Preprint of an article in Social Choice and Welfare 31(1): 59-78, 2008
Official Publication: https://link.springer.com/article/10.1007/s00355-007-0263-y

A Liberal Paradox for Judgment Aggregation

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May 2004, final version June 2007

In the emerging literature on judgment aggregation over logically connected propositions, expert rights or liberal rights have not been investigated yet. A group making collective judgments may assign individual members or subgroups with expert knowledge on, or particularly affected by, certain propositions the right to determine the collective judgment on those propositions. We identify a problem that generalizes Sen's 'liberal paradox'. Under plausible conditions, the assignment of rights to two or more individuals or subgroups is inconsistent with the unanimity principle, whereby unanimously accepted propositions are collectively accepted. The inconsistency can be avoided if individual judgments or rights satisfy special conditions.

1 Introduction

Groups frequently make collective judgments on certain propositions. Examples are legislatures, committees, courts, juries, expert panels and entire populations deciding what propositions to accept as true (thus forming *collective beliefs*) and what propositions to make true through their actions (thus forming *collective desires*). When a group forms collective beliefs, some group members or subgroups may have expert knowledge on certain propositions and may therefore be granted the right to be decisive on those propositions (an *expert right*). Legislatures or expert panels, for example, may grant such rights to specialist members or subcommittees so as to rely on their expertise or to achieve a division of labour. When a group forms collective desires, some group members or subgroups may be particularly affected by certain propositions, for example when those propositions concern their private sphere(s), and may also be granted the right to be decisive on those propositions (a *liberal right*).

How does the assignment of rights constrain a group's collective judgments? In this paper, we identify a problem that generalizes Sen's 'liberal paradox' (1970), the result that individual rights may conflict with the Pareto principle (for recent contributions, see Deb, Pattanaik and Razzolini 1997; van Hees 1999,

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2004; Dowding and van Hees 2003). Consider the following two examples.[2]

Example 1: expert rights.[3] An expert committee has to make judgments on the following propositions:

a: Carbon dioxide emissions are above some critical threshold.

b: There will be global warming.

$a \rightarrow b$: If carbon dioxide emissions are above the threshold, then there will be global warming.

Half of the committee members are experts on a, the other half experts on $a \rightarrow b$. So the committee assigns to the first half the right to determine the collective judgment on a and to the second a similar right on $a \rightarrow b$. The committee's constitution further stipulates that unanimous individual judgments must be respected. Now suppose that all the experts on a judge a to be true, and all the experts on $a \rightarrow b$ judge $a \rightarrow b$ to be true. In accordance with the expert rights, the committee accepts both a and $a \rightarrow b$. We may therefore expect it to accept b as well. But when a vote is taken on b, *all* committee members reject b. How can this happen? Table 1 shows the committee members' judgments on all propositions.

	a	$a \rightarrow b$	b
Experts on a	True	False	False
Experts on $a \rightarrow b$	False	True	False

Table 1: A paradox of expert rights

The experts on a accept a, but reject $a \rightarrow b$ and b. The experts on $a \rightarrow b$ accept $a \rightarrow b$, but reject a and b. So all committee members are individually consistent. Nonetheless, respecting the rights of the experts on a and $a \rightarrow b$ is inconsistent with respecting the committee's unanimous judgment on b. To achieve consistency, the committee must either restrict the expert rights or overrule its unanimous judgment on b.

Example 2: liberal rights.[4] The two members of a small society, Lewd and Prude, each have a personal copy of the book *Lady Chatterley's Lover*. Consider three propositions:

l: Lewd reads the book.

p: Prude reads the book.

$l \rightarrow p$: If Lewd reads the book, then so does Prude.

[2] In the expert rights example, accepted propositions are interpreted as propositions *believed* to be true; in the liberal rights example, as propositions *desired* to be true.

[3] A structurally similar example was given by Pauly and van Hees (2006).

[4] This example is inspired by Sen's example. While in Sen's example there is only one copy of the book – to be borrowed and read by at most one individual – in ours there are two copies; so the book may be read by both individuals, by one, or by neither.

Lewd desires to read the book himself, and that, if he reads it, then Prude read it too, as he anticipates that his own pleasure of reading the book will be enhanced by the thought of Prude finding the book offensive. Prude, by contrast, desires not to read the book, and that Lewd not read it either, as he fears that the book would corrupt Lewd's moral outlook. But he also desires that, if Lewd reads the book, then he read it too, so as to be informed about the dangerous material Lewd is exposed to. Table 2 shows Lewd's and Prude's desires on the propositions.[5]

	l	p	$l \to p$
Lewd	True	True	True
Prude	False	False	True

Table 2: A paradox of liberal rights

Society assigns to each individual the liberal right to determine the collective desire on those propositions that concern only the individual's private sphere. Since l and p are such propositions for Lewd and Prude, respectively, society assigns to Lewd the right to determine the collective desire on l, and to Prude a similar right on p. Further, according to society's constitution, unanimous desires of all individuals must be collectively respected. But because of Lewd's liberal right on l, l is collectively accepted; because of Prude's liberal right on p, p is collectively rejected; and yet, by unanimity, $l \to p$ is collectively accepted, an inconsistent collective set of desires. To achieve consistency, society must either restrict the liberal rights of the individuals or relax its constitutional principle of respecting unanimous desires.

In both examples, there is a conflict between some individuals' rights on some propositions and all individuals' unanimous judgments on others. This conflict is not accidental. We show that, as soon as the relevant propositions exhibit mild interconnections, no consistent mapping from individual to collective judgments can generally respect the rights of two or more individuals or subgroups and preserve unanimous judgments. Except in special cases, which we discuss later, respecting such rights may require overruling unanimity. We also derive Sen's original result as a corollary of our new result.

We present our result within the model of judgment aggregation on logically connected propositions, initially proposed by List and Pettit (2002), which combines axiomatic social choice theory and formal logic. Much of this literature has focused on generalizations of, and solutions to, another paradox, the

[5] Conditional desires, like Lewd's and Prude's desire of p given l, can be represented in various ways, which are controversially discussed in deontic logic. Our example represents a conditional desire of p given l as a desire of the implication $l \to p$, as distinct from a desire of p on the supposition/condition that l. A further question is whether '$\to$' should be a material or subjunctive conditional (our example works either way). See, e.g., Hintikka (1971), Wagner Decew (1981), Bradley (1999).

'doctrinal' or 'discursive' paradox (Kornhauser and Sager 1986, Pettit 2001), which is similar in spirit to Condorcet's famous paradox of cyclical majority preferences and consists in the fact that majority voting on logically connected propositions may lead to inconsistent majority judgments (for generalizations, see, e.g., List and Pettit 2002, 2004; Pauly and van Hees 2006; van Hees 2007; Dietrich 2006, 2007a; Nehring and Puppe 2006; Dietrich and List 2007; Dokow and Holzman 2005; for proposed solutions, see, e.g., List 2003, 2004a; Pigozzi 2006; Dietrich forthcoming, 2007b).[6] This paper, however, presents the first extension of Sen's liberal paradox to judgment aggregation. The use of formal logic illuminates the logical structure of the paradox and highlights its robustness. All proofs are given in the appendix.

