Efimov K-theory of Diamonds

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Abstract

Motivated by Scholze and Fargues' geometrization of the local Langlands correspondence using perfectoid diamonds and Clausen and Scholze's work on the K-theory of adic spaces using condensed mathematics, we present the Efimov K-theory of diamonds. We propose a pro-diamond, a large stable $(\infty, 1)$-category of diamonds $\mathcal{D}^{\diamond}$, diamond spectra and chromatic tower, and a localization sequence for diamond spectra.

Keywords: perfectoid spaces, diamonds, pro-diamond, Efimov K-theory, étale cohomology of diamonds, diamond spectra, localization sequence, shtuka, geometrization of local Langlands, Langlands correspondence

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1 Introduction

1.1 Efimov K-theory

K-theory is canonically defined on the category of small stable ∞-categories which are idempotent-complete, with morphisms exact functors [2]. There is a certain category of large compactly-generated stable ∞-categories equivalent to this small category. In Efimov K-theory, the idea is to weaken to dualizable the condition of being compactly generated, thus ensuring that K-theory is still defined. If a category $\mathcal{C}$ is dualizable, then $\mathcal{C}$ fits into a localization sequence $\mathcal{C} \to \mathcal{S} \to \mathcal{X}$, for $\mathcal{S}$ and $\mathcal{X}$ compactly generated. The Efimov K-theory is expected to be the fiber of the K-theory in the localization sequence $\mathcal{C} \to \mathcal{S} \to \mathcal{X}$. Clausen and Scholze propose the following:

Proposition 1.1.1. ([2] Proposition). Let Nuc_R be a full subcategory of solid modules [3]. Nuc_R is a presentable stable ∞-category closed under all colimits and tensor products. If R is a Huber ring, then Nuc_R is dualizable, making its K-theory well-defined. Nuc_R embeds into Mod_R. Let $\mathcal{X}$ be a Noetherian formal scheme and X the torsion perfect complexes of modules over R. Then we have the following:

Theorem 1.1.2. ([2] Theorem). $(R, R^+) \to \mathrm{Nuc}_R$ satisfies descent over $\mathrm{Spa}(R)$ and so does its Efimov K-theory. There exists a localization sequence $K(X) \to K^{\mathrm{Efimov}}(\mathcal{X}) \to K^{\mathrm{Efimov}}(\mathcal{X}^{\mathrm{rig}})$.

A diamond, in the sense of Scholze, is an "algebraic space for the pro-étale topology in Perf" [12], the category of perfectoid spaces of characteristic p, which is the full subcategory of the κ-small category of perfectoid spaces. Specifically, diamonds are pro-étale sheaves on Perf. Using diamond equivalence relations, we can construct similar quotients of perfectoid spaces called small v-sheaves. Therefore, diamonds are v-sheaves, mirroring Gabber's result that "algebraic spaces are fpqc sheaves" [17].

Perfectoid spaces, in the sense of Scholze, are adic spaces specially covered by affinoid adic spaces $\mathrm{Spa}(R, R^+)$ where R denotes a perfectoid ring and R^+ a ring of integral elements

[12]. Perfectoid spaces are best exemplified by:

- $S_{K^p} \sim \varprojlim_{K^p} (S_{K^pK_p} \bigotimes_E E_p)^{\text{ad}}$, the perfectoid shimura variety used in studying torsion in the cohomology of compact unitary Shimura varieties [18],

- any completion of an arithmetically profinite extension, in the sense of Fontaine and Wintenberger [15], and

- $\mathcal{M}_{LT,\infty} = \tilde{U}_x \times^{GL_2(\mathbb{Q}_P)_1} GL_2(\mathbb{Q}_P) \cong \bigsqcup_Z \tilde{U}_x$, the Lubin Tate tower at infinite level [14].

Perfectoid spaces are in correspondence with Artin v-stacks through the schema: [9]

- schemes $\hookrightarrow$ algebraic spaces $\hookrightarrow$ Artin stacks
- perfectoid spaces $\hookrightarrow$ locally spatial diamonds $\hookrightarrow$ Artin v-stacks.

Recall, a small v-stack is called Artin if

- $\Delta_X : X \to X \times X$ is representable in locally spatial diamonds, and
- A cohomologically smooth surjection $f : Y \to X$ exists with Y a locally spatial diamond.

The theme of this paper is that diamonds have many incarnations and appear in the global Langlands correspondence for function fields, in the geometric Langlands correspondence, and in Scholze and Fargues' geometrization of the local Langlands correspondence. Seven of the many incarnations are the following:

- $\mathcal{Y}^{\diamond}_{S,E} = S \times (\text{Spa}\mathcal{O}_E)^{\diamond}$: the diamond relative Fargues-Fontaine curve in the geometrization of the local Langlands correspondence [21],

- the moduli space $\text{Sht}_{\mathcal{G},b,\{\mu_i\}}$: the moduli space of mixed-characteristic local G-shtukas is a locally spatial diamond [17],

- $\pi_1((\text{Spd}Q_p)^n/p.Fr.) \simeq G^m_{Q_p}$: the diamond version of Drinfeld's lemma for the $n = 2$ case for global Langlands for function fields [17],

- $\text{Spd}Q_p = \text{Spd}(Q_p^{\text{cycl}})/\underline{Z_p^x}$ for $\underline{Z_p^x}$ the profinite group $Gal(Q_p^{\text{cycl}}/Q_p)$[17],

- $\text{Spd}Q_p \times_{\diamond} \text{Spd}Q_p = \mathcal{Y}^{\diamond}_{(0,\infty)}/\underline{G_{Q_p}}$: the diamond self product [17],

- all diamonds are v-sheaves in the v-topology [12], and
- every connected component of a spatial v-sheaf $\mathcal{F}$ is a geometric point $\mathrm{Spd}(C, C^+)$, for C an algebraically closed nonarchimedean field and $C^+ \subset C$ an open and bounded valuation subring [12].

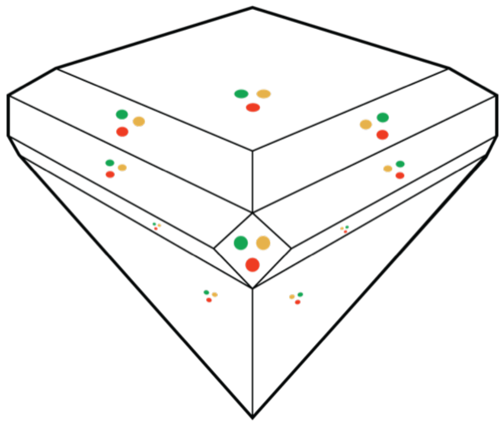


Figure 1: Diamond $\mathcal{D}$ with geometric point Spa $C \to \mathcal{D}$ pulled back through a quasi-proétale cover $X \to \mathcal{D}$, whose consecution is profinitely many copies of Spa C [17].

Studying diamonds' many mathematical guises all at once, similar to studying the profinite $Gal(\bar{Q}/Q)$ [11], may give us critical information globally about diamonds, a representation of which is diamonds and their Efimov K-theory. The goal is to get the lower $\mathrm{K}^{\mathrm{Efimov}}$-groups to return the datum of the mixed-characteristic shtukas and the higher $\mathrm{K}^{\mathrm{Efimov}}$-groups the moduli space of mixed-characteristic local G-shtukas, which is a spatial diamond [12]. Recall the global Langlands correspondence over number fields.

Conjecture 1.1.3. ([13] Conjecture I.1. Global Langlands (Clozel-Fontaine-Mazur) Conjecture)). Let F be a number field, p some rational prime, and fix an isomorphism $\mathbb{C} \simeq \bar{\mathbb{Q}}_p$. Then for any $n \geq 1$ there is a unique bijection between the set of L-algebraic cuspidal automorphic representations of $GL_n(\mathbb{A}_F)$, and the set of (isomorphism classes of) irreducible continuous representations $Gal(\bar{F}/F) \to GL_n(\bar{\mathbb{Q}}_p)$ which are almost everywhere unramified, and de Rham at places dividing p, such that the bijection matches Satake parameters with eigenvalues of Frobenius elements [1].

[1] Recall, $\mathbb{A}_F = \prod_v{}' F_v$ denotes the adeles of F, which is the restricted product of the completions F_v at

Drinfeld studies the moduli spaces of "X-shtukas" to obtain the $n = 2$ contribution to the global Langlands Correspondence over function fields, where X/F_p is a smooth projective curve and K is a function field ([Dri80] [17]). To wit, Scholze studies the moduli space of "mixed-characteristic local G-shtukas," $Sht_{(\mathcal{G},b,\{\mu_i\})}$, which Scholze identifies as a special class of diamond called a locally spatial diamond.

