Non-reflexive Nonsense: Proof Theory of Paracomplete Weak Kleene Logic

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Abstract

Our aim is to provide a sequent calculus whose external consequence relation coincides with the three-valued paracomplete logic 'of nonsense' introduced by Dmitry Bochvar and, independently, presented as the weak Kleene logic $\mathbf{K_3^w}$ by Stephen C. Kleene. The main features of this calculus are (i) that it is *non-reflexive*, i.e., Identity is not included as an explicit rule (although a restricted form of it with premises is derivable); (ii) that it includes rules where *no variable-inclusion conditions* are attached; and (iii) that it is *hybrid*, insofar as it includes both left and right operational introduction as well as elimination rules.

1 Introduction

The three-valued logics 'of nonsense' $\mathbf{K_3^w}$ and **PWK** are the paracomplete and paraconsistent members of the weak Kleene family, first developed by [2] and [14], respectively. These logics, extensively studied in the literature, are rendered by considering valuations complying with the weak Kleene truth-tables and, respectively, considering the set of designated values to be 1, or 1 and also ½.

	¬
1	0
½	½
0	1

∧	1	½	0
1	1	½	0
½	½	½	½
0	0	½	0

∨	1	½	0
1	1	½	1
½	½	½	½
0	1	½	0

Figure 1: weak Kleene truth-tables

Among their well-known distinctive features is the fact that they are negation duals of each other, i.e., for any sets of formulae Γ, Δ we have that $\Gamma \vDash_{\mathbf{PWK}} \Delta$ if and only if $\neg\Delta \vDash_{\mathbf{K_3^w}} \neg\Gamma$, and vice versa. Thus, for instance,

PWK contains all classical tautologies and no contradictions; conversely, $\mathbf{K_3^w}$ contains all classical contradictions and no tautologies. More generally, while the intersection of those inferences valid in Classical Logic and these systems is non-empty, it is known that when well-rehearsed variable-inclusion conditions are violated, both **PWK** and $\mathbf{K_3^w}$ lose some classical inferences, e.g., $\varphi \wedge \psi \nvDash_{\mathbf{PWK}} \varphi$ and $\varphi \nvDash_{\mathbf{K_3^w}} \varphi \vee \psi$, usually referred to as the rules of *Simplification* and *Addition*, respectively—for more, see [11, 17, 18].

Both logics have been studied under different presentations over the last decade, e.g. via abstract algebraic logic [3], logical matrices [4] and structural sequent calculi with explicit variable-inclusion conditions [5, 7]. More recently, [19] have offered a metainferential characterization of **PWK** in the form of a substructural sequent calculus, $\mathbf{LK}_W^-$. Distinctive features of this calculus are (i) that it is *non-transitive*, i.e., Cut is not included as an explicit rule (although a restricted form of Cut is derivable); (ii) that it includes the *fully classical rules for negation*, i.e., no variable-inclusion conditions are attached to $L\neg$ or $R\neg$; and (iii) that it is *hybrid*, insofar as it includes both (classical) left and right operational rules and their 'inverses', i.e. elimination rules for $\neg$, $\wedge$ and $\vee$. As a result, $\mathbf{LK}_W^-$ captures **PWK**-consequence from a metainferential perspective, in terms of its *external consequence relation*, as defined in [1]. For example, *Simplification* is invalid in **PWK**, and accordingly *Meta Simplification* (the metainference going from $\Rightarrow \varphi \wedge \psi$ to $\Rightarrow \varphi$) is not derivable in $\mathbf{LK}_W^-$.

Our aim in this paper is to offer a dual to $\mathbf{LK}_W^-$, in the form of a substructural calculus which we will call $\mathbf{LK}_W^{\setminus Id}$ whose external consequence relation coincides with $\mathbf{K_3^w}$-consequence.[1] Our target calculus will have a number of features, in accordance with the previous remarks. First, $\mathbf{LK}_W^{\setminus Id}$, like its paraconsistent counterpart $\mathbf{LK}_W^-$, will be hybrid. Secondly, in line with the aforementioned duality of the **PWK** and $\mathbf{K_3^w}$, and the duality between Cut and Reflexivity pointed out in [9], $\mathbf{LK}_W^{\setminus Id}$ will be non-reflexive —meaning that some classical metainferences will be lost— and that it will derive no classical inferences.

Some aspects of this calculus can be highlighted in order to see how it compares to formalisms already available in the literature. On the one hand, it is a sequent calculus, as opposed to a natural deduction one given in [20]. On the other hand, it is a two-sided sequent calculus, as opposed to the multi-sided sequent calculi presented for it in [12] and [5]. Furthermore, it is a two-sided sequent calculus for the target system which has no linguistic restrictions on its rules, distinct from the case of [8]. To conclude, it presents a two-sided sequent calculus with many inverted or "elimination" rules which

[1]We choose this nomenclature to be as descriptive as possible, inasmuch as our target calculus is an alternative presentation of **LK** without Identity or Reflexivity, and with a weakened set of inverse or elimination rules for the connectives. Furthermore, this nomenclature is in accordance with the recent practice in the specialized literature, for which see [10].

motivates an original extension of the reduction techniques presented in [15]—to account not only for the case of a general elimination rule like the Cut rule but also for more particular operational cases of such rules.

The article is structured as follows. In Section 2 we introduce the calculus $\mathbf{LK}_W^{\setminus Id}$, and highlight some of its properties. In Section 3 we prove that the derivability relation of $\mathbf{LK}_W^{\setminus Id}$ is sound and complete for the local validity with regard to the ts-consequence over the Weak Kleene valuations. Finally, using this result, in Section 4 we show that the external consequence relation of $\mathbf{LK}_W^{\setminus Id}$ coincides with the logic $\mathbf{K_3^w}$.

