Better Semantics for the Pure Logic of Ground

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Abstract

Philosophers have spilled a lot of ink over the past few years exploring the nature and significance of *grounding*. Kit Fine has made several seminal contributions to this discussion, including an exact treatment of the formal features of grounding [Fine, 2012a]. He has specified a language in which grounding claims may be expressed, proposed a system of axioms which capture the relevant formal features, and offered a semantics which interprets the language. Unfortunately, the semantics Fine offers faces a number of problems. In this paper, I review the problems and offer an alternative that avoids them. I offer a semantics for the pure logic of ground that is motivated by ideas already present in the grounding literature, and for which a natural axiomatization capturing central formal features of grounding is sound and complete. I also show how the semantics I offer avoids the problems faced by Fine's semantics.

Philosophers have spilled a lot of ink over the past few years exploring the nature and significance of *grounding*. Grounding is supposed to be a certain relation of dependence and determination among facts. This relation is linked with a certain kind of explanation. It is targeted by the sort of explanations we give sometimes when we say that some facts obtain *in virtue of* other facts and when we ask what *makes* something the case. When, for instance, ethicists ask what makes murder wrong, epistemologists ask what makes my belief that I have hands justified, chemists ask what makes alcohol miscible in water, and physicists ask what makes gravity so weak, they are asking after the worldly conditions on which the facts in question depend, and by which they are determined.

What, though, is the nature of grounding? Though there is much disagreement over the details, some rough points of consensus over some of its formal features have emerged. Kit Fine has offered an exact treatment of these formal features of grounding [Fine, 2012a]. He specifies a language in which grounding claims may be expressed, proposes a system of axioms which capture the relevant formal features, offers a semantics which interprets grounding claims expressible in the language, and shows that his axioms are sound and complete for his semantics.

As we shall see, however, there are reasons for dissatisfaction with Fine's semantics. We could, of course, avoid the unacceptable results by simply dropping the ambition to provide a semantics for those grounding claims expressible in Fine's language. We might then affirm the principles of the pure logic of ground without offering any formally-specified conception of what grounding claims say or under what conditions they are true.

In this paper I show that there is another approach available. I offer a formally specified, model-theoretic semantics for Fine's language, for which a certain natural axiomatization of the pure logic of ground is sound and complete. The semantics is motivated by ideas already present in the grounding literature, so it offers a plausible candidate for an exact specification of an intended interpretation of grounding claims. I also show how the semantics I offer avoids problems faced by Fine's semantics.

1 Some Formal Features of Grounding

What are the formal features of grounding? As one might expect, there is no unanimity on this question. Even so, three rough points of consensus have emerged.

First, many grounding enthusiasts agree that grounding is an explanatory relation among facts (perhaps *inter alia*).[1] In particular, the facts that ground

[1] Even theorists who express reservations in theory about the idea that grounding relates facts are happy to talk as if it does. Fine [2001], for instance, contends that, strictly speaking, we don't need to reify facts and claim that grounding is a relation between them in order to give a theory of ground; we may instead treat talk of grounding's being a relation between facts as a mere *façon de parler*. Fine argues that we should formalize our theory of ground by appeal to sentential operators which do not pick out any relation, and whose arguments, semantically speaking, do not pick out entities. Still, Fine's formal semantic treatment in [Fine, 2012a] appeals to an ontology of facts, sets of which serve as the values of an interpretation function applied to sentences; cf. n. 8. I will follow Fine in couching what follows in terms of facts and

a given fact are those facts in virtue of which it obtains. Though sometimes a fact is grounded in a single fact, more typically the grounds for a fact are a plurality. For instance, most enthusiasts of grounding agree that conjunctive facts are grounded in their conjuncts. So, the fact that it is cloudy and chilly is grounded in two facts: its being cloudy, and its being chilly. Similarly, assuming that the fact that interest rates are low is grounded in psychological facts, presumably the psychological facts in virtue of which interest rates are low are legion.

Second, grounding enthusiasts agree that these explanations can be and often are mediated. So, for instance, the fact that interest rates are low might be grounded in the antics of quarks and leptons via various intermediate grounding links to psychological, neurological, and chemical goings-on. Grounding links among facts chain together, connecting a fact to its grounds, the grounds of its grounds, and so on. Thus, grounding relations are transitive.[2]

Third, and more contentiously, grounding is an irreflexive relation on facts. The idea is that no explanation of a fact can appeal to that very fact, on pain of circularity. So, no fact obtains in virtue of itself. Given the transitivity of grounding, its irreflexivity implies that it is also asymmetric. So, there are no facts f and g, such that f grounds g and vice versa.

2 The Principles of the Pure Logic of Ground

Fine's Pure Logic of Ground (PLG) captures these formal features. In the language of PLG, the grounding relation is represented by an operator, '$<$'. The grounding operator can be combined with a set of sentences Δ on its left-hand side and a single sentence ϕ on its right-hand side to yield a more complicated syntactic object, a *sequent*, of the form '$\Delta < \phi$'. Intuitively, a sequent of this form says of the plurality of facts expressed by sentences in Δ that they (collectively) ground the fact stated by ϕ.[3] Thus, the syntax of Fine's language

the grounding relations they may bear to one another. I presume that, if Fine's semantics can be recast so as not to engage the ontology of facts, the alternative I propose can similarly be recast.

[2]Transitivity of grounding is somewhat controversial; see [Schaffer, 2012] and [Litland, 2013] for discussion. It is trivial, of course, that the ancestral of an ordinary binary relation is transitive. We can characterize the ancestral of a many-one relation like grounding as the result of closing grounding under the grafting operation described below. So, if grounding itself turns out to be intransitive, then we can construe Fine's system as giving the logic of the ancestral of ground in this sense.

[3]The grounding operator '$<$' can only join *sentences*. It cannot join sequents. So, PLG provides no way to represent the logic of embedded grounding claims. Also, the sentences of

captures the first formal feature, that grounding relates facts, and that a single fact may be grounded in a plurality of facts.[4]

There is another operator in the language of PLG, '$\prec$', for partial ground, which can be combined with a single sentence on both its left and right to yield a sequent of the form '$\phi \prec \psi$'. Intuitively, a sequent of this form says that there is a (perhaps empty) set of sentences that, together with ϕ, collectively ground ψ. Fine's axiom

SUBSUMPTION $$\frac{\Delta, \phi < \psi}{\phi \prec \psi.}$$

enforces the intended relation between '$<$' and '$\prec$'. The axioms

TRANSITIVITY $$\frac{\phi \prec \psi \qquad \psi \prec \theta}{\phi \prec \theta.}$$

CUT $$\frac{\Delta_0 < \phi_0 \qquad \Delta_1 < \phi_1 \qquad \ldots \qquad \Gamma, \phi_0, \phi_1, \cdots < \phi}{\Gamma, \Delta_0, \Delta_1, \cdots < \phi.}$$

encode the transitivity of ground, and

NON-CIRCULARITY $$\frac{\phi \prec \phi}{\perp.}$$

its irreflexivity.[5]

Given the formal features of ground that Fine's system aims to capture, one might expect that our four axioms exhaust Fine's treatment. But there is another axiom whose validity is derivable in Fine's system. The relevant axiom is

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the language of PLG do not have any significant structure to which the logic is sensitive. So, PLG concerns those aspects of the logic of ground that are independent of the structures of the sentences involved in grounding claims. This is Fine's point in calling his logic of ground *pure*.

[4]In what follows, I use lower-case Greek letters (often with a numerical subscript) to indicate sentences, and upper-case Greek letters (often subscripted) to indicate sets of sentences. Following Fine, I will sometimes use a comma-delimited list to indicate a set of sentences. The comma should be interpreted as set-theoretic union, and occurrences of lower-case Greek letters in the list should be interpreted as standing for the singletons of the indicated sentence. So, for instance, 'ϕ, Δ' indicates $\{\phi\} \cup \Delta$. I use lower-case latin letters (often subscripted) to indicate facts, and upper-case latin letters (often subscripted) to indicate sets of facts. The conventions for comma-delimited lists of facts are similar to those for comma-delimited lists of sentences.