To formally define $Sht_{(\mathcal{G},b,\{\mu_i\})}$, we first recall the definition of the adic space $S \dot{\times} SpaZ_p$.

Proposition 1.1.4. ([17] Proposition 11.2.1). If $S = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space of characteristic p, we can define an analytic adic space $S \dot{\times} \mathrm{Spa}Z_p$ such that there is a natural isomorphism

- $(S \dot{\times} \mathrm{Spa}Z_p)^\diamond = S \times \mathrm{Spd}Z_p$.

Proposition 1.1.5. ([17] Proposition 11.3.1). Let $S \in Perf$. The following sets are naturally identified:

- Sections of $S \dot{\times} \mathrm{Spa}Z_p \to S$.
- Morphisms $S \to \mathrm{Spd}Z_p$, and
- Untilts $S^\#$ of S.

Moreover, given these data, there is a natural map

- $S^\# \hookrightarrow S \dot{\times} \mathrm{Spa}Z_p$ of adic spaces over Z_p

that is the inclusion of a closed Cartier divisor.

A shtuka over a perfectoid space takes the following form [17]:

> For an object $S \in Perf$, a shtuka over S should be a vector bundle over an adic space $S \times SpaZ_p$ together with a Frobenius structure, where the product is a fiber product...It's associated diamond is the product of sheaves on $Perfd$.

The definition of a mixed-characteristic shtuka of rank n over S with legs $x_1, \cdots, x_m$ follows.

Definition 1.1.6. ([17] Definition 11.4.1). Let S be a perfectoid space of characteristic p. Let $x_1, \cdots, x_m : S \to \mathrm{Spd}Z_p$ be a collection of morphisms. For $i = 1, \cdots, m$ let

all finite or infinite places of F.

- $\Gamma_{x_i} : S_i^{\#} \to S \dot{\times} \mathrm{Spa} Z_p$

be the corresponding closed Cartier divisor. A mixed-characteristic shtuka of rank n over S with legs $x_1, \cdots, x_m$ is a rank n vector bundle $\mathcal{E}$ over $S \dot{\times} \mathrm{Spa} Z_p$ together with an isomorphism

- $\phi_{\mathcal{E}} : (Frob_S^* \mathcal{E})|_{S \dot{\times} \mathrm{Spa} Z_p \setminus \bigcup_{i=1}^m \Gamma_{x_i}} \to \mathcal{E}|_{S \dot{\times} \mathrm{Spa} Z_p \setminus \bigcup_{i=1}^m \Gamma_{x_i}}$

that is meromorphic along $\bigcup_{i=1}^m \Gamma_{x_i}$.

This formalism is extended to the moduli space of mixed-characteristic local G-shtukas.

Definition 1.1.7. ([17] Definition 23.1.1.) Let G be a reductive group over Q_p. G does not live over Z_p in the mixed characteristic setting. So we choose a smooth group scheme $\mathcal{G}$ over Z_p with generic fiber G and connected special fiber. Now let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space of characteristic p, with pseudouniformizer $\hat{\omega}$. Take a discrete algebraically closed field, and $L = W(k)[1/p]$. Let

- $(\mathcal{G}, b, \{\mu_i\})$

be a triple consisting of a smooth group scheme $\mathcal{G}$ with reductive generic fiber G and connected special fiber, an element $b \in G(L)$, and a collection $\mu_1, \cdots, \mu_m$ of conjugacy classes of cocharacters $G_m \to G_{\bar{Q_p}}$. For $i = 1, \cdots, m$, let E_i/Q_p be the field of definition of μ_i , and let $\hat{E_i} = E_i \cdot L$. The moduli space

- $\mathrm{Sht}_{\mathcal{G},\lfloor,\{\mu_\rangle} \to \mathrm{Spd}\hat{E}_1 \times_{Spdk} \cdots \times_{Spdk} Spd\hat{E}_m$

of shtukas associated with $(\mathcal{G}, b, \{\mu_i\})$ is the presheaf on $Perf_k$ sending $S = \mathrm{Spa}(R, R^+)$ to the set of quadruples $(\mathcal{P}, \{S_i^{\#}\}, \phi_{\mathcal{P}}, \iota_r\}$ where:

- $\mathcal{P}$ is a G-torsor on $S \dot{\times} \mathrm{Spa} Z_p$,
- $S_i^{\#}$ is an untilt of S to $\hat{E}_i$ for $i = 1, \cdots, m$,
- $\phi^{\mathcal{P}}$ is an isomorphism $\phi_{\mathcal{P}} : (Frob_S^* \mathcal{P})|_{(S \times_{F_q} X) \setminus \bigcup_{i=1}^m \Gamma_{x_i}} \tilde{\to} \mathcal{P}|_{(S \times_{F_q} X) \setminus \bigcup_{i=1}^m \Gamma_{x_i}}$ and finally
- ι_r is an isomorphism $\iota_r : \mathcal{P}|_{\mathcal{Y}_{[r,\infty)}(S)} \tilde{\to} G \times \mathcal{Y}_{[r,\infty)}(S)$ for large enough r, under which $\phi^{\mathcal{P}}$ gets identified with $b \times Frob_S$.

Theorem 1.1.8. ([17] Theorem 23.1.4.) The moduli space $(\mathcal{G}, b, \{\mu_i\})$ is a locally spatial diamond.

Drinfeld's lemma has a diamond reformulation, the consequence of replacing with the diamond SpdQ_p all the connected schemes X_i.

Recall Drinfeld's lemma for diamonds.

Theorem 1.1.9. ([17] Theorem 16.3.1). $(\pi_1((\text{Spd}Q_p)^n/\text{p.Fr.}) \simeq G^m_{Q_p}$.

We recall that the goal forthcoming is to get the lower K^{Efimov}-groups to return the datum of the mixed-characteristic shtukas and the higher K^{Efimov}-groups to return the moduli space of mixed-characteristic local G-shtukas, which is representable by a locally spatial diamond. [2]

We now recall the geometric Langlands correspondence.

Conjecture 1.1.10. ([38] Geometric Langlands Correspondence (Geometric Langlands Duality)). For G a reductive group, ${}^LG$ the Langlands dual group, and $\sum$ an algebraic curve, there is an equivalence of derived categories of D-modules on the moduli stack of G-principal bundles on σ and quasi-coherent sheaves on the ${}^LG$-moduli stack of local systems on σ:

- $\mathcal{O}Mod(Loc_{{}^LG}(\sum) \xrightarrow{\simeq} \mathcal{D}Mod(\text{Bun}_G(\sum))$.

Recall the definition of Bun_G from Fargues' conjecture, which "geometrizes" the local Langlands correspondence with commensurating v-sheaves on the diamond stack of specified G-bundles on the Fargues-Fontaine Curve.

Definition 1.1.11. ([21] Definition 2.1). Let E be either $\mathbb{F}_q((\pi))$ or a finite degree extension of Q_p with residue field $\mathbb{F}_q$. Let G be a reductive group over E. Let S be a perfectoid space. Consider PerfF_q equipped with the pro-étale topology. We note Bun_G is the fibered category over PerfF_q:

- $S \to$Groupoid of $\otimes$ exact functors from Rep(G) to Bun_{X_S}.

[2] We will then extend this correspondence to diamond spectra via the Lubin-Tate formal $\mathcal{O}_{\mathcal{F}}$-module law and an Efimov version of the Redshift Conjecture [24]

We have the conjectured diamond formulation of Bun_G.

Proposition 1.1.12. ([21] Proposition 2.2). Bun_G is a stack on $\mathrm{Perf}F_q$.

Conjecture 1.1.13. ([21] Hope 2.3). Bun_G is a "smooth diamond stack."