2 The model

We consider a group of individuals $N = \{1, 2, ..., n\}$ $(n \geq 2)$. The propositions on which judgments are made are represented in logic (following List and Pettit 2002, 2004; we use Dietrich's 2007a generalization).

Logic. Let $\mathbf{L}$ be a set of sentences, called *propositions*, closed under negation (i.e., if $p \in \mathbf{L}$ then $\neg p \in \mathbf{L}$, where $\neg$ denotes 'not'), and stipulate that each subset $S \subseteq \mathbf{L}$ is either *consistent* or *inconsistent*, subject to standard axioms.[7] In standard propositional logic, $\mathbf{L}$ contains propositions such as a, b, $a \wedge b$, $a \vee b$, $\neg(a \rightarrow b)$ (where $\wedge$, $\vee$, $\rightarrow$ denote 'and', 'or', 'if-then', respectively). Examples of consistent sets are $\{a, a \rightarrow b, b\}$ and $\{a \wedge b\}$, examples of inconsistent ones $\{a, \neg a\}$ and $\{a, a \rightarrow b, \neg b\}$. A proposition $p \in \mathbf{L}$ is a *tautology* if $\{\neg p\}$ is inconsistent and a *contradiction* if $\{p\}$ is inconsistent.

Agenda. The *agenda* is the set of propositions on which judgments are made, defined as a non-empty subset $X \subseteq \mathbf{L}$ expressible as $X = \{p, \neg p : p \in X_+\}$ for a set $X_+ \subseteq \mathbf{L}$ of unnegated propositions. We assume that X contains no tautologies or contradictions[8] and that double negations cancel each other out (i.e., $\neg\neg p$ stands for p).[9] In our examples, $X = \{a, \neg a, a \rightarrow b, \neg(a \rightarrow b), b, \neg b\}$ and $X = \{l, \neg l, l \rightarrow p, \neg(l \rightarrow p), p, \neg p\}$ (in standard propositional or conditional logic).

[6] Related contributions are those on abstract aggregation theory (Wilson 1975, Rubinstein and Fishburn 1986, Nehring and Puppe 2002) and belief merging in computer science (Konieczny and Pino-Perez 2002).

[7] C1: For any $p \in \mathbf{L}$, $\{p, \neg p\}$ is inconsistent. C2: If $S \subseteq \mathbf{L}$ is inconsistent, then so is any superset $T \supseteq S$ (in $\mathbf{L}$). C3: $\emptyset$ is consistent, and each consistent $S \subseteq \mathbf{L}$ has a consistent superset $T \supseteq S$ (in $\mathbf{L}$) containing a member of each pair $p, \neg p \in \mathbf{L}$. See Dietrich (2007a).

[8] This assumption is only needed in theorem 4 (where it could be avoided, for instance, by supposing that different individuals have disjoint rights sets).

[9] Hereafter, when we write $\neg p$ and p is already of the form $\neg q$, we mean q (rather than $\neg\neg q$).

Individual judgment sets. Each individual i's judgment set is the set $A_i \subseteq X$ of propositions that he or she accepts. On a belief interpretation, A_i is the set of propositions believed by individual i to be true; on a desire interpretation, the set of propositions desired by individual i to be true. A judgment set is *consistent* if it is a consistent set in $\mathbf{L}$ and *complete* if it contains a member of each proposition-negation pair $p, \neg p \in X$. A *profile* is an n-tuple $(A_1, ..., A_n)$ of individual judgment sets.

Aggregation functions. An *aggregation function* is a function F that maps each profile $(A_1, ..., A_n)$ from some domain of admissible ones to a collective judgment set $F(A_1, ..., A_n) = A \subseteq X$, the set of propositions that the group as a whole accepts. The collective judgment set A can be interpreted as the set of propositions collectively believed to be true or as the set collectively desired to be true. Below we impose minimal conditions on aggregation functions (including on the domain of admissible profiles). Standard examples of aggregation functions are *majority voting* (where $F(A_1, ..., A_n)$ is the set of propositions $p \in X$ for which the number of individuals with $p \in A_i$ exceeds that with $p \notin A_i$) and *dictatorships* (where $F(A_1, ..., A_n) = A_i$ for some antecedently fixed individual $i \in N$).

3 Impossibility results

We first state an impossibility result on the assignment of (expert or liberal) rights to individuals; we then state a similar result on the assignment of rights to subgroups. Following Sen's (1970) account of rights, we formalize rights in terms of a suitable notion of decisiveness. In the next section, we show that Sen's result is a corollary of ours.

Our impossibility results hold for all agendas exhibiting 'mild' interconnections in the following sense. Call propositions $p, q \in X$ *conditionally dependent* if there exist $p^* \in \{p, \neg p\}$ and $q^* \in \{q, \neg q\}$ such that $\{p^*, q^*\} \cup Y$ is inconsistent for some $Y \subseteq X$ consistent with each of p^* and q^*. The agenda X is *connected* if any two propositions $p, q \in X$ are conditionally dependent. Notice that the agendas in the two examples above are connected in this sense.

3.1 Individual rights

Call individual i *decisive* on a set of propositions $Y \subseteq X$ (under the aggregation function F) if any proposition in Y is collectively accepted if and only if it is accepted by i, formally

$$F(A_1, ..., A_n) \cap Y = A_i \cap Y.$$

Suppose we want to find an aggregation function with the following properties:

Universal Domain. The domain of F is the set of all possible profiles of consistent and complete individual judgment sets.

Minimal Rights. There exist (at least) two individuals who are each decisive on (at least) one proposition-negation pair $\{p, \neg p\} \subseteq X$.

Unanimity Principle. For any profile $(A_1, ..., A_n)$ in the domain of F and any proposition $p \in X$, if $p \in A_i$ for all individuals i, then $p \in F(A_1, ..., A_n)$.

Like Sen's (1970) condition of *minimal liberalism*, minimal rights is a weak requirement that leaves open which individuals have rights and to which propositions these rights apply. By using an undemanding rights requirement, our impossibility result becomes stronger. In a later section, we introduce explicit rights systems and state a stronger rights requirement.

Theorem 1 *If (and only if) the agenda is connected, there exists no aggregation function (generating consistent collective judgment sets) that satisfies universal domain, minimal rights and the unanimity principle.*[10]

So a group whose aggregation function has universal domain cannot *both* assign (liberal or expert) rights to more than one individual *and* respect unanimous judgments.

