Technical Notes

We here state some basic results used in the main body of "Abstraction and Grounding", forthcoming in *Philosophy and Public Affairs*.[1] We will assume that each of the pluralities we discuss is indexed to an ordinal. For the purposes of constructing explanatory arguments, we will also assume that we have a first-order language $\mathcal{L}$ with identity, names for every element of each of the pluralities of individuals under discussion, names for every natural number, and a relational predicate symbol for each of the relations-in-extenson among the pluralities that we will discuss. We write aa and bb for non-empty pluralities, and $\emptyset\emptyset$ for the empty plurality (if there is one). Let

$$T^+(aa) \;=\; \{\ulcorner c = c' \urcorner \mid c = c' \wedge (c, c' \in aa)\} \qquad T^-(aa) \;=\; \{\ulcorner c \neq c' \urcorner \mid c \neq c' \wedge (c, c' \in aa\}$$

$$\text{and} \quad T(aa) \;=\; T^+(aa) \cup T^-(aa).$$

Intuitively, $T(aa)$ is the set of *truths* concerning identities and distinctnesses among aa.

In what follows, we will refer to an indexed collection using standard notation, writing $(x_i)_{i<\alpha}$ for $\{x_i | i < \alpha\}$. To avoid clutter, we will write (x_i), omitting the subscripted restriction '$i < \alpha$' entirely. We indicate co-indexed sets by using the same subscripts. Where there are two subscripts, the first subscript may sometimes depend on the second subscript, and these abbreviations may be embedded. Some examples:

Abbreviation	**Expansion**
(x_i)	$x_0, x_1, \ldots$
$(\Delta_i \Rightarrow \phi_i)$	$\Delta_0 \Rightarrow \phi_0; \Delta_1 \Rightarrow \phi_1, \ldots$
(x_{ij})	$x_{00}, x_{10}, \ldots \quad x_{01}, x_{11}, \ldots, \quad x_{0j}, x_{1j}, \ldots, x_{ij}, \ldots, \quad , \ldots$

The notions of a *relevant derivation* of the formula ϕ from the set of formulas Δ and of $\Rightarrow$ are defined as in Appendix A. We will be sloppy about use-mention distinctions when more care will not improve clarity.

Where $f \in aa \otimes bb$, let the *domain* of f be the plurality $\mathscr{D}(f)$, such that $a \in \mathscr{D}(f)$ iff $f(a, b)$, for some b; and let the *range* of f be the plurality $\mathscr{R}(f)$ such that $b \in \mathscr{R}(f)$ iff $f(a, b)$, for some a.

Proposition 1 Let $f \in aa \otimes bb$, $f \neq \emptyset$, and $\neg f(a, b)$. Let $\mathscr{D}(f) \backslash \{a\} = (a_i)$ and $\mathscr{R}(f) \backslash b = (b_j)$. Then $(a \neq a_i), (b \neq b_j) \Rightarrow \neg f(a, b)$.

Proof We may suppose (wlog) that $a \in \mathscr{D}(f)$, $b \in \mathscr{R}(f)$, $f(a, b_1)$, and $f(a_1, b)$, so that $\neg f(x, y)$ if grounded in the same way as $\neg((x = a \wedge y = b_1) \vee (x = a_1 \wedge y = b \vee (x = a_i \wedge y = b_i)$, for $i \geq 2$. Then, since $\neg f(a, b)$,

$$\neg(a = a \wedge b = b_1), \neg(a = a_1 \wedge b = b), (\neg(a = a_i \wedge b = b_i)) \Rightarrow \neg f(a, b).$$

The result follows by CUT, since

$$b \neq b_1 \Rightarrow \neg(a = a \wedge b = b_1) \quad a \neq a_1 \Rightarrow \neg(a = a_1 \wedge b = b) \quad (a \neq a_i, b \neq b_1 \Rightarrow \neg(a = a_i \wedge b = b_i)).$$

□

[1] https://onlinelibrary.wiley.com/doi/10.1111/phpr.13036

Proposition 2 Let $f \in aa \otimes bb$, and $f(a,b)$. Then $a=a, b=b \Rightarrow f(a,b)$.