In addition to Proposition 1.1.12 and Conjecture 1.1.13, these foundational theorems are preliminaries for our construction:

Theorem 1.1.14. ([9] Theorem 11.23 (Fargues if E/Q_p, Anschütz in general)). If $S = Spa(C, C^+)$ where C is a complete algebraically closed field, then the functor

- $G \to \mathrm{Isoc} \to \mathrm{Bun}_G(S)$

sending a G-torsor on $\mathrm{Spa}\breve{E}/\sigma^Z$ to its pullback to $Y_S/\phi^Z = X_S$ induces a bijection on isomorphism classes $\mathrm{Bun}_G(S)/\sim \tilde{\to} B(G)$ which is a homeomorphism $|\mathrm{Bun}_G(S)| \simeq B(G)$. Let G/E be a reductive group. The moduli stack Bun_G of G-bundles on the Fargues-Fontaine curve represents the functor taking

- $\in \mathrm{Perf}\bar{F}_q \to \{G\text{-bundles on } X_S\}$.

Theorem 1.1.15. ([9] Theorem 16.1).

- Bun_G is an Artin v-stack, cohomologically smooth of dimension 0.
- The map $|\mathrm{Bun}_G| \to B(G)$ is a continuous bijection.
- For any $b \in B(G)$, we get a locally closed stratum $|\mathrm{Bun}_G^b| \subset Bun_G$. It has the form
  - $Bun_G^b = [*/\mathcal{G}_b]$

  where $\mathcal{G}_b$ fits into a short exact sequence
- $1 \to \underbrace{\text{unipotent group diamond}}_{\text{iterated extension of positive Banach-Colmez spaces}} \to \mathcal{G}_b \to G_b(E) \to 1.$

We now recall the local Langlands correspondence. For the set up, let G be split and let E be a non-archimedean local field, such as $F_q((t))$ or a finite extension of Q_p.

Conjecture 1.1.16. ([9] Local Langlands correspondence (Conjecture 1.6)). Consider representations over $L = \mathbf{C}$. There exists a natural map

- $\mathrm{Irrep}(G, E))/\sim \to \mathrm{Hom}(W_E, \hat{G}(\mathbf{C}))/G(\hat{\mathbf{C}})$

where $\hat{G}$ is the Langlands dual group, W_E is the Weil group of E which is surjective with finite fibers (called L-packets) defined as the pre-image of $Z \subset \hat{Z}$ under the surjection $Gal(\bar{E}/E) \to \bar{Z}$ corresponding to the maximal unramified extension of E.

In summary, to geometrize the local Langlands correspondence is to use the geometric structures of diamonds and perfectoid spaces to construct the geometric Langlands correspondence over the Fargues-Fontaine curve, whose diamond incarnation is

- $\mathcal{Y}^{\diamond}_{S,E} = S \times (\mathrm{Spa}\mathcal{O}_E)^{\diamond}$

and "to make Spec E geometric" for E a non-archimedean local field [9].

We conceive a link between geometric Langlands and Efimov K-theory via $\mathcal{Y}^{\diamond}_{S,E} = S \times (\mathrm{Spa}\mathcal{O}_E)^{\diamond}$. The expectation is that, in our forthcoming work, there will eventually exist an $(\infty, 1)$-categorification of the global Langlands program and geometrization of the local Langlands conjectures, else our work is antiphrastical. [3]

This is our main point. Moduli spaces of p-adic shtukas are diamonds, which are quotients of perfectoid spaces by pro-étale equivalence relations. Specifically, this moduli space of shtukas is a diamond that is fibered over a specific m-fold product [17]

- $SpaQ_p \times SpaQ_p \times \cdots_m \times SpaQ_p$.

We want to look at the Efimov K-theory of these moduli spaces.

The intent is to construct isomorphism classes of diamonds and build a K-theory framework to recover global information from the higher $\mathrm{K}^{\mathrm{Efimov}}$-groups of these specific diamonds:

[3] The connection supervenes the following: shtukas and Langlands (Drinfeld) → Drinfeld's lemma for diamonds (Scholze) → moduli spaces of mixed-characteristic local G-shtukas are representable by locally spatial diamonds (Scholze) → geometrization of local langlands using diamonds (Fargues-Scholze) → Banach-Colmez spaces are diamonds (Le Bras) → the category of Banach-Colmez spaces is equivalent to the full subcategory of the derived category of coherent sheaves on the Fargues-Fontaine Curve (Le Bras) → relative Fargues-Fontaine Curve is a diamond (Fargues-Scholze) → G torsors on relative Fargues-Fontaine Curve (Fargues-Scholze) → equivalence between the category of pro-étale $\underline{Q_p}$-local systems on a perfectoid space S and the category of vector bundles $\mathcal{E}$ on $\mathcal{X}_{FF,S}$ that are trivial at every geometric point of S (Kedlaya-Liu) →to link Efimov K-theory of $\mathcal{D}^{\diamond}$ and geometric Langlands via $\mathcal{Y}^{\diamond}_{S,E} = S \times (\mathrm{Spa}\mathcal{O}_E)^{\diamond}$ (Dobson).

[4] Lubin-Tate formal $\mathcal{O}_F$-module law [22]

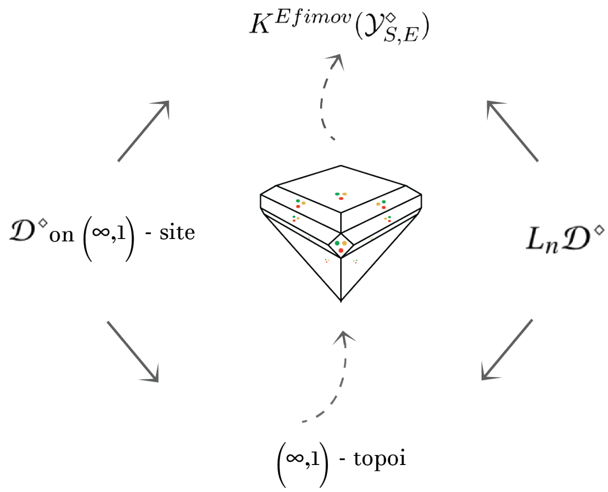


Figure 2: $K^{Efimov}(\mathcal{Y}^{\diamond}_{S,E})$ and Diamond $SpdQ_p = Spa(Q_p^{cycl})/\underline{Z_p^{\times}}$ in $LT/\mathcal{O}_F$ [4]

- $K^{\text{Efimov}}(\mathcal{Y}^{\diamond}_{S,E})$ for $\mathcal{Y}^{\diamond}_{S,E} = S \times (Spa\mathcal{O}_E)^{\diamond}$ and
- $\text{Spd}Q_p \times_{\diamond} \text{Spd}Q_p$, the diamond self-product.

The moduli spaces of shtukas in mixed characteristic live in the category of diamonds. Studying the isomorphism classes of moduli spaces of shtukas as the higher K^{Efimov}-groups of diamonds could link diamnods and global Langlands over function fields.

Our main conjectures follow.

2 Diamond Conjectures

The goal forthcoming is to get the lower K^{Efimov}-groups of the diamond spectra to encode the datum of the mixed-characteristic shtukas and the higher K^{Efimov}-groups to encode the moduli space of mixed-characteristic local G-shtukas, which is itself a locally spatial diamond.

We now state our main conjectures.

Conjecture 2.1. There exists a large, stable, presentable $(\infty, 1)$-category of diamonds

$\mathcal{D}^\diamond$ with spatial descent datum. $\mathcal{D}^\diamond$ is dualizable. Therefore, the Efimov K-theory is well-defined.

Conjecture 2.2. Let S be a perfectoid space, $\mathcal{D}^\diamond$ a stable dualizable presentable $(\infty, 1)$-category, and R a sheaf of E_1-ring spectra on S. Let $\mathcal{T}$ be a stable compactly generated $(\infty, 1)$-category and F: $\mathrm{Cat}_{St}^{idem} \to \mathcal{T}$ a localizing invariant that preserves filtered colimits. Then

- $F_{cont}(\mathrm{Shv}(\mathbb{S}^n, \mathcal{D}^\diamond)) \simeq \Omega^n F_{cont}(\mathcal{D}^\diamond)$.

Conjecture 2.3. Let $\mathcal{D}_\diamond$ be the complex of v-stacks of locally spatial diamonds. Let $\mathcal{D}^\diamond$ be the $(\infty, 1)$-category of diamonds. Let $\mathcal{Y}_{(R,R^+),E} = \mathrm{Spa}(R, R^+) \times_{\mathrm{Spa}F_q} \mathrm{Spa}F_q[[t]]$ be the relative Fargues-Fontaine curve. Let $(\mathcal{Y}^\diamond_{S,E})$ be the diamond relative Fargues-Fontaine curve. There exists a localization sequence

- $K(\mathcal{D}_\diamond) \to K^{\mathrm{Efimov}}(\mathcal{Y}^\diamond_{S,E}) \to K^{\mathrm{Efimov}}(\mathcal{Y}_{(R,R^+),E})$.