The result does not require complete collective judgment sets, only consistent ones. But, like all later results except theorem 4, it continues to hold if we add the completeness requirement on collective judgment sets. Further, theorem 1 continues to hold if decisiveness in minimal rights is weakened to *positive* decisiveness, where individual i is *positively decisive* on a set of propositions $Y \subseteq X$ (under the aggregation function F) if $F(A_1, ..., A_n) \cap Y \supseteq A_i \cap Y$. It also continues to hold if F is required to generate consistent *and* complete judgment sets and decisiveness in minimal rights is weakened to *negative* decisiveness (the presence of veto power), where individual i is *negatively decisive* on a set of propositions $Y \subseteq X$ (under the aggregation function F) if $F(A_1, ..., A_n) \cap Y \subseteq A_i \cap Y$. (Decisiveness *simpliciter* is the conjunction of positive and negative decisiveness.) Without a connected agenda, a modified impossibility holds in which minimal rights is strengthened to the requirement that there exist (at least) two individuals who are each decisive on (at least) one proposition-negation pair in X such that these two pairs are conditionally dependent.

3.2 Subgroup rights

A *subgroup* is a non-empty subset $M \subseteq N$. Call M *decisive* on a set of propositions $Y \subseteq X$ (under the aggregation function F) if any proposition in Y

[10] In this and later results, some parts are put in brackets in order to focus the attention on the other parts. The requirement of consistent collective judgment sets is left implicit in some of the informal discussion that follows.

accepted by all members of M is also collectively accepted and any proposition in Y rejected by all members of M is also collectively rejected, formally

$$\bigcap_{i \in M} (A_i \cap Y) \subseteq F(A_1, ..., A_n) \cap Y \text{ and } \bigcap_{i \in M} (Y \backslash A_i) \subseteq Y \backslash F(A_1, ..., A_n).$$

If M is singleton, this definition reduces to the one in the individual case. In the interest of strength of the next theorem, we have deliberately given an undemanding definition of subgroup decisiveness. For a subgroup to be decisive on a set of propositions, it suffices that the subgroup can determine the collective judgments on them when its members unanimously agree on them; without unanimity, there are no constraints. Stronger forms of subgroup decisiveness are imaginable. One may require, for example, that the subgroup can determine the collective judgment on the relevant propositions by taking majority votes on them. However, are there any aggregation functions that satisfy the following rights condition with decisiveness defined in the present weak sense?

Minimal Subgroup Rights. There exist (at least) two disjoint subgroups that are each decisive on (at least) one proposition-negation pair $\{p, \neg p\} \subseteq X$.

Theorem 2 *If (and only if) the agenda is connected, there exists no aggregation function (generating consistent collective judgment sets) that satisfies universal domain, minimal subgroup rights and the unanimity principle.*

So a group whose aggregation function has universal domain cannot *both* assign (liberal or expert) rights to more than one subgroup *and* respect unanimous judgments among its members. Theorem 2 strengthens theorem 1, because minimal subgroup rights is less demanding than minimal rights (the latter implies the former – take singleton subgroups – but not vice-versa).[11] As in the case of theorem 1, theorem 2 continues to hold if the notion of decisiveness in minimal subgroup rights is weakened to *positive* decisiveness (the first conjunct in the definition above) or (when collective judgment sets are also required to be complete) to *negative* decisiveness (the second conjunct in the definition).

4 Sen's liberal paradox

To show that our main result generalizes Sen's 'liberal paradox' (1970), we apply theorem 1 to the aggregation of (strict) preference relations (using a construction in Dietrich and List 2007; see also List and Pettit 2004). For this purpose, we define a simple predicate logic **L**, with

- a two-place predicate P (representing strict preference), and

[11] Except in the special case $n = 2$, where the two conditions are equivalent.

- a set of (two or more) constants $K = \{x, y, z, ...\}$ (representing alternatives),

where any set $S \subseteq \mathbf{L}$ is *inconsistent* if and only if $S \cup Z$ is inconsistent in the standard sense of predicate logic, with Z defined as the set of rationality axioms on strict preferences:

$$Z = \left\{ \begin{array}{c} (\forall v_1)(\forall v_2)(v_1 P v_2 \rightarrow \neg v_2 P v_1) \text{ (asymmetry)}, \\ (\forall v_1)(\forall v_2)(\forall v_3)((v_1 P v_2 \wedge v_2 P v_3) \rightarrow v_1 P v_3) \text{ (transitivity)}, \\ (\forall v_1)(\forall v_2)(\neg\, v_1 = v_2 \rightarrow (v_1 P v_2 \vee v_2 P v_1)) \text{ (connectedness)} \end{array} \right\}.$$ [12]

Thus the atomic propositions in $\mathbf{L}$ are binary ranking propositions of the form xPy, yPz etc.; examples of compound propositions are the axioms in Z. We discuss the interpretation in terms of preferences below. Sets such as $\{xPy, yPz\}$ are consistent, while sets such as $\{xPy, \neg xPy\}$, $\{xPy, yPx\}$, $\{xPy, yPz, zPx\}$, $\{\neg xPy, \neg yPx\}$ are inconsistent (the first set contains a proposition-negation pair; the second, third and fourth conflict with the first, second and third rationality axioms in Z, respectively).

The *preference agenda* is the set $X = \{xPy, \neg xPy \in \mathbf{L} : x, y \in K$ with $x \neq y\}$. The mapping that assigns to each fully rational (i.e., asymmetric, transitive and connected) preference relation $\succ$ on K the judgment set $A = \{xPy, \neg yPx \in X : x \succ y\}$ establishes a bijection between the set of all fully rational preference relations and the set of all consistent and complete judgment sets. More generally, any consistent judgment set $A \subseteq X$ represents an acyclic preference relation $\succ$ on K given by $x \succ y$ if and only if $xPy \in A$ or $\neg yPx \in A$ (for any $x, y \in K$).

What does accepting some binary ranking proposition xPy mean? On a belief interpretation, it means to believe that x is preferable to y; thus judgments on the preference agenda are beliefs on propositions of the form 'x is preferable to y'. On a desire interpretation, to accept xPy means to desire that, given a choice between x and y, x be chosen over y; here judgments on the preference agenda are desires on propositions of the form 'given a choice between x and y, x is chosen over y'.[13]

To represent Sen's original example in this way, let $N = \{1, 2\}$ be a two-member society consisting of Lewd and Prude, and let the set of alternatives be $K = \{l, p, n\}$, with the interpretation:

[12] For technical reasons, Z additionally contains, for each pair of distinct contants $x, y \in K$, $\neg\, x{=}y$ (exclusiveness).

