Against the countable transitive model approach to forcing

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Abstract: In this paper, I argue that one of the arguments usually put forward in defence of universism is in tension with current set theoretic practice. According to universism, there is only one set theoretic universe, V, and when applying the method of forcing we are not producing new universes, but only simulating them inside V. Since the usual interpretation of set generic forcing is used to produce a "simulation" of an extension of V from a countable set inside V itself, the above argument is credited to be a strong defence of universism. However, I claim, such an argument does not take into account current mathematical practice. Indeed, it is possible to find theorems that are available to the multiversists but that the advocate of universism cannot prove. For example, it is possible to prove results on infinite games in non-well-founded set-theories plus the axiom of determinacy (such as $ZF + AFA + PD$) that are not available in $ZFC + PD$. These results, I contend, are philosophically problematic on a strict universist approach to forcing. I suggest that the best way to avoid the difficulty is to adopt a *pluralist* conception of set theory and embrace a set theoretic multiverse. Consequently, the current practice of set generic forcing better supports a multiverse conception of set theory.

Keywords: Set theoretic multiverse, Foundations of Mathematics, Forcing, Set Theory

1 Introduction

Universism is the thesis that there is only one set theoretic universe, V, that instantiates the axioms of ZFC. Such a position has been argued by several authors, for example by Martin (2001), Woodin (2011), Isaacson (2011), Shelah (2014), Barton (2019), and Livadas (2020). This universe is the so called *canonical* model of set theory, as opposed to all the others different models (such as, for instance, the constructible universe L). Although it is true that set theorists make use of all kinds of non-canonical models, universists typically insist that each of these models can be "simulated" within

V and that, in the end, they are only “simulated universes”. Thus, universists typically argue against pluralist conception of set theory on the ground that the non-canonical universes that populate the so-called multiverses can be simulated within V. Before proceeding with the paper, I need to put forward a disclaimer. Throughout the paper, I will refer to “models” and “universes” interchangeably, as usually done in the multiverse literature.

Such a simulation is carried out through forcing. Forcing allows us to produce non-canonical models of V by extending its width. For example, we can use forcing to build models of $ZFC + \neg CH$, or models of $ZFC + CH$. In a very intuitive way, such an application of forcing produces an extended V that is a model of $ZFC + \neg CH$ (or $ZFC + CH$ as the case may be). Although this is the most intuitive explanation of forcing, one can understand forcing in at least three different ways:

1. the countable transitive model approach to forcing;
2. the Boolean evaluated approach to forcing;
3. the natural approach to forcing.

Among these, the first is the one that is usually accepted by set theorists. From a philosophical point of view, it is usually put forward as a defence of universism against pluralist conceptions, according to which set theory comprises a plurality of set-theoretic universes. (for a defence of this approach to forcing see Barton, 2019). According to this, forcing allows us to simulate different kind of models and universes, without actually producing them. This is because forcing is applied only to a countable transitive model inside V. With the countable transitive model approach to forcing we are producing a forcing extension of that model, not of the entire set theoretic universe V. Consequently, the entirety of set theoretic practice and forcing application are actually carried out inside the single universe V. On the other hand, the natural approach to forcing can be applied to the entire set theoretic universe, and thus is the most similar and respectful of the intuition that every application of forcing produces a new set theoretic universe (for example, one in which CH is true and one in which it is instead false). It is the one defended by Hamkins and is the basis of his set theoretic multiverse (see e.g. Hamkins, 2012), that accept every possible set theoretic universe as legitimate and existing. Finally, the Boolean evaluated approach can be carried out looking only at the forcing relation, without appealing to models, and moreover it is needed to prove some interesting results like the Intermediate

Model Theorem (this theorem says that if $V \models ZFC$ and $W \models ZFC$ is an intermediate model between V and one of its forcing extension then W is also a forcing extension of V, see Grigorieff, 1975 for details).

