Buying Logical Principles with Ontological Coin: The Metaphysical Lessons of Adding Epsilon to Intuitionistic Logic

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Abstract

We discuss the philosophical implications of formal results showing the consequences of adding the epsilon operator to intuitionistic predicate logic. These results are related to Diaconescu's theorem, a result originating in topos theory that, translated to constructive set theory, says that the axiom of choice (an "existence principle") implies the law of excluded middle (which purports to be a logical principle). As a logical choice principle, epsilon allows us to translate that result to a logical setting, where one can get an analogue of Diaconescu's result, but also can disentangle the roles of certain other assumptions that are hidden in mathematical presentations. It is our view that these results have not received the attention they deserve: logicians are unlikely to read a discussion because the results considered are "already well known," while the results are simultaneously unknown to philosophers who do not specialize in what most philosophers will regard as esoteric logics. This is a problem, since these results have important implications for and promise significant illumination of contemporary debates in metaphysics. The point of this paper is to make the nature of the results clear in a way accessible to philosophers who do not specialize in logic, and in a way that makes clear their implications for contemporary philosophical discussions. To make the latter point, we will focus on Dummettian

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discussions of realism and anti-realism.

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The goal of this paper is to argue for the philosophical, and in particular metaphysical, importance of certain results involving Hilbert's ε-operator. The results involve the addition of ε to intuitionistic (rather than classical) logic. More precisely, we will suggest that these results offer illumination of debates about realism and anti-realism, and so help us get a clearer picture of what it means to say that an individual or property is real, or objective, or is in some related way attributed a special status of this sort.

We suspect that the sort of discussion we will offer here tends to fall between the stools in contemporary philosophy. The formal results are not new, and so mathematicians and formal logicians might look past the philosophical discussion surrounding the results and conclude that "all this is already known." On the other hand, the formal results in question are in an area that many contemporary philosophers are likely to regard as the esoteric reaches of deviant logic, potentially inclining them not to read the paper either. We will suggest that this is a case where the interesting philosophy starts when the formal proof is completed—the philosophical importance is not something that can be read straight off the proof, but takes some showing. We hope that it repays the effort it might take to overcome whatever impatience (with philosophical niceties or with technical details, depending on one's background) one might bring to the task. Since this is a presentation of a basic idea that, if worthwhile at all, is worthy of a deeper study, we will attempt in the present paper not to overly try the patience of either sort of reader, keeping both philosophical and mathematical details to the minimum that will allow us to try to make our point clear.

The formal results in question show that the addition of ε to intuitionistic predicate logic is non-conservative.[1] As is well known, intuitionistic logic is a sub-system of classical logic in the sense that all validities of intuitionistic logic are also classically valid, but not conversely. As is also well known to people already familiar with the ε-operator, its addition to classical logic is conservative—this is the upshot of the so-called ε-theorems. To introduce some terminology that will facilitate discussion later on, we will call principles that are classically but not intuitionistically valid *superintuitionistic*.[2] We call the system that results by adding the ε-operator to intuitionistic predicate logic the *intuitionistic ε-calculus*. The basic form of the results

[1]The results we review below are from [1, 2, 3]; Bell's work was motivated by [4] and [5].

[2]This is fairly standard terminology, though we note that we intend to focus on consistent systems and to ignore consistent extensions of intuitionistic predicate logic that are not consistent

of interest is that certain superintuitionistic principles are valid in the intuitionistic ε-calculus, and that when we add in other, seemingly innocuous assumptions to that calculus we increase the number of interesting superintuitionistic principles that are valid, until we eventually make them all valid. That is, the addition of ε to intuitionistic logic is *always non-conservative*, and with the addition of some modest-seeming additional assumptions ε is sufficient to make over intuitionistic into classical logic.

The philosophical importance of these results is not, we submit, immediately apparent. The most direct and digestible way to make them clear, we think, is to consider the implications of the results for Michael Dummett's highly influential account of the relationship between realism and anti-realism. We will try to show that they offer us a way to improve Dummett's account, shoring up what might be regarded as a "soft spot" in his story. But we hope (and will suggest) that the lessons we draw are not entirely dependent on the details of Dummett's program for their interest, and that they offer more general lessons for how to think about notions such as reality and objectivity.

So far, we've described the project at a pretty high level of abstraction. In slightly more detail, we shall proceed as follows. First, we will present a sketch of Dummett's framework for understanding debates between realists and their opponents in different areas of philosophical dispute—realists and nominalists about universals, realists and behaviourists about mental states, realists and constructivists in mathematics, and so on. In our selective sketch we will draw attention to some key features of Dummett's framework for our purposes. One is his suggestion that all these debates are best re-cast as debates in philosophical logic, in the sense that the correctness of the principle of bivalence, and so the law of excluded middle, and so of classical logic, for a particular domain of discourse is a criterion for realism being correct for that domain. Another is that intuitionistic logic has a special status—it is, in fact, logic properly so-called, and so is metaphysically neutral. The classically but not intuitionistically valid principles, *i.e.* the superintutionistic principles, have a status akin to mathematical induction or laws of physics in that they can be employed perfectly legitimately in certain domains, but not others.

Secondly, we shall argue that the case Dummett makes for the link between realism and classical logic, while not unpersuasive, is a soft spot in his general account because it relies on a metaphor to link the metaphysical notion of a *mind-independent reality* to the acceptance of superintuitionistic principles. We will argue that the technical results to which we want to draw attention help fill in the details

with classical logic. We find the terminology, for instance, in [6, p.103]: "Extensions [of intuitionistic logic] are called superintuitionistic logics. Superintuitionistic logics which are contained in the classical logic are said to be intermediate. An intermediate propositional logic is the same as a consistent superintuitionistic logic; it is not true for predicate logics."

in Dummett's argument by providing a more direct link between metaphysical assumptions and the acceptance of superintuitionistic principles, including excluded middle.

The case in the second part has a few sub-parts. First, we note that the ε-operator, being a sort of choice principle, involves an "existence assumption." Choice principles and other metaphysically loaded principles, such as axioms of infinity, are commonly (if not universally) taken to be, for this very reason, non-logical. Next, we present a sketch of how the addition of these non-logical principles to intuitionistic logic make various superintuitionistic principles provable. Finally, we turn to the job of drawing philosophical lessons. We will contend that the results establish a clearer connection between metaphysical assumptions and the superintuitionistic principles than do vague suggestions about reality "fixing the truth values" of well-formed claims. We also argue that there is philosophical mileage in the fact that the superintuitionistic principles don't necessarily come as a package deal. As we shall see, adding ε together with different choices of additional assumptions yields different intermediate logics between the metaphysically neutral basis of intuitionistic logic and the classical logic that, according to Dummett, corresponds to full-blown realism. We will argue that these way-stations between the intuitionistic basis that antirealists should have no complaint about and full classical logic, and the assumptions that suffice to reach these different stations, link up in interesting ways to our intuitions surrounding notions like objectivity and reality. It is this final point, we think, that makes these results of interest whether or not one approves of the details of Dummett's story, since it offers the prospect of a more fine-grained categorization of metaphysical options than is available if one supposes that realism is an all or nothing matter, and it offers a way to discuss the question of reality and unreality, at least sometimes, at the level of individual, fairly homey, properties, rather than in terms of nebulous notions such as "discourses."