Conjecture 2.4. $\mathcal{D}^\diamond$ admits a topological localization, in the sense of Grothendieck-Rezk-Lurie $(\infty, 1)$-topoi.

Conjecture 2.5. There exists a diamond chromatic tower

- $\mathcal{D}^\diamond \to ... \to L_n\mathcal{D}^\diamond \to L_{n-1}\mathcal{D}^\diamond \to ... \to L_0\mathcal{D}^\diamond$

for L_n a topological localization for $K\mathcal{D}^\diamond$ the -theory spectrum of the diamond spectrum representative of the étale cohomology of diamonds.

Conjecture 2.6. The $(\infty, 1)$-category of perfectoid diamonds is an $(\infty, 1)$-topos.

2.1 Pro-Diamond

Our pro-diamond pro-object is constructed as follows:

Conjecture 2.1.1. Let $\mathcal{D}^{\diamond\diamond}$ be a small cofiltered category of diamonds with morphisms the diamond product [12]. Two objects in $\mathcal{D}^{\diamond\diamond}$ are diamonds (v-sheaves) and spatial v-sheaves. The pro-diamond pro-object in the category of pro-objects of $\mathcal{D}^{\diamond\diamond}$ is the formal cofiltered limit of objects of $\mathcal{D}^{\diamond\diamond}$. The $Hom_{\mathcal{D}^{\diamond\diamond}}(F(-), G(-))$ for pro-objects $F : D \to C$

and $G : E \to C$ is given by the pro-diamond functor, the pro version of the diamond functor. Recall

Proposition 2.1.2. ([9] Proposition 6.11). For an analytic adic space X/Z_p, the diamond functor

- $X^\diamond : S \in Perf \to \{S^\#/Z_p \text{ untilts of } S \text{ plus map } S^\# \to X\}$

defines a locally spatial diamond. There are canonical equivalences $|X| \simeq |X^\diamond|$ and $X_{\text{ét}} \simeq X^\diamond_{\text{ét}}$. With X perfectoid, $X^\diamond \simeq X^b$.

Recall that

> a pro-object of a category C is a formal cofiltered limit of objects of C [39].

A cofiltered category has the property that

> for every pair of objects c_1 and c_2 of C, there is an object c_3 of C such that there exists an arrow $c_3 \to c_1$ and there exists an arrow $c_3 \to c_2$ [39].

Recall the definition of the category of pro-objects in C.

Definition 2.1.3 [Definition 2.2 [39]]. Let C be a category. The category of pro-objects in C is the category defined as follows.

- The objects are pro-objects in C.
- The set of arrows from a pro-object $F : D \to C$ to a pro-object $G : E \to C$ is the limit of the functor $D^{op} \times E \to Set$ given by $Hom_C(F(-), G(-))$.
- Composition of arrows arises, given pro-objects $F : D_0 \to C$, $G : D_1 \to C$, and $H : D_2 \to C$ of C, by applying the limit functor for diagrams $D^{op} \times E \to Set$ to the natural transformation of functors $HomC(F(-), G(-)) \times HomC(G(-), H(-)) \to Hom_C(F(-), H(-))$ given by composition in C.
- The identity arrow on a pro-object $F : D \to C$ arises, using the universal property of a limit, from the identity arrow $Hom_C(F(c), F(c))$ for every object c of C.

We can also construct the pro-diamond pro-object of the category of diamonds by considering as pro-objects the isomorphism classes of diamonds under the diamond equivalence relation.

3 Diamonds

3.1 Souscompendium

The diamond construction mirrors the creation of an algebraic space as a quotient by an étale equivalence relation of a scheme. The main idea is the following.

Definition 3.1.1. ([21] Definition 1.12). A diamond is an algebraic space for the pro-étale topology in Perf.

We give examples of the many incarnations of diamonds and a souscompendium of main results from [12].

The motivation of diamonds is the construction of a functor

- $\{X$, analytic spaces over $Z_p\} \to \{X^\diamond$, diamonds$\}$

  that forgets the structure morphism to Z_p.

For X/Z_p an analytic adic space,

X is pro-étale locally perfectoid:

- $X = \text{Coeg}(\tilde{X} \times_X \tilde{X} \rightrightarrows \tilde{X})$

  where $\tilde{X} \to X$ is a pro-étale perfectoid cover.

The equivalence relation $R = \tilde{R} \times_X \tilde{X}$ is perfectoid because it is pro-étale over $\tilde{X}$.

We see this quotient in the following example.

Example 3.1.2. ([12] Example 8.1.1) If $X = Spa(Q_p)$, then a pro-étale perfectoid cover of X is $\tilde{X} = Spa(Q_p^{cycl})$. Thus $R = \tilde{X} \times_X \tilde{X}$ is essentially $\tilde{X} \times Z_p^x$. This is a perfectoid space, and so $X^\diamond$ should be the coequalizer of $\tilde{X}^b \times Z_p^x \rightrightarrows \tilde{X}$, which comes out to be the quotient $Spa(Q_p^{cycl})^b)/Z_p^x$, the meaning of which is curious.

At the least, this quotient $Spa((Q_p^{cycl})^b)/Z_p^x$ lives in a category of sheaves on the site of perfectoid spaces with pro-étale covers. Therein it is now apropos to recall the pro-étale topology and pro-étale morphisms between perfectoid spaces, as define the subcanonical v-topology, which is the finest topology on Perf, the category of perfectoid spaces of characteristic p.

Definition 3.1.3. ([12] Definition 1.1 (iii), (iv)). Let X be an analytic adic space X on which a fixed prime p is topologically nilpotent. We associate an étale site $X_{\text{ét}}$ for any X. There are several different topologies defined on Perf, successively refining what is previous, three of which are the following.

- The étale topology, where a cover $\{f_i : X_i \to X\}$ consists of étale maps $X_i \to X$ which jointly cover X.
- The Grothendieck pro-étale topology, where a cover $\{f_i : X_i \to X\}$ consists of pro-étale maps $X_i \to X$ such that for any quasicompact open subset $U \subset X$, there are finitely many indices i and quasicompact open subsets $U_i \subset X_i$ such that the U_i jointly cover U.
- The Grothendieck v-topology, where a cover $\{f_i : X_i \to X\}$ consists of any maps $X_i \to X$ such that for any quasicompact open subset $U \subset X$, there are finitely many indices i and quasicompact open subsets $U_i \subset X_i$ such that the U_i jointly cover U. The v-topology is generated by open covers and all surjective maps of affinoids.

Specifically, we define the two Grothendieck topologies as follows.

Definition 3.1.4. ([12] Definition 8.1). Let Perfd be the category of κ-small perfectoid spaces.

- The big pro-'etale site is the Grothendieck topology on Perfd for which a collection $\{f_i : Y_i \to X\}_{i\in I}$ of morphisms is a covering if all f_i are pro-étale, and for each quasicompact open subset $U \subset X$, there exists a finite subset $J \subset I$ and quasicompact open subsets $V_i \subset Y_i$ for $i \in J$, such that $U = \bigcup_{i\in J} f_i(V_i)$.
- Let X be a perfectoid space. The small pro-étale site of X is the Grothendieck topology on the category of perfectoid spaces $f : Y \to X$ pro-étale over X, with covers the same as in the big pro-étale site.

- The v-site is the Grothendieck topology on Perfd for which a collection $\{f_i : Y_i \to X\}_{i\in I}$ of morphisms is a covering if for each quasicompact open subset $U \subset X$, there exists a finite subset $J \subset I$ and quasicompact open subsets $V_i \subset Y_i$ for $i \in J$, such that $U = \bigcup_{i\in J} f_i(V_i)$.

To prove that the v-site and the pro-étale site are well-behaved, we summarize some properties of quasicompact and quasiseparated (qcqs) objects.

Definition 3.1.5. ([12] Definition 8.2). Let T be a topos.

- An object $X \in T$ is quasicompact if for any collection of objects $X_i \in T, i \in I$, with maps $f_i : X_i \to X$ such that $\sqcup f_i : \sqcup X_i \to X$ is surjective, there is a finite subset $J \subset I$ such that $\sqcup_{i\in J} X_i \to X$ is surjective (cf. SGA 4 VI Définition 1.1).