[13] The two proposed interpretations – which correspond to *cognitivist* and *emotivist* interpretations of preferences – thus differ both in the meaning of the predicate P and in the meaning of 'accepting' a proposition. On a cognitivist interpretation, xPy means that x is preferable to/better than y, and the question is whether or not to *believe* such a proposition. On an emotivist interpretation, xPy means that x is chosen over y in a binary choice, and the question is whether or not to *desire* such a proposition. The two interpretations illustrate our broader point that judgment aggregation can be viewed either as the aggregation of belief sets or as that of desire sets.

l: Lewd reads the book.
p: Prude reads the book.
n: No-one reads the book.[14]

Table 3 shows the two individuals' judgments on the ranking propositions lPn, nPp and pPl; the preference relations represented by these judgments are shown in brackets.

	lPn	nPp	pPl
Lewd ($p \succ l \succ n$)	True	False	True
Prude ($n \succ p \succ l$)	False	True	True

Table 3: Sen's example

Society assigns to Lewd the right to determine the collective judgment on lPn. On a belief interpretation, this means that Lewd is given an expert right on whether or not Lewd-reading-the-book is preferable to no-one-reading-the-book; on a desire interpretation, that he is given a liberal right on whether or not, in a choice between these two alternatives, Lewd-reading-the-book is chosen over no-one reading the book. Similarly, society assigns to Prude the right to determine the collective judgment on nPp, interpretable analogously. Given the individual judgments in table 3, respecting these rights means that society must accept both lPn and nPp; and since both individuals accept pPl, the Pareto principle requires the collective acceptance of pPl. But the resulting judgment set $\{lPn, nPp, pPl\}$ is inconsistent: it represents a cyclical preference relation. More generally, we can apply theorem 1 to the preference agenda.

Lemma 1 *The preference agenda is connected.*

This lemma has a straightforward proof (given in the appendix); for instance, propositions xPy and $x'Py'$ for pairwise distinct alternatives $x, y, x', y' \in K$ are conditionally dependent, as is seen by conditionalizing on $Y = \{yPx', y'Px\}$.

Corollary 1 *(Sen 1970) For the preference agenda, there exists no aggregation function (generating consistent collective judgment sets) that satisfies universal domain, minimal rights and the unanimity principle.*

Note that an aggregation function for the preference agenda with universal domain and generating consistent collective judgment sets represents a preference aggregation function that maps any possible profile of fully rational preference relations to an acyclic one, and the conditions of minimal rights and the unanimity principle correspond to Sen's conditions of minimal liberalism and the Pareto principle.

[14] For convenience, we use the symbol n here, which elsewhere in the paper denotes the group size.

5 Possibility results

We now consider conditions under which the conflict between (expert or liberal) rights and the unanimity principle does not arise. For simplicity, we focus on individual rights, but our results can be generalized to subgroup rights too. To state our possibility results, we first refine our account of rights. The condition of minimal rights above does not specify which individuals have rights on which propositions. We now make the assignment of rights more 'targeted' by introducing explicit rights systems.

A *rights system* is an n-tuple $(R_1, ..., R_n)$, where each R_i is a (possibly empty) subset of X containing pairs $p, \neg p$. For each i, we call R_i individual i's *rights set*. On a belief interpretation, the elements of R_i are the propositions on which individual i is the expert; on a desire interpretation, the propositions that belong to i's private sphere. An aggregation function respects a rights system if it satisfies the following condition.

Rights. Every individual i is decisive on the rights set R_i.

It is easy to see that this condition can be met by a well-behaved aggregation function only if the rights system is consistent in a minimal sense. Call a rights system $(R_1, ..., R_n)$ *consistent* if $B_1 \cup ... \cup B_n$ is consistent for any consistent subsets $B_1, ..., B_n$ of $R_1, ..., R_n$, respectively.

Proposition 1 *If and only if the rights system is consistent, there exists an aggregation function F (generating consistent collective judgment sets) that satisfies universal domain and rights.*

But even for a consistent rights system, theorem 1 immediately implies that, if the agenda is connected and two or more distinct R_i's each contain at least one proposition-negation pair, respecting rights is inconsistent with universal domain and the unanimity principle in an aggregation function generating consistent collective judgment sets. We now show that the inconsistency can be avoided if individual judgments fall into a suitably restricted domain or the rights system (together with the agenda) has a particular property.[15]

5.1 Special domains: deferring/empathetic judgments

Let a rights system be given. When one individual adopts the judgments of another whenever those judgments concern propositions in the other's rights set, we say that the first individual *defers* to the judgments of the second (if the rights in question are expert rights) or is *empathetic* towards them (if the rights are liberal rights). Formally, individual i is *deferring/empathetic* in profile $(A_1, ..., A_n)$ if $A_i \cap R_j = A_j \cap R_j$ for all $j \neq i$, and a profile $(A_1, ..., A_n)$ is

[15] For an overview of domain restrictions in response to the original liberal paradox in preference aggregation, including preference-based definitions of 'empathy' and 'tolerance', see Sen (1983); see also Craven (1982), Gigliotti (1986).

deferring/empathetic if every individual is deferring/empathetic in it. Deferring/empathetic profiles exhibit unanimous agreement on every proposition in some individual's rights set, a strong restriction. Our possibility theorem, however, is based on a less demanding restriction. A profile $(A_1, ..., A_n)$ is *minimally deferring/empathetic* if some individual is deferring/empathetic in it.

Minimally Deferring/Empathetic Domain. The domain of F is the set of all minimally deferring/empathetic profiles of consistent and complete individual judgment sets.

If more than one individual i has a non-empty rights set R_i, the minimally deferring/empathetic domain is a proper subset of the universal domain.[16]

Theorem 3 *For any agenda and any rights system, there exists an aggregation function (generating consistent collective judgment sets) that satisfies minimally deferring/empathetic domain, rights and the unanimity principle.*

Surprisingly, the result does not require a consistent rights system $(R_1, ..., R_n)$. But if $(R_1, ..., R_n)$ is inconsistent, how could a single deferring/empathetic individual prevent the other individuals from exercising their rights in an inconsistent way, leading to an inconsistent collective judgment set by respecting rights? The answer is that individual i's deferral/empathy *does* prevent such inconsistencies, albeit in a technical sense. Inconsistencies in the exercise of the others' rights would (by the definition of deferral/empathy) lead individual i to have an inconsistent judgment set A_i, something excluded by the minimally deferring/empathetic domain. Our definition of this domain thus restricts individuals $j \neq i$ in their exercise of rights so as to allow individual i to be both deferring/empathetic and consistent. To avoid this feature of the definition, one could redefine a deferring/empathetic individual as one who adopts the others' judgments (where they have rights) unless these judgments are mutually inconsistent; formally, one may define individual i to be *deferring/empathetic* in profile $(A_1, ..., A_n)$ if $[A_i \cap R_j = A_j \cap R_j$ for all $j \neq i]$ whenever $\bigcup_{j \neq i}[A_j \cap R_j]$ is consistent. Under this modified definition, theorem 3 continues to hold provided the rights system $(R_1, ..., R_n)$ is consistent.