1 Realism and Its Opponents

Michael Dummett was one of the most influential philosophers of the second half of the 20th Century, so many philosophers are likely to have at least a vague idea of his views. As well-known as any of these would be his commitment to the idea that the acceptance of the principle of bivalence, and so of the law of excluded middle, and therefore of classical logic, was a "criterion of realism."[3] Given what we said in the introduction, it will not surprise anyone that this is an important part of

[3]While commentators on Dummett perhaps used the phrase "criterion of realism" more often than Dummett did himself, he does use it occasionally, for instance in [7, p.379,467].

Dummett's view for our purposes. We will provide only a quick sketch—we hope not a caricature—of the case Dummett makes for this view, highlighting strands that will be of use in the ensuing discussion. The argument was presented many times with differences of emphasis and detail throughout Dummett's long career, and the full-dress presentation involves discussion of theories of meaning, the learnability of language, and more, that are fascinating, much disputed, and far too intricate for what we are hoping to do in this paper. We ask forgiveness from Dummett scholars who regard our simplified presentation as too simple. We rely largely on Dummett's own restrospective descriptions of what was central to his account of realism and antirealism from late in his career, especially his inaugural address upon taking up the Wykeham Chair at Oxford, [8].

Realism has two parts:

- First, the idea that the things we say in a particular area of discussion, a *discourse*, are properly regarded realistically involves a commitment to the idea that we are making claims about a reality that is in an important way "independent of us," and in particular is independent of our ability to know about it. Of course, few would deny that in some sense reality is independent of us. What is distinctive about realism is the suggestion that if realism is correct for a discourse, in spite of this independence our language somehow links up with the reality in a special way: the truth values of our statements in the discourse are fixed by that reality, independently of whether we can come to know those truth values. Indeed, in discourses about which realism is correct, the prospect that there are claims whose truth values we *cannot* come to know, even in principle, cannot be ruled out. While there is some fussing to be done about difficult details (for instance, what to say about vague statements that don't appear to be either definitely true or definitely false), the presence of a mind- and language-independent reality to fix the truth values of our statements is the link between metaphysics and bivalence.

- Secondly, some of our claims are true.[4] We may regard the language we use to discuss unicorns as purporting to refer to a mind- and language-independent reality, but think that there are no unicorns. Presumably that should suffice for us to count as anti-realists about unicorns.

Since realism, according to this story, has a two part definition, speaking at a very general level there are two ways of rejecting realism. As the example of unicorns

[4]More precisely, this should be formulated as the requirement that some of the atomic claims we make in the discourse in question must be true, since we don't want the condition to be met simply because of vacuous quantificational claims or negative claims turning out true.

suggests, if none of the (atomic) statements of a domain are true, then realism is not true for that domain. But there are, of course, more interesting versions of this sort of antirealism, such as the various "error theories" in ethics (most famously [9]) or mathematics (*e.g.*, Field's account of arithmetic in [10]). Various versions of fictionalism in these domains (starting with Mackie's and Field's own positive accounts) are plausibly viewed as antirealist for similar reasons—the claims in the domain are not *literally* true (or true *when taken at face value*), but there is some other story about what makes the claims involved seem so important to us in spite of their (literal) falsity. We set aside the question of whether a proper formulation of realism requires that we find a way to unpack the work that *literalness* (or some similar notion) plays in explaining why fictionalism is not a sort of realism.

Anti-realisms of another sort for some reason reject the other component of the definition of realism. Emotivists in ethics say that our statements are not in the business of saying true and false things at all, and kindred *expressivist* accounts in various areas of philosophy (about conditionals, laws of nature, *etc.*) similarly deny that the apparent statements of some domain are properly regarded as statements (if by statement we mean a claim that could be either true or false) at all. Others, though, do not want to go so far as saying that the apparent statements of a domain are not actually in the true/false game, but deny in some way that their truth values are suitably "independent" of us. Such, for instance, are the views of constructivists in mathematics who hold that a mathematical statement is true precisely if it is provable, and false if it is refutable. Such a view certainly fails to make the truth values of statements in this domain independent of our ability to know them (presuming that by "provable" we have in mind, somehow, provability by agents like us and not, for instance, mathematicians with infinite capacities of some sort), but it allows that the statements in this domain are in the game of making true or false claims.

Finally, many antirealist views have traditionally taken on the guise of reduction ism—for instance, the phenomenalist claim that claims about the physical world were somehow indirect ways of talking about perceptions, or logical behaviourist claims that talk of mental states were really indirect descriptions of behaviour.

2 The status of superintutionistic principles

We are now in a position to see Dummett's reasons for emphasizing the role of superintuitionistic principles in debates between realists and antirealists. If, as has been suggested, realism involves a commitment to bivalence, then it is (barring some esoteric further maneuvering) a short step to the acceptance of classical logic, since bivalence more-or-less implies classical logic. (See [11] for some of the nuances ignored in this statement.)

From the other direction, Dummett has little good to say about expressivist or reductionist versions of antirealism. Indeed, he sometimes suggests that, due to the prevalence of such versions of antirealism in the history of philosophy, his review of historical debates between realists and antirealists might have seemed to not even be worth pursuing, for the merits of the respective views "threatened that all the contests would end in victory for the realist before the comparative study began" [7, p.470]. Constructivist antirealism about mathematics, he judged, was the only antirealism not to fall prey to arguments likely to be fatal to expressivist and reductionist views. Presuming that there were substantial philosophical insights behind traditional antirealist positions in other realms to which their expressivist or reductionisist presentations failed to do justice, Dummett suggested that the best road forward was to recast the antirealist views in all the traditional disputes along the lines suggested by constructivism in mathematics. That is, they should take as their starting point the idea that the truth of a claim in the disputed domain consists in whatever counts as conclusively establishing that claim, just as constructivism takes truth to be provability, while falsity amounts to the possibility of conclusively ruling the claim out. Dummett's suggestion is that evidentially constrained notions of truth of this sort will share the same basic logic as one finds in the mathematical antirealism that serves as its model—namely, intuitionistic logic.

It is important to recognize that on this account the question of realism and antirealism is one that varies by domain, or at least could do so. It is an open possibility that one ought to be a realist about tables and chairs but not about mental states, for instance. Since intuitionistic logic is a subsystem of classical logic, we can say that the intuitionistically valid logical principles are *metaphysically neutral* in the sense that they are not in dispute between the participants in any of the realism/antirealism debates, once those debates are recast according to Dummett's recommendations. The superintuitionistic principles, on the other hand, arguably do not deserve the label "logic" at all, on this account. The case for saying so turns one of the grounds Frege used to defend the claim that the truths of arithmetic are logical to the opposite purpose. Frege's case appeals to the idea that properly logical inferences are the ones that apply in *every realm of human thought*, something he claimed was true of arithmetic priniciples. If Dummett is right, superintuitionistic principles are correct in domains where realism is the correct view, but not in general. Nowadays we take the same criterion to show that, for instance, mathematical induction is not a principle of logic because it holds when talking about countable, discrete objects but not when talking about real numbers. Similar reasoning seems to show that superintuitionistic principles should be regarded as non-logical, and, as we have noted above, can be seen as similar to mathematical induction, or the laws of physics which apply to inferences about physical objects but not in every domain.

