}, v \rangle \Vdash \psi$,

$\langle \mathcal{I}, v \rangle \Vdash \varphi \wedge \psi$ iff $\langle \mathcal{I}, v \rangle \Vdash \varphi$ and $\langle \mathcal{I}, v \rangle \Vdash \psi$,

$\langle \mathcal{I}, v \rangle \Vdash \varphi \supset \psi$ iff $\langle \mathcal{I}, v \rangle \nVdash \varphi$ or $\langle \mathcal{I}, v \rangle \Vdash \psi$,

$\langle \mathcal{I}, v \rangle \Vdash \forall x\varphi$ iff $\langle \mathcal{I}, v[x := k] \rangle \Vdash \varphi$ for all $k \in \mathcal{D}$,

$\langle \mathcal{I}, v \rangle \Vdash \forall \mathbf{x}\varphi$ iff $\langle \mathcal{I}, v[\mathbf{x} := k], \rangle \Vdash \varphi$ for all $k \in \mathcal{D}$,

$$v(\iota x\varphi) = \begin{cases} v'(x) \text{ if } \langle \mathcal{I}, v[\iota x\varphi := x] \rangle \Vdash \varphi \text{ for a unique } v', \\ \text{otherwise it is } O \end{cases}$$

$$v(\iota \mathbf{x}\varphi) = \begin{cases} v'(\mathbf{x}) \text{ if } \langle \mathcal{I}, v[\iota \mathbf{x}\varphi := \mathbf{x}] \rangle \Vdash \varphi \text{ for a unique } v', \\ \text{otherwise it is } O \end{cases}$$

3.4 A Slingshot for Full Plural Logic

Our strategy for proving the slingshot argument for Full Plural Logic is similar to what we have done in Two Ways of Reconstructing Gödel's Slingshot and again borrows some ideas from [Shramko and Wansing, 2011]. To talk about the denotation of a sentence in our language, we take the conservative extension of $\mathcal{L}_{PL}$ with a binary connective $\equiv$ such that intuitively $\varphi \equiv \psi$ means that the situation (or fact) that φ denotes is the same situation that ψ denotes. Let $\mathcal{L}^* = \mathcal{L}_{PL} \cup \{\equiv\}$. Let us extend the theory of Full Plural Logic to include the following new axioms:

Axioms

21 $\varphi \equiv \varphi$

22 $\varphi \equiv \psi \supset \psi \equiv \varphi$

23 $(\varphi \equiv \psi \wedge \psi \equiv \chi) \supset \varphi \equiv \chi$

24 $P(a) \equiv (a = \iota\mathbf{x}(\mathbf{x} = a \wedge P(\mathbf{x})))$

Note that from Axiom 6 above we have the following fact:

(Fact 1) $\vdash \iota\mathbf{x}\varphi = \iota\mathbf{x}\psi \supset (A(\iota\mathbf{x}\varphi) \supset (A(\iota\mathbf{x}\psi))$

Lemma 0: $\vdash F(a) \supset a = \iota\mathbf{x}(\mathbf{x} = a \wedge F(\mathbf{x}))$.

(i) $\vdash F(a)$ hypothesis

(ii) $\vdash \forall\mathbf{y}(\mathbf{y} = \iota\mathbf{x}(\mathbf{x} = a \wedge F(\mathbf{x})) \leftrightarrow \forall\mathbf{x}(\mathbf{x} = a \wedge F(\mathbf{x}) \leftrightarrow \mathbf{x} = \mathbf{y})$ by Axiom 19

(iii) $\vdash a = a$ by Axiom 16

(iv) $\vdash a = a \supset \exists\mathbf{z}\ \mathbf{z} = a \wedge \exists\mathbf{w}\ \mathbf{w} = a$ by Axiom 18

(v) $\vdash \exists\mathbf{z}\ \mathbf{z} = a \wedge \exists\mathbf{w}\ \mathbf{w} = a$ from (iii) and (iv) by MP