[5]The '$\perp$' in the statement of NON-CIRCULARITY is not a distinguished sequent standing for falsity, but rather indicates that any sequent whatsoever is derivable from a sequent representing circularity.

$$\textbf{AMALGAMATION} \quad \frac{\Delta_0 < \phi \qquad \Delta_1 < \phi \qquad \ldots}{\Delta_0, \Delta_1, \cdots < \phi.}$$

CUT, SUBSUMPTION, TRANSITIVITY, NON-CIRCULARITY, and AMALGAMATION are the axioms of a fragment of PLG that we might call the *logic of strict ground* (LSG). I have collected them in fig. 1 for convenient reference. One of the main theorems of [deRosset, 2014] is that PLG is a conservative extension of LSG. It follows that its principles fully characterize that fragment of PLG which concern the familiar (*i.e.*, strict) grounding claims we have been considering.[6] PLG's axiom system also includes other axioms, governing claims involving weak ground. But the extra complication attending that axiomatization is unnecessary to characterize the pure logic of ground.[7]

$$\textbf{SUBSUMPTION} \quad \frac{\Delta, \phi < \psi}{\phi \prec \psi}$$

$$\textbf{TRANSITIVITY} \quad \frac{\phi \prec \psi \qquad \psi \prec \theta}{\phi \prec \theta}$$

$$\textbf{CUT} \quad \frac{\Delta_0 < \phi_0 \qquad \Delta_1 < \phi_1 \qquad \ldots \qquad \Gamma, \phi_0, \phi_1, \cdots < \phi}{\Gamma, \Delta_0, \Delta_1, \cdots < \phi}$$

$$\textbf{NON-CIRCULARITY} \quad \frac{\phi \prec \phi}{\perp}$$

$$\textbf{AMALGAMATION} \quad \frac{\Delta_0 < \phi \qquad \Delta_1 < \phi \qquad \ldots}{\Delta_0, \Delta_1, \cdots < \phi}$$

Figure 1: The logic of strict ground

3 The Semantics of PLG

What interpretation should we give to sequents in the language of PLG? Fine offers a semantics which provides at least a partial answer to that question. Fine's semantics relies on two metaphysical ideas. First, the semantics presup-

[6]See [deRosset, 2014, pp. 733-4] for a proof of the result.

[7]It should be noted that in Fine's axiomatization, TRANSITIVITY, CUT, and AMALGAMATION are derived rules, rather than basic axioms.

poses that we are given a set of *facts*, thought of as parts of the actual world. This notion of "fact" is evidently metaphysically thick: every fact in this sense is a part of reality, and the structure of grounding relations among facts is part of the structure of reality.[8] Second, there is a *fusion operation* on these facts: the fusion of some facts $a, b, c, \ldots$ is a "conjunctive fact" $a.b.c. \ldots$ that obtains just in case each of the fused facts obtains.[9]

The fundamental semantic idea on which Fine draws is the idea of *truth-making*. It is presumed that sentences of the language are true. For each such truth, there may be facts which *make it true*. The set of truth-makers for a sentence is its *verification set*. It is presupposed that the set of truth-makers for any sentence is closed under fusion: if $a, b, c, \ldots$ are each truth-makers for ϕ, then so is their fusion.[10] The model gives a stock of potential verification sets, not all of which need be assigned by the model to sentences of the language. So, a model for the language of the pure logic of ground comprises a set of facts, a fusion operation, a set of potential verification sets, and a specification, for each sentence of the language, of a verification set for that sentence.

We can now specify conditions for the truth of the grounding claims represented by the sequents of the language. It is useful for stating the conditions to define the notion of a *component-wise fusion* of some sets of facts $F_0, F_1, \ldots$: any fusion of $f_0, f_1, \ldots$, where each of the f_i is a member of F_i, is a component-wise fusion of $F_0, F_1, \ldots$. Intuitively, a component-wise fusion of some sets of facts is what you get by taking a representative from each set and fusing. The most basic notion of grounding in Fine's semantics is the notion of *weak ground*: $\phi_0, \phi_1, \ldots$ are (collectively) a *full weak ground* for ψ just in case ev-

[8]To be clear, the *formal semantic system itself* is just a set-theoretic construction, carrying no commitments to "thick" facts. I speak here of the metaphysical ideas which motivate Fine's specification and, presumably, help distinguish intended models for an interpreted language from unintended models. There is a metaphysically thin use of 'fact' in ordinary English on which P entails 'It is a fact that P.' The notion of "fact" I am deploying in the paper is not expressed by this metaphysically thin use. Thanks to Martin Glazier and Jon Erling Litland for independently stressing the need for this clarification.

[9][Fine, 2012a, p. 7]. The fusion operation Π is defined on subsets of the set of facts, and is constrained to be associative, in the sense that

$$\Pi(\{\Pi(G_1), \Pi(G_2), \ldots\}) = \Pi(G_1 \cup G_2 \cup \ldots).$$

Given the set-theoretic background, fusion is also commutative: if $\{f_1, f_2, ...\} = \{g_1, g_2, ...\}$, then $f_1.f_2.... = g_1.g_2.....$ Finally, I will assume that fusion is idempotent, in the sense that $f.f.f.... = f$. Though Fine does not assume idempotence, it clearly does not conflict with his intentions, since the fusion operation in the canonical model constructed in his completeness proof is idempotent.

[10]This closure condition is the analogue of AMALGAMATION in Fine's semantics.

ery component-wise fusion of the verification sets for the ϕ_i's is a member of ψ's verification set. Given plausible assumptions, this implies that the ϕ is a weak ground for ψ iff ϕ's verification set is a subset of ψ's.[11] ϕ is a *partial weak ground* for ψ iff there is a (potential) verification set G such that every component-wise fusion of ϕ's verification set and G is a member of ψ's verification set. The familiar sort of grounding, strict grounding, is then said to be irreversible weak grounding: $\phi_0, \phi_1, \ldots$ (strictly) grounds ψ iff $\phi_0, \phi_1, \ldots$ fully weakly grounds ψ, and ψ does not even partially weakly ground any of the ϕ_i's.

Fine shows that the axioms of PLG are sound and complete for this semantics. Thus, the sequents derivable by the axioms of PLG from a set of sequents S are exactly those grounding claims that are true in every model that verifies S. As we have seen, PLG is a conservative extension of LSG [deRosset, 2014, pp. 733-4]. So, LSG is also sound and complete for this semantics.

4 Some Problems for Fine's Semantics

There are three reasons for dissatisfaction with Fine's semantics. The first reason is impressionistic, but it looms largest in my mind. The explication of grounding in terms of truth-making does not seem to capture the metaphysical idea that animates grounding enthusiasts. As I have briefly mentioned, grounding enthusiasts contend that grounding is a relation of dependence and determination: a fact depends on and is determined by the facts in virtue of which it obtains. But on the interpretation of (strict) grounding claims given by the semantics, P (strictly) grounds Q whenever, roughly, every truth-maker for P is also a truth-maker for Q, but not vice versa.[12] This is a reason to think that Q (or, perhaps, the fact that Q is true) bears a relation of dependence and determination to every truth-maker for P, and also to something which isn't a truth-maker for P. This is enough for there to be a relation of *co*-dependence and *co*-determination between Q and P. But it provides no reason to think that Q depends on or is determined by P itself.

A symptom of this difficulty emerges when we consider the most natural way to extend Fine's semantics to handle conjunctions. It is natural to suggest

[11] Only one extra assumption, already flagged in n. 9, is needed: that fusion is idempotent, so that $\Pi(\{f\}) = \Pi(\{f, f\}) = f.f = f$.

[12] This is a slight simplification. On Fine's interpretation, '$P < Q$' can be false even though Q has a truthmaker that P lacks, so long as this extra truthmaker fuses with some other fact to yield a truthmaker for P.

that the truthmakers for a conjunction are exactly the component-wise fusions of the verification sets for each of its conjuncts. Suppose, for illustration, that the sole truthmaker for 'it is snowy' is some fact f concerning the trajectories of certain snowflakes, and that for 'it is muddy' is some fact g concerning the mixture of water and dirt. Then 'it is snowy and muddy' will have, as its truthmaker, the fusion $f.g$, a more compendious fact concerning both the snowflake trajectories and the mixture. This link between conjunction and fusion is well-nigh inevitable on the understanding of the fusion operation as the analogue of conjunction for facts. Let us suppose, then, that the truthmakers for a conjunction are the component-wise fusions of truthmakers for its conjuncts. On this assumption, it turns out that, in a broad class of cases, if χ grounds ϕ, then, for any unrelated claim ψ, the conjunction $(\chi \wedge \psi)$ grounds $(\phi \wedge \psi)$.[13] For instance, on plausible assumptions, Fine's semantics will allow us to conclude

(1) (it is muddy $\wedge$ certain snowflakes are moving along certain trajectories) $<$ (it is muddy $\wedge$ it is snowy).

This makes perfect sense if the relation indicated by '$<$' is not itself a relation of dependence and determination, but a relation of co-dependence and co-determination on something else. It is implausible, however, if we interpret the relation indicated by '$<$' as itself the relation of dependence and determination tracked by 'in virtue of' explanations. For instance, it is implausible to think that it is both muddy and snowy in virtue of its being both muddy and ϕ, for any claim ϕ.

It would be nice, then, to offer a semantics which sticks closer to the core metaphysical idea to which grounding enthusiasts appeal, on which the expressions for grounding indicate relations of dependence and determination.[14]

Second, it is discomfiting that the semantics requires that (strict) grounding satisfy AMALGAMATION in addition to being transitive and irreflexive.