It is also possible to consider a purely syntactical approach to forcing, where we focus only on the partial order $(P, \leq)$. In this case we avoid the appeal to models, universes, generic filters and the like, and instead investigate only the forcing relation. In other words, we don't say that there is a model that satisfies φ, but instead only that there is a $p \in P$ which *forces* φ.

In this paper, I argue that the common interpretation of forcing is actually inadequate, since it does not align with current set theoretic practice and it restricts the range of possible results. In particular, I argue against its use in the universist's argument that the multiverse can be simulated inside the single universe V. I will do this by first arguing that, although in the single universe V we can actually simulate any non-canonical model of set theory, we cannot efficiently compare them all. In particular, we cannot resort to model-theoretic techniques, or to the tools of MAXIMIZE and restrictiveness (for example as presented by Maddy, 1996 and Incurvati & Löwe, 2014).

The paper is structured as follows. First of all I introduce the countable transitive model approach to forcing (section 2), both from the mathematical perspective (section 2.1) and the philosophical one (section 2.2). After this, I present my arguments against it (section 3), with particular attention to some examples regarding determinacy and ill-founded set theory (section 3.1). In the end, some concluding remarks summarise the paper (section 4).

2 The countable transitive model approach to forcing

Forcing was first developed in the 60's by Cohen as a method to produce non-canonical models of set theory (see Cohen, 2003). These models take the form of *width* extensions of V, i.e. enlargement of V "to the outside". Since its inception, forcing has been used mainly to prove the independence of certain set theoretic statements from ZFC, with the two most notable examples being the Axiom of Choice AC and the Continuum Hypothesis CH. To prove the independence of such statements, forcing is used to build two different extensions of V, one in which $ZFC + CH$ holds, and the other in which $ZFC + \neg CH$ holds (or, in the case of AC, one in which ZFC holds and one in which $ZF + \neg AC$ holds).

2.1 Mathematical details

As mentioned in the introduction, I am mainly interested in arguing against the countable transitive model approach to forcing and its use in the universist's argument that the multiverse can be simulated inside the single universe V. Before giving the details of such an argument, I need to clarify in what sense the set theoretic multiverse can be "simulated" in the single universe V. In set theory, we have the classical axiomatization ZFC and its canonical model, the cumulative hierarchy V. Through the application of set generic forcing, it is possible to produce a non-canonical model of ZFC from V. That is to say, we can "create" a new model of ZFC + some other statement: the usual example is the mutually incompatible models of $ZFC + CH$ and $ZFC + \neg CH$. In this case, we are creating two new models, V' and V^*, in which the Continuum Hypothesis is, respectively, true and false. These two models are "fatter", i.e. larger than the original V: they are usually considered *width extensions* of V, produced by the addition of new subsets to the cumulative hierarchy. There are also *height extensions* of V, produced by the additions of new ordinals, but they are not relevant for this particular argument. Crucially, set forcing cannot be applied to the whole V, but only to countable sets. Consequently, according to such approach, in order to produce a model of, for example, $ZFC + CH$, we take a countable set *in* V that "simulates" the whole universe, and apply set forcing to it to produce its width extension. But since we started with a countable set *inside* V, we are not producing a whole new universe, but only a slighter larger countable set inside the canonical universe (for a detailed account of forcing, see Nik, 2014).

The countable transitive model approach to forcing interprets the procedure just described at face value (see for example Barton, 2019). On such a view, there is nothing outside V, and when forcing a given model M we are not simulating something that could be outside V. Rather, the alternative universes that we are "seeing" are nothing but a technical artifact. This is the usual answer that universists give considering the fact that the cumulative hierarchy of sets V is usually conceived as already containing all possible subsets. Consequently, how could we add new subsets to it if it already contains all of them? The countable transitive model approach solves this impasse by appealing to the fact that all consistent first order theories have a countable model. Thus, there is a countable set $M \subset V$ that it is a model of ZFC. From Cantor's diagonal argument, we know that no countable set can contain all subsets of any infinite set it contains, so our

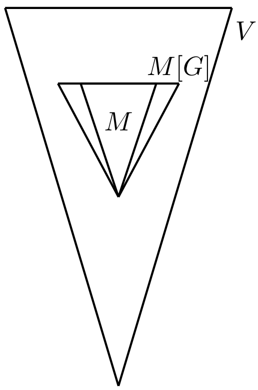

Figure 1: A representation of the countable transitive model approach to forcing

countable set M must miss some of these subsets. Forcing (following this approach) helps to "complete" the set M, by adding some of those missing sets. Moreover, we choose which subsets we want to add in such a way that the new expanded M' will be a model of the theory we are trying to study (for example $ZFC + \neg CH$).