5.2 Special domains: agnostic/tolerant judgments

When one individual makes no judgment on propositions in another's rights set, we say that the first individual is *agnostic* about the judgments of the second (if the rights in question are expert rights) or *tolerant* towards them

[16] If there exists only one individual i with $R_i \neq \emptyset$, then i is trivially deferring/empathetic in every profile. If there exists no individual i with $R_i \neq \emptyset$, then every individual is trivially deferring/empathetic in every profile. So, if $R_i \neq \emptyset$ for at most one individual i, then the minimally deferring/empathetic domain coincides with the universal domain.

(if the rights are liberal rights). We define agnosticism/tolerance as the requirement that an individual's judgment set be consistent with any possible consistent exercise of rights by others. Formally, individual i with judgment set A_i is *agnostic/tolerant* if A_i is consistent with every consistent set of the form $B_1 \cup ... \cup B_{i-1} \cup B_{i+1} \cup ... \cup B_n$, where, for each individual $j \neq i$, $B_j \subseteq R_j$. A profile $(A_1, ..., A_n)$ is *agnostic/tolerant* if every individual is agnostic/tolerant in it. A profile $(A_1, ..., A_n)$ is *minimally agnostic/tolerant* if some individual is agnostic/tolerant in it. Our possibility theorem requires only minimally agnostic/tolerant profiles.

Minimally Agnostic/Tolerant Domain. The domain of F is the set of all minimally agnostic/tolerant profiles of consistent individual judgment sets.

The minimally agnostic/tolerant domain does not require complete judgment sets, and hence is not a subset of the universal domain. In fact, an agnostic/tolerant individual cannot have a complete judgment set (unless all other individuals have an empty rights set), since agnosticism/tolerance forces an individual to make no judgments on propositions in other individuals' rights sets. If *at least two* individuals have a non-empty rights set, then the universal domain neither contains, nor is contained by, the minimally agnostic/tolerant domain.[17]

Theorem 4 *For any agenda and any consistent rights system, there exists an aggregation function (generating consistent collective judgment sets) that satisfies minimally agnostic/tolerant domain, rights and the unanimity principle.*

Unlike our result on the minimally deferring/empathetic domain, the present result explicitly requires a consistent rights system. Also, in this theorem (unlike in all others) it is essential that we allow incomplete collective judgment sets: respecting rights forces the collective to take over any incompleteness of any individual's judgments within his or her rights set. If we wish to ensure complete collective judgment sets in theorem 4 we may *either* weaken people's rights by making each individual i merely *positively* decisive on R_i *or* restrict the domain by allowing only those minimally agnostic/tolerant profiles $(A_1, ..., A_n)$ in which each A_i is complete within R_i (i.e., each A_i contains a member of every proposition-negation pair in R_i). In such a restricted domain, each individual may refrain from making judgments only outside his or her rights set.

5.3 Special agendas and rights systems

Instead of restricting the domain, we now consider special rights systems, namely ones we call *disconnected*. We have seen in proposition 1 that consistency of

[17] Again, if $R_i \neq \emptyset$ for only one individual i, then i is trivially agnostic/tolerant in every profile; and if $R_i \neq \emptyset$ for no individual i, then every individual is trivially agnostic/tolerant in every profile. So, if $R_i \neq \emptyset$ for at most one individual i, then the minimally agnostic/tolerant domain contains the universal domain.

a rights system is sufficient for the existence of aggregation functions satisfying universal domain and rights, yet the unanimity principle may be violated. We now strengthen the consistency requirement on the rights system so as to make it sufficient for the existence of aggregation functions satisfying universal domain, rights and the unanimity principle.

For a finite agenda or compact logic,[18] our definition of a disconnected rights system can be stated as follows (in the appendix we give a more general statement). The rights system $(R_1, ..., R_n)$ is *disconnected* (in X) if no proposition in any R_i is conditionally dependent of any proposition in any R_j $(j \neq i)$. Informally, a disconnected rights system is one in which the rights of different individuals are not 'entangled' with each other conditional on other propositions in the agenda. Note that a disconnected rights system in which more than one individual has a non-empty rights set can exist only if the agenda is *not* connected. The following theorem holds.

Theorem 5 *If (and only if) the rights system is disconnected, there exists an aggregation function (generating consistent collective judgment sets) that satisfies universal domain, rights and the unanimity principle.*

However, while the domain is not restricted – there need not be any deferring/empathetic or agnostic/tolerant individuals – disconnectedness is a severe constraint on a rights system and satisfiable (if more than one individual is to have a non-empty rights set) only for special agendas.

6 Discussion

We have identified a liberal paradox for judgment aggregation. If the agenda of propositions under consideration is connected, then, under universal domain, the assignment of (expert or liberal) rights to two or more individuals or subgroups is inconsistent with the unanimity principle. The inconsistency arises because propositions on which unanimous judgments are reached are sometimes logically constrained by other propositions that lie in some individual's or subgroup's sphere of rights. The inconsistency does not arise for the restricted domains of deferring/empathetic judgments or agnostic/tolerant judgments or for a disconnected rights system – which requires an agenda that is not connected, if more than one individual or subgroup is to have rights. For example, if different individuals (or subgroups) each live on their own Robinson Crusoe island, where the propositions relevant to different islands are not conditionally dependent on each other, then rights can be assigned to them without violating the unanimity principle. But such scenarios are rare; almost all realistic collective decision problems presuppose some interaction between different agents, which makes it plausible to expect connections between different individuals' rights sets.

[18] A logic is *compact* if every inconsistent set of propositions has a finite inconsistent subset.

Our results have implications for the design of mechanisms that groups (societies, legislatures, committees, expert panels, management boards, organizations) can use for making decisions on multiple interconnected propositions. For some groups or decision problems, the existence of agnostic/tolerant or deferring/empathetic group members may avoid the paradox. But there is no guarantee that such attitudes will exist, and constitutional provisions may be needed to deal with the possible occurrence of the paradox. Ultimately, the group faces the constitutional choice between either relaxing the (democratic) unanimity principle or relaxing (expert or liberal) rights of individuals or subgroups. Let us briefly discuss each option.

If it is deemed unacceptable to weaken any rights, violations of the unanimity principle will have to be allowed in collective decision making – an option advocated, among others, by Sen (1976) in the context of preference aggregation. The overruling of unanimous judgments may be defended on the grounds of unacceptable individual *motivations* behind such judgments, which disregard the rights of other individuals. Individual judgments driven by such unacceptable motivations may be seen as the counterpart in judgment aggregation of the so-called *meddlesome* preferences in preference aggregation (Blau 1975).