There is another line of argument to be found in Dummett's writings for the claim that logic, properly so called, is intuitionistic logic. Probably the most explicit discussion of this matter occurs in *The Logical Basis of Metaphysics* [12]. The discussion there considers candidates for the status of "logical operator" in natural deduction terms, taking the meaning of any proposed operator to be specified in terms of its introduction and elimination rules.[5] Building from the intuititive idea that a properly valid deductive inference must not allow us to infer in a conclusion anything that was not already "contained in the premises," Dummett argues that it is a necessary condition of being a properly logical operator that there be "harmony" between the introduction and elimination rules for the operator. That is, if P is a sentence with a particular operator as its main logical operator, all and only the things required to make an inference of P using the introduction rule(s) are things we can extract from P using the elimination rule(s).[6] In *Logical Basis* we get Dummett's most explicit argument that the problem with some of the classical logical operators is a lack of harmony: that is, the introduction and elimination rules for, for instance, classical negation are not in harmony (and so are not properly logical). The principles of intuitionistic logic are by contrast, he suggests after considerable discussion, in harmony.

To summarize: we find in Dummett two different lines of argument for the view that, in spite of what we teach students in their first formal logic courses, such classically valid principles as $\forall x(Px \vee \neg Px)$, $P \vee \neg P$, $(P \to Q) \vee (Q \to P)$, and $\neg(P \wedge Q) \to (\neg P \vee \neg Q)$ are not really *logical* principles at all. Instead, Dummett argues, the principles of intuitionistic logic are logic, properly so-called.

It follows that in circumstances in which the superintuitionistic principles are correct, their correctness must have some further basis besides logical correctness, for they are not logically correct. As we have seen, Dummett's suggestion is that what justifies them, when they are justified, is the existence of a suitably mind-independent realm to which the statements of the discourse in question hook on in the right way so that the principle of bivalence is true for that domain. Given their dependence on a metaphysical truth, it is perhaps not too much of a stretch to call them "metaphysical laws."

But perhaps this is too hasty. One might buy the suggestion that intuition-

[5]The reasons for this approach have primarily to do with a particular sort of theory of meaning that Dummett also argues for in the book. We think it is harmless to leave those details aside.

[6]We set aside interesting details again here: for instance, whether an introduction/elimination pair is in harmony depends on what logical operators are presumed to be in place already.

istic logic is logic properly so-called without accepting the suggestion that extra metaphysical commitment warrants additional principles of reasoning. Dummett suggests that the presence of a "mind-independent reality," onto which our language hooks appropriately, suffices to establish the correctness of the superintuitionisic principles—because in those circumstances reality is supposed to determine one of two truth values for each well-formulated statement, and bivalence implies classical logic. This picture has not universally been found to be compelling. For instance, one might well be puzzled, on roughly Dummettian grounds, by how we could come legitimately to think that we've successfully latched onto reality in this way. Occasionally, commentators on Dummett's work raise this concern in the form of a question: how, if the standard antirealist arguments he canvasses ever work, can they fail to *always* work—can Dummett be arguing for anything but *global* antirealism? [13] We next will suggest alternative grounds that help bolster Dummett's suggestion that metaphysical, and so non-logical, commitments can ground the acceptance of these metaphysical laws.

3 Existence and Logic

The basic structure of the help we propose to offer Dummett is this: certain formal results make clear that adding in clearly non-logical, plausibly metaphysical (because ontological) principles to "logic properly-so-called" make various superintuitionistic principles correct. We therefore have a more direct, rigorous link between metaphysical assumptions and the superintuitionistic principles than is provided by a suggestive detour through talk of mind-independent realities. As a next step, we consider the question of whether we're right to say that the principles being added are metaphysical and non-logical.

We begin by noting that the principles in question all assert the existence of something, and it is quite a prevalent view that existence claims are *ipso facto* not logical. As long ago as his 1919 *Introduction to Mathematical Philosophy* [14], Russell complained about the "impurity" of sentences like $\exists x.x = x$, which are valid in usual formulations of predicate logic but which, he complains, are true only if at least one object exists (and so are not truly logical). Carnap, in *The Logical Syntax of Language* [15, §38a], sketches a method for constructing a logical system which, he says, does not make such assumptions. We find here early steps towards the development of what became known as *free logic*.[7]

[7] Neither Russell nor Carnap made provision for names (properly so-called) which do not refer to existent objects, preferring to explain away "Pegasus" and "The King of France" as not genuine names. It wasn't until the 1950s and 1960s that a sustained effort was made to provide suitable

Now let us consider what we take to be the standard attitude towards free logics. While some of the advocates of free logic were quite militant about the lessons to be drawn from their work—and correspondingly militant in their opposition to continued use of the usual classical predicate calculus—few logicians work in free logic nowadays except in special cases where the role of the existence assumptions involved in standard predicate logic are especially salient (*e.g.*, type theories in which some types might be empty, or in some quantified modal logics). It is easy enough to keep in mind that the validity of $\exists .x = x$ in standard predicate logic is merely an artifact of a simplifying assumption. We work in a system that considers only models with non-empty domains in order to simplify our system of rules; nobody is thereby tempted to such reasoning as "aha, something necessarily exists, so let's call it 'God' ..." so the simplification is harmless. We are all clear that not every validity in the first order predicate calculus as usually presented is *really* a *logical truth.*

Consider next a perhaps more familiar example, a potted version of the standard story of the demise of logicism in the philosophy of mathematics.[8] The story comes in two parts. First, Frege's logicism. It had the considerable virtue of deriving the existence of the natural numbers from self-evident principles. Alas, it also had the even more considerable demerit of being inconsistent, and thus serves as an early pothole in the rough ride the 20th Century provided for the notion of self-evidence. The second chapter is Russell and Whitehead's logicism, where they make a valiant attempt to formulate all of mathematics within a type-theoretic logic in *Principia Mathematica.* But their formidable technical achievement does not count as a vindication of logicism because along the way they must appeal to certain axioms which are manifestly non-logical—the usual culprits pointed to being the axioms of infinity, of reducibility, and of choice. And the reason these principles (especially the first and third) are regarded as manifestly non-logical is that they imply the existence of particular entities.

It is worth pointing out that it is not the unanimous opinion in the history of logic and philosophy that having existential implications is enough to rule a principle out as a principle of logic. Indeed, any vindication of logicism as it seems to have been conceived in the late 19th and early 20th Centuries seems to have

formulations of classical predicate logic innocent of existential assumptions and which allowed for singular terms which do not refer to existing objects, and nowadays "free logic" usually refers to systems meeting both those conditions. An impressive cast of logicians contributed to the effort to develop free logics, including Henry Leonard, Hugues Leblanc, Theodore Halperin, Jaako Hintikka, Dana Scott, Bas van Fraassen, Robert Meyer, Karel Lambert and many others.