2 The calculus $\mathbf{LK}_W^{\setminus Id}$

In this section, we will introduce the calculus $\mathbf{LK}_W^{\setminus Id}$ whose external consequence relation we will later show to coincide with $\mathbf{K_3^w}$. Before doing that, we present the calculus **LK** for classical logic.[2]

Definition 2.1. Let Γ, Δ be sequences[3] of formulas, and φ, ψ formulas. **LK** is the sequent calculus defined by the following rules:

Structural rules

$$\frac{}{\varphi \Rightarrow \varphi}\ Id$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ LW \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ RW \quad \frac{\Gamma_1, \psi, \varphi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \varphi, \psi, \Gamma_2 \Rightarrow \Delta}\ LE \quad \frac{\Gamma \Rightarrow \Delta_1, \psi, \varphi, \Delta_2}{\Gamma \Rightarrow \Delta_1, \varphi, \psi, \Delta_2}\ RE$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta}\ Cut \quad \frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ LC \quad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ RC$$

Operational Rules

$$\frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma, \neg\varphi \Rightarrow \Delta}\ L\neg \quad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}\ R\neg$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\ R\wedge \quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ L\wedge$$

[2] For the sake of convenience, the calculus we will present and call **LK** is not exactly Gentzen original one, but an equivalent one.

[3] Sequences are ordered collections of objects (in this case, formulas). To avoid any confusion, the elements of a sequence will be denoted as tuples $\langle \varphi_1, \dots, \varphi_n \rangle$, to distinguish them from sets that will be denoted as usual with brackets $\{\varphi_1, \dots, \varphi_n\}$.

$$\frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi \vee \psi}\ R\vee$$

Based on this calculus, we will build $\mathbf{LK}_W^{\setminus Id}$ as follows.

Definition 2.2. Let Γ, Δ be sequences of formulas, and φ, ψ formulas. $\mathbf{LK}_W^{\setminus Id}$ is the sequent calculus defined by LW, RW, LE, RE, Cut, LC, RC, $L\neg$, $R\neg$, $R\wedge$, $L\vee$ plus the following rules:

$$\frac{\Gamma, \neg\varphi \Rightarrow \Delta, \neg\varphi}{\Gamma, \varphi \Rightarrow \Delta, \varphi}\ E\neg$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta, \varphi \quad \Gamma, \psi \Rightarrow \Delta, \psi}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ L\wedge^*$$

$$\frac{\Gamma, \varphi \wedge \psi \Rightarrow \Delta, \varphi \wedge \psi}{\Gamma, \varphi \Rightarrow \Delta, \varphi}\ E\wedge \qquad \frac{\Gamma, \varphi \wedge \psi \Rightarrow \Delta, \varphi \wedge \psi}{\Gamma, \psi \Rightarrow \Delta, \psi}\ E\wedge$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \psi \quad \Gamma, \varphi \Rightarrow \Delta, \varphi \quad \Gamma, \psi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ R\vee^*$$

$$\frac{\Gamma, \varphi \vee \psi \Rightarrow \Delta, \varphi \vee \psi}{\Gamma, \varphi \Rightarrow \varphi, \Delta}\ E\vee \qquad \frac{\Gamma, \varphi \vee \psi \Rightarrow \Delta, \varphi \vee \psi}{\Gamma, \psi \Rightarrow \psi, \Delta}\ E\vee$$

Notice that there are many differences between **LK** and $\mathbf{LK}_W^{\setminus Id}$. Firstly, in **LK** there are only introduction rules, while in $\mathbf{LK}_W^{\setminus Id}$ we also add elimination rules for the connectives. Secondly, while **LK** contains all the structural rules, $\mathbf{LK}_W^{\setminus Id}$ is not reflexive since the rule Id is not in it. This implies, among other things, that $\mathbf{LK}_W^{\setminus Id}$ proves no sequent. Finally, the rules $L\wedge$ and $R\vee$ of **LK** are not included in $\mathbf{LK}_W^{\setminus Id}$, but they are replaced by the weaker $L\wedge^*$ and $R\vee^*$. As we will see in the next section, we need these modifications in order to obtain soundness and completeness regarding the semantics we will present. More comments on these will need to wait to the end of the next section.

Given any sequent calculus we can extract, at least, two different consequence relations from it. We focus on them for the case of our target system, next.

Definition 2.3. The *derivability* relation $\vdash^{seq}_{\mathbf{LK}_W^{\setminus Id}}$ is defined as follows: given a set of sequents S and a sequent s, $S \vdash^{seq}_{\mathbf{LK}_W^{\setminus Id}} s$ if and only if s is derivable in the sequent calculus resulting from adding the set of sequents S as axioms to $\mathbf{LK}_W^{\setminus Id}$.

Notice that when $S = \varnothing$, $\vdash^{seq}_{\mathbf{LK}^{\smallsetminus Id}_W}$ denotes the so-called internal-consequence relation, i.e. that describing the provable sequents in the calculus $\mathbf{LK}^{\smallsetminus Id}_W$. Also, we can define the external consequence relation from [1], with a slight adaptation.[4]

Definition 2.4. The *external consequence* relation $\vdash^{E}_{\mathbf{LK}^{\smallsetminus Id}_W}$ is defined as follows: given two sequences of formulas Γ, Δ, $\Gamma \vdash^{E}_{\mathbf{LK}^{\smallsetminus Id}_W} \Delta$ if and only if $\{\Rightarrow \gamma_1, \ldots, \Rightarrow \gamma_n\} \vdash^{seq}_{\mathbf{LK}^{\smallsetminus Id}_W} \{\Rightarrow \Delta\}$, where $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$.