[13] Say that, for facts f, g, $f \sqsubseteq g$ iff $g = f.g$; $f \sqsubset g$ iff $(f \sqsubseteq g \wedge f \neq g)$; and f *overlaps* g $(f \circ g)$ iff $(\exists h)(h \neq \Pi(\emptyset) \wedge h \sqsubseteq f \wedge h \sqsubseteq g)$. A claim ψ is *unrelated* in the relevant sense iff none of its truthmakers overlap any truthmaker for ϕ. The fusion relation is *gap-preserving* iff $(a \sqsubset b \wedge c \not\circ b \Rightarrow a.c \neq b.c)$, for all facts a, b, and c. Suppose we are given a model $\mathscr{M}$ in which the fusion relation is gap-preserving and verification sets for conjunction satisfy the assumption noted in the main text. It is then easy to show that if $\chi < \phi$ is true in $\mathscr{M}$, then $(\chi \wedge \psi) < (\phi \wedge \psi)$ is too, for any arbitrary unrelated sentence ψ.

[14] Alternatively, one could offer a semantics on which what *appears* to be a relation of dependence and determination among facts turns out to be something else instead. It would then be nice if we could show why it might be easy to mistake the relation expressed by grounding operators in LSG with a relation of dependence and determination among facts. The semantics of sparse models described below has this feature.

AMALGAMATION, recall, is not motivated by the idea that grounding is a many-one, transitive, and irreflexive relation of dependence and determination. To my mind, something similar holds for the alleged irreflexivity of ground, which has been challenged by some authors.[15]

The third reason one might be dissatisfied with the semantics for PLG is that it appears to rule out a view on which the facts are "sparse" in such a way that there are truths which do not report any fact. A plausible example of such a truth is a disjunction with a false disjunct. It seems that the only truth-makers for such a disjunction are the truth-makers for its true disjunct. Such a view is intuitively attractive, and many in the truthmaker literature endorse it.[16] But, on Fine's semantics, we cannot maintain both that the disjunction is grounded in its true disjunct and that it has no verifiers that aren't also verifiers for its true disjunct. For then the disjunction would, according to the semantics, be a weak ground for its true disjunct, and so would not be (strictly) grounded by it.[17] It would be nice to have a development of the pure logic of ground that did not require additional truth-makers for every additional grounded truth.

What we would like to have, then, is a semantics for LSG that cleaves more closely to the idea that grounding is a relation of dependence and determination among facts, neither requires nor disallows failures of AMALGAMATION and irreflexivity, and can accommodate a sparse conception of facts. Such a semantics is easily lifted from the metaphysical ideas found in the writings of grounding enthusiasts.

5 A Picture and Corresponding Semantics

Consider again the alleged role of grounding in explicating the idea that some facts are dependent on and determined by congeries of other facts. Facts involving interest rates obtain in virtue of certain psychological facts. Each of these psychological facts obtains (at least in part) in terms of certain neurological facts. These, in turn, obtain in virtue of chemical facts. And so it goes.

On the resulting picture, each fact f that has a ground occupies a node at

[15] Della Rocca [2010] contends that ground is or may not be asymmetric, and Jenkins [2011] develops a sophisticated view that is consistent with there being symmetric relations of ground among facts.

[16] See, for instance, [Rodriguez-Pereyra, 2006, pp. 965-8] for discussion and defense.

[17] See [deRosset, 2014, pp. 716-7] for a detailed argument for this conclusion.

the top of one or more trees.[18] Each of those trees branches from its top node to nodes collectively containing the facts in virtue of which f obtains. The tree may then branch from each of those nodes to nodes (collectively) containing further facts in virtue of which they obtain, and so on. A fact f may be connected directly by branches to a number of other facts, reflecting the idea that f is often directly grounded in a congeries of facts. The children of f are, collectively, facts on which f directly depends and by which it is directly determined. We assume that every such tree that can be extended to yield another such tree is so extended. So, if there is a tree with a leaf g that has an unmediated ground, then there is another tree extending the original tree on which the children of g are, collectively, facts that immediately ground g. Given the totality of grounding trees, we can specify a relation that g and f stand in just in case g occurs lower on a grounding tree than f. This relation is transitive, since if h occurs lower than g on some tree T and g occurs lower than f on some other tree, then h occurs lower than f on some extension of T. And, it might be held, it is irreflexive: no fact occurs lower than itself on any tree. So, each of the grounding trees has the structure of an acyclic directed graph.[19]

This is a pretty picture, but how can we obtain a semantics for a pure logic of ground from it? We need only add the simple semantical idea that every sentence can be mapped to a fact. These ideas give us the raw materials for a semantics for the language of LSG.

The semantics for the language of LSG takes as basic the notion of a fact, and a two-place relation between a set of facts Δ and a fact ϕ, which corresponds to the idea of an *unmediated ground* for ϕ. Mediated grounds for ϕ are facts that explain ϕ only by way of explaining some other facts that also explain ϕ. Unmediated grounds for ϕ explain ϕ without relying on any such explanatory detour. Plausible examples of unmediated ground are fairly easy to come by. Its being either chilly or windy outside is immediately explained by the fact that it is chilly outside. Its being both cloudy and chilly outside is immediately explained (collectively) by two facts: (i) its being cloudy outside; and (ii) its

[18] Formally speaking, what the picture requires is that we are given a forest of labeled, rooted trees; a node *contains* a fact iff the fact labels the node. I will often suppress mention of nodes in what follows, saying, for instance, that the root of a tree T is a fact f, and its leaves are $f_0, f_1, ...$ instead of saying that the root and leaves are nodes containing $f, f_0, f_1, ...$, respectively.

[19] This picture is lifted from [Rosen, 2010] and [Schaffer, 2009].

being chilly outside. If, as seems likely, my jacket has a mass greater than one kilogram in virtue of the antics of quarks and leptons, the relevant facts about quarks and leptons are mediated grounds for the fact that my jacket has a mass greater than one kilogram.[20] These two metaphysical ideas are represented in our definition of a *model frame*, which consists of a set of facts and a relation from pluralities of members of that set into that set:

Definition 5.1 A *model frame* is a pair $\langle F, R\rangle$ such that

- F is a non-empty set; and
- R is a relation from $\mathscr{P}(F)$ (the power set of F) to F.

Formally speaking, a model frame is a directed hypergraph.[21]

It's now easy to see how a model frame gives us a structure of trees along the lines suggested in the writings of grounding enthusiasts. This is done by closing R under an operation which I will call *grafting*. Let T be a tree with leaves $f_0, f_1, ..., G$, and for each i, T_i a tree with root f_i. Then the *graft* of $T_0, T_1, ...$ into T is the result of replacing each of the leaves f_i with T_i. Suppose, for instance, that we are given a tree T

$$\begin{tikzcd} & f \arrow[dl] \arrow[d] \arrow[dr] & \\ g_0 & g_1 & \cdots \end{tikzcd}$$

and two trees:

$$\begin{tikzcd} & g_0 \arrow[dl] \arrow[d] \arrow[dr] & & & g_1 \arrow[dl] \arrow[d] \arrow[dr] & \cdots \\ h_0^0 & h_1^0 & \cdots & h_0^1 & h_1^1 & \cdots \end{tikzcd}$$

Then the tree:

[20] I do not mean to suggest that unmediated grounds for a fact f don't also ground facts that ground f; Fine agrees [Fine, 2012b, p. 51]. In fact, the idea of unmediated ground is consistent with the claim that unmediated ground induces a dense ordering on facts, so that any unmediated ground for f is also an unmediated ground of an unmediated ground of f. If g is an unmediated ground of f, one *needn't* make a detour through something grounded by g to explain f. This is consistent with the claim that one *can* make such a detour. In slogan form, we might say that the immediacy of unmediated ground is *explanatory*, not structural.

[21] A directed hypergraph is a pair $\langle N, A\rangle$, such that N is a non-empty set and A is a relation on $\mathscr{P}(N) \times N$. The members of N are the *nodes*, and the members of A are *hyperedges*, each connecting a node to a set of nodes whose members are, collectively, its children.

$$
\begin{tikzcd}
 & & & f \arrow[dll] \arrow[d] \arrow[drrr] & & & \\
 & g_0 \arrow[dl] \arrow[d] \arrow[dr] & & g_1 \arrow[dl] \arrow[d] \arrow[dr] & & & \cdots \\
h^0_0 & h^0_1 & \cdots h^1_0 & h^1_1 & \cdots & &
\end{tikzcd}
$$

is a graft.

We can now define the grounding trees for a given model frame.