- An object $X \in T$ is quasiseparated if for all quasicompact objects $Y, Z \in T$, the fibre product $Y \times_X Z \in T$ is quasicompact, (cf. SGA 4 VI D´efinition 1.13).

- A map $f : Y \to X$ is quasicompact if for all quasicompact $Z \in T$, the fibre product $Z \times_X Y$ is quasicompact, (cf. SGA 4 VI D´efinition 1.7).

- A map $f : Y \to X$ is quasiseparated if $\Delta_f : Y \to Y \times_X Y$ is quasicompact, (cf. SGA 4 VI D´efinition 1.7).

T is presumed an algebraic topos (cf. SGA 4 VI Définition 2.3). Certain topoi of sheaves on Perfd are also algebraic.

Proposition 3.1.6. ([12] Proposition 8.3). The topoi of sheaves on Perfd for either the (big) pro-étale or v-topology, and the topos of sheaves on $X_{\text{proét}}$ for a perfectoid space X, are algebraic. A basis of qcqs objects stable under fibre products is in all cases given by affinoid perfectoid spaces. Moreover, a perfectoid space X is quasicompact respectively quasiseparated in any of these settings if and only if $|X|$ is quasicompact respectively quasiseparated.

Remark 3.1.7. ([12] Definition 1.3 (Discussion)). The v-topology is the finest. It resembles the fpqc topology on schemes in that the structure sheaf is a sheaf for the v-topology on Perfd, mirroring the same for the fpqc topology on schemes (cf. [12] Theorem 8.7, Proposition 8.8). The v-topology is powerfully useful in proving results about diamonds.

Theorem 3.1.8. ([12] Theorem 1.2). The v-topology on Perf is subcanonical, and for any affinoid perfectoid space $X = Spa(R, R^+)$, $H^0_v(X, \mathcal{O}^+_X) = R^+$ and for $i > 0, H^i_v(X, \mathcal{O}_X) = 0$ and $H^i_v(X, \mathcal{O}^+_X)$ is almost zero.

What is immediate is that

- All diamonds are v-sheaves [12].

We can define sites $\mathcal{D}_v$ and $\mathcal{D}_{\text{ét}}$ for a diamond $\mathcal{D}$. We revisit this topic after formally introducing diamonds and their examples.

We now discuss affinoid pro-étale morphisms.

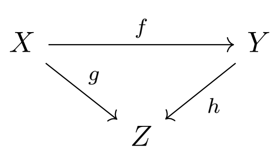
Definition 3.1.9. ([17] Definition 8.2.1). A morphism $f : Spa(B, B^+) \to Spa(A, A^+)$ of affinoid perfectoid spaces is affinoid pro-étale if

- $(B, B^+) = \widehat{\varinjlim}(A_i, A_i^+)$ is a completed filtered colimit of pairs (A_i, A_i^+) with A_i perfectoid, such that
- $Spa(A_i, A_i^+) \to Spa(A, A^+)$ is étale.

A morphism $f : X \to Y$ of perfectoid spaces is pro-étale if it is locally on the source and target affinoid pro-étale.

Nice properties of pro-étale morphisms are the following.

- **Proposition 3.1.10**. ([17] Proposition 8.2.5). Compositions of pro-étale maps are pro-étale.

  is a diagram of perfectoid spaces where g and h are pro-étale, then f is pro-étale.

- Pullbacks of pro-étale morphisms are pro-étale.

Important for our diamond spectrum and diamond cryptography is the example of pro-étale morphisms from profinite sets.

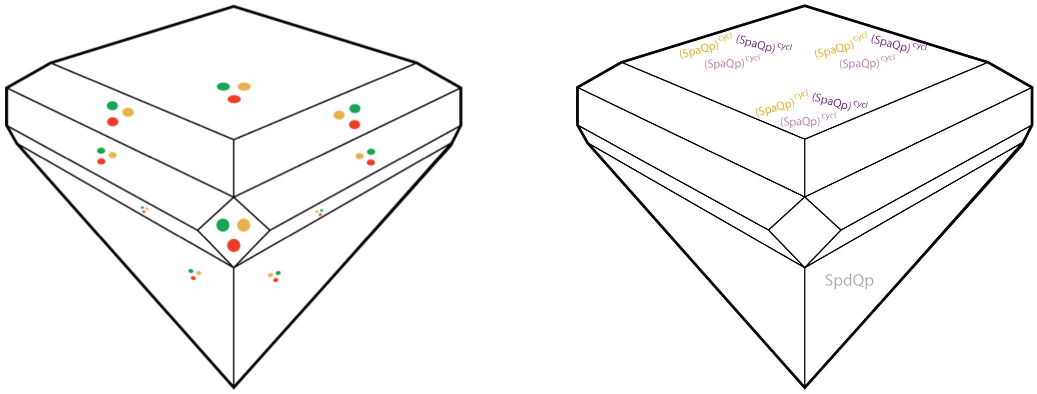


Figure 3: Diamond $SpdQ_p = Spa(Q_p^{cycl})/\underline{Z_p^\times}$ with geometric point $Spa\ C \to \mathcal{D}$

Example 3.1.11. ([17] Example 8.2.2). If X is any perfectoid space and S is a profinite set, we can define a new perfectoid space $X \times \underline{S}$ as the inverse limit of $X \times S_i$, where $S = \underset{\leftarrow}{lim} S_i$ is the inverse limit of finite sets S_i . Then $X \times \underline{S} \to X$ is pro-étale. This construction extends to the case that S is locally profinite.

We now have the definition of diamond.

Definition 3.1.12. ([17] Definition 1.3). Let Perfd be the category of perfectoid spaces. Let Perf be the subcategory of perfectoid spaces of characteristic p. Let Y be a pro-étale sheaf on Perf. Then Y is a diamond if Y can be written as the quotient X/R with X a perfectoid space of characteristic p and R a pro-étale equivalence relation $R \subset X \times X$.

The meaning of the pro-étale equivalence relation is developed in the following proposition.

Proposition 3.1.13 ([17] Proposition 11.3). Let X be a perfectoid space of characteristic p, and let R be a perfectoid space with two pro-étale maps $s, t : R \to X$ such that the induced map $R \to X \times X$ is an injection making R an equivalence relation on X. Then $D = X/R$ is a diamond and the natural map $R \to X \times X$ is an isomorphism.

Additionally, we have the diamond accompanying an analytic adic space over Z_p.

Definition 3.1.14. ([17] Definition 15.5). Let Y be an analytic adic space over Z_p. The diamond associated to Y is the v-sheaf defined by

- $Y^{\diamond} : X \to \{((X^{\#}, \iota), f : X^{\#} \to Y)\}/ \simeq$,
- where $X^{\#}$ is a perfectoid space with an isomorphism $\iota : (X^{\#})^b \simeq X$.

Remark 3.1.15. Diamonds are so named because their geometric points resemble mathematical mineralogical impurities. Let C be an algebraically closed affinoid field and $\mathcal{D}$ a diamond. A geometric point $Spa(C) \to \mathcal{D}$ is made "visible" by pulling it back through a quasi-pro-étale cover $X \to \mathcal{D}$, the consecution of which is profinitely many copies of $Spa(C)$. Multiple representations of the geometric points of $\mathcal{D}$ can be made based on multiple quasi-pro-étale covers $X \to \mathcal{D}$.

3.2 Examples of Diamonds

Examples of diamonds are the following:

- **Example 3.2.1** [17]. For $X = Spa(R, R^+)$, we say $Spd(R, R^+) = Spa(R, R^+)^{\diamond}$.
- **Example 3.2.2** [17]. The Fargues-Fontaine Curve X_{FF} is a regular noetherian scheme of Krull dimension 1 which is locally the spectrum of a principal ideal domain. The set of closed points of X_{FF} is identified with the set of characteristic 0 untilts of C^b modulo Frobenius. For C an algebraically closed perfectoid field of characteristic $p > 0$ and ϕ the Frobenius automorphism of C we have
  - $X_{FF}^{\diamond} \cong (SpdC \times SpdQ_p)/(\phi \times id)$.
- **Definition 3.2.3** [17]. The diamond equation is
  - $\mathcal{Y}_{S,E}^{\diamond} = S \times (Spa\mathcal{O}_E)^{\diamond}$
- **Example 3.2.4** [17]. Let $\mathcal{D}$ and $\mathcal{D}'$ be diamonds. Then the product sheaf $D \times_{\diamond} D'$ is also a diamond.
- **Example 3.2.5** [17]. $SpdQ_p = Spd(Q_p^{cycl})/\underline{Z^{x_p}}$ where $\underline{Z^{x_p}}$ is the profinite group $Gal(Q_p^{cycl}/Q_p)$.
- **Example 3.2.6** [17]. $SpdQ_p \times_{\diamond} SpdQ_p$.
- **Example 3.2.7** [17]. $Sht_{\mathcal{G},b,\{\mu_i\}}$: moduli spaces of mixed-characteristic local G-shtukas is a locally spatial diamond.