(vi) $\vdash \exists\mathbf{z}\ \mathbf{z} = a$ from (v) by FOL

(vii) $\vdash \forall\mathbf{y}(\mathbf{y} = \iota\mathbf{x}(\mathbf{x} = a \wedge F(\mathbf{x})) \leftrightarrow \forall\mathbf{x}(\mathbf{x} = a \wedge F(\mathbf{x}) \leftrightarrow \mathbf{x} = \mathbf{y}) \supset \exists\mathbf{z}\ \mathbf{z} = a \supset (a = \iota\mathbf{x}(\mathbf{x} = a \wedge F(\mathbf{x}))$ by Axiom 15

(viii) $\vdash \exists\mathbf{z}\ \mathbf{z} = a \supset (a = \iota\mathbf{x}(\mathbf{x} = a \wedge F(\mathbf{x})$ from(ii) and (vii) by MP

(ix) $\vdash (a = \iota\mathbf{x}(\mathbf{x} = a \wedge F(\mathbf{x}))$ from (vi) and (viii) by MP

Now, using Lemma 0 we are going to prove the first half of the Slingshot for Full Plural Logic, that is, prove that from two distinguished terms, a and b, the situation denoted by asserting that a has a certain property is the same situation denoted by asserting b has another different property. Actually, for the first positive half of the argument we are going to prove that $F(a), a \neq b, G(b) \vdash F(a) \equiv G(b)$ and $F(a), a = b, G(b) \vdash F(a) \equiv G(b)$.

$F(a), a \neq b, G(b) \vdash F(a) \equiv G(b)$

Suppose the following sentences (i), (ii) and (iii) are given as hypothesis:

(i) $F(a)$ hypothesis

(ii) $G(b)$ hypothesis

(iii) $a \neq b$ hypothesis

(iv) $\vdash a = \iota\mathbf{x}(\mathbf{x} = a \wedge F(\mathbf{x}))$ from (i) by Lemma 0

(v) $\vdash b = \iota\mathbf{x}(\mathbf{x} = b \wedge G(\mathbf{x}))$ from (ii) by Lemma 0

(vi) $\vdash a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} \neq b)$ from (iii) by Lemma 0

(vii) $\vdash b = \iota\mathbf{x}(\mathbf{x} = b \wedge a \neq \mathbf{x})$ from (iii) by Lemma 0

(viii) $\vdash \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} \neq b) = a$ from (vi) by symmetry of identity

(ix) $\vdash \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} \neq b) = \iota\mathbf{x}(\mathbf{x} = a \wedge F(\mathbf{x}))$ from (viii) and (iv) by transitivity of identity

(x) $\vdash \iota\mathbf{x}(\mathbf{x} = b \wedge G(\mathbf{x})) = b$ by symmetry in (v)

(xi) $\vdash \iota\mathbf{x}(\mathbf{x} = b \wedge G(\mathbf{x})) = \iota\mathbf{x}(\mathbf{x} = b \wedge a \neq \mathbf{x})$ from (x) and (vii) by transitivity of identity

(xii) $\vdash F(a) \equiv (a = \iota\mathbf{x}(\mathbf{x} = a \wedge F(\mathbf{x})))$ by Axiom 24

(xiii) $\vdash (a \neq b) \equiv (a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} \neq b))$ by Axiom 24

(xiv) $\vdash F(a) \equiv (a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} \neq b))$ from (ix) and (xii) by (Fact 1)

(xv) $\vdash (a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} \neq b)) \equiv (a \neq b)$ from (xiii) by Axiom 22 and FOL

(xvi) $\vdash (F(a)) \equiv (a \neq b)$ from (xiv) and (xv) by Axiom 23

(xvii) $\vdash G(b) \equiv (b = \iota\mathbf{x}(\mathbf{x} = b \wedge G(b))$ by Axiom 24

(xviii) $\vdash (a \neq b) \equiv (b = \iota\mathbf{x}(\mathbf{x} = b \wedge a \neq \mathbf{x}))$ by Axiom 24

(xix) $\vdash G(b) \equiv (b = \iota\mathbf{x}(\mathbf{x} = b \wedge \mathbf{x} \neq a))$ from (xi) and (xvii)by (Fact 1)

(xx) $\vdash (b = \iota\mathbf{x}(\mathbf{x} = b \wedge a \neq \mathbf{x})) \equiv G(b)$ from (xix) by Axiom 22 and FOL