On the other hand, if the unanimity principle is deemed indispensible, then some weakening of rights is necessary. One possibility is to assign such rights in a suitably disconnected way, so that different rights never conflict with each other or with unanimous judgments on other propositions. Alternatively, rights can be made *alienable*, i.e., conditional on not conflicting with other rights or unanimous judgments. Dowding and van Hees (2003) have suggested that rights may sometimes be overruled by other considerations; in particular, different rights may carry a different threshold of being respected, which may vary from right to right and from context to context.

The choice of whether or not to give rights priority over the unanimity principle also depends on whether these rights are expert rights or liberal rights. In the case of liberal rights, the choice is ultimately a normative one, which depends on how much weight we give to individual liberty as a value relative to other values such as certain democratic decision principles. In the case of expert rights, by contrast, the choice is not just normative. If the propositions are factually either true or false, then it becomes an epistemological question which aggregation function is better at tracking their truth-values: one that respects expert rights or one that satisfies the unanimity principle. The answer to this question – which we cannot provide here – depends on several factors, such as how competent the experts and non-experts are on the various propositions and whether different individuals' judgments are mutually dependent or independent. The literature on the Condorcet jury theorem can be modified to address this question (Bovens and Rabinowicz 2006, List 2004b).

As the liberal paradox continues to be discussed in social choice theory and game theory, we hope that our findings will help to extend this discussion to the emerging theory of judgment aggregation and inspire further work.

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A Appendix: proofs

We write $Domain(F)$ for the domain of F. As mentioned earlier, theorems 1, 2, 3 and 5 and proposition 1 continue to hold if completeness of collective judgment sets is also required. To turn our proofs of these results into proofs of the results with the added completeness condition, one must modify the constructed aggregation function F in each proof (specifically, in one direction of the implication) by replacing every consistent output $F(A_1, ..., A_n)$ by a consistent *and complete* superset of it.

Proof of theorem 1. 1. First assume the agenda X is connected. Suppose the aggregation function F satisfies minimal rights, the unanimity principle and universal domain. We show that F generates an inconsistent collective judgment set on some profile. By minimal rights, some individual i is decisive on some $\{p, \neg p\} \subseteq X$, and some other individual j is decisive on some $\{q, \neg q\} \subseteq X$. As X is connected, there exist propositions $p^* \in \{p, \neg p\}$ and $q^* \in \{q, \neg q\}$ and a set

$Y \subseteq X$ inconsistent with the pair p^*, q^* but consistent with p^* and with q^*. As the sets $\{p^*\} \cup Y$ and $\{q^*\} \cup Y$ are each consistent, they can each be extended to a consistent and complete judgment set. Consider a profile $(A_1, ..., A_n)$ of complete and consistent judgment sets such that A_i extends $\{p^*\} \cup Y$, A_j extends $\{q^*\} \cup Y$, and each A_k, $k \neq i, j$, extends either $\{p^*\} \cup Y$ or $\{q^*\} \cup Y$. By universal domain, $(A_1, ..., A_n) \in Domain(F)$. $F(A_1, ..., A_n)$ contains p^* by i's decisiveness on $\{p, \neg p\}$, contains q^* by j's decisiveness on $\{q, \neg q\}$, and contains all $y \in Y$ by the unanimity principle. So $\{p^*, q^*\} \cup Y \subseteq F(A_1, ..., A_n)$. Hence $F(A_1, ..., A_n)$ is inconsistent.

2. Now assume X is not connected. Then there are propositions $p, q \in X$ that are not conditionally dependent. Let F be the aggregation function with universal domain given by

$$F(A_1, ..., A_n) := (A_1 \cap \{p, \neg p\}) \cup (A_2 \cap \{q, \neg q\}) \cup (A_1 \cap ... \cap A_n)$$

for all $(A_1, ..., A_n) \in Domain(F)$. We show that F satisfies all requirements.

First, F satisfies the unanimity principle because, for all $(A_1, ..., A_n) \in Domain(F)$, $A_1 \cap ... \cap A_n \subseteq F(A_1, ..., A_n)$.

To show that F satisfies minimal rights, we show that individuals 1 and 2 are decisive, respectively, on $\{p, \neg p\}$ and $\{q, \neg q\}$. For all $(A_1, ..., A_n) \in Domain(F)$, we have

$$F(A_1, ..., A_n) \cap \{p, \neg p\} = A_1 \cap \{p, \neg p\}$$

because $\{p, \neg p\} \cap \{q, \neg q\} = \emptyset$ (otherwise p and q would be conditionally dependent, in fact dependent conditionally on $\emptyset$). So individual 1 is decisive on $\{p, \neg p\}$. For analogous reasons, individual 2 is decisive on $\{q, \neg q\}$.

Finally, we consider any profile $(A_1, ..., A_n) \in Domain(F)$ and show that $F(A_1, ..., A_n)$ is consistent. Note that $F(A_1, ..., A_n) = \{p^*, q^*\} \cup Y$, where p^* is the member of $A_1 \cap \{p, \neg p\}$, q^* the member of $A_2 \cap \{q, \neg q\}$, and Y the set $A_1 \cap ... \cap A_n$. By $\{p^*\} \cup Y \subseteq A_1$, $\{p^*\} \cup Y$ is consistent. By $\{q^*\} \cup Y \subseteq A_2$, $\{q^*\} \cup Y$ is consistent. So, as p and q are not conditionally dependent, $\{p^*, q^*\} \cup Y$ is consistent, i.e. $F(A_1, ..., A_n)$ is consistent. ■

Proof of theorem 2. If the agenda X is not connected then there exists an aggregation function with the relevant properties, by Theorem 1 and since minimal rights implies minimal subgroup rights (take singleton subgroups). The converse implication follows by straightforwardly adapting part 1 of the proof of Theorem 1. ■

Proof of lemma 1. Consider any two proposition p and q in the preference agenda $X = \{xPy, \neg xPy : x, y \in K, x \neq y\}$. Without loss of generality, we may assume that p and q are of the non-negated form xPy, because any negated proposition $\neg xPy \in X$ is logically equivalent to the non-negated proposition yPx. So let p be xPy, and y be $x'Py'$. To show that xPy and $x'Py'$ are conditionally dependent, we have to choose propositions $p^* \in \{xPy, \neg xPy\}$ and $q^* \in \{x'Py', \neg x'Py'\}$ and a set $Y \subseteq X$ such that $\{p^*\} \cup Y$ and $\{q^*\} \cup Y$

are consistent, and $\{p^*, q^*\} \cup Y$ is inconsistent (in fact, represents a cycle). The choices of p^*, q^*, Y depend on whether $x \in \{x', y'\}$ and whether $y \in \{x', y'\}$.