Proposition 3 Suppose $f \in aa \otimes bb$, and $f : aa \xrightarrow[onto]{1-1} bb$. Then,

1. For some $S \subseteq T(aa)$, $S, T(bb) \Rightarrow f$ is 1-1; and

2. For some $S \subseteq T(bb)$, $S, T(aa) \Rightarrow f$ is functional.

Proof For each $a_i, a_j \in aa$, $b_k \in bb$, let $\phi_{ijk} = (f(a_i, b_k) \wedge f(a_j, b_k) \rightarrow a_i = a_j)$. If $a_i \neq a_j$, then, since f is 1-1, either $f(a_i) \neq b_k$ or $f(a_j) \neq b_k$. Suppose (wlog) $f(a_i) \neq b_k$. By P1, for some $S \subseteq T(aa)$, $S, (b_k \neq b'_m) \Rightarrow f(a_i \neq b_k) \Rightarrow \phi_{ijk}$, for $bb \setminus \{b\} = b'_1, b'_2, \ldots$. If $a_i = a_j$, then, by P2, $a_i = a_i, b_k = b_k \Rightarrow \phi_{ijk}$. Now, for each $b_k \in bb$, there are $a_i, a_j \in aa$ such that $a_i = a_j$ and $f(a_i, b)$, and so $a_i = a_i, b_k = b_k \Rightarrow \phi_{ijk}$. And, for each $b_k, b'_k \in bb$ such that $b_k \neq b'_k$, there are $a_i, a_j \in aa$ such that $a_i \neq a_j$, and thus $S, (b_k \neq b'_k) \Rightarrow \phi_{ijk}$. So, $S, T(bb) \Rightarrow (\forall a_i, a_j \in aa)(\forall b_k \in bb)\phi_{ijk} = f$ is 1-1, for some $T \subseteq T(aa)$. This proves (1). An exactly similar argument yields (2).

□

Proposition 4 Suppose $f : aa \xrightarrow[onto]{1-1} bb$, and let $aa = (a_i)$ and $bb = (b_i)$. Then

1. $(a_i = a_i)(b_j = b_j) \Rightarrow f$ is onto; and

2. $(a_i = a_i)(b_j = b_j) \Rightarrow f$ is total.

Proof Let $b_j \in bb$. Then, for some $a_{k_j} \in aa$, $a_{k_j} = a_{k_j}, b_j = b_j \Rightarrow (\exists a \in aa)f(a) = b$. So, $(a_{k_j} = a_{k_j}), (b_j = b_j) \Rightarrow (\forall b \in bb)(\exists a \in aa)f(a) = b$. Since f is total, $(a_{k_j} = a_{k_j}) = (a_i = a_i)$. This proves (1). A similar argument proves (2).

□

Proposition 5 Suppose $f(aa \xrightarrow[onto]{1-1} bb)$. Then:

1. $T(aa), T(bb) \Rightarrow f : aa \xrightarrow[onto]{1-1} bb$;

2. $T(aa) \Rightarrow f : aa \xrightarrow[onto]{1-1} aa$; and

3. $T(aa) \Rightarrow aa \approx aa$.

Proof (1) follows by P3 and P4. (2) follows immediately from (1), and (3) from (2).

□

Proposition 6

1. Suppose $f \in aa \otimes bb$, $a_i, a_j \in aa$, $b_k \in bb$, $f(a_i, b_k)), f(a_i, b_k))$, and $a_i \neq a_j$. Then

(a) $a_i = a_i, a_j = a_j, b_k = b_k, a_i \neq a_j \Rightarrow \neg(f(a_i, b_k) \wedge f(a_j, b_k) \to a_i = a_j)$; and

(b) $a_i = a_i, a_j = a_j, b_k = b_k, a_i \neq a_j \Rightarrow \neg(f \text{ is 1-1})$.