- **Example 3.2.8** [17]. All Banach-Colmez spaces are diamonds.
- **Example 3.2.9** [17]. Any closed subset of a diamond is a diamond.

Remark 3.2.10. To understand Example 3.2.4, we recall that the definition $SpdQ_p = Spa(Q_p^{cycl})/\underline{Z_p^\times}$ means $SpdQ_p$ is the coequalizer of

- $\underline{Z_p^\times} \times Spa(Q_p^{cycl})^\flat \rightrightarrows Spa(Q_p^{cycl})^\flat$,

for one map the projection and the other the action ([17] Definition 9.4.1).

This construction is well-defined.

Lemma 3.2.11. ([17] Lemma 9.4.2). Let $g : \underline{Z_p^\times} \times SpdQ_p^{cycl} \to SpdQ_p^{cycl} \times SpdQ_p^{cycl}$ be the product of the projection onto the second factor and the group action. Then g is an injection.

Proof. Let (K, K^+) be a perfectoid affinoid field. Then $Z_p^\times$ acts freely on $\mathrm{Hom}(F_p((t^{\frac{1}{p^\infty}})), K)$. By Proposition 8.3.3, this implies the result. □

Corollary 3.2.12. ([17] Corollary 9.4.3). The map $SpaQ_p^{cycl} \to SpdQ_p$ is a $\underline{Z_p^\times}$-torsor and the description of $SpdQ_p$ in Proposition 8.4.1 holds true.

Proof. Given any map $Spa(R, R^+) \to SpdQ_p$, via pullback we get a $\underline{Z_p^\times}$-torsor over $Spa(R, R^+)$, which are described by Proposition 9.3.1. □

Remark 3.2.13. To understand Example 3.2.5, we need to know what it means for a profinite group G to act continuously on a perfectoid space X. Scholze explains that this is an action of the

> pro-étale sheaf of groups $\underline{G}$ on X, where $\underline{G}(T)$ is the set of continuous maps from $|T|$ to G, for any $T \in Perf$. Let G act continuously on X. Then one can define $R = X \times \underline{G}$...in concert with the definition of the perfectoid space $X \times \underline{S}$ for a profinite set S. $R = X \times \underline{G}$ comes with two maps $R \rightrightarrows X$: the projection to the first component and the action map. If the induced map $R \to X \times X$ is an injection, then X/R is a diamond also denoted by $X/\underline{G}$ [16].

Essentially, $\underline{Z_p^x}$ remembers descent datum from the cyclotomic field to Q_p.

Fortuitously, the sheaf $SpdQ_p$ attaches to any perfectoid space S of characteristic p the set of all untilts $S^{\#}$ over Q_p. Specifically, we have the following:

Proposition 3.2.14. ([17] Proposition 8.4.1). If $X = Spa(R, R^+)$ is an affinoid perfectoid space of characteristic p, then $(SpdQ_p)(X)$ is the set of isomorphism classes of data of the following shape.

- A $\underline{Z_p^x}$-torsor $R \to \tilde{R}$; that is, $\tilde{R}= (\widehat{\lim_{\leftarrow} R_n})$, where R_n/R is finite étale with Galois group $(Z/p^n Z)^\times$.

  A topologically nilpotent element unit $t \in \tilde{R}$ such that for all $\gamma \in Z_p^\times$, $\gamma(t) = (1+t)^\gamma - 1$.

The following theorem results.

Theorem 3.2.15. ([17] Theorem 9.4.4). The following categories are equivalent:

- The category of perfectoid spaces over Q_p.
- Perfectoid spaces X of characteristic p equipped with a structure morphism $X \to SpdQ_p$.

Remark 3.2.16. To understand Example 3.2.6, we must recall that a diamond $\mathcal{D}$ has "multiple incarnations," as $\mathcal{D}$ can be represented as $X^\diamond/\underline{G}$ in many ways. In the expression $X^\diamond/\underline{G}$, X is taken to be an analytic adic space over $SpaZ_p$, and G is a profinite group [17]. For the diamond self-product $SpdQ_p \times_\diamond SpdQp$,

there are (at least) the following two incarnations:

- 1. $X = \tilde{D}^*_{Q_p}$, $G = Z_p^\times$.
- 2. $X = SpaA_{inf}/\{p[p^b] = 0\}$ with $A_{inf} = W(\mathcal{O}_{C_p^b})$, and $G = G_{Q_p}$.

Remark 3.2.17. From the first incarnation, there results an isomorphism of diamonds $(\tilde{D}^*_{Q_p})^\diamond/\underline{Z_p^x} \simeq SpdQ_p \times SpdQ_p$. The $n = 2$ case of Drinfeld's lemma for diamonds proclaims as such:

Lemma 3.2.18. ([17] Theorem 16.3.1 Lemma (Drinfeld)). $\pi_1((\tilde{D}^*_{Q_p})^\diamond/\underline{Q^x_p}) \simeq G_{Q_p} \times G_{Q_p}$.

Lemma 3.2.19. ([17] Lemma 16.3.2). For any complete and algebraically closed field C/Q_p, one has $\pi_1((\tilde{D}^*_C)^\diamond/\underline{Q^x_p}) \simeq G_{Q_p}$.

G_{Q_p} takes the form of a geometric fundamental group.

Remark 3.2.20. For Example 3.2.7, recall the following definition.

Definition 3.2.21. ([17] Definition). Fix an algebraically closed nonarchimedean field C/Q_p. The category of Banach-Colmez spaces over C is the thick abelian subcategory of the category of pro-étale sheaves of $\underline{Q_p}$-modules on $Perfd_C$ generated by $\underline{Q_p}$ and $G^\diamond_{a,C}$.

Remark 3.2.22. Diamonds indeed have many incarnations. For a diamond model of neurological agency as profinite geometric-mathematical impurities, see [7].

We now discuss pro-étale and spatial descent for $\mathcal{D}^\diamond$.

3.3 Descent

To construct $\mathcal{D}^\diamond$ we examine pro-étale descent for perfectoid spaces and spatial descent for spatial diamonds. We summarize the descent discussion in [17].

> Consider the following notion of descent along an fpqc covering $X' \to X$. Take a morphism of schemes $Y' \to X'$ together with a descent datum over $X' \times_X X'$. To ask if Y' descends to X is to ask if there exists a unique morphism $Y \to X$ for which $Y' = Y \times_X X'$. We say that the fibered category $X \to \{Y/X \text{ affine}\}$ on the category of schemes is a stack for the fpqc topology [17].

Scholze asks if there exist similar descent results for perfectoid spaces using the pro-étale topology.

Question 3.3.1. (Question 9.1.1). Is the fibered category

- $X \to \{\text{morphisms } Y \to X \text{ with } Y \text{ perfectoid}\}$

on the category of perfectoid spaces a stack for the pro-étale topology? This is to ask:

Question 3.3.2. If $X' \to X$ is a pro-étale cover and $Y' \to X'$ a morphism together with a descent datum over $X' \times_X X'$ then does $Y' \to X'$ descend to $Y \to X$? If not, is this true under stronger hypothesis on the morphisms? What if all spaces are assumed to be affinoid perfectoid? (Such a descent is unique up to unique X-isomorphism if it exists, by [17] Proposition 8.2.8).

The answer is in general no, but under stronger conditions, it is yes.

Theorem 3.3.3. ([17] Theorem 9.1.3). Descent along a pro-étale cover $X' \to X$ of a perfectoid space $f : Y' \to X'$ is effective in the following cases.

- 1. If X, X' and Y, Y' are affinoid and X is totally disconnected.
- 2. If f is separated and pro-étale and X is strictly totally disconnected.
- 3. If f is separated and étale.
- 4. If f is finite étale.

Moreover, the descended morphism has the same properties [17].

Recall the notion of (strictly) totally disconnected spaces.

Definition 3.3.4. ([12] Definition 1.14). Let X be a perfectoid space. Then X is totally disconnected (respectively, strictly totally disconnected) if X is quasicompact, and every open cover of X splits (respectively, and every étale cover of X splits).