[8] We are well aware that the actual history of logicism is longer, more complicated, and more interesting than it would be useful to detail here.

presupposed that we would be able to prove the existence of infinitely many objects (*e.g.*, the natural numbers) on purely logical grounds, and Frege famously argued for the existence of these "logical objects." Frege has at least two distinguishable methods for getting to this claim. First, he *derives* the axioms of arithmetic from "self-evident" principles. Alas, among these was the notorious Basic Law V, which rendered his system inconsistent. On the other hand, he argues that the truths of arithmetic are logical because they are among the principles that "extend to everything that is thinkable; and a proposition that exhibits this kind of generality is justifiably assigned to logic" (Frege, "On Formal Theories of Arithmetic," as quoted in [16, p.44]). The first path clearly does not have many advocates today. Nor, though, does the latter: as discussed above, at least some of the basic principles of arithmetic, including mathematical induction, are standardly viewed as applying in some domains and not others, and so these principles are categorized as non-logical by appeal to essentially the same criterion Frege uses to classify them as part of logic. So while there may be grounds for believing in "logical objects," not many today are likely to think they find those grounds in Frege—and we know of no other compelling alternative arguments.

It's tempting to state the lesson as follows: there had to be something wrong with Frege's account, since it extracted such rich ontological information from putatively logical principles; and we ought not to be surprised that to get mathematics out of logic Russell would have had to smuggle the non-logical existence assumptions in somewhere.

As a final remark for this section, we note that the results we will consider begin with principles that are all versions of (or relatives of) the axiom of choice. The axiom of choice, of course, has its own long and contentious history in the philosophy of mathematics. Its legitimacy has at times been hotly disputed, usually because of its awkward or implausible consequences—for instance, it's classical equivalence to the well-ordering principle implies that there is a well-ordering of the real numbers, whose well-ordering is somewhat difficult to imagine, and it is the key to proving the theorems that lie behind things with names like "The Banach-Tarski Paradox" and "Skolem's Paradox."[9] What matters for us, though, is that the Axiom of Choice

[9] In standard form, AC is the claim that for any family of non-empty, disjoint sets there is a function that chooses an element from each. Most famously, this turns out to be classically equivalent to the well-ordering theorem (the claim that every set can be well ordered), and Zorn's Lemma. But it is also equivalent to the upward and downward Löwenhiem-Skolem Theorem, and to the claim that every onto function has a "section" (*i.e.*, "epis split"): that is, if $h : A \to B$ is an onto function, there is a function $s : B \to A$ such that $h \circ s$ is the identity function on B. There are weaker choice principles that are also much studied, and which also come in classically equivalent families: König's Lemma is equivalent to the completeness of first order logic, which is equivalent to the Prime Ideal Theorem, *etc.* Many of the principles which are equivalent in classical set theory

is an existence principle; depending on formulation, it might assert the existence of functions or of sets (*e.g.* a set that includes a single member from each of a family of sets). But it is an "existence principle" in the sense of asserting that for each of one sort of thing that exists, there is a thing of another sort that exists, too. At the risk of too much repetition, it was precisely this existential import that marked the version of Russell's axiom of choice off as properly mathematical, and so not logical, when people point to the need to invoke it as a sign of the failure of *Principia Mathematica* to achieve its logicist goals.[10]

4 Choice Principles and Classical Logic

We turn, at last, to the technical results. The results in question are all relatives of a result that has been known for a while: that in intuitionistic set theory and related mathematical systems, the axiom of choice implies the law of excluded middle (and hence all of classical logic).[11] That the axiom of choice in a constructive setting implies the law of excluded middle, and so all of classical logic, is often called Diaconescu's Theorem. Diaconescu's original proof was in the context of Topos Theory, and so required somewhat formidable mathematical machinery to formulate and explain. It has since become clear that the heavy machinery is not necessary to get this result. Starting in the mid-1990s with the work of John Bell [1, 2], more illuminating versions of and variations on this result began to appear in the philosophical literature.

Let's look first at the "stripped down" version of the proof of Diaconescu's The-

are not equivalent in intuitionistic systems.

[10]It is not uncommon to hear it claimed that the Axiom of Choice is, in fact, a principle of constructive logic, since its truth follows from the meaning of the constructive existential quantifier. ("A choice is implied in the very meaning of existence," as Bishop and Bridges say in *Constructive Analysis.*) Indeed, in the early 1990s there were two very different research programs travelling under the name "intuitionistic type theory," in one of which the axiom of choice implied classical logic while in the other the axiom of choice was said to be a principle of constructive logic. It would take us too far afield to review this fascinating history here. See [17, 18] for discussion. While we would contend that the principle that goes by the name AC in the constructive systems where it is said to be logically valid doesn't really deserve the name, the key point for the present is that the case made for calling that principle logical involves showing that it does not have existential import in the relevant sense (*i.e.*, the claim is that the existence of the choice function is implied by the truth of the existential claim in the antecedent because the existential quantifier requires the existence of a "witness" for its truth).

[11]While true, this claim hides some hedging in the "related systems" clause. As noted in an earlier footnote, some systems of constructive mathematics can't be counted as related systems since in them a version of AC is valid—unless, of course, one argues instead that the valid principle itself isn't really the Axiom of Choice. We steer clear of this debate for present purposes.

orem presented in [1]. One virtue of this version of the theorem for present purposes is that it removes complications about the specific version of intuitionistic set theory or the axiom of choice in question—those versions where something called "choice" holds in an intuitionstic set theory without implying excluded middle somehow do not satisfy the assumptions of the theorem. A second virtue it will ease our transition from the set theoretic to the logical context.

Theorem 4.1 (Diaconescu's Theorem)**.** *The core of the argument*

Proof. Assume the following:

(1) There are two terms c and d such that $\vdash c \neq d$

(2) For any A, we can find an s and t, such that:

  (a) $\vdash A \to s = t$

  (b) $\vdash (s = c \vee A) \wedge (t = d \vee A)$

We can then reason as follows:

$\vdash (s = c \wedge t = d) \vee A$ (distributivity)

$\vdash (s \neq t) \vee A$ (from 1)

$\vdash (s \neq t) \to \neg A$ (2 (a), contraposition)

$\vdash A \vee \neg A$

□

Since the validity of excluded middle is enough to make all of classical logic valid, this proof provides us with an easy way to show that a particular intuitionistic theory is powerful enough to prove all the principles of classical logic—we need only show that it allows us to prove conditions (1) and (2) from theorem 4.1.

Consider, for instance, why this should be expected to hold in an intuitionistic set theory.[12] If we assume that we have the Axiom of Choice, and assume that sets and functions behave in what will strike classical mathematicians as a natural way, then we have LEM.

Theorem 4.2 ("Intuitionisitic set theory" plus (ε) implies LEM)**.**

[12]We do not present this discussion in terms of any particular intuitionistic set theory, instead simply flagging important assumptions that should be familiar to anyone with a passing acquaintance with classical set theory. Once again, we justify this approach as a way to avoid getting bogged down in details that do not advance the narrative.