Definition 5.2 The *grounding trees* given by a model frame $\langle F, R\rangle$ are the members of the smallest set $\mathbf{T}$ closed under grafts and containing a tree

$$
\begin{tikzcd}
 & f \arrow[dl] \arrow[d] \arrow[dr] & \\
g_0 & g_1 & \cdots
\end{tikzcd}
$$

whenever $R(\{g_0, g_1, ...\}, f)$.[22]

I will sometimes refer to the set of grounding trees for a model frame $\mathscr{F}$ using the expression '$\mathbf{T}$', with the reference to $\mathscr{F}$ implicit. Note that the set of grounding trees $\mathbf{T}$ for a model frame $\langle F, R\rangle$ also defines a set of grounding trees for each member f of F: the set of trees whose root is f. I'll call this set $\mathbf{T}_f$. It will be convenient to call the set whose members are exactly the occupants of the leaf nodes of a grounding tree the tree's *floor*. A subset D of F is a *floor for* f (in a model frame $\mathscr{F}$) iff D is the floor of some tree in $\mathbf{T}_f$.

So much for the metaphysical presuppositions of the semantics. Now we need to offer interpretations of grounding claims. A language $\mathscr{L}$ of LSG is given by a specification of a non-empty set of *sentences*, all of which are presumed to be true. To ensure that sequents of $\mathscr{L}$ have a unique parsing, we stipulate that the operators are pairwise distinct and not among the sentences. Following Fine, the *sequents* of $\mathscr{L}$ are all of the expressions of the form $\Delta < \phi$ and $\psi \prec \phi$.

A *model* of a language $\mathscr{L}$ is given by a model frame and an interpretation that specifies, for every sentence of $\mathscr{L}$, which fact that sentence expresses.

Definition 5.3 A *model* for a language $\mathscr{L}$ of LSG is a triple $\langle F, R, [\,]\rangle$ such that

[22] Thanks to an anonymous referee for suggesting a significant improvement in the statement of this definition.

- $\langle F, R\rangle$ is a model frame; and
- $[\,]$ is a function from sentences of $\mathscr{L}$ into F.

I will write '$[\phi]$' to indicate the value of $[\,]$ for the sentence ϕ, and I will sometimes write '$[\Delta]$' to indicate $\{[\delta]|\delta \in \Delta\}$. The semantics is completed by specifying the conditions under which a sequent is true in a model. First, define the grounding relations given by a model frame.

Definition 5.4 Let $\mathscr{M}$ be a model with model frame $\mathscr{F}$:

- $G <_{\mathscr{M}} f$ iff G is a floor for f in $\mathscr{F}$.
- $g \prec_{\mathscr{M}} f$ iff there is an H such that $\{g\} \cup H <_{\mathscr{M}} f$.

Defining truth in a model is now straightforward.

Definition 5.5 Let Δ be a set of sentences, and ϕ, ψ be sentences. Let $\mathscr{M}$ be a model $\langle F, R, [\,]\rangle$.

- $\Delta < \phi$ is *true in* $\mathscr{M}$ ($\mathscr{M} \vDash \Delta < \phi$) iff $[\Delta] <_{\mathscr{M}} [\phi]$.
- $\psi \prec \phi$ is *true in* $\mathscr{M}$ ($\mathscr{M} \vDash \psi \prec \phi$) iff $[\psi] \prec_{\mathscr{M}} [\phi]$.

I'll say that a model *verifies* a sequent when that sequent is true in the model, and that a model *verifies* a set of sequents when it verifies every member. A tree T *verifies* $\Delta < \phi$ (in a model $\mathscr{M}$) iff its floor is $[\Delta]$ and its root contains $[\phi]$; T *verifies* $\phi \prec \psi$ iff $[\phi]$ is a member of T's floor and T's root contains $[\psi]$.

6 Soundness and Completeness Results

We seek a way of treating the semantics for a pure logic of ground that is neutral on the validity of AMALGAMATION and does not assume that grounding is irreflexive. For this purpose, it is useful to speak of certain sub-logics of LSG. Call the logic given by SUBSUMPTION, TRANSITIVITY, and CUT the *base logic B* for LSG. Adding NON-CIRCULARITY to B yields *BNC*. Adding AMALGAMATION to B yields *AB*. Of course, all of the axioms together (*ABNC*) yield LSG. The notions of derivations in each of these sub-logics are defined in the obvious way.

We now define notions of semantic consequence corresponding to B and each of its extensions. First, we define the restrictions on models that we will need.

Definition 6.1 Let $\mathscr{M}$ be a model $\langle F, R, [\,]\rangle$.

i. $\mathscr{M}$ is *acyclic* iff the digraph reduction of the directed hypergraph $\langle F, R\rangle$ is acyclic;[23] and

ii. R is *additive* iff it is closed under unions in its first argument: for all $G_0, G_1, ...$, if $R(G_0, f)$ and $R(G_1, f)$ and ..., then $R(G_0 \cup G_1 \cup ..., f)$.

$\mathscr{M}$ is *cyclic* if it is not acyclic, and $\mathscr{M}$ is *additive* iff R is.

We can now define the notions of semantic consequence we'll be working with:

Definition 6.2 Let S be a set of sequents and σ a sequent.

i. $S \vDash_B \sigma$ iff every model that verifies S also verifies σ;

ii. $S \vDash_{BNC} \sigma$ iff every acyclic model that verifies S also verifies σ;

iii. $S \vDash_{AB} \sigma$ iff every additive model that verifies S also verifies σ; and

iv. $S \vDash \sigma$ iff every additive, acyclic model that verifies S also verifies σ.

Then the following results are straightforward to prove:

Theorem 6.3 *The base logic B of LSG is sound and complete: for all sets of sequents S and sequents σ, $S \vdash_B \sigma$ iff $S \vDash_B \sigma$.*

Theorem 6.4 *BNC is sound and complete for all acyclic models: $S \vdash_{BNC} \sigma$ iff $S \vDash_{BNC} \sigma$*

Theorem 6.5 *AB is sound and complete for all additive models: $S \vdash_{AB} \sigma$ iff $S \vDash_{AB} \sigma$*

Theorem 6.6 *LSG is sound and complete for all additive, acyclic models: $S \vdash \sigma$ iff $S \vDash \sigma$*

Proofs are in appendices A-C.

[23] The digraph reduction of a hypergraph is an ordinary directed graph (*i.e.*, a digraph) which contains an edge from n_1 to n_2 iff there is a set nodes N containing n_1 related by the hypergraph to n_2.

7 Grounding the Unreal

Our semantics for the language of the pure logic of ground avoids the first two problems we identified for Fine's semantics. By taking the notion of an unmediated ground as basic, our model frames capture the idea that grounding is a relation of dependence and determination. By allowing for separate treatment of AMALGAMATION and NON-CIRCULARITY, we can isolate and treat separately the commitments of views which allow self-grounding or deny AMALGAMATION. But the third problem is still with us. A model assigns a fact to every truth, and so will not accommodate a sparse conception of facts.

Let's focus again, for illustration, on the case of disjunction. A view of the sort we wish to accommodate holds that there isn't really any further fact corresponding to disjunctive truths. There are just the original, non-disjunctive facts, and one or more of those non-disjunctive facts somehow ground the disjunctive truth without grounding any further fact stated by that truth. More generally, on the sparse view we are exploring, there are some facts; those facts might be expressed by appropriate claims; but there are truths that don't express new facts, but somehow are grounded in congeries of the old facts. Disjunctive truths provide a plausible example. The idea here is that every disjunctive *truth* is grounded in each of its true disjuncts, but the whole truth about disjunctive *facts* is: there aren't any.

This is an appealing metaphysical idea. Fortunately, there is a natural extension of the logical treatment offered above which can accommodate it. Suppose we are given a domain of facts F, and the relation R of unmediated ground on those facts. We need a way to represent what above I called "congeries" of facts. So, we'll need to constructively extend our domain.

Recall that R, the relation of unmediated ground, relates subsets of F to members of F. Thus every member of F is associated by R with a set of subsets of F: the set of subsets of F that are R-related to that member of F. Suppose, to illustrate, that we wanted to add a new fact f to our given domain of facts F, and we were sure that f is grounded somehow in congeries of facts in F. Intuitively, the set of all subsets of the power set of F specifies all the ways of extending R to specify unmediated grounds of this new fact: any such extension will determine a set of subsets of F that are unmediated grounds of f. Suppose instead that we *didn't* want to extend our domain by adding a new fact, but simply wanted to say that a certain *truth* was grounded in certain congeries of members of F. Then the set of all subsets specifies all the ways in which members of F might be combined (into a "congeries") to ground truths. The idea obviously generalizes: the ways in which truths at higher levels might be grounded in lower-level congeries of the original facts can be given recursively.