The external consequence relation is a relation between formulas, while the derivability relation is a relation between sequents. Notice, however, that both relation are closely related, and in particular providing a semantics for the derivability relation can be used to provide a semantics for the external consequence relation. In the next section, we do exactly that.

Before ending this section, notice that the choice of the rules of $\mathbf{LK}^{\smallsetminus Id}_W$ is at some extent arbitrary, and more related to the way we will prove completeness than to some deep conceptual reason. In this sense, there are many equivalent (and maybe interesting) ways of defining $\mathbf{LK}^{\smallsetminus Id}_W$. As an example, the usual inverse rules of the classical connectives are derivable in $\mathbf{LK}^{\smallsetminus Id}_W$.

Fact 2.5. *The following rules are derivable in* $\mathbf{LK}^{\smallsetminus Id}_W$*:*

$$\frac{\Gamma, \neg\varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ \neg L^{\delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma, \varphi \Rightarrow \Delta}\ \neg R^{\delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}{\Gamma \Rightarrow \Delta, \varphi(\psi)}\ \wedge R^{\delta} \qquad \frac{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}{\Gamma, \varphi, \psi \Rightarrow \Delta}\ \wedge L^{\delta}$$

$$\frac{\Gamma, \varphi \vee \psi \Rightarrow \Delta}{\Gamma, \varphi(\psi) \Rightarrow \Delta}\ \vee L^{\delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \vee \psi}{\Gamma \Rightarrow \Delta, \varphi, \psi}\ \vee R^{\delta}$$

Proof. Let's show the derivability of $\wedge R^{\delta}$ and$\vee L^{\delta}$. The other cases are similar. For conjunction $\wedge R^{\delta}$, we only show the derivability of $\Gamma \Rightarrow \Delta, \varphi$, since showing the derivability of $\Gamma \Rightarrow \Delta, \psi$ is identical to it. To improve readability, we omit the application of the rules LC, RC, RE and LE:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta, \varphi \wedge \psi}\,LW}{\Gamma, \varphi \Rightarrow \Delta, \varphi}\,E\wedge}{\Gamma, \varphi \Rightarrow \Delta, \varphi, \varphi}\,RW \quad \dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta, \varphi \wedge \psi}\,LW}{\Gamma, \psi \Rightarrow \Delta, \psi}\,E\wedge}{\Gamma, \psi \Rightarrow \Delta, \psi, \varphi}\,RW \quad \dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta, \varphi \wedge \psi}\,LW}{\Gamma, \varphi \Rightarrow \Delta, \varphi}\,E\wedge}{\Gamma, \varphi, \psi \Rightarrow \Delta, \varphi}\,LW}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta, \varphi}\,L\wedge^{*} \quad \dfrac{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi, \varphi}\,RW}{\Gamma \Rightarrow \Delta, \varphi}\,Cut$$

[4] In [1], the external consequence relation is defined as a Tarskian consequence relation, i.e. only with single conclusions. Here we are straightforwardly adapting the definition in order to accommodate multiple conclusions.

For the case of the rule of the disjunction $\vee L^{\delta}$, we only show the derivability of $\Gamma, \varphi \Rightarrow \Delta$, since showing the derivability of $\Gamma, \psi \Rightarrow \Delta$ is identical to it. To improve readability, once again, we omit the application of the rules LC, RC, RE and LE:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \varphi \vee \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta, \varphi \vee \psi}\,RW}{\Gamma, \varphi \Rightarrow \Delta, \varphi}\,E\vee}{\Gamma, \varphi, \varphi \Rightarrow \Delta, \varphi}\,LW \quad \dfrac{\dfrac{\dfrac{\Gamma, \varphi \vee \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta, \varphi \vee \psi}\,RW}{\Gamma, \psi \Rightarrow \Delta, \psi}\,E\vee}{\Gamma, \varphi, \psi \Rightarrow \Delta, \psi}\,LW \quad \dfrac{\dfrac{\dfrac{\Gamma, \varphi \vee \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta, \varphi \vee \psi}\,RW}{\Gamma, \varphi \Rightarrow \Delta, \varphi}\,E\vee}{\Gamma, \varphi \Rightarrow \Delta, \varphi, \psi}\,RW}{\Gamma, \varphi \Rightarrow \Delta, \varphi \vee \psi}\,R\vee^{*} \quad \dfrac{\Gamma, \varphi \vee \psi \Rightarrow \Delta}{\Gamma, \varphi, \varphi \vee \psi \Rightarrow \Delta}\,LW}{\Gamma, \varphi \Rightarrow \Delta}\,Cut$$

□

Other very important derivable rules related with the completeness proof are the following:

Fact 2.6. *The following all-purpose elimination rules are derivable in* $\mathbf{LK}_{W}^{\smallsetminus Id}$*:*

$$\frac{\Gamma, \neg\varphi, \varphi \wedge \psi, \varphi \vee \psi \Rightarrow \Delta, \neg\varphi, \varphi \wedge \psi, \varphi \vee \psi}{\Gamma, \varphi \Rightarrow \varphi, \Delta}$$

$$\frac{\Gamma, \neg\psi, \varphi \wedge \psi, \varphi \vee \psi \Rightarrow \Delta, \neg\psi, \varphi \wedge \psi, \varphi \vee \psi}{\Gamma, \psi \Rightarrow \psi, \Delta}$$

Proof. Straightforward by applying $E\vee$, $E\wedge$ and $E\neg$ in this order and then LC and RC three times each. □

3 Soundness and completeness

In this section, we will provide a semantics for the derivability relation $\vdash^{seq}_{\mathbf{LK}_{W}^{\smallsetminus Id}}$, and prove soundness and completeness for it. For this purpose, the appropriate notion of satisfaction of a sequent by a valuation will be the notion of ts-satisfaction, as we shall see next.[5]

Definition 3.1. A weak Kleene valuation v *ts-satisfies* a sequent $\Gamma \Rightarrow \Delta$ if and only either $v(\gamma) \in \{0\}$ for some $\gamma \in \Gamma$, or $v(\delta) \in \{1\}$ for some $\delta \in \Delta$.