Proof. Clearly, in any reasonable set theory we'll have $\vdash 0 \neq 1$, so $c \neq d$ is very easy. To get (2), we choose a y not free in A and define $B(y)$ to be $A \vee y = 0$ and $C(y)$ to be $A \vee y = 1$. This defines two non-empty subsets of $\{0, 1\}$, namely

$$z = \{y \in \{0, 1\} | B(y)\}$$

and

$$w = \{y \in \{0, 1\} | C(y)\}.$$

Since we assume the axiom of choice, let f be a choice function on the power set of $\{0, 1\}$. Then $f(z)$ and $f(w)$ will serve as the terms s and t in (2). For if A is true then by extensionality $z = w$, and so since f is a function we have $A \to f(z) = f(w)$. Moreover, since $\vdash f(z) = 0 \vee f(z) = 1$ and $f(z) = 1 \to A$, we have $\vdash f(z) = 0 \vee A$, and similarly $\vdash f(w) = 1 \vee A$, so we have (b) as well. $\square$

Note that we didn't require the full power of the Axiom of Choice to get s and t, only a choice function on $\mathcal{P}(\{0, 1\})$. There are two obvious lessons in this fact. First, we might expect other principles weaker than the Axiom of Choice to give us the required terms. Secondly, the existence of a choice function on the power set of a two-element set cannot be the same triviality in intuitionistic set theory that it is in classical set theory—where the only non-empty subsets are $\{0\}$, $\{1\}$ and $\{0, 1\}$ after all—since there are perfectly good intuitionistic set theories in which the law of excluded middle does not hold, even though all the other elements of the proof just sketched are in place.

For the purposes of drawing metaphysical lessons, though, it will be helpful to move from the mathematical to a more straightforwardly logical setting. The tool that will allow us to do is is Hilbert's ε-operator. Loosely speaking, the ε–operator adds, for each predicate Φ of a language, a new term $\varepsilon x \Phi$ to the language, one in which x does not occur free.[13] This makes ε a (variable-binding) term-forming operator of a familiar sort, similar to a definite description operator, for instance. What distinguishes one such operator from another are the logical rules governing them. The logical rules for ε are give by *the epsilon axiom*, which is the following scheme:

$$\exists x \Phi(x) \to \Phi(\varepsilon x \Phi(x)), \qquad \text{for all } \Phi(x). \tag{ε}$$

A moment's reflection will make clear why ε is sometimes called a "logical choice function," and so one might expect that it would be a useful tool for translating

[13] More precisely, a clause to this effect needs to be added to the recursive definition of the well-formed expressions of the language, because we want to allow for the presence of ε–terms in the formulas from which new such terms are formed.

facts about the axiom of choice from a mathematical to a logical setting. As we shall see, it allows us to do more than that.

It will be useful for what follows if we specify (and in some cases re-introduce) some terminology. We will use "IPC" to refer to the intuitionistic predicate calculus (with identity), and will often refer to the addition of the epsilon axiom to a theory using locutions such as "with (ε)" or "+ (ε)." The logical theory that results from adding (ε) to IPC we call the *intuitionistic epsilon calculus*, and we sometimes designate it as "IPC(ε)." We will often have occasion to refer to *Ackermann's extensionality principle* as "(Ack)."

$$\forall x(\Phi(x) \leftrightarrow \Psi(x)) \to \varepsilon x\Phi = \varepsilon x\Psi, \qquad \text{(Ack)}$$

IPC(ε) + (Ack) we will refer to as the *extensional intuitionistic epsilon calculus.*

Theorem 4.3 (The extensional intuitionistic epsilon calculus + the existence of two provably distinct individuals implies LEM)**.** *Let T be a theory in the extensional intuitionistic epsilon calculus in which we can prove $c \neq d$. Then $T \vdash LEM$.*

Proof. For the present, we write $\vdash$ for $T \vdash$. Recall that, according to theorem 4.1, to prove LEM it suffices that the following hold:

(1) $\vdash c \neq d$

(2) For any A, we can find an s and t, such that:

(a) $\vdash A \to s = t$

(b) $\vdash (s = c \vee A) \wedge (t = d \vee A)$

We have assumed that (1) holds.
To establish that we also have (2), first, for any A, choose a variable y not free in A and define:

$$B(y) \equiv (A \vee (y = c)) \text{ and } C(y) \equiv (A \vee (y = d))$$

Let $\varepsilon y B(y) = s$ and $\varepsilon y C(y) = t$. Since obviously $\vdash \exists x B(x)$ and $\vdash \exists x C(x)$, using the epsilon axiom we readily derive:

$$\vdash (s = c \vee A) \wedge (t = d \vee A),$$

that is we get (2b).

To see that we also have (2a), note that $\vdash A \to \forall y(B(y) \leftrightarrow C(y))$, so by (Ack) we have that $\vdash A \to s = t$.
The result follows by theorem 4.1. $\square$

The parallel between this proof and the proof of the preceding proof that the axiom of choice implies excluded middle in intuitionistic set theory is obvious. There are virtues, though, in the present proof. It makes clear, for instance, the role that the assumption of extensionality plays in the proof of excluded middle. In a set theoretic context, especially if one's familiarity with set theory is based on the classical versions, appeals to extensionality hardly need to be noticed. We shall return to this point in our philosophical discussion below.[14]

Thus the addition of (Ack) and (ε) to first-order intuitionistic logic is non-conservative in a most striking way (in any situation in which there are provably distinct objects). However, even the addition of (ε) without (Ack) is non-conservative in the sense that in intuitionistic logic with (ε) we can prove ε–free formulas we cannot prove in ε–free intuitionistic logic. For instance, it is easy to see that the (ε) principle implies the validity of the scheme

$$\exists x(\exists y \Phi(y) \to \Phi(x)), \tag{$\dagger$}$$

which is not provable in the usual formulations of intuitionistic logic. This is a striking contrast to the classical case, where Hilbert's 'Second ε–Theorem' tells us that adding (ε) to classical first-order logic is a conservative extension. So, summarizing roughly, ε and extensionality is dramatically non-conservative, but ε alone is (less dramatically) non-conservative.

Of course, one striking difference is that in the presence of (Ack) and the modest assumption that two provably distinct entities exist we make valid both additional quantifier laws and additional propositional principles, while our only example of an additional principle made valid by epsilon alone is a quantificational law. For the philosophical discussion to follow it is interesting that ε without extensionality also implies new propositional laws. To get the result we again need some assumptions. We continue to assume the existence of two provably distinct objects, and while we no longer assume (Ack) we replace it with the assumption that one of the terms is "decidable," *i.e.* that $\forall y(y = c \vee y \neq c)$. With these assumptions we can no longer prove excluded middle, but we *can* prove important superintuitionistic principles, including the intuitionistically invalid De Morgan's law, $\neg(A \wedge B) \to (\neg A \vee \neg B)$.

[14]But it is perhaps worth noting immediately that in constructive mathematical settings in which something called "Choice" is provable, its failure to imply excluded middle can often be traced to some failure of extensionality. See, for instance, [19, 20].

Indeed, we can prove the stronger principle sometimes called "Linearity" or "Dummett's scheme,"

$$(P \to Q) \vee (Q \to P), \tag{LIN}$$

from which DeMorgan's law follows.