Proposition 3.3.5. ([12] Proposition 1.15). Let X be a perfectoid space. Then X is totally disconnected (respectively strictly totally disconnected) if and only if X is affinoid, and every connected component of X is of the form $Spa(K, K^+)$, where K is a perfectoid field with an open and bounded valuation subring $K^+ \subset K$ (resp. and K is algebraically closed).

Thus, (strictly) totally disconnected perfectoid spaces are most aptly characterized as profinite sets of (geometric) points.

The following two lemmas are descent conditions.

Lemma 3.3.6. ([12] Lemma 1.16). Let X be any quasicompact perfectoid space. Then there is a pro-étale cover $\tilde{X} \to X$ such that $\tilde{X}$ is strictly totally disconnected.

Lemma 3.3.7. ([12] Lemma 1.17). Let $X = Spa(R, R^+)$ be a totally disconnected perfectoid space, and $f : Y = Spa(S, S^+) \to X$ be any map of perfectoid spaces. Then $R^+/\bar{\omega} \to S^+/\bar{\omega}$ is flat, for any pseudouniformizer $\bar{\omega} \in R$. v-descent reduces to the case of strictly totally disconnected spaces. Then the second lemma reduces this to faithfully flat descent.

Remark 3.3.8. A "puncture criterion" resolves an issue becoming pro-étale morphisms which are not local on the target in the pro-étale topology.

Lemma 3.3.9. ([12] Lemma 9.1.4). Let $f : X \to Y$ be a morphism of affinoid perfectoid spaces. The following are equivalent.

- 1. There exists $Y' \to Y$ which is affinoid pro-étale surjective, such that the base change $X' = X \times_Y Y'$ is affinoid pro-étale.
- 2. For all geometric points $SpaC \to Y$ of rank 1, the pullback $X \times_Y SpaC \to SpaC$ is affinoid pro-étale; equivalently, $X \times_Y SpaC = SpaC \times \underline{S}$ for some profinite set S.

Consider this excellent example of a non-pro-étale morphism which is locally pro-étale, featuring a base space torn apart severely. It is one of our most important constructions.

Example 3.3.10. ([12] Example 9.1.5 (A non-pro-étale morphism which is locally pro-étale)). Assuming $p \neq 2$, let

- $Y = SpaK < T^{\frac{1}{p^\infty}} >$, and
- $X = SpaK < T^{\frac{1}{2p^\infty}} >$.

Then $X \to Y$ appears to be ramified at 0, and it is not pro-étale. However, consider the following pro-étale cover of Y. Let

- $Y' = \underleftarrow{lim} Y'_n, Y'_n = \{x \in Y || x| \leq |\bar{\omega}^n|\} \sqcup \amalg_{i=1}^n \{x \in Y || \bar{\omega}|^i \leq |x| \leq |\bar{\omega}|^{i-1}\}$.

Let $X' = X \times_Y Y'$. We claim that the pullback $X' \to Y'$ is affinoid pro-étale.
As a topological space, $\pi_o(Y') = \{1, \frac{1}{2}, \frac{1}{3}, \cdots, 0\} \subset \mathbb{R}$. The fiber of Y' over $\frac{1}{i}$ is

- $\{x \in Y || \bar{\omega}|^i \leq |x| \leq |\bar{\omega}|^{i-1}\}$,

and the fiber over 0 is just 0. Let

- $X'_n = \{x||x| \leq |\bar{\omega}|^n\} \sqcup \amalg_{i=1}^n \{|\bar{\omega}|^i| \leq |x| \leq |\bar{\omega}|^{i-1}\} \times_Y X$

so that $X'_n \to Y'_n$ is finite étale. Then $X' \simeq \underleftarrow{lim}(X'_n \times_{Y'_n} Y') \to Y'$ is pro-étale.

Remark 3.3.11. To prove Proposition 9.1.4 requires the notion of strictly disconnected perfectoid spaces, wherein the associated base space is even more severely ripped apart .

Definition 3.3.12. ([12] Definition 9.1.6). A perfectoid space X is totally disconnected (resp. strictly totally disconnected) if it is qcqs and every open (resp. ´etale) cover splits.

Lemma 3.3.13. ([12] Lemma 7.5). Any totally disconnected perfectoid space is affinoid.

Lemma 3.3.14. ([12] Lemma 7.18, Proposition 7.16). The connected components of a totally disconnected space are of the form $Spa(K, K^+)$ for a perfectoid affinoid field (K, K^+).

Lemma 3.3.15. ([12] Lemma 7.13, Proposition 7.16). The connected component of a strictly totally disconnected spaces are of the form $Spa(K, K^+)$ for a perfectoid affinoid field (K, K^+) where K is algebraically closed.

Lemma 3.3.16. ([12] Lemma 7.18). For any affinoid perfectoid space X one can find an affinoid pro-étale map $\tilde{X} \to X$ such that $\tilde{X}$ is strictly totally disconnected.

Enlarging the class of pro-étale morphisms of Proposition 9.1.4, we have the quasi-proétale morphisms.

Definition 3.3.17. ([12] Definition 9.2.1). A morphism $f : X \to Y$ of perfectoid spaces is quasi-pro-étale if for any strictly totally disconnected perfectoid space Y' with a map $Y' \to Y$ the pullback $X' = X \times_Y Y' \to Y'$ is pro-étale.

Remark 3.3.18. In general, $f : X \to Y$ is quasi-pro-étale if and only if it is so when restricted to affinoid open subsets, in which case it is equivalent to the condition of Proposition 9.1.4 [17].

The following definition will produce a a characterization of diamonds equivalent to Definition 1.3.

Definition 3.3.19. ([12] Definition 9.2.2). Consider the site Perf of perfectoid spaces of characteristic p with the pro-étale topology. A map $f : \mathcal{F} \to \mathcal{G}$ of sheaves on Perf is quasi-pro-étale if it is locally separated and for all strictly totally disconnected perfectoid spaces Y with a map $Y \to \mathcal{G}$ (i.e., an element of $\mathcal{G}(Y)$), the pullback $\mathcal{F} \times_{\mathcal{G}} Y$ is representable by a perfectoid space X and $X \to Y$ is pro-étale.

The equivalent characterization is as follows.

Proposition 3.3.20. ([12] Proposition 11.5). A pro-étale sheaf Y on Perf is a diamond if and only if there is a surjective quasi-pro-étale map $X \to Y$ from a perfectoid space X.

3.4 $\underline{G}$-torsor

In order to fully understand the important diamond examples $SpdQ_p \times SpdQ_p$ and $\mathcal{Y}^{\diamond}_{(S,E)}$, as well as our conjectured $(\infty, 1)$-diamond topos, we recall the notion of a G-torsor, define a $\underline{G}$-torsor, and summarize main properties of both.

Definition 3.4.1. ([17] Definition Section 9.3). If G is a finite group, we have the notion of G-torsor on any topos. A G-torsor on a topos is a map $f : \mathcal{F}' \to \mathcal{F}$ with an action $G \times \mathcal{F}' \to \mathcal{F}'$ over $\mathcal{F}$ such that locally on $\mathcal{F}$, one has a G-equivariant isomorphism $\mathcal{F}' \simeq \mathcal{F} \times G$.

Definition 3.4.2. ([17] Definition Section 9.3). There is another notion if G is a group object in a topos. The example apropos to our discussion is a group object in the category of pro-étale sheaves on Perf [17]. The construction is the following. For any topological space T, we can introduce a sheaf $\underline{T}$ on Perf, by

- $\underline{T}(X) = C^0(|X|, T)$.
- $\underline{T}$ is a pro-étale sheaf because pro-étale covers induce quotient mappings (Proposition 4.3.3).
- If T is a profinite set, this agrees with the definition of $\underline{T}$ given in Example 8.2.2.

- If G is a topological group, then $\underline{G}$ is a sheaf of groups.
- If $G = \underset{\leftarrow i}{lim} G_i$ is a profinite group, then in $\underline{G} = \underset{\leftarrow i}{lim} \underline{G_i}$.
- $\underline{G}$ is not representable, even if G is finite.

Remark 3.4.3. $\underline{G}$ is not representable because Perf does not contain a final object X, which, in the positive, would take the form of a base. Let us imagine Perf had a final object.