When no confusion arises we will talk about satisfaction in a valuation, simpliciter (omitting the ts part). Given this, we are in position to define a semantics for the derivability relation, as preservation of satisfaction at every valuation.

Definition 3.2. $S \vDash^{seq}_{\mathbf{WK}_{ts}} s$ if and only if for every weak Kleene valuation, if it ts-satisfies every sequent in S, then it ts-satisfies s.

[5]The notion of ts-satisfaction, as well as other notions of validity, counterexample, or satisfaction, can be found, e.g., in [6].

This relation is referred to in, e.g. Humberstone's [16], and related literature as the relation of local consequence or *local validity* given by a fixed notion of satisfaction for sequents (in this case ts-satisfaction), and a fixed set of valuations (in this case weak Kleene valuations, symbolized as **WK**). Thus, we will prove that: for any set of sequents S and sequent $\Gamma \Rightarrow \Delta$ over the propositional language $\mathcal{L}$.

$$S \vdash^{seq}_{\mathbf{LK}^{\smallsetminus Id}_{W}} \Gamma \Rightarrow \Delta \quad \text{if and only if} \quad S \vDash^{seq}_{\mathbf{WK}_{ts}} \Gamma \Rightarrow \Delta$$

In other words, we will show that the derivable rules in $\mathbf{LK}^{\smallsetminus Id}_{W}$ correspond to the locally valid metainferences with regard to the ts-consequence relation over the weak Kleene valuations. This will suffice in providing our target semantics for the derivability relation of our resident calculus.

3.1 Soundness

We can easily deal with the right-to-left direction of our target result, as follows.

Lemma 3.3. *For any set of sequents S and any sequent $\Gamma \Rightarrow \Delta$, if $S \vdash^{seq}_{\mathbf{LK}^{\smallsetminus Id}_{W}}$ $\Gamma \Rightarrow \Delta$, then $S \vDash^{seq}_{\mathbf{WK}_{ts}} \Gamma \Rightarrow \Delta$.*

Proof. Assume $S \vdash^{seq}_{\mathbf{LK}^{\smallsetminus Id}_{W}} \Gamma \Rightarrow \Delta$, and suppose there is a weak Kleene valuation v that ts-satisfies all the sequents in S. Since there is a derivation in $\mathbf{LK}^{\smallsetminus Id}_{W}$ from S to $\Gamma \Rightarrow \Delta$, and given the fact that the rules from $\mathbf{LK}^{\smallsetminus Id}_{W}$ preserve ts-satisfaction over the weak Kleene valuations, we can prove (by induction on the length of the derivation) that the aforementioned weak Kleene valuation v, in fact, ts-satisfies $\Gamma \Rightarrow \Delta$. □

3.2 Completeness

The hard part, as always, is coming from the proof theory to the semantics. In order to prove the main result of our paper, the completeness theorem, we will follow [15], but instead of using *deduction chains* [21], we will use the equivalent method of *reduction trees*, see [13, 22]. On another note, we will need to introduce some modifications regarding the method presented in [15], since the calculus under consideration contains additional elimination rules besides Cut.

Let S be any finite set of sequents. We will, then, prove that given any S and a particular sequent $\Gamma \Rightarrow \Delta$, either $S \vdash^{seq}_{\mathbf{LK}^{\smallsetminus Id}_{W}} \Gamma \Rightarrow \Delta$ or there is a weak Kleene valuation which ts-satisfies all the inferences in S but doesn't ts-satisfy the sequent $\Gamma \Rightarrow \Delta$.

For this purpose, assume some enumeration of the formulas of the language $F_0, F_1, \ldots$. Let's start with any sequent $\Gamma \Rightarrow \Delta$ (different from the empty sequent), denoted by $\Gamma_0 \Rightarrow \Delta_0$ in the tree.

Let $set(\Gamma)$ be the set of formulas in Γ, with Γ any sequence of formulas. We say that a sequent $\Gamma \Rightarrow \Delta$ is in the Weakening-closure (W-closure, for short) of a theory S if there exists a sequent $\Sigma \Rightarrow \Pi$ in S such that $set(\Sigma) \subset set(\Gamma)$ and $set(\Pi) \subset set(\Delta)$.

The idea is to build a tree in steps in such a way that at each step n we transform the topmost sequent of each branch of the tree ($\Gamma_n \Rightarrow \Delta_n$).[6] The tree so built will be called a *reduction tree*. If the topmost sequent of a branch of the tree belongs to the W-closure of S, then we say that the branch is closed (and otherwise is open). If all of the branches of a tree are closed, we say the tree is closed (and otherwise is open). The tree is built according to the following instructions:

n = 4m If $\Gamma_n \Rightarrow \Delta_n$ is not in the W-closure of S, then extend the tree with $F_m, \Gamma_n \Rightarrow \Delta_n$ and $\Gamma_n \Rightarrow F_m, \Delta_n$.

n = 4m + 1 If $\Gamma_n \Rightarrow \Delta_n$ is not in the W-closure of S, then, if $set(\Gamma_n) \cap set(\Delta_n) = \{\varphi_0, \dots, \varphi_l\}$, extend the tree with one only one node of the form:
$\neg\varphi_j, F_i \vee \varphi_j, \varphi_j \vee F_i, F_i \wedge \varphi_j, \varphi_j \wedge F_i, \Gamma_n \Rightarrow \neg\varphi_j, F_i \vee \varphi_j, \varphi_j \vee F_i, F_i \wedge \varphi_j, \varphi_j \wedge F_i, \Delta_n$
for each $i \in \{0, \dots, m\}$, and each $j \in \{0, \dots, l\}$. If $set(\Gamma_n) \cap set(\Delta_n) = \varnothing$ go to the next step.