(xxi) $\vdash (a \neq b) \equiv G(b)$ from (xviii) and (xx) by Axiom 23 and FOL

(xxii) $\vdash F(a) \equiv G(b)$ from (xvi) and (xxi) by Axiom 23 and FOL

$F(a), a = b, G(b) \vdash F(a) \equiv G(b)$

Suppose the following sentences (i), (ii) and (iii) are given as hypothesis:

(i) $F(a)$ hypothesis

(ii) $G(b)$ hypothesis

(iii) $a = b$ hypothesis

(iv) $\vdash a = \iota\mathbf{x}(\mathbf{x} = a \wedge F(\mathbf{x}))$ from (i) by Lemma 0

(v) $\vdash b = \iota\mathbf{x}(\mathbf{x} = b \wedge G(\mathbf{x}))$ from (ii) by Lemma 0

(vi) $\vdash a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} = b)$ from (iii) by Lemma 0

(vii) $\vdash b = \iota\mathbf{x}(\mathbf{x} = b \wedge a = \mathbf{x})$ from (iii) by Lemma 0

(viii) $\vdash \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} = b) = a$ from (vi) by symmetry of identity

(ix) $\vdash \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} = b) = \iota\mathbf{x}(\mathbf{x} = a \wedge F(\mathbf{x}))$ from (viii) and (iv) by transitivity of identity

(x) $\vdash \iota\mathbf{x}(\mathbf{x} = b \wedge G(\mathbf{x})) = b$ by symmetry in (v)

(xi) $\vdash \iota\mathbf{x}(\mathbf{x} = b \wedge G(\mathbf{x})) = \iota\mathbf{x}(\mathbf{x} = b \wedge a = \mathbf{x})$ from (x) and (vii) by transitivity of identity

(xii) $\vdash F(a) \equiv (a = \iota\mathbf{x}(\mathbf{x} = a \wedge F(\mathbf{x})))$ by Axiom 24

(xiii) $\vdash (a = b) \equiv (a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} = b))$ by Axiom 24

(xiv) $\vdash F(a) \equiv (a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} = b))$ from (ix) and (xii) by (Fact 1)

(xv) $\vdash (a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} = b)) \equiv (a = b)$ from (xiii) by Axiom 22 and FOL

(xvi) $\vdash (F(a)) \equiv (a = b)$ from (xiv) and (xv) by Axiom 23

(xvii) $\vdash G(b) \equiv (b = \iota\mathbf{x}(\mathbf{x} = b \wedge G(b))$ by Axiom 24

(xviii) $\vdash (a = b) \equiv (b = \iota\mathbf{x}(\mathbf{x} = b \wedge a = \mathbf{x}))$ by Axiom 24

(xix) $\vdash G(b) \equiv (b = \iota\mathbf{x}(\mathbf{x} = b \wedge \mathbf{x} = a))$ from (xi) and (xvii)by (Fact 1)

(xx) $\vdash (b = \iota\mathbf{x}(\mathbf{x} = b \wedge a = \mathbf{x})) \equiv G(b)$ from (xix) by Axiom 22 and FOL

(xxi) $\vdash (a = b) \equiv G(b)$ from (xviii) and (xx) by Axiom 23 and FOL

(xxii) $\vdash F(a) \equiv G(b)$ from (xvi) and (xxi) by Axiom 23 and FOL

Of course this is just half of the argument. The above argument proves that any two true sentences denote the same fact (or situation). As to the other half we need to prove $\neg F(a), a = b, \neg G(b) \vdash \neg F(a) \equiv \neg G(b)$ and $\neg F(a), a \neq b, \neg G(b) \vdash \neg F(a) \equiv \neg G(b)$, i.e, that any false sentence denotes the same situation.