Theorem 4.4. *IPC(ε) plus two provably distinct objects, one decidable, implies LIN.*

Proof. Assume:

(1) $\vdash c \neq d$
(2) $\vdash (\forall x)(x = c \vee x \neq c)$

Now, chooosing an x free in neither P nor Q, we define:

$$A(x) \equiv (P \wedge x = c) \vee (Q \wedge x \neq c) \tag{*}$$

We have:

$$A(c) \leftrightarrow P, \text{ and } x \neq c \vdash A(x) \to Q$$

Since: $(\exists x)A(x) \leftrightarrow P \vee Q$, we have:

$$P \vee Q \leftrightarrow A(\varepsilon x A(x))$$

and by (2) we have:

$$[(P \vee Q) \to (A(\varepsilon x A(x)) \wedge \varepsilon x A(x) = c)] \vee [(P \vee Q) \to (A(\varepsilon x A(x)) \wedge \varepsilon x A(x) \neq c)]$$

by the definition of $A(x)$ and $(*)$ we have:

$$((P \vee Q) \to P) \vee ((P \vee Q) \to Q)$$

and so:

$$((P \to P) \wedge (Q \to P)) \vee ((Q \to Q) \wedge (P \to Q))$$

Simplifying we have:

$$(Q \to P) \vee (P \to Q)$$

□

Of course, as it stands these proofs only show that epsilon plus these other conditions are *sufficient* to get these results, not that they are necessary. To show necessity, we want a semantics for intuitionistic epsilon calculus. Unfortunately, this can be a tricky business. We will therefore satisfy ourselves with a few simple remarks, and point readers to, for instance, [3] for details.

In, for instance, [21], a very simple form of semantics is employed for the classical ε-calculus. We add a choice function f to each interpretation, and $\varepsilon x A(x)$ is interpreted by whatever f chooses from the "truth set" for $A(x)$, *i.e.* the set of elements of the domain that make $A(x)$ true when x is assigned to them under the interpretation in question; if the truth set is empty, then the epsilon term gets assigned to an arbitrary but fixed element of the domain. This is problematic in the intuitionistic case for several reasons.

First, it is not hard to see that such an approach will make (Ack) come out valid. In the classical case this arguably doesn't matter very much, since ε, with or without (Ack), is conservative over classical logic. We have seen, though, that ε + (Ack) is as far from conservative over intuitionistic logic as anybody is going to want to go.

A second problem is that intuitionistic logic cannot have a bivalent semantics. Whatever semantics we use for intuitionistic logic, one way or another we are going to have to confront the prospect that many formulas (for a given interpretation) are not "completely true" nor are they "completely false." Since all such formulas will have the same "truth set" as, for instance, $P(x) \wedge \neg P(x)$, namely $\emptyset$, they will all have the same object as the referent of their ε term.

Relatedly, to get the ε principle to come out valid, we need to ensure that for each φ, the truth value of $\varphi(\varepsilon x.\varphi)$ is always equal to the truth value of $\exists x\varphi$. It is easy to see how the Leisenring semantics can ensure this in the classical case, since there is a sufficient supply of saturated models in classical predicate logic. We can therefore restrict attention to interpretations under which if $\exists x\varphi$ is true, then there is some element of the domain d that makes $\varphi(x)$ true when x is interpreted as d, *i.e.*, by restricting attention to interpretations where φ's truth set is non-empty. In the intuitionistic case we obviously can't restrict attention to truth sets, given what was said in the preceding paragraph, but we do need to ensure that some element d gives $\varphi(x)$ the same truth value as $\exists x\varphi$, *i.e.*, we need to ensure that the "as true as possible" set is non-empty.

Solving these problems in detail is messy. The approaches we will focus on are built on a standard algebraic semantics for intuitionistic logic. The basic idea is this: interpretations of predicates in classical logic take them to be "propositional functions" in the sense of taking tuples of members of the domain of interpretation into the set $\{0, 1\}$. But the standard truth tables for the classical $\wedge$, $\vee$ and $\neg$ operations correspond exactly to the algebraic operations of meet, join and complement if we set $0 < 1$ and consider this a two element Boolean algebra. Algebraic semantics, generally speaking, starts with the question "what's special about the two element Boolean algebra?" If we allow our interpretations to be other Boolean algebras, we get *Boolean valued semantics*, but this turns out not to change which principles count as valid. But if we allow *other types of algebras* besides Boolean ones, we get more interesting, non-classical logics. In particular, if we allow the algebra of truth values to be Hetying algebras (of which Boolean algebras are a special case), we have a semantics for intuitionistic logic.[15]

Bell solves the third problem by restricting attention to interpretations under which the algebra of truth values is an inversely well-ordered set—that is, every subset of the set of truth values has a maximal element. The result is a sound but not complete semantics for intuitionistic ε calculus. It allows him to prove several interesting independence results, including that while ε + (Ack) implies the law of excluded middle, ε alone does not. His relatively simple semantics also makes (Ack) turn out valid. To get a non-extensional semantics, [3] makes the value of $\varepsilon x\varphi$ depend not only on the truth values φ takes when the various members of the domain are used to interpret x, but also on the syntax of φ.

We do not need to pursue the details here. For our purposes it is enough to note that this sort of semantics allow us to demonstrate independence results that establish that the proofs above don't just give us sufficient conditions for, *e.g.*, deriving excluded middle, but that the various suppositions in the proofs each play an essential role (*e.g.* Theorem 4.3 doesn't go through without (Ack)). Thus, the algebra of truth values for ε + (Ack) + two-provably-distinct-objects must be a Boolean algebra, while ε plus two provably distinct objects and one decidable object assures that the truth values form and L-algebra, *i.e.*, a Heyting algebra in which $(a \to b) \vee (b \to a) = 1$ for all a, b.

The most obvious examples of non-Boolean L-algebras are chains; if a chain has more than two members, it is Heyting but not Boolean. However there are other more interesting examples, Horn, for example, constructs an L-algebra that is neither Boolean nor a simple linear ordering by considering a lattice composed of a selection of infinitely long sequences of 0s, $\frac{1}{2}$s and 1s compared component-wise [24, p.404]. We can construct a much simpler lattice of ordered pairs to illustrate the basic idea

[15] A Heyting algebra is sometimes defined as a Brouwerian Lattice with a bottom element. A Brouwerian Lattice, or implicative lattice, is a lattice with relative pseudo-complementation. However the terminology is not uniform, in some of the literature a Brouwerian Algebra is taken to mean the same thing as a Co-Heyting Algebra, the dual of Heyting Algebra. Heyting Algebras, Co-Heyting Algebras, and Brouwerian Algebras are all also referred to collectively as Pseudo-Boolean Algebras. For a comprehensive explication of Heyting algebras see [22, pp.58*ff.*] or [23, pp.33*ff.* and pp.128*ff.*]

Horn expands on. Consider a algebra of ordered pairs compared component wise (*i.e.*, $\langle a, b\rangle \leq \langle c, d\rangle$ iff $a \leq c$ and $b \leq d$). The pairs in question include all elements of $(\{1,2,3\} \times \{1,2,3\}) \cup \{\langle 0,0\rangle\}$. Clearly, the bottom element $\bot = \langle 0,0\rangle$, while the top $\top = \langle 3,3\rangle$.

$$
\begin{tikzcd}
 & \langle 3,3\rangle \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle 2,3\rangle \arrow[d] \arrow[dr] & \langle 3,2\rangle \arrow[r] \arrow[d] & \langle 3,1\rangle \arrow[d] \\
\langle 1,3\rangle \arrow[dr] & \langle 2,2\rangle \arrow[r] \arrow[d] & \langle 2,1\rangle \arrow[d] \\
 & \langle 1,2\rangle \arrow[r] & \langle 1,1\rangle \arrow[d] \\
 & & \langle 0,0\rangle
\end{tikzcd}
$$

For every pair $x \neq \bot$, $\neg x$ is $\bot$ (while $\neg\bot = \top$), and yet $x \vee \neg x = x$ (*e.g.* $\langle 2,3\rangle \vee \neg\langle 2,3\rangle = \langle 2,3\rangle$) and for any two pairs x, and y we get $(x \to y) \vee (y \to x) = \top$.[16] We shall discuss such L-algebras and their philosophical interest briefly below.