Without delving too deeply in the details, the following is a sketch of how the forcing technique works. We start by specifying an infinite partial order $\mathbb{P} \in M$. Since M is a countable set, it follows from Cantor's diagonal argument that there is some $G \subset \mathbb{P}$ that it is not in M. We pick a G such that it is *generic filter* over M (so that the resulting operation is called *set-generic* forcing). The key properties of a generic filter are that:

1. G is a subset of $\mathbb{P}$;

2. $1 \in G$;

3. if p is a forcing condition (forcing conditions are members of $\mathbb{P}$, the smaller they are the stronger they are) weaker than q and $q \in G$, then $p \in G$;

4. if $p, q \in G$, then there exists in G a forcing condition smaller than both of them;

5. G meets all dense subsets of $\mathbb{P}$ that are in M.

The first four properties ensure that G is a filter, the last one that it is generic. We add such a G to M, and write $M[G]$ to denote the resulting extended M with the missing subset G. Let φ be the statement we want to prove. With forcing we can prove that for any such generic missing subsets of $\mathbb{P}$, $M[G] \models ZFC + \varphi$. At this point, we consider the forcing relation $\Vdash$, that connects facts about M to facts about $M[G]$. From this perspective, we are proving $ZFC + \varphi$ by proving a fact about M, i.e. that M, provided the poset $\mathbb{P}$, *forces* $ZFC + \varphi$, in symbols $M \Vdash_{\mathbb{P}} ZFC + \varphi$. Furthermore, note that this is only a claim about sets in M, and it does not carry any assumption about the whole V. However, it is possible to generalise such result, and prove that if there exists any M-generic G that is one of the "missing subsets" of $\mathbb{P}$, then a statement φ is forced if and only if for every such G, $M[G] \models \varphi$.

So, in other words, with forcing we can prove facts about M by adding to it some of the subsets that it is missing. Obviously such a method is more general: we are proving that any countable model $M \in V$ can be extended to the countable model $M[G] \models ZFC+$ some statement. Consequently, while the facts of the form $M \Vdash \varphi$ that we prove with forcing apply only inside V, we can interpret them as a "simulation" of an extended universe $V[G]$.

2.2 The philosophical argument in favour of universism

The appeal of this method for universists lies in the fact that it allows us to bypass the limitation of a single V that contains all possible subsets, while still being able to use forcing to produce non-canonical models of set theory. Their opposition to the multiverse conception of set theory is rooted in the fact that there are no V-generic filters, simply because, according to universists, V already has all the possible subsets, so we cannot apply our method of forcing to add some more subsets (the situation with class forcing is more complex, see for example Antos, 2018).

Consequently, the advocate of universism argues that it is not possible to apply set-generic forcing to the whole V, so that the very notion of "extension of V" is meaningless. All we can do is to use countable transitive sets inside V as simulations of the whole V, and consider the extensions of such countable models as "simulacra" of those elusive extensions of V.