$\neg F(a), a = b, \neg G(b) \vdash \neg F(a) \equiv \neg G(b)$ (i) $\neg F(a)$

(ii) $\neg G(b)$

(iii) $a = b$

(iv) $\vdash a = \iota\mathbf{x}(\mathbf{x} = a \wedge \neg F(\mathbf{x}))$ from (i) by Lemma 0

(v) $\vdash b = \iota\mathbf{x}(\mathbf{x} = b \wedge \neg G(\mathbf{x}))$ from (ii) by Lemma 0

(vi) $\vdash a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} = b)$ from (iii) by Lemma 0

(vii) $\vdash b = \iota\mathbf{x}(\mathbf{x} = b \wedge \mathbf{x} = a)$ from (iii) by Lemma 0

(viii) $\vdash \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} = b) = a$ from (vi) by symmetry of identity

(ix) $\vdash \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} = b) = \iota\mathbf{x}(\mathbf{x} = a \wedge \neg F(\mathbf{x}))$ from (viii) and (iv) by transitivity of identity

(x) $\vdash \iota\mathbf{x}(\mathbf{x} = b \wedge \neg G(\mathbf{x})) = b$ by symmetry in (v)

(xi) $\vdash \iota\mathbf{x}(\mathbf{x} = b \wedge \neg G(\mathbf{x})) = \iota\mathbf{x}(\mathbf{x} = b \wedge \mathbf{x} = a)$ from (x) and (vii) by transitivity of identity

(xii) $\vdash \neg F(a) \equiv (a = \iota\mathbf{x}(\mathbf{x} = a \wedge \neg F(\mathbf{x})))$ by Axiom 24

(xiii) $\vdash (a = b) \equiv (a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} = b))$ by Axiom 24

(xiv) $\vdash \neg F(a) \equiv (a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} = b))$ from (ix) and (xii) by (Fact 1)

(xv) $\vdash (a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} = b)) \equiv (a = b)$ from (xiii) by Axiom 22 and FOL

(xvi) $\vdash \neg F(a) \equiv (a = b)$ from (xiv) and (xv) by Axiom 23

(xvii) $\vdash \neg G(\mathbf{x}) \equiv (b = \iota\mathbf{x}(\mathbf{x} = b \wedge \neg G(b))$ by Axiom 24

(xviii) $\vdash (a = b) \equiv (b = \iota\mathbf{x}(\mathbf{x} = b \wedge \mathbf{x} = a))$ by Axiom 24

(xix) $\vdash \neg G(\mathbf{x}) \equiv (b = \iota\mathbf{x}(\mathbf{x} = b \wedge \mathbf{x} = a))$ from (xi) and (viii)by (Fact 1)

(xx) $\vdash (b = \iota\mathbf{x}(\mathbf{x} = b \wedge \mathbf{x} = a)) \equiv (\neg G(b))$ from (xvii) by Axiom 22 and FOL

(xxi) $\vdash (a = b) \equiv \neg G(b)$ from (xviii) and (xx) by Axiom 23 and FOL

(xxii) $\vdash \neg F(a) \equiv \neg G(b)$ from (xvi) and (xxi) by Axiom 23 and FOL

$\neg F(a), a \neq b, \neg G(b) \vdash \neg F(a) \equiv \neg G(b)$

(i) $\neg F(a)$

(ii) $\neg G(b)$

(iii) $a \neq b$

(iv) $\vdash a = \iota\mathbf{x}(\mathbf{x} = a \wedge \neg F(\mathbf{x}))$ from (i) by Lemma 0

(v) $\vdash b = \iota\mathbf{x}(\mathbf{x} = b \wedge \neg G(\mathbf{x}))$ from (ii) by Lemma 0

(vi) $\vdash a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} \neq b)$ from (iii) by Lemma 0

(vii) $\vdash b = \iota\mathbf{x}(\mathbf{x} = b \wedge \mathbf{x} \neq a)$ from (iii) by Lemma 0

(viii) $\vdash \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} \neq b) = a$ from (vi) by symmetry of identity

(ix) $\vdash \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} \neq b) = \iota\mathbf{x}(\mathbf{x} = a \wedge \neg F(\mathbf{x}))$ from (viii) and (iv) by transitivity of identity

(x) $\vdash \iota\mathbf{x}(\mathbf{x} = b \wedge \neg G(\mathbf{x})) = b$ by symmetry in (v)

(xi) $\vdash \iota\mathbf{x}(\mathbf{x} = b \wedge \neg G(\mathbf{x})) = \iota\mathbf{x}(\mathbf{x} = b \wedge \mathbf{x} = a)$ from (x) and (vii) by transitivity of identity

(xii) $\vdash \neg F(a) \equiv (a = \iota\mathbf{x}(\mathbf{x} = a \wedge \neg F(\mathbf{x})))$ by Axiom 24