These reflections give us a very natural conception of a constructive extension of F. For simplicity, assume that the members of F are ur-elements:

Definition 7.1 The α^{th} *constructive extension* of a domain F ($\mathcal{D}_\alpha(F)$) is defined by recursion:

i. $\mathcal{D}_0$ = the union of F and the non-empty sets of non-empty subsets of F; and

ii. $\mathcal{D}_{\alpha+1}$ = the union of $\mathcal{D}_\alpha$ and the non-empty sets of non-empty subsets of $\mathcal{D}_\alpha$.

iii. $\mathcal{D}_\lambda$ = the union of $\bigcup(\{\mathcal{D}_\alpha | \alpha < \lambda\})$ and the non-empty sets of non-empty subsets of $\bigcup(\{\mathcal{D}_\alpha | \alpha < \lambda\})$ (for limit λ).

The *total constructive extension* of F ($\mathcal{D}_F$) is the union over α of all of the $\mathcal{D}_\alpha(F)$ for $\alpha \leq$ the cardinality of the power set of the set of sentences of our language. A *constructive extension of* F is a transitive set $\mathcal{D}$ such that $F \subseteq \mathcal{D} \subseteq \mathcal{D}_F$.

Intuitively, a constructive extension of F says not only what facts there are, but also how those facts compose "congeries" to ground truths that do not themselves state facts.

The sparse semantics I am developing relies, as we have just seen, on a hierarchy of congeries. It may be helpful to say a little more about the nature of congeries of facts, congeries of congeries of facts, *etc.* 'Congeries' is just a covering term I have used (and which is conveniently spelled the same in both the singular and plural) for a bunch of things. I have not said anything about how the bunching is to be understood, but the proposal does impose constraints on any such understanding. For instance, if, inspired by [Fine, 2012a], we identify a congeries of facts with the fusion of those facts, then the proposal will need to distinguish the fusion of the fusion of f from f itself. Similarly, if, inspired by [Boolos, 1984], we interpret talk of a congeries of facts as plural talk of those facts, then the proposal will have to distinguish the plurality of the plurality whose sole member is f from f itself. (Of course if, inspired by the formal semantic treatment, we interpret talk of congeries set-theoretically, then there is no problem distinguishing $\{\{f\}\}$ from f.) In any case, any sparse theorist who wants to avail herself of the technical resources specified here will find herself saddled with this commitment.[24]

Given our relation R, the domain of facts F on which it is defined, and a constructive extension of F, R may be extended naturally to a relation on the constructive extension:

Definition 7.2 Suppose R is a relation from $\mathscr{P}(F)$ to F, and $\mathcal{D}$ is a constructive extension of F. The *constructive extension of* R to $\mathcal{D}$ ($R_F(\mathcal{D})$) is a relation from $\mathscr{P}(\mathcal{D})$ to $\mathcal{D}$ such that:

i. $R_F(\mathcal{D})(G, f)$ iff $R(G, f)$, if $f \in F$; and

ii. $R_F(\mathcal{D})(G, f)$ iff $G \in f$, otherwise.

This extension of R just adds to R a specification of how elements of the constructive extension that don't belong to the given domain of facts are (immediately) grounded. Now we can define the sort of model which is consistent with a sparse conception of facts:

Definition 7.3 A *sparse model* for a language $\mathscr{L}$ of LSG is a 4-tuple $\langle F, \mathcal{D}, R, [\,]\rangle$ such that

- F is a non-empty set;
- $\mathcal{D}$ is a constructive extension of F;
- R is a relation from $\mathscr{P}(F)$ to F; and
- $[\,]$ is a function from sentences of $\mathscr{L}$ into $\mathcal{D}$.

If $\langle F, \mathcal{D}, R, [\,]\rangle$ is a sparse model for some language $\mathscr{L}$, then $\langle F, \mathcal{D}, R\rangle$ is a *sparse model frame*.

We can then define the domain of *grounding** *trees* for a sparse model $\mathscr{M}^*$ in the same way we defined the domain of grounding trees for an ordinary model $\mathscr{M}$, except that we use $R_F(\mathcal{D})$ in place of R.

[24]Thanks to Dan Korman, Mark Moyer, and an anonymous referee for emphasizing the need to clarify this commitment. My principal aim here is not to defend the sparse view, but to limn its features. See [deRosset, manuscript, §4.4] for an argument that this commitment to congeries of facts, congeries of congeries of facts, *etc.*, is one the sparse theorist should be happy to make.

How might a sparse model accommodate the sparse conception of facts? Suppose, to illustrate, that

(2) e is negatively charged $<$ (e is negatively charged or the weather is sunny)

is true, even though there is no fact stated by

(3) e is negatively charged or the weather is sunny.

Suppose that e is negatively charged, the weather is not sunny, and 'e is negatively charged' states a fact. Let f be that fact. Then there is no fact stated by (3), but that sentence can be assigned the member $\{\{f\}\}$ of the constructive extension of our domain of facts. Intuitively, we may think of the assignment of $\{\{f\}\}$ to (3) as indicating that the only unmediated ground of that truth is the fact f.

Is this cheating? Are we denying that (3) states a fact, while our interpretation surreptitiously supplies a fact for (3) to state? No. It would be a mistake to identify the congeries of facts which immediately ground (3) with the fact expressed by it. If, in contravention to the sparse conception we are exploring, (3) were to state a fact, it would be a fact which could obtain even if e did not exist, and even if f did not obtain. The congeries of facts that immediately grounds the truth (3) is represented in our sparse model by $\{\{f\}\}$. This congeries of facts could not exist if e did not exist, and would not obtain if f did not obtain. At least, that is a very plausible view to take, and there is no reason for the proponent of the sparse conception of facts not to take it. So, there's no reason to think that the elements introduced when we constructively extend the domain of facts are themselves further facts, and so no reason to think that an interpretation assigning those elements to sentences is secretly supplying facts for those sentences to state.

To define the truth conditions for grounding relations, it will be convenient to extend the notion of a floor. Call the set whose members are exactly the occupants of the leaf nodes of a grounding* tree the tree's *floor*. A subset D of $\mathcal{D}$ is an *extended floor for* f (in a sparse model frame $\mathscr{F}$) iff D is the floor of some grounding* tree whose root contains f. Now we can easily define the grounding relations given by a sparse model frame:

Definition 7.4 Let $\mathscr{M}$ be a sparse model with sparse model frame $\mathscr{F}$:

- $G <^*_{\mathscr{M}} f$ iff G is an extended floor for f in $\mathscr{F}$; and
- $g \prec^*_{\mathscr{M}} f$ iff there is an H such that $\{g\} \cup H <^*_{\mathscr{M}} f$.

Defining truth in a sparse model is now straightforward.

Definition 7.5 Let Δ be a set of sentences, and ϕ, ψ be sentences. Let $\mathscr{M}$ be a sparse model $\langle F, \mathcal{D}, R, [\,]\rangle$.

- $\Delta < \phi$ is *true in* $\mathscr{M}$ ($\mathscr{M} \vDash^* \Delta < \phi$) iff $[\Delta] <^*_{\mathscr{M}} [\phi]$.
- $\psi \prec \phi$ is *true in* $\mathscr{M}$ ($\mathscr{M} \vDash^* \psi \prec \phi$) iff $[\psi] \prec^*_{\mathscr{M}} [\phi]$.

A sparse model $\langle F, \mathcal{D}, R, [\,]\rangle$ is *acyclic* iff the digraph reduction of $\langle F, R\rangle$ is acyclic, and it is *additive* iff the constructive extension of R to $\mathcal{D}$ is additive. Thus, we can now specify the appropriate notions of semantic consequence for our sparse semantics:

Definition 7.6 Let S be a set of sequents and σ a sequent.

i. $S \vDash^*_B \sigma$ iff every sparse model that verifies S also verifies σ;

ii. $S \vDash^*_{BNC} \sigma$ iff every acyclic sparse model that verifies S also verifies σ;

iii. $S \vDash^*_{AB} \sigma$ iff every additive sparse model that verifies S also verifies σ; and

iv. $S \vDash^* \sigma$ iff every additive, acyclic sparse model that verifies S also verifies σ.

The following results are very easy to prove:

Theorem 7.7 *The base logic B of LSG is sound and complete: for all sets of sequents S and sequents σ, $S \vdash_B \sigma$ iff $S \vDash^*_B \sigma$.*

Theorem 7.8 *BNC is sound and complete for all acyclic sparse models: $S \vdash_{BNC} \sigma$ iff $S \vDash^*_{BNC} \sigma$.*

Theorem 7.9 *AB is sound and complete for all additive sparse models: $S \vdash_{AB} \sigma$ iff $S \vDash^*_{AB} \sigma$.*

Theorem 7.10 *LSG is sound and complete for all additive, acyclic sparse models: $S \vdash \sigma$ iff $S \vDash^* \sigma$.*

The proofs are in appendix D.