Even though it may seem that the countable transitive model approach curtails the power of forcing, it is still versatile and powerful enough. For instance, in V we can construct both models of $ZFC + V = L$ and models of $ZFC + LCs$ (e.g. $ZFC+$ "there exists a measurable cardinal"), even though these two models are incompatible ($V = L$ is the Axiom of Constructability,

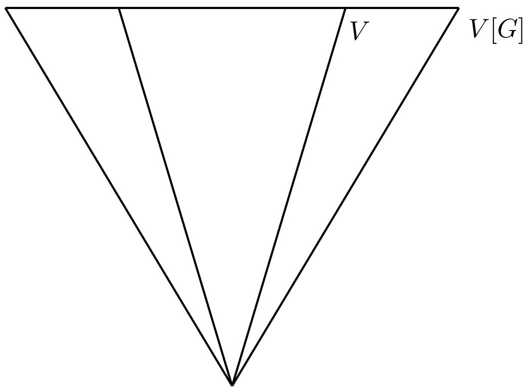

Figure 2: A representation of the natural interpretation to forcing

that says that all the sets of the universe can be built from simpler sets, and it is incompatible with the existence of most large cardinals). A point in favour of such an interpretation of forcing is the fact that every result achieved regarding forcing extensions and non-canonical models of ZFC can be achieved using such restrictive interpretation of forcing.

Moreover, even other interpretations of forcing, e.g.. the Boolean evaluated method and the natural forcing interpretations, can still be used as arguments for universism. With the Boolean evaluated method, instead of assigning a truth value to the statement φ we want to force, we pick a truth value from an atomless Boolean Algebra. We then pick an ultrafilter $\mathbb{P}$ in this algebra that assignes truth values to statements in our theory T. We can finally interpret the resulting model of the theory $T + \mathbb{P}$ as extending the old one with this new ultrafilter. The natural forcing method on the other hand admits the existence of a V-generic filter missing a subset G and thus of a generic extension $V[G]$. However, both alternative interpretations of forcing can still be considered compatible with the Single Universe view. The Boolean evaluated method can also be seen as being carried out entirely within V, since there is no appeal to its extensions, while the natural forcing interpretation (for details see Hamkins, 2012) can be interpret as a techical tool that creates the illusion of real extensions of V. Moreover, all the results that can be obtained using one of these forcing interpretations can be re-formulated using another one. So, for example, if I prove φ using the countable transitive model approach, I can also prove it using the Booleans evaluated method instead.

Thus, we have an intuitively natural way to interpret forcing from the universism perspective: forcing is actually carried out completely inside V, and even when it seems that this is not the case (with the natural forcing interpretation), we are in fact only using a technical artefact to make things easier and more intuitive.

3 Against the countable transitive model approach

The countable transitive model approach, while enticing, has some problems. First of all suppose that we accept the natural interpretation of forcing only for its instrumental value. This means not committing to the existence of the non-canonical models, however without commiting to their complete non-existence either. They are useful and allows us to gain insights on the true and unique universe of sets, and as such they are admitted in our set theoretic discourse. Although this objection is compelling, I claim that a purely instrumental view of these canonical models is already a pluralist and multiversistic view of the set theoretic universe. It may well be that, if asked, a universist will profess that there exists only one set theoretic universe (albeit indeterminate and incomplete). However, if her day to day practice is actually multiversist, her argument lose strenght.