5 More philosophy

We return now to a more explicitly philosophical discussion, trying (briefly) to make good our suggestion that there are metaphysical lessons in these formal results—both lessons for how to fill in some gaps in Dummett's story linking metaphysics to logical principles and more general lessons for those not persuaded of the details of Dummett's account.

Let us draw together some strands of the discussion. First, we will recall some key features of (our potted version of) Dummett's account. Intuitionistic logic is logic properly-so-called, and so is metaphysically neutral in the sense that everyone should accept it, regardless of their metaphysical commitments. Superintuitionistic principles, if justified, must be justified on extra-logical grounds. Dummett suggests

[16]For example consider two non-comparable elements $\langle 2,3\rangle$ and $\langle 3,2\rangle$ of the algebra presented above:

$$
\begin{aligned}
(\langle 2,3\rangle \to \langle 3,2\rangle) \vee (\langle 3,2\rangle \to \langle 2,3\rangle) &= \bigvee\{x | x \wedge \langle 2,3\rangle \leq \langle 3,2\rangle\} \vee \bigvee\{y | y \wedge \langle 3,2\rangle \leq \langle 2,3\rangle\} \\
&= \langle 3,2\rangle \vee \langle 2,3\rangle \\
&= \langle 3,3\rangle
\end{aligned}
$$

that these grounds, if they ever are available, will be metaphysical ones, *i.e.* reason for believing that the truth-values of the statements in some domain are fixed by reality in some way that is suitably "independent of us." In such cases, the mind- and language-independent reality will justify a commitment to bivalence, and so to classical logic.

What the formal results above show is that there are ways to make the connection between metaphysical commitment and logical principles less nebulous. Choice principles, and in particular the epsilon principle, are metaphysical assumptions, because they encode claims about conditions under which we can assert the existence of "objects" of some sort, and we can see from the results that these lead fairly directly to the validity of superintuitionistic principles. This strikes us as less metaphorical than the detour through "mind-independent reality fixing truth values," and so already as more philosophically illuminating. But the proofs that show the role of an extensionality assumption in getting all of classical logic, while weaker assumptions lead us to superintuitionistic but non-classical systems, allow us to make connections between metaphysical assumptions and logical principles that give interesting ways of seeing that the question of realism is not an all-or-nothing thing.

As a preliminary step, consider what the ε axiom says. Recall that when teaching classical logic, it's not uncommon to have to try to explain what $\exists x(\exists y A(y) \to A(x))$ is saying, by way of trying to convince students that it's not crazy that it's valid. A common way to do so is to use examples like "suppose $A(x)$ means 'x will pass the test'; then the formula is saying that there is someone who will pass the test if anyone does." And, indeed, this is what the ε-axiom says: that for any property there is an object which is *the likeliest thing* to have the property, or perhaps the $\varepsilon x A(x)$ is the *paradigm example* of the As. As noted, this is already a constructively invalid principle, for interesting reasons that we cannot pursue here. (See [18] for discussion.) For the present, it is more important to ask: do our intuitions about when we find it reasonable to think that *there is always a likeliest* and when we don't track our thoughts about the reality or objectivity of the subject under discussion? We think it does. We will not argue for the claim, but only offer what we hope are some suggestive comments. It is no accident that we use an example like "will pass the test if anyone does" because in most classes there is a student or a small group of students who are more diligent in preparing for tests, and diligence is a good predictor of success on tests. On the other hand, there are reasons we don't instead use examples like "suppose $A(x)$ means 'x will win the lottery,' " as there is no reason to think in advance of the draw that there is (already) someone who is will win if anyone does, for if we think the lottery is fair we don't think there is any fact grounding such a claim. We think similar intuitions can be generated for other standard examples where antirealist intuitions are reputed to be especially common—is there really a "likeliest to be funny" joke?

What, then, do we make of theorem 4.3? It shows that in domains where we not only assume that every property is such that some object is likeliest to have it, but that that "likeliest to have" is determined *extensionally* (and there are at least two provably distinct objects), then we have classical logic. For instance, if the students who pass are precisely the students who study, then the likeliest-to-pass and the likeliest-to-study will be the same student. If Dummett is right about the link between realism and classical logic, this result shows us that discourses in which we have grounds to believe that all the properties come with extensionally-determined "likeliest" objects are ones about which we have reason to accept realism. In this connection, it is worth noting that it is actually not news to think that there is a link between extensionality and *objectivity*, a notion clearly important to our thinking about realism. Famously, in the middle of the past century the need for the grammar of our attributions of intentional states to be non-extensional was regarded by some as reason to question the appropriateness of intentional states for inclusion among the features of the world apt for scientific description. But there are examples that are both homier and more current. When teaching decision theory, it is important to draw students' attention to reasons for doubting whether "preference functions" are tracking something real. Would you prefer chocolate or broccoli? Would you prefer something that will give you a heart attack or broccoli? Since the answers to such questions depend on the description and not just what is described, we have reason for scepticism about whether the answers about what someone prefers are "objective" or not.

What do we get if we (continue to assume that there are two distinct objects and) remove extensionality while assuming that one of the objects is "decidable"? We no longer have classical logic, and so according to the Dummettian account we must accept some sort of antirealism for the domain in question. But we think this intermediate way-station is one which gives us grounds for saying "well, maybe not realism, but not really antirealism either."

Consider the models as described in the previous chapter, for the "shape" of the "algebras of truth-values" can provide us with some idea how the properties in a domain must behave. As noted, the obvious L-algebras are linear. While the two-valued L-algebra case is precisely the one Dummett pointed to as encoding realist assumptions, there is some reason to regard any situation in which the truth-values of claims are arranged linearly as one where something "objective" is in question. For it is natural in such cases to think in terms of "degrees of truth,"[17] so for instance

[17]Though one needs to be cautious not to transfer over ideas from other discussions where that

for any pair of objects there will be a fact of the matter about which of them has any property P to the greater extent. But more complex L-algebras suggest other interesting possibilities to do with "multi-dimensional" properties, where each dimension is "objective," but taken together they generate a property with an in-between status, as reflected in our example of a non-linear L-algebra.