n = 4m + 2 If $\Gamma_n \Rightarrow \Delta_n$ is not in the W-closure of S, then what to do is determined by the rightmost formula in Δ_n:

- If $\Delta_n = \Delta'_n, p$ with p a propositional letter, extend the tree with $\Gamma_n \Rightarrow p, \Delta'_n$.
- If $\Delta_n = \Delta'_n, \varphi \wedge \psi$, extend the tree with two branches: $\Gamma_n \Rightarrow \varphi \wedge \psi, \Delta'_n, \varphi$ and $\Gamma_n \Rightarrow \varphi \wedge \psi, \Delta'_n, \psi$.
- If $\Delta_n = \Delta'_n, \varphi \vee \psi$ extend the tree with three branches:
  - the first one with $\Gamma_n \Rightarrow \varphi \vee \psi, \Delta'_n, \varphi, \psi$
  - the second one with $\Gamma_n, \varphi \Rightarrow \varphi \vee \psi, \Delta'_n, \varphi$
  - the third one with $\Gamma_n, \psi \Rightarrow \varphi \vee \psi, \Delta'_n, \psi$
- If $\Delta_n = \Delta'_n, \neg\varphi$, extend the tree with $\Gamma_n, \varphi \Rightarrow \neg\varphi, \Delta'_n$.

n = 4m + 3 If $\Gamma_n \Rightarrow \Delta_n$ is not in the W-closure of S, then what to do is determined by the rightmost formula in Γ_n and it's left to the reader.

Stop condition If all the topmost sequents $\Gamma_n \Rightarrow \Delta_n$ are in the W-closure of S, stop the process.

[6] In order to simplify the notation, by $\Gamma_n \Rightarrow \Delta_n$ we refer to possibly different sequents, i.e. to the different topmost sequents belonging to each branch of the tree at the step n.

Now, we will prove that if the reduction tree is closed, then we can build a proof of the root sequent from the set of sequents S. In order to do that, we need the following definition:

Definition 3.4. The height of a finite tree is the number of nodes in the longest branch of the tree (i.e. the branch containing more nodes).

Lemma 3.5. *Given some set of sequents S and a sequent $\Gamma \Rightarrow \Delta$, if the reduction tree is closed then $S \vdash^{seq}_{\mathbf{LK}^{\vee Id}_{W}} \Gamma \Rightarrow \Delta$.*

Proof. Notice first that if a tree is closed then all the topmost sequents of all the branches are in the W-closure of S. So, take any of these topmost sequents, say $\Gamma \Rightarrow \Delta$ and let $\Sigma \Rightarrow \Pi \in S$, such that $set(\Sigma) \subset set(\Gamma)$ and $set(\Pi) \subset set(\Delta)$. It's straightforward to check that then $\Gamma \Rightarrow \Delta$ can be obtained from $\Sigma \Rightarrow \Pi$ by successive applications of Weakening (LW and/or RW), and possibly Exchange (LE and RE) and Contraction (LC and RC). In other words, there is a derivation from S to the topmost sequents of all the closed branches. Now we will prove by induction on the height of the tree that from the topmost sequents of any finite tree we can build a derivation of the root sequent. Of course we will only focus on the interesting cases.

The base case is trivial and consists in a tree with height 1. By the way the tree is constructed, this means that the tree has only one node and therefore the topmost sequent of the tree coincides with the root sequent (and is a sequent in S).

For the inductive step, assume all the topmost sequents in any branch of any tree with height lesser or equal than n are such that from all of them it is possible to build a proof of the root sequent. Now, we need to prove that from the topmost sequents of a tree with height lesser or equal than $n+1$ it is possible too. Thus, we only need to consider how the topmost sequents of the subtree of height $n+1$ are obtained. So, we need to take into account the last instruction applied in order to generate each topmost sequent. Most of the instructions read from top to bottom are simple applications of rules of the calculus $\mathbf{LK}^{\vee Id}_{W}$ with some Contraction and Exchange, and therefore are left to the reader. The most interesting cases are the sequents obtained by the application of the reduction instruction for a conjunction on the left, a disjunction on the right and the instructions applied when some formula is both on the antecedent and the succedent of a sequent.

Let's start then with the first case. Assume we have the following situation in the topmost sequents of the subtree with height $n+1$ (Case i):

$$\dfrac{\varphi \wedge \psi, \Gamma, \varphi, \psi \Rightarrow^{n+1} \Delta \qquad \varphi \wedge \psi, \Gamma, \varphi \Rightarrow^{n+1} \Delta, \varphi \qquad \varphi \wedge \psi, \Gamma, \psi \Rightarrow^{n+1} \Delta, \psi}{\dfrac{\Gamma, \varphi \wedge \psi \Rightarrow^{n} \Delta}{\vdots}}$$

where the superscript in the $\Rightarrow^{n+1}$ means that the sequent is the topmost sequent of a tree of height $n+1$. Notice, that read from top to bottom the tree above is not a derivation in $\mathbf{LK}_W^{\setminus Id}$.

The second case is somewhat similar to the previous one. Suppose we get the following situation in the topmost sequents of the subtree with height $n+1$ (Case ii):

$$\dfrac{\dfrac{\Gamma \Rightarrow^{n+1} \varphi \vee \psi, \Delta, \varphi, \psi \qquad \Gamma, \varphi \Rightarrow^{n+1} \varphi \vee \psi, \Delta, \varphi \qquad \Gamma, \psi \Rightarrow^{n+1} \varphi \vee \psi, \Delta, \psi}{\Gamma \Rightarrow^{n} \Delta, \varphi \vee \psi}}{\vdots}$$

It is not the case that read from top to bottom the above tree is a derivation in $\mathbf{LK}_W^{\setminus Id}$.