(xiii) $\vdash (a \neq b) \equiv (a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} \neq b))$ by Axiom 24

(xiv) $\vdash \neg F(a) \equiv (a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} = b))$ from (ix) and (xii) by (Fact 1)

(xv) $\vdash (a = \iota\mathbf{x}(\mathbf{x} = a \wedge \mathbf{x} \neq b)) \equiv (a = b)$ from (xiii) by Axiom 22 and FOL

(xvi) $\vdash \neg F(a) \equiv (a \neq b)$ from (xiv) and (xv) by Axiom 23

(xvii) $\vdash \neg G(\mathbf{x}) \equiv (b = \iota\mathbf{x}(\mathbf{x} = b \wedge \neg G(b))$ by Axiom 24

(xviii) $\vdash a \neq b \equiv (b = \iota\mathbf{x}(\mathbf{x} = b \wedge \mathbf{x} \neq a))$ by Axiom 24

(xix) $\vdash \neg G(\mathbf{x}) \equiv (b = \iota\mathbf{x}(\mathbf{x} = b \wedge \mathbf{x} \neq a))$ from (xi) and (viii)by (Fact 1)

(xx) $\vdash (b = \iota\mathbf{x}(\mathbf{x} = b \wedge \mathbf{x} \neq a)) \equiv (\neg G(b))$ from (xvii) by Axiom 22 and FOL

(xxi) $\vdash a \neq b \equiv \neg G(b)$ from (xviii) and (xx) by Axiom 23 and FOL

(xxii) $\vdash \neg F(a) \equiv \neg G(b)$ from (xvi) and (xxi) by Axiom 23 and FOL

If we are right about the slingshot for Full Plural Logic, the above arguments force the authors of *Plural Logic* to put sentences under the same umbrella of proper names. Given that, all the true sentences denote only one object, The True, and all the false ones denote The False. Although Oliver and Smiley don't comment on slingshots, besides the quote on fregean proper names, they do comment on Russell's interpretation of definite descriptions. However, Oliver and Smiley reject completely the strategy of contextually eliminating a plural term. This matter was explored in *Plural Logic* (see [Oliver and Smiley, 2013a, p. 124]) in which they argue, for instance, that the equivalence following equivalence is not the case:

$G(\iota\mathbf{x}F\mathbf{x}) \leftrightarrow \exists\mathbf{x}(\forall\mathbf{y}(F\mathbf{y} \leftrightarrow \mathbf{x} = \mathbf{y}) \wedge G\mathbf{x})$

Given that is possible that $G(\iota\mathbf{x}F\mathbf{x})$ is true over a non-existent object, and the non-existence of it would make $\exists\mathbf{x}(\forall\mathbf{y}(F\mathbf{y} \leftrightarrow \mathbf{x} = \mathbf{y}) \wedge G\mathbf{x})$ false.

> At this point readers brought up on Russell's theory of descriptions might expect that the various sorts of descriptions will all turn out to be contextually eliminable, by analogy with his contextual elimination of singular description. They would be wrong. [Oliver and Smiley, 2013a, p. 124]

First of all, the claim that there is no contextual elimination of plural descriptions is completely different than the claim that the equivalence above is not a good candidate for a contextual elimination of plural descriptions. Simply so because it could be a different suitable equivalent formulae that has no plural description in it. Secondly, even if there was an way of eliminating plural terms, if we were right in the second chapter about the elimination of descriptions, this would not be a way of evading the slingshot. Given that if there was an elimination of plural descriptions, then it would be a prove that whatever was proved in a language with plural descriptions, it is still provable in a language without it. Therefore, if the slingshot is an argument proved in a language with plural descriptions, there would be a proof of a slingshot in a language without plural descriptions.

This chapter presented Plural Logic under the view of Oliver and Smiley. This logic was created to deal with plurals in the language. We presented a slingshot for Plural Logic and argued that is an undesirable result for the authors. This points against Gödel's idea that definite descriptions are necessary for the slingshot. In the conclusion we will comment on other ways to evade the slingshot by recalling recipes presented in Chapter One.