2. Suppose $f \in aa \otimes bb$, $b_i, b_j \in bb$, $a_k \in aa$, $f(a_k, b_i)), f(a_k, b_j))$, and $b_i \neq b_j$. Then

(a) $b_i = b_i, b_j = b_j, a_k = a_k, b_i \neq b_j \Rightarrow \neg(f(a_k, b_i) \wedge f(a_k, b_j) \to b_i = b_j)$; and

(b) $b_i = b_i, b_j = b_j, a_k = a_k, b_i \neq b_j \Rightarrow \neg(f \text{ is functional})$.

Proof By P2, we have $a_i = a_i, b_k = b_k \Rightarrow f(a_i, b_k)$ and

$$a_j = a_j, b_k = b_k \Rightarrow f(a_j, b_k).$$

(1a) follows by an application of CUT. (1b) follows immediately from (1a). The proof of (2) is similar.

□

Proposition 7

1. Suppose $f \in aa \otimes bb$, f is nonempty, $b \in bb$, and, for all $a \in aa$, $\neg f(a, b)$. Then, letting $bb \setminus \{b\} = (b'_j)$, $T^-(aa), (b \neq b'_j) \Rightarrow \neg(f \text{ is onto } bb)$.

2. Suppose $f \in aa \otimes bb$, f is nonempty, $a \in aa$, and, for all $b \in bb$, $\neg f(a, b)$. Then, letting $aa \setminus \{a\} = (a'_i)$, $T^-(bb), (a \neq a'_i) \Rightarrow \neg(f \text{ is total on } aa)$.

Proof For each $a_i \in aa$, P1 implies $(a_i \neq a'_i), (b \neq b'_j) \Rightarrow \neg f(a_i, b)$, where $aa \setminus \{a_i\} = (a'_i)$. So,

$$T^-(aa), (b \neq b'_j) \Rightarrow \neg(\exists a \in aa) f(a, b) \Rightarrow \neg(\forall b \in bb)(\exists a \in aa) f(a, b) \Rightarrow \neg(f \text{ is onto } bb).$$

This proves (1). The proof of (2) is similar.

□

Let B be any individual not in $\mathbb{N}^+$. Inductively define aa_n for $n \in \mathbb{N}^+$ so that $aa_1 = B, B$ and $aa_{n+1} = aa_n \cup n, n$.

Proposition 8

1. Suppose $m, n \in \mathbb{N}$, $m > n$, and $aa_n \not\approx aa_m$. Then $T(aa_m) \Rightarrow aa_n \not\approx aa_m$.

2. Suppose $m, n \in \mathbb{N}$, $m > n$, and $aa_m \not\approx aa_n$. Then $T(aa_m) \Rightarrow aa_m \not\approx aa_n$.

Proof To prove (1), note that, since $\neg f : aa_n \xrightarrow[onto]{1-1} aa_m$ for every non-empty $f \in aa_n \otimes aa_m$, P6 and P7 imply that $S \Rightarrow \neg f : aa_n \xrightarrow[onto]{1-1} aa_m$, for some $S \subseteq T(aa_m)$. For the empty function

$\emptyset$, $B = B \Rightarrow \neg\emptyset(B, B) \Rightarrow \neg(\emptyset : aa_n \xrightarrow[onto]{1-1} aa_m)$. So, we have $S \Rightarrow aa_n \not\approx aa_m$, for some $S \subseteq T(aa_m)$. Now, there is a $g \in aa_n \otimes aa_m$ such that $g(B, b)$ for each $b \in aa_m$. Moreover, g is not functional, since $m > n$. So, by P6, for each $a_i, a_j \in aa_m$, where $a_i \neq a_j$ $a_i = a_i, a_j = a_j, B = B, a_i \neq a_j \Rightarrow \neg(g$ is functional). By AMALGAMATION, $T(aa_m) \Rightarrow aa_n \not\approx aa_m$. (2) is proved similarly, using a $g \in aa_m \otimes aa_n$ that is a constant, non-injective function.