The key result for proving these theorems is that there are systematic correspondences between ordinary models in which every sentence of $\mathscr{L}$ expresses a fact (*i.e.*, a member of F), and sparse models, in which some sentences of $\mathscr{L}$ express no such fact, but are instead associated by our interpretation with congeries of such facts, or congeries of congeries, *etc.*

Definition 7.11 Let $\mathscr{M}^*$ be a sparse model $\langle F, \mathcal{D}, R, [\,]\rangle$. The *ordinary counterpart* of $\mathscr{M}^*$ is $\langle \mathcal{D}, R_F(\mathcal{D}), [\,]\rangle$.

Definition 7.12 Let $\mathscr{M}$ be an ordinary model $\langle F, R, [\,]\rangle$. The *sparse counterpart* of $\mathscr{M}$ is $\langle F, F, R, [\,]\rangle$.

The two relevant lemmas say that any sparse model verifies the same grounding claims as its ordinary counterpart, and any ordinary model verifies the same grounding claims as its sparse counterpart.

Lemma D.1 *Let $\mathscr{M}^*$ be a sparse model, and $\mathscr{M}$ its ordinary counterpart. Then $\Delta <^*_{\mathscr{M}^*} f$ iff $\Delta <_{\mathscr{M}} f$.*

Lemma D.4 *Let $\mathscr{M}$ be an ordinary model, and $\mathscr{M}^*$ its sparse counterpart. Then $\Delta <^*_{\mathscr{M}^*} f$ iff $\Delta <_{\mathscr{M}} f$.*

One immediate consequence of these two lemmas is that, if $\mathscr{M}^*$ is a sparse model $\langle F, \mathcal{D}, R, [\,]\rangle$, then the sparse counterpart of its ordinary counterpart is $\langle \mathcal{D}, \mathcal{D}, R_F(\mathcal{D}), [\,]\rangle$ and verifies *exactly* the same grounding claims as $\mathscr{M}^*$. This latter model also is one in which every sentence of the language is mapped to a member of the domain of facts for the model. Thus, the specification of what grounds what in the language of the pure logic of ground will not distinguish the sparse view from its corresponding full view. In this sense, the theory of grounding cannot, by itself, settle the question of what facts there really are. From inside a metalanguage suited only to express grounding claims, *we can't tell the difference* between a sparse world and a corresponding full one. We need a richer object language to specify which facts there really are.[25]

[25] This result won't hold for richer object languages. Most obviously, if we enrich the object language by adding an operator $\mathcal{R}$ such that $\mathcal{R}\phi$ is true in a model iff ϕ expresses a fact in that model, then we can distinguish a sparse world from a full one. More subtly, if we introduce truth-functional operators as part of an impure logic of ground, then natural syntactic and semantic assumptions force difficult choices for proponents of the sparse view. See [deRosset, manuscript] for extended discussion of this second issue. Using the resources of the graph-theoretic semantics sketched here to develop an impure logic of ground is attempted in work under development. Thanks to an anonymous referee for noting this wrinkle.

What has been accomplished? We discerned in Fine's [2012a] treatment an axiomatization LSG of the pure logic of ground that does not rely on any notion of weak ground and noted that that it exactly captures the pure logic of (strict) ground. We found in the writings of grounding enthusiasts a metaphysical picture which could easily be worked up into a formal semantics for LSG. I then showed that the axiomatization and the picture fit one another exactly. We then asked whether we could jettison the commitment of the picture to the idea that each grounded truth expressed a fact and found that we could without affecting the logic at all. It must be admitted that, mathematically speaking, the soundness and completeness results that demonstrate the perfect fit between LSG and the structure of grounding trees are not particularly surprising. Both the derivations of LSG and the universe of grounding trees can be represented as directed hypergraphs, so one would expect there to be a simple way to show that their structures correspond. Similar remarks apply to the soundness and completeness results demonstrating fit between LSG and the structure of grounding* trees.

Still, I have argued, the results have philosophical significance. We now have a common formal framework for exploring the structure of views which accept cases of reflexive grounding and competitors which reject them. The same goes for AMALGAMATION. Also, we have tools we can use to develop and assess views on which reality is sparse, but the claims that accurately describe it are abundant. Such a view accepts that certain higher-level claims are true in virtue of the underlying fundamental facts, without postulating higher-level facts over and above congeries of fundamental facts. In these respects, I have argued, the semantics for the pure logic of ground developed here does better than Fine's truthmaker semantics.

Appendices

A Soundness of B and AB

A *derivation* in LSG of a sequent σ from a set of sequents S is a well-founded converse tree[26] whose root is σ, whose leaves are members of S, and whose nodes follow from their predecessors by one of the inference rules. The *depth* of a derivation is defined recursively. Any derivation consisting of a single node (*i.e.* of a premise) has a depth of 1. Otherwise, the depth of a derivation is the least ordinal greater than the depth of every immediate subderivation. Notice that infinite depths are allowed, since, *e.g.*, an instance of CUT may have infinitely many premises, and there may be no finite upper bound on the depths of the derivations of the premises. For instance, any derivation witnessing

$$\{\phi_0 \leq \phi_1, \qquad \phi_1 \leq \phi_2, \qquad ..., \qquad \phi_1, \phi_2, ... \leq \psi\} \vdash \phi_0 \leq \psi$$

will have infinite depth. Many of the proofs that follow deploy an induction on the depth of derivations.

Theorem A.1 ***(Soundness of B)*** *if $S \vdash_B \sigma$, then $S \vDash_B \sigma$.*

Proof The result is proved by induction on the depth of derivations d witnessing $S \vdash_B \sigma$. CUT is the only interesting case.

CUT: suppose σ comes by

$$\frac{\Delta_0 < \phi_0 \quad \Delta_1 < \phi_1 \quad \ldots \quad \phi_0, \phi_1, \ldots, \Gamma < \phi}{\Delta_0, \Delta_1, \ldots, \Gamma < \phi}$$

Suppose $\mathscr{M}$ verifies S. Application of the inductive hypothesis (IH) yields:

(i) $\mathscr{M} \vDash \Delta_i < \phi_i$ for all i; and

(ii) $\mathscr{M} \vDash \phi_0, \phi_1, \ldots, \Gamma < \phi$.

By (ii), there is a grounding tree $T \in \mathbf{T}$ whose floor is $[\phi_0, \phi_1, \ldots, \Gamma]$ and whose root is $[\phi]$. By (i), for each i, there is a grounding tree $T_i \in \mathbf{T}$

[26]A well-founded converse tree is a tree for which the converse of the "child of" relation on the nodes of any derivation is well-founded. That is, there are no infinite chains starting at the root and extending up along any branch: every branch terminates in finitely many steps.

with floor $[\Delta_i]$ and root $[\phi_i]$. So, the graft of all of the T_i's into T has floor $[\Delta_0, \Delta_1, \ldots, \Gamma]$ and root $[\phi]$. By D5.2, this graft is a grounding tree verifying σ.

□

Theorem A.2 ***(Soundness of AB)*** *if* $S \vdash_{AB} \sigma$*, then* $S \vDash_{AB} \sigma$*.*

Proof We need only add to the proof of TA.1 an argument for the case in which σ comes by an application of AMALGAMATION.

AMALGAMATION: Suppose σ comes by

$$\frac{\Delta_0 < \phi \quad \Delta_1 < \phi \quad \ldots}{\Delta_0, \Delta_1, \cdots < \phi}$$

Let $\mathscr{M}$ be an additive model verifying S. IH implies that $\mathscr{M}$ also verifies $\Delta_i < \phi$, for each i. By D5.5, there is grounding tree T_i verifying $\Delta_i < \phi$. Let Γ_i be the set whose members are contained in immediate children of of the root of T_i. By, D5.2, $R(\Gamma_i, \phi)$. By the additivity of R, $R(\bigcup_i(\Gamma_i), \phi)$. It's easy to see, then that there is a member T of $\mathbf{T}$ with the floor $[\Delta_0 \cup \Delta_1, \ldots]$ and a root containing $[\phi]$. This tree verifies σ.

□

B Completeness of B

The proof of the completeness of B will be Henkin-style.

Definition B.1 Let x be an individual that is not among the sentences of $\mathscr{L}$, and S be a set of sequents. The *witnessed extension of* S ($S^<$) is the set containing:

i. $\Delta < \phi$, if $\Delta < \phi \in S$;

ii. $\phi, x < \psi$, if $\phi \prec \psi \in S$; and

iii. nothing else.[27]

[27] The witnessed extension does not correspond to an intended interpretation of the partial sequents, since there is a single fact x which "completes" each of them. This means that the canonical model (defined below) is massively unintended. Of course, almost every Henkin-style canonical model is unintended, since we do not typically intend to use expressions of the object language to talk about themselves. In both cases, this makes no difference to the cogency of the completeness proofs. Thanks to an anonymous referee for pointing this out.