In both cases, the proofs of the bottom sequents from the topmost sequents can be easily obtained just by applying $L\wedge^*$ and $R\vee^*$, plus some applications of ER, EL, CL, CR.

The last interesting case is the following (Case iii):

$$\dfrac{\dfrac{\neg\varphi_j, F_i \vee \varphi_j, \varphi_j \vee F_i, F_i \wedge \varphi_j, \varphi_j \wedge F_i, \Gamma_n \Rightarrow^{n+1} \neg\varphi_j, F_i \vee \varphi_j, \varphi_j \vee F_i, F_i \wedge \varphi_j, \varphi_j \wedge F_i, \Delta_n}{\Gamma' \Rightarrow^{n} \Delta'}}{\vdots}$$

with $j \in \{1, \dots, l\}$ and $\{\varphi_1, \dots, \varphi_l\} = set(\Gamma') \cap set(\Delta')$ and $i \in \{0, \dots, m\}$, where $n = 4m+1$. Again, this tree read from top to bottom is not a legitimate derivation in $\mathbf{LK}_W^{\setminus Id}$.

However, it is very easy to notice that in order to obtain the sequent of level n from the sequent of level $n+1$ one needs to apply several times $E\neg$, $E\vee$, $E\wedge$, LE, RE, LC and RC. Of course, the order and the number of the application of these rules will ultimately depend on l and the formulas in Γ_n, Δ_n.

By Inductive Hypothesis we know that there is a proof from the conclusion sequents of each of these derivations and all the other topmost sequents of the tree of height n to the root sequent (since they are topmost sequents of a tree of height n). Since we have built derivations of these sequents from the topmost sequents of trees of height $n+1$, we can combine the two proofs and obtain a proof from the topmost sequents of a tree of height $n+1$ to the root sequent. Notice that any closed tree is a finite tree, and therefore in any closed tree we have a proof from S to the topmost sequents of the tree and from them to the root sequent, i.e. we have a proof from S to the root sequent. □

Now, we need to prove that if a tree is open, then we can build a counterexample. In order to do that, we need to introduce the following definition:

Definition 3.6. The complexity of a formula φ is the number of connectives of the formula.

Lemma 3.7. *Given some set of sequents S and a sequent $\Gamma \Rightarrow \Delta$, if the reduction tree is open, then there is a weak Kleene valuation v which ts-satisfies all the sequents in S but does not ts-satisfy $\Gamma \Rightarrow \Delta$.*

Proof. Suppose the reduction tree for $\Gamma \Rightarrow \Delta$ from S is open and then choose any open branch of it. We will now consider the result of collecting in Γ_ω all the formulas appearing to the left of any sequent of this branch and in Δ_ω all the formulas appearing to the right of any sequent of this branch. Naturally, we can consider the sets $set(\Gamma_\omega)$ and $set(\Delta_\omega)$ having all the formulas appearing in these collections. Moreover, we know that no sequent appearing on any node of this branch is in the W-closure of the theory S. Furthermore, by construction, we know that every formula appears either to the left or to the right, or both, in one of the sequents of this branch. Therefore, we know that for every formula φ it is such that $\varphi \in \Gamma_\omega$ or $\varphi \in \Delta_\omega$. Now, consider a valuation v such that

$$v(p) = \begin{cases} 1 & \text{if } p \in \Gamma_\omega \text{ and } p \notin \Delta_\omega \\ 0 & \text{if } p \notin \Gamma_\omega \text{ and } p \in \Delta_\omega \\ 1/2 & \text{if } p \in \Gamma_\omega \text{ and } p \in \Delta_\omega \end{cases}$$

We now show a number of things regarding this valuation. First, that it is a weak Kleene valuation such that for all φ, $v(\varphi) = 1$ if and only if $\varphi \in \Gamma_\omega$ and $\varphi \notin \Delta_\omega$, $v(\varphi) = 0$ if and only if $\varphi \notin \Gamma_\omega$ and $\varphi \in \Delta_\omega$, whereas $v(\varphi) = 1/2$ if and only if $\varphi \in \Gamma_\omega$ and $\varphi \in \Delta_\omega$. Secondly, that it will constitute a ts-counterexample to the root sequent $\Gamma \Rightarrow \Delta$ while ts-satisfying all of the sequents in the theory S.

Let's start by proving the first thing above. We prove this by induction on the complexity of the formulas. The base case where φ has no connectives i.e. is a propositional variable, is obvious by definition. Let's move then to the inductive step. Assume it holds for formulas with complexity n or less, and prove that it also holds for formulas of complexity $n+1$. So, φ is either a negation, a conjunction, or a disjunction. We will consider each of these cases.

If $\varphi = \neg\psi$ we have to consider three cases. Firstly, suppose $\neg\psi \in \Gamma_\omega$ and $\neg\psi \notin \Delta_\omega$. By the former, we know that $\psi \in \Delta_\omega$ given the reduction rules, but we also know that $\psi \notin \Gamma_\omega$ because otherwise $\neg\psi \in \Delta_\omega$ would be the case given the reduction rules, but contrary to our assumptions. Therefore, by the inductive hypothesis, we know that $v(\psi) = 0$ and furthermore that $v(\neg\psi) = 1$. Secondly, suppose $\neg\psi \notin \Gamma_\omega$ and $\neg\psi \in \Delta_\omega$. By the latter, we know that $\psi \in \Gamma_\omega$ given the reduction rules, but we also know that $\psi \notin \Delta_\omega$ because otherwise $\neg\psi \in \Gamma_\omega$ would be the case given the reduction rules, but contrary to our assumptions. Therefore, by the inductive hypothesis, we know that $v(\psi) = 1$ and furthermore that $v(\neg\psi) = 0$. Thirdly, suppose $\neg\psi \in \Gamma_\omega$ and $\neg\psi \in \Delta_\omega$. By the former, we know that $\psi \in \Delta_\omega$ given the reduction rules, and by the latter we know that $\psi \in \Gamma_\omega$ given the reduction rules. Thus, by the inductive hypothesis, we know that $v(\psi) = 1/2$ and furthermore that $v(\neg\psi) = 1/2$.