□

Let $S_1 = \{\ulcorner B = B \urcorner\}$, and, for $n \in \mathbb{N}^+$, let

$$S_{n+1} \quad = \quad \{\ulcorner B = B \urcorner, \ulcorner B \neq 1 \urcorner, \ulcorner 1 \neq B \urcorner, \ldots, \ulcorner B \neq n \urcorner, \ulcorner n \neq B \urcorner\}.$$

Proposition 9 For all $m, n \in \mathbb{N}^+$, $m < n$:

1. $S_m \Rightarrow m = m$;
2. $S_n \Rightarrow n \neq m$; and
3. $S_n \Rightarrow m \neq n$.

Proof We prove the result by induction. The basis case for (1) follows immediately from P5, since $T(aa_1) = \ulcorner B = B \urcorner$, and so $B = B \Rightarrow aa_1 \approx aa_1 \Rightarrow 1 = 1$.. The result in the basis cases for (2) and (3) follows from P8 and the basis case of (1). For the induction step, assume that each of (1)-(3) are true for each $k < m, j < n$. To see that (1) is true for n, notice that P5 implies $T(aa_n) \Rightarrow n = n$, and every member ϕ of $T(aa_n) \setminus S_n$ has one of the forms $\ulcorner j = j \urcorner$, $\ulcorner j \neq j' \urcorner$, or $\ulcorner j' \neq j \urcorner$ for some $j, j' < n$, $j > j'$. By IH, $S_j \Rightarrow k = k$, $S_j \Rightarrow j \neq j'$, and $S_j \Rightarrow j' \neq j$. Also, $S_j \subseteq S_n$. So, CUT yields (1). The arguments for (2) and (3) are similar, using P8 in place of P5.

□

Since the specification of explanatory inferences and the grounding principles in (deRosset and Linnebo, ming, §5) are exactly parallel, and strict ground, like $\Rightarrow$ is closed under CUT, Proposition 1 in (deRosset and Linnebo, ming, §5) can be proved by substituting '$<$' for '$\Rightarrow$' in the proof of P9.

Proposition 10 Suppose that $\emptyset\emptyset$ is an empty plurality, i.e., $(\forall x) x \not\in \emptyset\emptyset$. For all $n \in \mathbb{N}^+$:

1. $\emptyset \Rightarrow 0 = 0$;
2. $\emptyset \Rightarrow 0 \neq n$; and
3. $\emptyset \Rightarrow n \neq 0$.

Proof $\emptyset\emptyset \otimes \emptyset\emptyset$ has exactly one member, the empty function f, and we have $\emptyset \Rightarrow (\forall x \in \emptyset\emptyset)\phi$, for any ϕ. So, $\emptyset \Rightarrow f : \emptyset\emptyset \xrightarrow[onto]{1-1} \emptyset\emptyset \Rightarrow 0 = 0$, yielding (1). Let $bb_{n+1} = 0, 1, \ldots, n$, for $n \in \mathbb{N}$. To show (2), note that $\emptyset\emptyset \otimes bb_n$ has exactly one member, the empty relation f. Also, we have $\emptyset \Rightarrow \neg(\exists a \in \emptyset\emptyset)\phi$, for all ϕ. So,

$$\emptyset \quad \Rightarrow \quad \neg(\exists a \in \emptyset\emptyset) f(a, 0) \quad \Rightarrow \quad \neg(\forall b \in bb_n)(\exists a \in \emptyset\emptyset) f(a, b) \quad \Rightarrow \quad \neg(f \text{ is onto } bb_n)$$

$$\Rightarrow \quad \neg(f : \emptyset\emptyset \xrightarrow[onto]{1-1} bb_n) \quad \Rightarrow \quad \neg(\exists g \in \emptyset\emptyset \otimes bb_n)(g : \emptyset\emptyset \xrightarrow[onto]{1-1} bb_n) \quad \Rightarrow \quad \emptyset\emptyset \not\approx bb_n \quad \Rightarrow \quad 0 \neq n.$$

The proof of (3) is similar, using the failure of the empty relation in $bb_n \otimes \emptyset\emptyset$ to be total on bb_n in place of the failure of the empty relation in $\emptyset\emptyset \otimes bb_n$ to be onto bb_n.