Nevertheless, suppose instead that we don't want anything to do with these non-canonical extensions of V, and instead claim that we can investigate the countable models just as easily as the multiversist. In doing so, we appeal to the fact that we can use methods from model theory and methods from interpretability theory (see Visser, Enayat, Kalantari, & Moniri, 2006). However, neither of these methods is ideal, and there are some problems in using them. Very briefly, interpretability theory uses the notion of isomorphism to study the *sameness* of theories (e.g. mutual interpretability, synonymy, etc.). In particular, it is possible to use it to prove that a theory is restrictive over another theory, i.e. it proves fewer isomorphisms. This approach is quite promising, but it has one important drawback: it can only be used at its full force to compare models of theories that are *mutually bi-interpretable* (two theories are mutually bi-interpretable if and only if in any model M of the first theory T we can define a domain N in such a way that it is a model of the other theory S and the other way around, and if and only if these two new models N and N' are isomorphic, see Visser et al., 2006 for details). In our example, both the theories $ZFC + V = L$ and $ZFC+$ "there exists a measurable cardinal" are only mutually interpretable, but not *bi*-interpretable (no two theories extending ZFC are bi-interpretable, see Visser et al., 2006 and Enayat, 2016), so we cannot compare them using restrictiveness methods. The model-theoretic tools are also problematic. In general, it is preferable to use object-linguistic methods, i.e. methods that are carried out directly in the object language . However, model-theoretic methods are essentially meta-theoretic, since they take models of a given theory to be the objects of the investigation, and we cannot do this in the theory itself. In general, a method that works directly in the theory is to be preferred, since in this way we avoid having to justify the legitimacy of our meta-theory and our methods. The main consequence of this fact is that we cannot efficiently compare two non-canonical models. It would be better if we could compare them in a common context. In principle, ZFC could be such a context, but other than the technical problems just hinted there are also philosophical and interpetative problems. In particular, if we take a universist stance regarding V, we have to commit to it: is it equal to L, but without measurable cardinals, or not (but with measurable cardinals)? This forced choice will restrict the range of possible results we can prove. So the problem for the universist still stands: committing to a single universe, in which the full power of forcing techniques cannot be used, or accept a pluralist conception (even if only for instrumental reasons). Indeed, all these non-canonical models are available in the set theoretic multiverse, and we can prove all possible results comparing them, especially isomorphisms between their structures. This can be done in a better way because in the set theoretic multiverse we can define a different language for each of the universes (see Hamkins (2012) for details). This enable us to use the methods of restrictiveness without any limitation. Moreover, we could even easily use model theoretic methods without having to retreat to the meta-theory: we can do all sorts of model theoretic constructions directly in the theory of the multiverse, in which the various universes are only objects. Clearly this cannot be done in *all* multiverses: there are multiverse conceptions perfect for this purpouses and conceptions less so. However, in general, compared with a pluralistic conception of set theory, such as the ones advocated by Hamkins (2012), Steel (2014), Antos, Friedman, Honzik, and Ternullo (2018) and Gorbow and Leigh (2020), the universist conception has a lot less power in comparing these non-canonical models.

The other problem with this account is that it does not allow to compare two non-canonical models that are *mutually incompatible* in an efficient way (as already mentioned). Moreover, it is not possible to efficiently use one model in the investigation of the other one. For example, it is possible to first produce by forcing a model M of $ZFC + CH$ and then a model N of $ZFC + CH + PD$ such that $M \subseteq N$. And in this case we can use all the tools available in the model of $ZFC + CH$ in our second model of $ZFC + CH + PD$. However, consider the situation in which we first force a model M of $ZFC + \neg CH$, and then a model N of $ZFC + CH$ such that $M \subseteq N$. In this case, we cannot use any added object or method from M in the new model N (in particular, we cannot use the existence of a set x such that $|\mathbb{N}| < |x| < |\mathbb{R}|$ from the model $M \models ZFC + \neg CH$ in the model $N \models ZFC + CH$). For a less trivial example, consider $M \models ZFC+$ "there exists a measurable cardinal" and $N \models ZFC + V = L$: we cannot use the existence of a measurable cardinal in M to prove something about N, since the measurable cardinal cannot be found in N.