Case $x \neq x', y' \& y \neq x', y'$: $p^* = xPy$, $q^* = x'Py'$, $Y = \{yPx', y'Px\}$.
Case $y = y' \& x \neq x', y'$: $p^* = xPy$, $q^* = \neg x'Py$ ($\equiv yPx'$), $Y = \{x'Px\}$.
Case $y = x' \& x \neq x', y'$: $p^* = xPy$, $q^* = yPy'$, $Y = \{y'Px\}$.
Case $x = x' \& y \neq y', x'$: $p^* = \neg xPy$ ($\equiv yPx$), $q^* = xPy'$ $Y = \{y'Py\}$.
Case $x = y' \& y \neq x', y'$: $p^* = xPy$, $q^* = x'Px$, $Y = \{yPx'\}$.
Case $x = x' \& y = y'$: $p^* = xPy$, $q^* = \neg xPy$ ($\equiv yPx$), $Y = \emptyset$.
Case $x = y' \& y = x'$: $p^* = xPy$, $q^* = yPx$, $Y = \emptyset$. ■

Proof of proposition 1. (i) First, assume the rights system $(R_1, ..., R_n)$ is consistent. Let F be the aggregation function with universal domain defined by

$$F(A_1, ..., A_n) = (A_1 \cap R_1) \cup ... \cup (A_n \cap R_n)$$

for any profile $(A_1, ..., A_n) \in Domain(F)$. Obviously, F satisfies rights. To show collective consistency, note that, for any consistent sets $A_1, ..., A_n \subseteq X$, also $A_1 \cap R_1, ..., A_n \cap R_n$ are consistent, hence have a consistent union as the rights system is consistent.

(ii) Now assume the aggregation function F has all properties. To show that the rights system $(R_1, ..., R_n)$ is consistent, let $B_1, ..., B_n$ be consistent subsets of, respectively, $R_1, ..., R_n$. As each B_i is consistent, it may be extended to a consistent and complete judgment set A_i. The so-defined profile $(A_1, ..., A_n)$ belongs to the (universal) domain of F. By rights, $B_i \cap F(A_1, ..., A_n) = B_i$ for all individuals i, and so

$$\begin{aligned} B_1 \cup ... \cup B_n &= [B_1 \cap F(A_1, ..., A_n)] \cup ... \cup [B_n \cap F(A_1, ..., A_n)] \\ &= [B_1 \cup ... \cup B_n] \cap F(A_1, ..., A_n). \end{aligned}$$

So $B_1 \cup ... \cup B_n$ is a subset of the consistent set $F(A_1, ..., A_n)$, hence is itself consistent. ■

Proof of theorem 3. For each minimally deferring/empathetic profile $(A_1, ..., A_n)$, define $F(A_1, ..., A_n)$ as the judgment set A_i of some deferring/empathetic individual i (if there are several such individuals, choose any one of them). The so-defined aggregation function satisfies all conditions, because the collective judgment set, by being the judgment set of a deferring/empathetic individual, is consistent, matches the judgments of any individual within this individual's rights set (so that F satisfies rights), and contains each proposition that every individual accepts (so that F satisfies the unanimity principle). ■

Proof of theorem 4. Suppose the rights system $(R_1, ..., R_n)$ is consistent. For every minimally agnostic/tolerant profile $(A_1, ..., A_n)$, since each A_i is consistent, so is each $A_i \cap R_i$. Hence, by the consistency of the rights system, the union $\cup_i (A_i \cap R_i)$ is consistent. So, as $(A_1, ..., A_n)$ is minimally agnostic/tolerant, there exists an (agnostic/tolerant) individual j such that A_j is consistent with $\cup_{i \neq j}(A_i \cap R_i)$, i.e. such that the set

$$A_j \cup [\cup_{i \neq j}(A_i \cap R_i)]$$

is consistent. Let $F(A_1, ..., A_n)$ be this set. To show that the so-definied aggregation function F satisfies all properties, note first that F by construction satisfies minimally agnostic/tolerant domain, and consistent collective judgment sets. Also the unanimity principle holds: for all minimally agnostic/tolerant profiles $(A_1, ..., A_n)$, $F(A_1, ..., A_n)$ is by definition a superset of $A_1 \cap ... \cap A_n$.

To show rights, consider a minimally agnostic/tolerant profile $(A_1, ..., A_n)$. Then there is an agnostic/tolerant individual j such that

$$F(A_1, ..., A_n) = A_j \cup [\cup_{i \neq j}(A_i \cap R_i)].$$

Individual j's rights are respected since

$$F(A_1, ..., A_n) \cap R_j = A_j \cap R_j,$$

where we use the fact that the sets $R_1, ..., R_n$ are pairwise disjoint by the consistency of the rights system (and since we have excluded tautologies and contradictions). To see that the rights of any individual $k \neq j$ are also respected, note first that

$$F(A_1, ..., A_n) \cap R_k = (A_j \cap R_k) \cup (A_k \cap R_k),$$

again using that $R_1, ..., R_n$ are pairwise disjoint. But $A_j \cap R_k$ is empty: otherwise A_j would not be consistent with all consistent subsets of R_k, hence j would not be agnostic/tolerant. Hence

$$F(A_1, ..., A_n) \cap R_k = A_k \cap R_k,$$

as desired. ■

In the main text, we have stated the definition of a disconnected rights system in the case that X is finite or the logic is compact. The general definition is as follows. The rights system $(R_1, ..., R_n)$ is *disconnected* (in X) if there are no sets $B \subseteq R_i$ and $C \subseteq R_j$ with $i \neq j$ such that $B \cup C$ is inconsistent with some set $Y \subseteq X$ that is consistent with B and with C. This definition is closely related to the previous one: if we restrict the sets B and C to be singletons, we obtain the previous definition. We now prove the equivalence of the two definitions.

Lemma 2 *For a rights system* $(R_1, ..., R_n)$,

(a) *if X is finite or belongs to a compact logic, the two disconnectedness definitions are equivalent;*

(b) *in general, disconnectedness in the new sense implies disconnectedness in the old sense, and is equivalent to the following condition:*

- *the sets $R_1, ..., R_n$ are logically independent conditional on any set $B \subseteq X \backslash (R_1 \cup ... \cup R_n)$, i.e., for every set $B \subseteq X \backslash (R_1 \cup ... \cup R_n)$, $B_1 \cup ... \cup B_n$ is consistent with B whenever each $B_i \subseteq R_i$ is.*

Proof of lemma 2. We denote by D1 the condition defining disconnectedness in the main text, by D2 the condition defining disconnectedness in the appendix, and by D3 the condition stated in lemma 2.

We first prove part (b).

'D2⇒D1'. Assume D1 does not hold. We show that D2 does not hold. As D1 is violated, there are $p \in R_i$ and $q \in R_j$ $(i \neq j)$ that are conditionally dependent, that is: for some $p^* \in \{p, \neg p\}$, $q^* \in \{q, \neg q\}$ and $Y \subseteq X$, $\{p^*, q^*\} \cup Y$ is inconsistent but each of $\{p^*\} \cup Y$ and $\{q^*\} \cup Y$ is consistent. So D2 is violated: take $B := \{p^*\}$ and $C := \{q^*\}$.