Lemma B.2 *$S^< \vdash_B \phi \prec \psi$ iff, for some Γ, $S^< \vdash_B \phi, \Gamma < \psi$.*

Proof

> $\Leftarrow$: The result is trivial, requiring a single application of SUBSUMPTION.
>
> $\Rightarrow$: The result is proved by an easy induction on the depth of derivations d witnessing $S^< \vdash_B \phi \prec \psi$. TRANSITIVITY is the only non-trivial case. Suppose $\phi \prec \psi$ comes by
>
> $$\frac{\phi \prec \chi \quad \chi \prec \psi}{\phi \prec \psi}$$
>
> IH implies $S \vdash_B \phi, \Gamma < \chi$ and $S \vdash_B \chi, \Delta < \psi$ for some Γ, Δ. So, an application of CUT yields the result.

□

It is easy to show that $S^<$ is a conservative extension of S:

Lemma B.3 *If σ is a sequent of $\mathscr{L}$, then $S \vdash_B \sigma$ iff $S^< \vdash_B \sigma$.*

Proof The only even slightly tricky case is when σ is $\phi \prec \psi$ and we need to show the result in the right-to-left direction. For this case, we prove by a simple induction on the depth of derivations the stronger result that, if $S^< \vdash_B \phi \prec \psi$ or $S^< \vdash_B \phi, \Gamma < \psi$, then $S \vdash_B \phi \prec \psi$.
□

Definition B.4 The closure of a set of sequents S (C_S) is defined by recursion:

i. if $\Delta < \phi \in S^<$, then $\Delta < \phi \in C_S$;

ii. if $\Delta_0 < \phi_0$, $\Delta_1 < \phi_1$, $\ldots$, and $\phi_0, \phi_1, \ldots, \Gamma < \phi \in C_S$, then $\Delta_0, \Delta_1, \ldots, \Gamma < \phi \in C_S$; and

iii. nothing else is in C_S.

Intuitively, C_S is the result of closing the witnessed extension of S under applications of CUT. It is easy to establish the adequacy of this intuitive characterization:

Lemma B.5 *$S^< \vdash_B \Delta < \phi$ iff $\Delta < \phi \in C_S$.*

We now use C_S to define a model.

Definition B.6 The *canonical model* of a set of sequents S ($\mathscr{M}_S$) is a triple $\langle F, R, [\,]\rangle$ such that:

- F is the union of $\{x\}$ and the set of sentences of $\mathscr{L}$;
- $[\phi] = \phi$ for all $\phi \in F$; and
- $R(\Delta, \phi)$ iff $\Delta < \phi \in C_s$ for all $\Delta \subseteq F$ and $\phi \in F$.

Lemma B.7 $\Delta <_{\mathscr{M}_S} \phi$ *iff* $\Delta < \phi \in C_S$.

Proof

$\Leftarrow$: The result is a trivial consequence of DB.6.

$\Rightarrow$: Suppose $\Delta <_{\mathscr{M}_S} \phi$. Then there is a tree T' in $\mathbf{T}$ whose floor is Δ and whose root is ϕ. We'll prove the result by induction on the recursive definition D5.2 of the set of grounding trees $\mathbf{T}$ for $\mathscr{M}_S$.

Basis: Suppose $R(\Delta, \phi)$. Then, by DB.6, $\Delta < \phi \in C_S$.

Induction step: Suppose $T, T_0, T_1, \cdots \in \mathbf{T}$ and T' is the graft of $T_0, T_1, \ldots$ into T. Let Γ be leaves of T that are also leaves of T'. Let Φ^* be the leaves of T replaced in the graft to yield T'. Then, for each T_i, T_i's root is some $\phi_i \in \Phi^*$. Let T_i's leaves be Δ_i. So, IH applies to yield the result that $\Delta_i < \phi_i \in C_S$. Similarly, the tree T verifies $\Phi^*, \Gamma < \phi$ ($= \phi_0, \phi_1, \ldots, \Gamma < \phi$), so IH implies that $\phi_0, \phi_1, \ldots, \Gamma < \phi \in C_S$. By DB.4, $\Delta_0, \Delta_1, \ldots, \Gamma < \phi \in C_S$.

□

Lemma B.8 $\mathscr{M}_S \vDash S$.

Proof Suppose $\sigma \in S$. If σ is full, then LB.3, LB.5, and LB.7 imply $\mathscr{M}_S \vDash \sigma$. If σ is a partial sequent $\psi \prec \phi$, then $\phi, x < \psi \in C_S$ by DB.1 and DB.4. So, by LB.7, $\phi, x <_{\mathscr{M}_S} \psi$. By D5.4 and D5.5, $\mathscr{M}_S \vDash \phi \prec \psi$.
□

Lemma B.9 *If* $S \nvdash_B \sigma$*, then* $\mathscr{M}_S \nvDash \sigma$*.*

Proof If σ is full, then LB.3, LB.2, and LB.7 imply that $\mathscr{M}_S \not\models \sigma$. If σ is a partial sequent $\phi \prec \psi$, then LB.3, LB.2, LB.5, and LB.7 imply that, for all Γ, $\phi, \Gamma \not<_{\mathscr{M}_S} \psi$. So, by D5.4 and D5.5, $\mathscr{M}_S \not\models \sigma$.
□

Theorem B.10 ***(B-Completeness)*** *If $S \models_B \sigma$, then $S \vdash_B \sigma$.*

Proof The contrapositive is immediate from LB.8 and LB.9.
□

Similar techniques suffice to prove completeness for AB.

Theorem B.11 ***(AB-Completeness)*** *If $S \models_{AB} \sigma$, then $S \vdash_{AB} \sigma$.*

C Soundness and Completeness for BNC and LSG

Definition C.1 An *S-chain* is a sequence of sentences $\langle \phi_0, \phi_1, \ldots, \phi_n \rangle$ such that, for every pair $\langle \phi_i, \phi_{i+1} \rangle$, either $\phi_i, \Gamma < \phi_{i+1} \in S$ or $\phi_i \prec \phi_{i+1} \in S$.

For readability, I will often omit angle brackets when referring to chains.[28]

A correspondence between the existence of S-chains and derivability of partial sequents in our systems is established by arguments of [Fine, 2012a, p. 12, Lemmas 4.3, 4.4] and [deRosset, 2014, Theorem B.3]

Lemma C.2

i. *$S \vdash_B \phi \prec \psi$ iff there is an S-chain of the form $\phi, \ldots, \psi$.*

ii. *$S \vdash_{AB} \phi \prec \psi$ iff there is an S-chain of the form $\phi, \ldots, \psi$.*

iii. *$S \vdash_{BNC} \phi \prec \psi$ without use of NON-CIRCULARITY iff there is an S-chain of the form $\phi, \ldots, \psi$.*

iv. *$S \vdash \phi \prec \psi$ without use of NON-CIRCULARITY iff there is an S-chain of the form $\phi, \ldots, \psi$.*

Definition C.3

[28] Fine introduces the notion of an S-chain at [Fine, 2012a, p. 12].

i. An S-*loop* is an S-chain of the form $\phi, \ldots \phi$;

ii. S is *acyclic* iff there is no S-loop; and

iii. S is *cyclic* iff S is not acyclic.

Lemma C.4

i. $S \vdash_{BNC} \sigma$ *iff either* $S \vdash_B \sigma$ *or* S *is cyclic.*

ii. $S \vdash \sigma$ *iff either* $S \vdash_{AB} \sigma$ *or* S *is cyclic.*

Proof

> $(i)\Rightarrow$: Suppose d witnesses $S \vdash_{BNC} \sigma$. If d contains no application of NON-CIRCULARITY, then d witnesses $S \vdash_B \sigma$, and we're done. Suppose that d contains an application of NON-CIRCULARITY, and let d' be a subderivation of d which contains an application of NON-CIRCULARITY, and is such that no subderivation of d of lesser depth contains an application of NON-CIRCULARITY. (Intuitively, d' is the "shallowest" derivation containing an application of NON-CIRCULARITY.) Then d' terminates in an application of NON-CIRCULARITY whose premise is $\phi \prec \phi$. The derivation of $\phi \prec \phi$ contains no application of NON-CIRCULARITY. So, LC.2(*iii*) implies that S is cyclic. Notice that this argument, together with LC.2(*iv*) implies $(ii)\Rightarrow$.
>
> $(i)\Leftarrow$: Suppose $S \vdash_B \sigma$. Then $S \vdash_{BNC} \sigma$. Suppose S is cyclic. By LC.2(i), $S \vdash_B \phi \prec \phi$. So, $S \vdash_{BNC} \phi \prec \phi$, and a single application of NON-CIRCULARITY suffices. Notice that replacing '$\vdash_B$' in this argument with '$\vdash_{AB}$' suffices for $(ii)\Leftarrow$.