Intelligence, for instance, might be this sort of thing. While not uncontroversial, it is common to hear people speak of intelligence as having various dimensions.[18] Maybe both culinary smarts and strategic ability are real things, and each is part of what we mean by "intelligence." And perhaps Yotam Ottolenghi has more culinary smarts, but less strategic ability, than Magnus Carlsen. In that case, perhaps there is just no answer to the question of which of the two is smarter—to be smarter means being at least as smart on every dimension and smarter on some. Both, though, might be smarter than the present authors, having both more strategic sense and more culinary ability than we do. (Of course, there might be other dimensions that are part of intelligence on which we can pin our hopes for blunting this judgment: we couldn't be *more intelligent* than those two, but were we to rank ahead on another dimension we could at least be judged *non-comparable* with them with respect to intelligence, rather than less intelligent.)

If we accept that domains in which superintuitionistic principles are valid are ones in which some, so-to-speak, realistically-inclined metaphysical presuppositions are legitimate, and if the discussion above shows that discussions of intelligence are such a domain, we should regard intelligence in ways different from how we judge discourses where realism is truly implausible—for instance, perhaps, humour or beauty. And yet we should not regard it in the same way in which we regard domains about which we are fully realists, either. This strikes us as very much how intelligence *is* regarded by those who defend multi-dimensional views. Critics of the view, on the other hand, often argue that there is a single factor ("general intelligence") that underlies strong performance in any dimension, and so that the apparent multi-dimensionality is an illusion—in effect, arguing that reducing it to a single dimension is to show that intelligence is a "real thing."[19]

We think this discussion does a few useful things for discussions of realism and antirealism in the Dummettian tradition. First, as noted, it puts some additional

phrase is used; for instance, for all formulas P with a non-$\perp$ truth value in a linear L-algebra, the truth value of $\neg P$ is $\perp$, rather than 1 minus the truth-value of P, as in probability semantics, for instance.

[18] We diliberately choose somewhat flippant "dimensions" rather than opting for some seriously offered, for instance, by advocates of Multiple Intelligence theories. We do not intend to be wading into this debate, merely using obvious aspects of what is presumably a familiar example to most readers.

[19] There is more to be said here. [25] contains a fuller discussion.

flesh on the bones of Dummett's suggestion that metaphysical commitments can give rise to commitment to logical principles. It also gives us good reason to say that the question of realism is not an all-or-nothing thing—intelligence may not be as objective as mass, but it's not as ephemeral as humour, either, even if the Multiple Intelligence folks are right. Finally, this sort of discussion can help bring the discussion of realism and antirealism down a couple of levels of abstraction. Rather than discussing things like what logic applies to a "domain of discourse" (whatever that is), this approach offers a way to discuss relatively familiar concepts and to see what is at issue in ways that reflect debates as we actually see them occurring between theorists we can actually see debating the status of those familiar concepts.

6 Conclusion

Of course, there are other ways we can get from intuitionistic to classical logic than via choice principles, or indeed to get part way from one to the other. Indeed, there are other ways to make the transition using variants on ε (such as the one encoding "dependent choice" included in [2]), or using different term-forming operators. What such formal results offer, we think, is a variety of pathways for investigating relationships between metaphysical commitments and principles of reasoning. By focusing on just a few such results we've tried to sketch one way one might try to spell out a strategy for showing how realist commitments imply logical principles, but it is just one among many. One thing that we hope attention to such matters would do is put some flesh on the bones of the Dummettian suggestion that realists about a particular domain must commit themselves to at least one of these pathways.

References

[1] J. Bell, "Hilbert's ε-operator and classical logic," *Journal of Philosophical Logic*, vol. 22, pp. 1–18, 1993.

[2] J. Bell, "Hilbert's ε-operator in intuitionistic type theories," *Mathematical Logic Quarterly*, vol. 39, pp. 323–337, 1993.

[3] D. DeVidi, "Intuitionistic ε- and τ-calculi," *Mathematical Logic Quarterly*, vol. 41, pp. 523–546, 1995.

[4] R. Diaconescu, "Axiom of choice and complementation," *Proceedings of the American Mathematical Society*, vol. 51, no. 1, pp. 176–178, 1975.

[5] N. D. Goodman and J. Myhill, "Choice implies excluded middle," *Zeitschrift fur Mathematische Logik und Grundlagen der Mathematik*, vol. 24, p. 461, 1975.

[6] D. Gabbay and L. Maksimova, "Interpolation and definability," in [26], Springer, 2011.

[7] M. Dummett, *The Seas of Language.* Clarendon Press, Oxford, 1993.

[8] M. Dummett, "Realism and anti-realism," in *Seas of Language*, pp. 462–472, Clarendon Press, Oxford, 1993.

[9] J. Mackie, *Ethics: Inventing Right and Wrong.* Penguin UK, 1977.

[10] H. Field, *Science Without Numbers: The Defence of Nominalism.* Princeton University Press, 1980.

[11] D. DeVidi and G. Solomon, "On confusions about bivalence and excluded middle," *Dialogue: Canadian Philosophical Review*, vol. 38, pp. 785–899, 1999.

[12] M. Dummett, *The Logical Basis of Metaphysics.* Cambridge, Mass.: Harvard University Press, 1991.

[13] G. Rosen, "Review article: The shoals of language," *Mind*, vol. 104, pp. 599–609, 1995.

[14] B. Russell, *Introduction to Mathematical Philosophy.* London: George Allen and Unwin, 1919.

[15] R. Carnap, *The Logical Syntax of Language.* London: Routledge and Kegan Paul, 1937. transl. by A. Smeaton.

[16] M. Dummett, *Frege: Philosophy of Mathematics.* Cambridge, Mass.: Harvard University Press, 1991.

[17] D. DeVidi, "Choice principles and constructive logics," *Philosophia Mathematica*, vol. 12, pp. 222–243, 2004.

[18] D. DeVidi, "Assertion proof and the axiom of choice," in [27], Springer-Verlag, 2006.

[19] M. Maietti, "About effective quotients in constructive type theory.," in *Types for Proofs and Programs, International Workshop "Types 98"* (T. Altenkirch, W. Naraschewski, and B. Reus, eds.), no. 1657 in Lecture Notes in Computer Science, pp. 164–178, Springer-Verlag, 1999.

[20] M. Maietti and S. Valentini, "Can you add power-set to Martin-Löf intuitionistic type theory?," *Mathematical Logic Quarterly*, vol. 45, pp. 521–532, 1999.

[21] A. Leisenring, *Mathematical Logic and Hilbert's ε-Symbol.* New York: Gordon & Breach Science Publishers, 1969.

[22] H. Rasiowa and R. Sikorski, *The Mathematics of Metamathematics.* Monografie Matematyczne, Państwowe Wydawn. Naukowe, 1963.

[23] B. Davey and H. Priestly, *Introduction to Lattices and Order.* Cambridge University Press, second ed., 2002.

[24] A. Horn, "Logic with truth values in a linearly ordered Heyting algebra," *Journal of Symbolic Logic*, vol. 34, pp. pp. 395–408, 1969.

[25] C. Mulvihill, *Existence Assumptions and Logical Prinicples: Choice Operators in Intuitionistic Logic.* PhD thesis, University of Waterloo, 2015.

[26] D. Gabbay and F. Guenthner, eds., *Handbook of Philosophical Logic*, vol. 15. Kluwer Academic Publishers, 2011.

[27] D. DeVidi and T. Kenyon, *A Logical Approach to Philosophy Essays in Honour of Graham Solomon.* Springer-Verlag, 2006.

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