'D2⇒D3'. Suppose D3 does not hold. We show that D2 does not hold. As D3 does not hold, there are sets $B_1 \subseteq R_1, ..., B_n \subseteq R_n, B \subseteq X \backslash (R_1 \cup ... \cup R_n)$ such that each $B_i \cup B$ is consistent but $(\cup_{i=1,...,n} B_i) \cup B$ is inconsistent. Among all sets of individuals $K \subseteq \{1, ..., n\}$ such that $(\cup_{k \in K} B_k) \cup B$ is inconsistent (there is at least one), let K be one of smallest size. We have $|K| \geq 2$, since otherwise some $B_k \cup B$ would be inconsistent. So there are distinct individuals $i, j \in K$. To find a counterexample to D2, let $C := B_i$, $D := B_j$ and $Y := (\cup_{k \in K \backslash \{i,j\}} B_k) \cup B$. The sets $Y \cup C = (\cup_{k \in K \backslash \{j\}} B_k) \cup B$ and $Y \cup D = (\cup_{k \in K \backslash \{i\}} B_k) \cup B$ are each consistent (by the minimality of K), but the set $Y \cup C \cup D = (\cup_{k \in K} B_k) \cup B$ is inconsistent, as desired.

'D3⇒D2'. Assume D3. Suppose for a contradiction that $B \subseteq R_i$, $C \subseteq R_j$ $(i \neq j)$, and $Y \subseteq X$, and that $B \cup C \cup Y$ is inconsistent but $B \cup Y$ and $C \cup Y$ are consistent. Put $Z := B \cup C \cup Y$. Then (*) Z is inconsistent, and (**) $Z \backslash B$ and $Z \backslash C$ are each consistent. By D3, the sets $R_1, ..., R_n$ are pairwise disjoint: otherwise they would be logically dependent conditional on $B = \emptyset$ (since some pair $p, \neg p$ would belong to two of the sets $R_1, ..., R_n$, so that we could choose consistent subsets of $R_1, ..., R_n$, respectively, whose union contains the pair $p, \neg p$, hence is inconsistent). So, among the sets $B_1 := Z \cap R_1, ..., B_n := Z \cap R_n$, all except B_i are disjoint with B, and all except B_j are disjoint with C. Hence each of $B_1, ..., B_n$ is a subset of $Z \backslash B$ or of $Z \backslash C$. So, as $D := Z \backslash (R_1 \cup ... \cup R_n)$ is a subset of $Z \backslash B$ *and* of $Z \backslash C$, each of the sets $B_1 \cup D, ..., B_n \cup D$ is a subset of $Z \backslash B$ or of $Z \backslash C$, hence is consistent by (**). But the union

$$\begin{aligned} B_1 \cup ... B_n \cup D &= [(Z \cap R_1) \cup ... \cup (Z \cap R_n)] \cup [Z \backslash (R_1 \cup ... \cup R_n)] \\ &= [Z \cap (R_1 \cup ... \cup R_n)] \cup [Z \backslash (R_1 \cup ... \cup R_n)] = Z \end{aligned}$$

is inconsistent by (*). This contradicts D3.

To prove part (a), it remains to show the following implication, assuming that X is finite or the logic compact.

'D1⇒D2'. Suppose for a contradiction that D1 holds but D2 does not. As D2 is violated, there are sets $B \subseteq R_i$ and $C \subseteq R_j$ with $i \neq j$ and $Y \subseteq X$ such that $B \cup C \cup Y$ is inconsistent but $B \cup Y$ and $C \cup Y$ are each consistent. As X is finite or the logic compact, $B \cup C \cup Y$ has a minimal inconsistent subset Z. By Z's inconsistency, Z is neither a subset of $C \cup Y$ nor of $B \cup Y$. So there is a $p \in B \cap Z$ and a $q \in C \cap Z$. Let $Z' := Z \backslash \{p, q\}$. By D1, p and q are not conditionally dependent, hence are distinct. So $\{p\} \cup Z'$ and $\{q\} \cup Z'$ are each proper subsets of Z, so are consistent; but $\{p, q\} \cup Z' = Z$ is inconsistent. Hence p and q are conditionally dependent, violating D1. ■

Proof of theorem 5. 1. First let the rights system $(R_1, ..., R_n)$ be disconnected. Define F as as the aggregation function with universal domain given, for all $(A_1, ..., A_n) \in Domain(F)$, by

$$F(A_1, ..., A_n) := B_1 \cup ... \cup B_n \cup B,$$

where

$$B_i := A_i \cap R_i, \, i = 1, ..., n,$$

and

$$B := (A_1 \cap ... \cap A_n) \backslash (R_1 \cup ... \cup R_n).$$

We now show that F satisfies all relevant properties.

First, each outcome $F(A_1, ..., A_n)$ is consistent: defining $B_1, ..., B_n, B$ as before, each $B_i \cup B$ is consistent (by being a subset of the consistent set A_i), whence the union $B_1 \cup ... \cup B_n \cup B$ $(= F(A_1, ..., A_n))$ is consistent by part (b) of lemma 2.

Second, F satisfies rights since $F(A_1, ..., A_n) \cap R_i = A_i \cap R_i$ for all individuals i and profiles $(A_1, ..., A_n) \subseteq Domain(F)$.

Finally, F satisfies the unanimity principle since, for all profiles $(A_1, ..., A_n) \in Domain(F)$, $F(A_1, ..., A_n)$ contains each member of $A_1 \cap ... \cap A_n$, whether it belongs to some R_i (hence to $A_i \cap R_i$) or to no R_i (hence to $(A_1 \cap ... \cap A_n) \backslash (R_1 \cup ... \cup R_n)$).

2. Conversely, assume that F is an aggregation function with all the required properties. To prove that the rights system is disconnected, it suffices by part (b) of lemma 2 to consider sets $B_1 \subseteq R_1, ..., B_n \subseteq R_n$ consistent with a set $B \subseteq X \backslash (R_1 \cup ... \cup R_n)$, and to show that $B_1 \cup ... \cup B_n \cup B$ is consistent. As each $B_i \cup B$ is consistent, it can be extended to a complete and consistent judgment set $A_i \subseteq X$. The collective judgment set $F(A_1, ..., A_n)$ contains all $p \in B_1 \cup ... \cup B_n$ (by rights) and all $p \in B$ (by the unanimity principle). So $B_1 \cup ... \cup B_n \cup B \subseteq F(A_1, ..., A_n)$. Hence, as $F(A_1, ..., A_n)$ is consistent, so is $B_1 \cup ... \cup B_n \cup B$, as desired. ■