Conclusion and Unfinished Business

This dissertation aimed at arguing against the sufficiency of descriptions for obtaining the slingshot. This was done, on a first part, by analysing Gödel's solution to the slingshot, by way of Russell's theory of definite descriptions, and on a second part, by analysing Plural Logic, which formalizes a language with plural descriptions. In both cases a slingshot was found to follow.

In Chapter One we presented the debate between Frege and Russell about the nature of definite descriptions. In Chapter Two we presented Gödel's suspicion that Russell's position on definite descriptions would evade the slingshot, and we shown that this was wrong. In Chapter Three we proved a slingshot in Plural Logic and showed how this would be an undesirable result for Oliver and Smiley's system.

There are many directions to take after this, and here we are going to point out some of them. For instance, one recipe given by Dunn stated that what is needed for the slingshot are three things: first a term-forming operator, second a certain notion of Replacement and third the Indiscernibility of Identicals. The first ingredient has being shown to be unnecessary since the slingshot can be reached without it.

Another option for evading the slingshot would be to get rid of one of the remaining ingredients. This two notions are related to identity. One particular notion of identity that was used through the proof of the slingshot was the transitivity, thus, it seems possible to block the argument if we drop this property of identity. One example of study on non-transitive identity is [Priest, 2010].

We recall Gödel's recipe for the slingshot requires a specific term-forming operator and compositionality. Seeing that Gödel's attempt of getting rid of the term-forming operator failed at avoiding the slingshot, we take it that changing the notion of compositionality would be a new interesting approach for evading the slingshot. One interesting project is to use non-deterministic semantics [Avron, 2011], which extends the notion of compositionality to such end. We see this formal method as a way to explore the limits of compositionality within the recipe for the slingshot. Even though the propositional case is very well behaved[10], it was noticed in [Avron and Zamansky, 2005] that when extending non-deterministic matrices to the first-order case, some issues arise, such as the failure of α-equivalence between formula, namely, it is possible that two formulae differing only in the names of their bound variables do not have the same truth value. Another problem is the failure of equivalence between formulas differing only on codenotative terms, for example $P(a)$ and $P(b)$ where $a = b$. There are two proposals for solving those problems, one by using a substitutional interpretation of the quantifiers (see [Avron and Zamansky, 2005] and [Avron, 2005]), and [Ferguson, 2014] as well. We suggest as an interesting future project to explore the other half of Gödel's proposal to evade the slingshot by changing the notion of compositionality using non-deterministic semantics.

In a very loose sense, this dissertation was able to falsify two common statements about slingshot arguments. The first statement is in the work of Barwise and Perry where they state that "[t]he argument is so small, seldom encompassing more than half a page, and employs such a minimum ammunition [...] that we dub it *the slingshot*" [Barwise and Perry, 1981, p. 395]. We believe this statement is confusing given that, if the length of the argument was such an important property for baptising the argument as slingshot, than either the arguments showed in this dissertation are not slingshots, or the size of the argument is not an important feature for its name. We stand by the position that the size of the argument is not relevant to characterize a slingshot, but rather its conclusion. The second statement is in the work of Neale where he states that "[t]his essay should, I believe, answer all technical questions raised by slingshot arguments" [Neale, 1995, p. 764]. Here we disagreed with Neale given that we believe the technical questions about the slingshot have flourish over the years rather than settle.

Of course, there are other proposals of how to evade the slingshot, for example, Jon Barwise and John Perry developed their so called *Situation Semantics* (in [Barwise and Perry, 1983]) as a way of evading this kind of arguments. In recent years the philosopher Oswaldo Chateaubriand had attacked the slingshot arguments, most notably in Chapter 4 of the first volume of his book *Logical Forms* [Chateaubriand, 2001] and he reconsider the issue in [Chateaubriand, 2015]. Chateaubriand's main argument against slingshots is that, given a third value logic, is possible to create to sentences that are synonymous, in Church's sense, but differ in truth value. Indicating, perhaps, that there

[10] See for example [Avron, 2007] where the author provides non-deterministic semantics for a number of systems of paraconsistent logic.

is more on the slingshot that lay down in the notion of synonymity then it seems.

Lasty, we should state that it was never the goal of this project to defend that slingshot arguments are some kind of an undeniable truth, but rather it was our goal to try to challenge the solution proposed by Gödel. We recognize, nevertheless, that other possible solutions exist.

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