Another drawback of the countable transitive model approach is that it can be applied only to a very small subset of all the possible models of set theory, i.e. to only the countable transitive ones, and even this application is problematic, mainly because there are meta-mathematical problems regarding them (as Hamkins, 2012 points out). First, it follows from the Incompleteness Theorem, that if ZFC is consistent, then it cannot prove the existence of these models. A possible solution would be to assume that ZFC is consistent (so we assume $Con(ZFC)$). In this new theory, we can indeed prove that there are countable models of ZFC. However, we still cannot prove that they are *transitive*, since if they exist then $ZFC + Con(ZFC)$ is consistent (so $Con(ZFC + Con(ZFC))$). Consequently, we can either simply assume their existence (but this is not really satisfactory), or work in a stronger theory than ZFC that can serve as a suitable meta-theory. However, this move is hardly justifiable, since the same arguments will still apply to the meta-theory. Indeed, the strength of the universist's argument comes from the fact that everything is carried out inside V, and in the *theory* itself. However, the insurmountable problem of incompleteness compels the universist to carry out forcing in the meta-theory. Being a meta-theory, it has its own model, in which V is an object. Obviously this "meta-model" cannot be located inside V itself, but it has to be"outside" it. For the multiversist this is obviously not a problem (see for example Gorbow & Leigh, 2020 for an interesting construction around this problem). Even a very moderate potentialist (potentialism is one of the positions, the other being actualism, regarding the "expandability" of V: actualism states that V is not, while potentialism states that it is possible, with radical potentialism admitting expansion both in height, adding new ordinals, and in width, adding new subsets, while for moderate potentialism only one of those expansions is possible) that does not claim the existence of a full multiverse can still be able to operate “outside” V with the help of admissible sets (for details on how admissible sets work, see Barwise, 1975). On the other hand, the universist can no longer claim that everything he is doing is done inside V, and has to concede that there is actually something other than V.

Nevertheless, let’s ignore this point, and concede to the universist that their countable transitive model approach to forcing can actually be carried out inside V in its entirety without any of the problems mentioned above. Even in this case, the universist argument incurs in one major problem, caused by the fact that it cannot compare two incompatible models and use them to prove something in one another. Without this possibility, the range of possible results and theorems is restricted.

3.1 Determinacy and ill-founded set theory

The inability to consider (in the sense explained above) two mutually incompatible models produced by forcing means that there are results that cannot be proved in the Single Universe context described above. In contrast, in a multiverse conception of set theory (or a “Single Universe” with an instrumental acceptance of the natural approach to forcing) we can prove a theorem in a particular model using tools and methods from an incompatible model. Obviously we should proceed with caution. Suppose we have produced two models $N \subseteq M$, such that $M \models ZFC+$ “there exists a huge cardinal” and $N \models ZFC + V = L$. In this case, we cannot use the existence of a huge cardinal in M to prove the existence of a measurable cardinal in N, even though N is contained in M and the existence of a huge cardinal implies the existence of a measurable one.

For another, more complex, example, consider the relation between the Axiom of Determinacy (AD), ill-founded set theory, and infinite games. An infinite game is a series of plays in which two players, player I and player II, alternately pick a number (this could be a natural number or, in the case of the Banach-Mazur game, a real number). After infinite many such moves, a sequence of natural numbers $(n_i)_{i\in\omega}$ (or of real numbers, but here I will stick to the simplest case) is generated: if $(n_i)_{i\in\omega} \in A$, then player I wins, otherwise if $(n_i)_{i\in\omega} \notin A$ player II wins (figure 1 is an example of such games). The set A is the *outcome* set, or winning set. Player I has a *winning strategy* if there is a sequence of plays that he can make such that the overall sequence ends up in the outcome set, while player II has a winning strategy if he can make plays that avoids player I winning. The Axiom of Determinacy

I	r_0		r_2		$\ldots$	r_n	
II		r_1		r_3	$\ldots$		r_{n+1}

Table 1: An infinite game

states that every infinite game is determined, i.e. one of the players has a winning strategy (for details on AD see Woodin, 1999). Note that these does not entail that both players can have a winning strategy. We know that this axiom is incompatible with the Axiom of Choice, since using AC it is possible to build a game in which both players lose. However, it is possible to restrict ourselves to the Axiom of Projective Determinacy (PD), that instead states that the winning sets, i.e. the victory conditions, are projective sets, that is compatible with AC.

Intuitively, we can arrange and represent these infinite games as trees. All the possible sequences of the game are the various branches of the tree, and each play determines which branch we are following. The winning conditions does not change, and we can interpret the winning strategy in terms of branches: a winning strategy for player I is a branch that is in the outcome set, while a winning strategy for player II is a branch that is not in the winning set (see figure 3).