□

Lemma C.5 *Let* $\mathscr{M}$ *be a model* $\langle F, R, [\,]\rangle$, *and let* $\langle F, R^*\rangle$ *be the digraph reduction of* $\langle F, R\rangle$. *Then* $f, G <_{\mathscr{M}} h$ *(for some* G*) iff* f *bears the ancestral of* R^* *to* h.

Proof

> $\Rightarrow$: Suppose there is a G such that $f, G <_{\mathscr{M}} h$. Then, for some $T \in \mathbf{T}$, f is in the floor of T and h is the root of T. We'll prove the result by induction on the depth of trees $T \in \mathbf{T}$.

Basis: Suppose $R(\{f\} \cup G, h)$. Then $R^*(f, h)$.

Induction step: Suppose $T', T_0, T_1, \dots \in \mathbf{T}$ and T is the graft of trees $T_0, T_1, \dots$ into T'. IH and the transitivity of the ancestral of R^* yield the result.

$\Leftarrow$: Suppose f bears the ancestral of R^* to h. We'll prove the result by induction on the definition of the ancestral of R^*.

Basis: Suppose $R^*(f, h)$. Then, by the definition of R^*, for some G, $R(\{f\} \cup G, h)$.

Induction step: Suppose, for some g, f bears the ancestral of R^* to g, and g bears the same relation to h. IH yields $f, G <_{\mathscr{M}} g$ and $g, D <_{\mathscr{M}} h$, for some G and D. Let T be a tree verifying $g, D <_{\mathscr{M}} h$, and T' be a tree verifying $f, G <_{\mathscr{M}} g$. Then g is in T's floor and is T''s root. By D5.2, the graft of T' into T is also a member of $\mathbf{T}$, and verifies $f, H <_{\mathscr{M}} h$, for some H.

□

Lemma C.6 *S is cyclic iff $\mathscr{M}_S$ is cyclic.*

Proof

$\Rightarrow$: Suppose S is cyclic. By LC.2(i), $S \vdash_B \phi \prec \phi$. By LB.3, LB.2, LB.5, and LB.7, for some Γ, $\phi, \Gamma <_{\mathscr{M}_S} \phi$. By LC.5, $\mathscr{M}_S$ is cyclic.

$\Leftarrow$: Suppose $\mathscr{M}_S$ is cyclic. By LC.5, for some Γ, $\phi, \Gamma <_{\mathscr{M}_S} \phi$. By LB.7, LB.5, LB.2, and LB.3, $S \vdash_B \phi \prec \phi$. So, LC.2($i$) implies that S is cyclic.

□

Theorem C.7 ***(BNC Completeness)*** *If $S \vDash_{BNC} \sigma$, then $S \vdash_{BNC} \sigma$.*

Proof The contrapositive is an immediate consequence of LB.8, LB.9, LC.4(i), and LC.6.

□

A similar argument suffices to show

Theorem C.8 ***(Completeness of LSG)*** *If $S \vDash \sigma$, then $S \vdash \sigma$.*

Theorem C.9 ***(Soundness of BNC)*** *if* $S \vdash_{BNC} \sigma$*, then* $S \vDash_{BNC} \sigma$*.*

Proof Suppose $S \vdash_{BNC} \sigma$. By LC.4(i), either $S \vdash_B \sigma$ or S is cyclic. By TA.1, if $S \vdash_B \sigma$, then $S \vDash_B \sigma$, and, *a fortiori*, $S \vDash_{BNC} \sigma$. It is easy to show, using LC.2, TA.1, and LC.5, that if S is cyclic, then S has no acyclic model, and so $S \vDash_{BNC} \sigma$ is vacuously true.
□

A similar argument suffices to show

Theorem C.10 ***(Soundness of LSG)*** *If* $S \vdash \sigma$*, then* $S \vDash \sigma$*.*

D Soundness and Completeness Results for Sparse Models

Lemma D.1 *Let* $\mathscr{M}^*$ *be a sparse model* $\langle F, \mathcal{D}, R, [\,]\rangle$*, and* $\mathscr{M}$ *its ordinary counterpart* $\langle \mathcal{D}, R_F(\mathcal{D}), [\,]\rangle$*. Then* $D <^*_{\mathscr{M}^*} f$ *iff* $D <_{\mathscr{M}} f$*.*

Proof By D7.4 and D5.4, it is enough to show that D is an extended floor for f in $\mathscr{M}^*$ iff it is also a floor for f in $\mathscr{M}$. But this is immediate from the fact that the grounding trees in $\mathscr{M}$ are given by the very same directed hypergraph $\langle \mathcal{D}, R_F(\mathcal{D})\rangle$ as the grounding* trees in $\mathscr{M}^*$.
□

Lemma D.2 *Let* $\mathscr{M}^*$ *be a sparse model* $\langle F, \mathcal{D}, R, [\,]\rangle$*, and* $\mathscr{M}$ *its ordinary counterpart* $\langle \mathcal{D}, R_F(\mathcal{D}), [\,]\rangle$*. Then:*

i. $\mathscr{M}$ *is acyclic iff* $\mathscr{M}^*$ *is; and*

ii. $\mathscr{M}$ *is additive iff* $\mathscr{M}^*$ *is.*

Proof (ii) is an immediate result of the definitions of additivity for sparse and ordinary models. (i) is an immediate consequence of the definition of acyclicality for ordinary models and the axiom of foundation.
□

Theorem D.3 ***(Sparse Soundness)***

i. *If* $S \vdash_B \sigma$*, then* $S \vDash^*_B \sigma$*;*

ii. *If* $S \vdash_{AB} \sigma$*, then* $S \vDash^*_{AB} \sigma$*;*

iii. *If* $S \vdash_{BNC} \sigma$*, then* $S \vDash^*_{BNC} \sigma$*; and*

iv. *If* $S \vdash \sigma$*, then* $S \vDash^* \sigma$*;*

Proof The cases are all proved very similarly, using the relevant soundness theorems, LD.1, and, for (*ii*)-(*iv*), LD.2. I'll prove (*ii*) for illustration.

(*ii*): Suppose $S \vdash_{AB} \sigma$. Let $\mathscr{M}^*$ be an additive sparse model verifying S. By LD.1, its ordinary counterpart $\mathscr{M}$ is an ordinary model that verifies S. By LD.2 and the additivity of $\mathscr{M}^*$, $\mathscr{M}$ is additive. By the soundness (for the ordinary semantics) of AB (TA.2), $\mathscr{M}$ verifies σ. By LD.1 again, $\mathscr{M}^*$ verifies σ.

□

Lemma D.4 *Let* $\mathscr{M}$ *be an ordinary model* $\langle F, R, [\,]\rangle$*, and* $\mathscr{M}^*$ *its sparse counterpart* $\langle F, F, R_F(F), [\,]\rangle$*. Then* $D <^*_{\mathscr{M}^*} f$ *iff* $D <_{\mathscr{M}} f$*.*

Proof Notice that $R_F(F) = R$ by D7.2. Thus, the result is an immediate consequence of D7.4, and D5.4.
□

Lemma D.5 *Let* $\mathscr{M}$ *be an ordinary model* $\langle F, R, [\,]\rangle$*, and* $\mathscr{M}^*$ *its sparse counterpart* $\langle F, F, R_F(F), [\,]\rangle$*. Then:*

i. $\mathscr{M}$ *is acyclic iff* $\mathscr{M}^*$ *is; and*

ii. $\mathscr{M}$ *is additive iff* $\mathscr{M}^*$ *is.*

Proof The argument for LD.2(*i*) also establishes (*i*). Since $R_F(F) = R$, (*ii*) is trivial.
□

Theorem D.6 ***(Sparse Completeness)***

i. *If* $S \vDash^*_{B} \sigma$*, then* $S \vdash_{B} \sigma$*;*

ii. *If* $S \vDash^*_{AB} \sigma$*, then* $S \vdash_{AB} \sigma$*;*

iii. *If* $S \vDash^*_{BNC} \sigma$*, then* $S \vdash_{BNC} \sigma$*; and*

iv. *If* $S \vDash^* \sigma$*, then* $S \vdash \sigma$*;*

Proof The cases are all proved very similarly, using the relevant completeness theorems, LD.4, and, for (*ii*)-(*iv*), LD.5. I'll prove (*ii*) for illustration.

(*ii*): We'll prove the contrapositive. Suppose $S \nvdash_{AB} \sigma$. By the completeness (for the ordinary semantics) of AB (TB.11), $S \nvDash_{AB} \sigma$. So, there is an additive ordinary model $\mathscr{M}$ that verifies S and does not verify σ. By LD.4, there is a sparse model $\mathscr{M}^*$ that verifies S and does not verify σ. By the additivity of $\mathscr{M}$ and LD.5, $\mathscr{M}^*$ is additive.

□

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