If $\varphi = \psi \wedge \chi$ we have to consider three cases. Firstly, suppose $\psi \wedge \chi \in \Gamma_\omega$ and $\psi \wedge \chi \notin \Delta_\omega$. By the former, and given the reduction rules we know that either $\psi \in \Gamma_\omega$ and $\chi \in \Gamma_\omega$, but none belongs in Δ_ω, or $\psi \in \Gamma_\omega$ and $\psi \in \Delta_\omega$, or $\chi \in \Gamma_\omega$ and $\chi \in \Delta_\omega$. But given the reduction rules, the latter two cases would imply that $\psi \wedge \chi \in \Delta_\omega$, contrary to our assumptions. Thus, we are necessarily in the case where $\psi \in \Gamma_\omega$ and $\chi \in \Gamma_\omega$, but none belongs in Δ_ω. Whence, by the inductive hypothesis, we know that $v(\psi) = v(\chi) = 1$ and furthermore $v(\psi \wedge \chi) = 1$. Secondly, suppose $\psi \wedge \chi \notin \Gamma_\omega$ and $\psi \wedge \chi \in \Delta_\omega$. By the latter, we know that $\psi \in \Delta_\omega$ or $\chi \in \Delta_\omega$, given the reduction rules. Moreover, we know that in each of the respective cases, $\psi \notin \Gamma_\omega$ or $\chi \notin \Gamma_\omega$, because otherwise $\psi \wedge \chi \in \Gamma_\omega$ would be the case given the reduction rules, contrary to our assumptions. Therefore, by the inductive hypothesis, we know that $v(\psi) = 0$ or $v(\chi) = 0$, while also being the case that $v(\psi) \neq 1/2 \neq v(\chi)$, and furthermore that $v(\psi \wedge \chi) = 0$. Thirdly, suppose $\psi \wedge \chi \in \Gamma_\omega$ and $\psi \wedge \chi \in \Delta_\omega$. By the former, given the reduction rules we know that either $\psi \in \Gamma_\omega$ and $\chi \in \Gamma_\omega$, or $\psi \in \Gamma_\omega$ and $\psi \in \Delta_\omega$, or $\chi \in \Gamma_\omega$ and $\chi \in \Delta_\omega$. By the latter, we know that $\psi \in \Delta_\omega$ or $\chi \in \Delta_\omega$. Thus, by inductive hypothesis, either $v(\psi) = 1/2$ or $v(\chi) = 1/2$. Whence, $v(\psi \wedge \chi) = 1/2$. The case for $\varphi = \psi \vee \chi$ is analogous to the previous case and is left to the reader as an exercise.

Given this, we now prove the second fact mentioned above, that it will constitute a *ts*-counterexample to the root sequent $\Gamma \Rightarrow \Delta$ while not constituting a *ts*-counterexample of this kind for any of the sequents in the theory S. The former is guaranteed by the fact that $set(\Gamma) \subseteq set(\Gamma_\omega)$ and $set(\Delta) \subseteq set(\Delta_\omega)$, thus for all $\gamma \in \Gamma$ and all $\delta \in \Delta$, v is such that $v(\gamma) \in \{1, 1/2\}$ while $v(\delta) \in \{1/2, 0\}$. The latter is guaranteed by the fact that no sequent appearing on any node of this branch is in the W-closure of any sequent in S. This further guarantees that v is not a valuation such that for any sequent $\Sigma \Rightarrow \Pi$ in S, and for all $\sigma \in \Sigma$ and all $\pi \in \Pi$, $v(\sigma) \in \{1, 1/2\}$ while $v(\pi) \in \{1/2, 0\}$. Therefore, v is a valuation that *ts*-satisfies all of the sequents in S. This concludes the proof. □

Theorem 3.8. *For any set of sequents S and any sequent $\Gamma \Rightarrow \Delta$, if $S \vDash^{seq}_{\mathbf{WK}_{ts}} \Gamma \Rightarrow \Delta$, then $S \vdash^{seq}_{\mathbf{LK}^{\setminus Id}_{W}} \Gamma \Rightarrow \Delta$.*

Proof. From Lemmas 3.5 and 3.7. □

Before ending this section, let's go back to the differences between **LK** and $\mathbf{LK}^{\setminus Id}_{W}$. First, now it should be clear why the rules $L\wedge$ and $R\vee$ are not included in $\mathbf{LK}^{\setminus Id}_{W}$: they are not locally valid with regard to the *ts*-

consequence relation over the weak Kleene valuations. Take for example the following application of $L\wedge$:

$$\frac{p, q \Rightarrow r}{p \wedge q \Rightarrow r}\ L\wedge$$

It's easy to note that this instance is not locally valid (e.g. the valuation $v(p) = 1/2$, $v(q) = v(r) = 0$).

On the other hand, for the sake of completeness we need to replace them by the weaker, but sound, $L\wedge^*$ and $R\vee^*$. Regarding the elimination rules, as it becomes clear from the proofs in this section, these are crucial to obtain completeness. Of course, these rules are derivable in **LK**, but once Id is subtracted from **LK**, the elimination rules for the connectives are not derivable anymore, and that's why we need to add them.