□

Proposition 11 Suppose that $\emptyset\emptyset$ is an empty plurality, i.e., $(\forall x) x \notin \emptyset\emptyset$. For all $n, m \in \mathbb{N}$, where $n \neq m$, $\emptyset \Rightarrow n = n$ and $\emptyset \Rightarrow n \neq m$.

Let bb_{n+1} be defined as in the proof of Prop 10, and recall that $\textsc{Prec}(xx, yy)$ abbreviates

$$(\exists yy' \subseteq yy)(\exists y \in yy)(\ (\forall z \in yy)(z \in yy' \leftrightarrow z \neq y)\ \wedge\ xx \approx yy\ ).$$

Given an empty plurality $\emptyset\emptyset$ and the explanatory inferences for quantifications restricted to a plurality, it is easy to see that there will be explanatory arguments witnessing $\Delta \Rightarrow \textsc{Prec}(bb_k, bb_{k+1}) \Rightarrow P(k, k+1)$ for all $k \in \mathbb{N}^+$, where all members of Δ have one of the forms: $n = n$, $n \neq m$, or $bb_{n+1} \approx bb_{n+1}$, for some $n, m \in \mathbb{N}$. Similarly, it is easy to see that there will be explanatory arguments witnessing $\Delta \Rightarrow \textsc{Prec}(\emptyset\emptyset, bb_1) \Rightarrow P(0, 1)$, where Δ's members are all either $0 = 0$ or $\emptyset\emptyset \approx \emptyset\emptyset$. So, the application of Prop 11 yields:

Proposition 12 Suppose that $\emptyset\emptyset$ is an empty plurality, i.e., $(\forall x) x \notin \emptyset\emptyset$. For all $n, m \in \mathbb{N}$, where $n + 1 \neq m$, $\emptyset \Rightarrow P(n, n+1)$ and $\emptyset \Rightarrow \neg P(n, m)$.

Now we can sketch how all of the facts expressible in second-order Peano arithmetic are grounded, assuming the existence of an empty plurality $\emptyset\emptyset$. Second-order Peano arithmetic can be formulated in a language, $\mathcal{L}_{\text{PA2}}$, whose only primitive predicates are '$=$' and 'P' (Boolos, 1995). We wish to proceed to show, by induction on syntactic complexity, that, so long as there is an empty plurality $\emptyset\emptyset$, for every formula φ of $\mathcal{L}_{\text{PA2}}$ (relative to a variable assignment), either φ or $\neg\varphi$ is derivable in an explanatory argument from the empty set of premises and so zero-grounded. (To simplify the exposition, we elide the variable assignments and talk directly about natural numbers and relations-in-extension based on these.) Propositions 11 and 12 ensure that the claim holds for atomic formulas involving '$=$' and 'P'. The same goes for atomic formulas involving plural membership or predication of a relation-in-extension (cf. Appendix A). The induction step for disjunction, conjunction, and negation is straightforward. As for the quantifiers, the key is first to define the plurality nn of all natural numbers as the least plurality containing 0 and closed under the successor relation. (This plurality exists according to our Critical Plural Logic by its axiom of Infinity; see Appendix B.) We can now use plurality-restricted quantifiers of the form '$(\exists x \in nn)$' and '$(\forall x \in nn)$' to interpret the first-order quantifiers of $\mathcal{L}_{\text{PA2}}$, and analogously for quantification over pluralities and relations-in-extension. The induction step for true existential generalizations and negated universal generalizations involving these quantifiers is straightforward, while that of true universal generalizations (or negated existential generalizations) requires that true *plurality-restricted* generalizations of these forms can be derived in explanatory arguments from the collection of their instances (or negated instances).

References

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