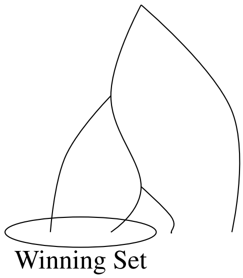

Figure 3: An infinite game as a tree

Now, since these infinite games are representable as trees, it is very natural to study them by means of a non-well-founded set theory (for an introduction to non-well founded set theory, see Aczel, 1988). Non-well founded set theory admits ill-founded sets, that is, sets that contains themselves as members. These sets are not admitted in ZFC (their existence is ruled out by the Axiom of Foundation), but they can be admitted in a non-well founded set theory by replacing the Axiom of Foundation with one of the several anti-foundation axioms. Among them, the most important is the Anti-Foundation Axiom (AFA), that states that every rooted digraph (a graph in which a vertex is recognised as the root) corresponds to a unique set. For example, the loop graph (a graph with only one vertex and an edge to itself) corresponds to the set $\{x\}$. Non-well founded set theory (for example ZFA, i.e. $ZFC-$ Foundation $+AFA$) is very similar to classical set theory, since all the proofs and definitions in which the Axiom of Foundation is not needed still go through as normal (for example, Russell's Paradox, the Axiom of Choice, relations, pairs, natural numbers, transfinite recursion etc.). The main difference is on how we can approach objects that could have some circularity in their definition (for example, graphs, infinite chains etc.).

Another possible formulation of AFA is that every graph as a unique decoration. A *decoration* d of a graph G is a function from the vertices to natural numbers, with the following property:

$$d(g) = \{d(h) : g \to h\}.$$

In other words, the decoration function takes a vertex as an input, and gives the set corresponding to its children as the output. For example, take the leaf of a digraph: since it has no children, its decoration is the empty set. Suppose instead that we want to calculate the decoration of a vertex just before the leaf: in this case, that vertex has one children (the leaf), so its decoration is the singleton $\{\emptyset\}$. A graph can be *extended* by adding to it the decoration of its children.

It is possible to approach questions about infinite games from the perspective of non-well-founded set theory. In particular, such a perspective enables us to use tools and methods available only in non-well-founded set theory (mainly abbreviations of proofs and definitions regarding graphs and infinite trees) to the investigation of infinite games. For example, we can prove that there is a point in which player I has a winning strategy and player II no longer has one appealing to extended graphs and decorations: if that particular tree can be extended by adding a subset of the winning set, then player I has a winning strategy while player II no longer has one (since every move that she can make are already in the subset of the winning set). This is only one of the possible examples of interaction between determinacy and non-well-founded set theory, but it is possible to use the axiom that every tree corresponds to a unique set to prove more interesting results (the existence of unique winning strategies, for example). However, to get these results, we need to assume AFA. Consequently, if we believe only in the countable transitive model approach to forcing sketched above, we would not be able to prove these results, since in that case we cannot use objects and tools from ZFA in $ZFC + PD$ (we could try to work directly in $ZFC + PD$, but this could be problematic). By contrast, even in a very simplified toy multiverse composed of only two universes, one well founded and one non-well founded, those results are instead available.

4 Concluding remarks

In conclusion, appealing to the fact that we can simulate any non-canonical model of set theory (an thus set theoretic universe) in the Single Universe V is in tension with current set theoretic practice. Thus, it cannot be used to defend the universist position. In particular, I claim that with the countable transitive model approach non-canonical incompatible models cannot be efficiently compared and that we cannot use them in the investigation of one another, which in turn makes it impossible to prove a number of results in set theory - results, however, that by contrast are attainable in a multiverse conception of set theory. For the sake of definitness, I have introduced some results that cannot be proved in the Single Universe context with the appeal to ill-founded sets, but that instead can only be proved in a set theoretic multiverse.. These results can be proved using the tools and methods of ill-founded set theory, and in particular the fact that every extended graph has a unique decoration. However to prove this we need both the Anti-Foundation Axiom and at least Projective Determinacy, and such a setting it is not possible in the Single Universe with only the countable transitive model approach to forcing at our disposal.

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