4 The consequences for paracomplete weak Kleene

Now that we have shown that the derivability relation of our calculus coincides with the preservation of ts-satisfaction by a weak Kleene valuation (i.e., that $\vDash^{seq}_{\mathbf{WK}_{ts}} = \vdash^{seq}_{\mathbf{LK}^{\setminus Id}_{W}}$) we will use these facts to show that our target calculus can, indeed, represent a proof-theory for $\mathbf{K_3^w}$. Notice first the following:

Definition 4.1. $\Gamma \vDash_{\mathbf{K_3^w}} \Delta$ if and only if for all weak Kleene valuations v, if $v(\gamma) = 1$ for all $\gamma \in \Gamma$, then $v(\delta) = 1$ for some $\delta \in \Delta$.

Lemma 4.2. $\Gamma \vDash_{\mathbf{K_3^w}} \Delta$ *if and only if* $\{\Rightarrow \gamma_1, \dots, \Rightarrow \gamma_n\} \vDash^{seq}_{\mathbf{WK}_{ts}} \{\Rightarrow \Delta\}$.

Proof. Straightforward, by noticing that $\{\Rightarrow \gamma_1, \dots, \Rightarrow \gamma_n\} \vDash^{seq}_{\mathbf{WK}_{ts}} \{\Rightarrow \Delta\}$ if and only if either some of the $\Rightarrow \gamma_i$ is not ts-satisfied by a weak Kleene valuation, or $\Rightarrow \Delta$ is ts-satisfied by a weak Kleene valuation. If the former, then this means that $v(\gamma_i) \in \{1/2, 0\}$, whereas the latter would mean that $v(\delta) = 1$ for some $\delta \in \Delta$. It is immediate to notice that this would amount to this valuation not being a counterexample to $\Gamma \vDash_{\mathbf{K_3^w}} \Delta$, whence the satisfaction conditions for $\{\Rightarrow \gamma_1, \dots, \Rightarrow \gamma_n\} \vDash^{seq}_{\mathbf{WK}_{ts}} \{\Rightarrow \Delta\}$ and $\Gamma \vDash_{\mathbf{K_3^w}} \Delta$ are equivalent. □

Finally, this suffices to show that the external consequence relation of our non-reflexive sequent calculus appropriately characterizes the consequence relation of $\mathbf{K_3^w}$.

Theorem 4.3. $\Gamma \vDash_{\mathbf{K_3^w}} \Delta$ *if and only if* $\Gamma \vdash^{E}_{\mathbf{LK}^{\setminus Id}_{W}} \Delta$.

Proof. Immediate from Lemma 4.2, Lemma 3.3, and Theorem 3.8. □

Before turning the page on these observations, one may reflect a bit about how the involvement of variable-inclusion characteristic of $\mathbf{K_3^w}$ is present in our calculus. In this vein, it is well known as reported, e.g., in [18], that $\Gamma \vDash_{\mathbf{K_3^w}} \varphi$ if and only if $\Gamma \vDash_{\mathbf{CL}} \varnothing$, or $\Gamma \vDash_{\mathbf{CL}} \varphi$ and $Var(\varphi) \subseteq Var(\Gamma)$. This could lead some to wonder how a similar fact is to be represented within the $\mathbf{LK}_W^{\setminus Id}$. For that purpose, observe the following general fact, and the more concrete reflection on the limited validity of reflexivity within said calculus:

Theorem 4.4. *The following rule is derivable in* $\mathbf{LK}_W^{\setminus Id}$

$$\frac{\Gamma \Rightarrow \Delta}{\varphi \Rightarrow \varphi}$$

if $var(\varphi) \subseteq var(\gamma)$ *and* $var(\varphi) \subseteq var(\delta)$, *for every* $\gamma \in \Gamma$ *and* $\delta \in \Delta$.

Proof. Let Γ, Δ be two sequences of formulas and φ a formula such that $var(\varphi) \subseteq var(\gamma)$ and $var(\varphi) \subseteq var(\delta)$ for every $\gamma \in \Gamma$ and $\delta \in \Delta$. Let v be a weak Kleene valuation that it is a ts-counterexample of $\varphi \Rightarrow \varphi$. It is straightforward to notice that it should be the case that $v(p) = 1/2$ for some $p \in var(\varphi)$ (otherwise it couldn't be a counterexample of φ). But, by hypothesis, $p \in var(\gamma)$ and $p \in var(\delta)$, for every $\gamma \in \Gamma$ and $\delta \in \Delta$. Hence, since v is a weak Kleene valuation, $v(\gamma) = v(\delta) = 1/2$ for every $\gamma \in \Gamma$ and $\delta \in \Delta$. By definition this means that v is a counterexample of $\Gamma \Rightarrow \Delta$. This implies that $\Gamma \Rightarrow \Delta \vDash^{seq}_{\mathbf{WK_{ts}}} \varphi \Rightarrow \varphi$. By completeness, i.e., by Theorem 3.8, $\Gamma \Rightarrow \Delta \vdash^{seq}_{\mathbf{LK}_W^{\setminus Id}} \varphi \Rightarrow \varphi$. In other words, the rule

$$\frac{\Gamma \Rightarrow \Delta}{\varphi \Rightarrow \varphi}$$

it is derivable in $\mathbf{LK}_W^{\setminus Id}$.

□

Notice that with this observation we do not intend to exhaust the characterization of all the cases in which Identity or Reflexivity is derivable in our calculus, but just to emphasize that in some instances the variable-inclusion requirements have a connection with it being so. Furthermore, a fully detailed account of the link between variable-inclusion requirements and rule derivability in our calculus (beyond the restricted case of the external consequence relation) is also worth exploring, but will take much more space than what we are granted here.

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