> Then the sheaf $\underline{G}$ for finite G would be representable by G copies of X. Once supplied with a base, $\underline{G}$ becomes representable. Given X a perfectoid space and G a profinite group, then $X \times \underline{G}$ is representable by a perfectoid space,
>
> - $X \times \underline{G} = \underset{\leftarrow}{lim} X \times G/H$.
>
> The notation $X \times G/H$ means a finite disjoint union of copies of X [17].

Remark 3.4.4. If G is rather a profinite set, the same conclusions hold (cf. [17] Example 8.2.2).

We now define a $\underline{G}$-torsor.

Definition 3.4.5. ([17] Definition Section 9.3). A $\underline{G}$-torsor is a morphism $f : \mathcal{F}' \to \mathcal{F}$ with an action $\underline{G} \times \mathcal{F}' \to \mathcal{F}'$ such that locally on $\mathcal{F}$ we have a G-equivariant isomorphism $\mathcal{F}' \simeq \mathcal{F} \times G$.

We have the following general result for torsors under locally profinite groups G.

Lemma 3.4.6. ([12] Lemma 10.13). Let $f : \mathcal{F}' \to \mathcal{F}$ be a $\underline{G}$-torsor, with G profinite. Then for any affinoid $X = Spa(B, B^+)$ and any morphism $X \to \mathcal{F}$, the pullback $\mathcal{F}' \times_{\mathcal{F}} X$ is representable by a perfectoid affinoid $X' = Spa(A, A^+)$. Furthermore, A is the completion of $\underset{\to H}{lim} A_H$, where for each open normal subgroup $H \subset G$, A_H/B is a finite étale G/H-torsor in the algebraic sense.

3.5 $|\mathcal{D}|$

We now discuss the underlying topological space of diamonds. We begin with the following proposition.

Proposition 3.5.1. ([17] Proposition 10.2.3). Universal homeomorphisms induce isomorphisms on diamonds.

Definition 3.5.2. ([17] Definition 11.14). Let $\mathcal{D}$ be a diamond, and choose a presentation $\mathcal{D} = X/R$. The underlying topological space of $\mathcal{D}$ is the quotient $|\mathcal{D}| = |X|/|R|$.

Remark 3.5.3. Highly pathological is the topology of $\mathcal{D}$, which may not even be T_0.

Example 3.5.4. ([17] Example). The quotient of the constant perfectoid space Z_p over a perfectoid field by the equivalence relation "congruence modulo Z" produces a diamond with underlying topological space Z_p/Z.

Remark 3.5.5. ([17] Remark 10.3.2). There is a special class of diamonds, qcqs, which are better behaved. We now recall the meaning of qcqs. Any topos contains a quasicompact object, which means that any covering family has a finite subcover. We say an object Z is quasiseparated if for any quasicompact X, $Y \to Z$, the fiber product $X \times_Z Y$ is quasicompact. We say an object that is both quasicompact and quasiseparated is qcqs.

What is important for our topos construction is the following.

Remark 3.5.6[17]. If a topos has a generating family B for the topos, this means that every object is a colimit of objects in the generating family. If B consists of quasicompact objects which is stable under fiber products, then.

- All objects of B are qcqs,
- Z is quasicompact if and only if it has a finite cover by objects of B, and
- Z is quasiseparated if and only if for all $X, Y \in B$ with maps $X, Y \to Z$, the fiber product $X \times_Z Y$ is quasicompact.

We consider the topos of sheaves on the pro-étale site of Perf. Let B represent the class of all affinoid perfectoid spaces. Maps connecting analytic adic spaces are adic, which implies B is closed under fiber products.

Proposition 3.5.7. ([17] Proposition 10.3.4). Let $\mathcal{D}$ be a qcqs diamond. Then $|\mathcal{D}|$ is T_0. For all distinct $x, y \in |\mathcal{D}|$, there is an open subset $U \subset |\mathcal{D}|$ that contains exactly one of x and y.

Proof (Sketch [17]). Let $X \to \mathcal{D}$ be a quasi-pro-étale surjection from a strictly totally disconnected space, and let $R = X \times_{\mathcal{D}} X$ which is qcqs pro-étale over X, and so itself a perfectoid space. As $\mathcal{D}$ is qcqs, so too are R and the map $R \to X$ qcqs. Therefore, we can apply [Lemma 2.7 [12]] to $|X|/|R|$. □

$|\mathcal{D}|$ is linked with open immersions, as in the following definition and immediate proposition.

Definition 3.5.8. ([17] Definition 10.3.5). A map $\mathcal{G} \to \mathcal{F}$ of pro-étale sheaves on Perf is an open immersion if for any map $X \to \mathcal{F}$ from a perfectoid space X, the fiber product $\mathcal{G} \times_{\mathcal{F}} X \to X$ is representable by an open subspace of X. In this case we say that $\mathcal{G}$ is an open subsheaf of $\mathcal{F}$.

Proposition 3.5.9. ([17] Proposition 11.15). Let $\mathcal{F}$ be a diamond. The category of open subsheaves of $\mathcal{F}$ is equivalent to the category of open immersions into $|\mathcal{F}|$, via $\mathcal{G} \to |\mathcal{G}|$, where $\mathcal{G}$ is an open subsheaf of $\mathcal{F}$.

Remark 3.5.10. This means that the "diamond remembers the underlying topological space" and this is the basis for our diamond cryptography.

Lemma 3.5.11. ([17] Lemma 15.6). Let X be an analytic pre-adic space over Z_p, with associated diamond $X^\diamond$. There is a natural homeomorphism $|X^\diamond| \simeq |X|$.

Proof. (Sketch). One reduces to the case that $X = Spa(R, R^+)$ is affinoid, so that $X = \tilde{X}/\underline{G}$, where $\tilde{X} = Spa(\tilde{R}, \tilde{R^+})$ is affinoid perfectoid and a $\underline{G}$-torsor over X. Then $|X| = |\tilde{X}|/\underline{G} = |\tilde{X}|^b/G = |X^\diamond|$, as desired. □

3.6 Étale Site for Diamonds

We now recall the étale site for diamonds.

Definition 3.6.1. ([12] Definition 10.1(ii)). A map $f : \mathcal{G} \to \mathcal{F}$ of pro-étale sheaves on $Perf$ is étale (resp. finite étale) if it is locally separated and for any perfectoid space X with a map $X \to \mathcal{F}$, the pullback $\mathcal{G} \times_{\mathcal{F}} X$ is representable by a perfectoid space Y étale (resp. finite étale) over X.

Remark 3.6.2. ([12] Proposition 11.7). Anything étale over Y is automatically itself a diamond [12]. If Y is a diamond, we consider the category $Y_{\text{ét}}$ of diamonds étale over Y and turn it into a site by declaring covers to be collections of jointly surjective maps.

Theorem 3.6.3. ([12] Theorem 10.42, Lemma 15.6). The functor $X \to X^{\diamond}$ from analytic pre-adic spaces over Z_p to diamonds induces an equivalence of sites $X_{\text{ét}} \simeq X^{\diamond}_{\text{ét}}$, restricting to an equivalence $X_{\text{fét}} \simeq X^{\diamond}_{\text{fét}}$.

Remark 3.6.4. ([17] Remark). Moreover, for any locally spatial diamond Y, one can define a quasi-pro-étale site $Y_{\text{qproét}}$ by looking at quasi-pro-étale maps into Y which are all diamonds. In particular, for any analytic pre-adic space X, we can define $X_{\text{qproét}} = X^{\diamond}_{\text{qproét}}$.

Proposition 3.6.5. ([12] Proposition 10.4.3). Let X be an analytic pre-adic space. The pullback functor

- $\nu_X^* : \tilde{X_{\text{ét}}} \to \tilde{X_{\text{qproét}}}$

from étale sheaves to quasi-pro-étale sheaves on X is fully faithful.

3.7 v-topology

Recall our earlier discussion of the v-topology. Fix a prime p on an analytic space X and let p be topologically nilpotent.

Definition 3.7.1. ([12] Definition 1.1). The v-topology, where a cover $\{f_i : X_i \to X\}$ consists of any maps $X_i \to X$ such that for any quasicompact open subset $U \subset X$, there are finitely many indices i and quasicompact open subsets $U_i \subset X_i$ such that the U_i jointly cover U.

Remark 3.7.2. The v-topology is the finest.

Theorem 3.7.3. ([12] Theorem 1.2). The v-topology on Perf is subcanonical, and for any affinoid perfectoid space $X = Spa(R, R^+)$, $H^0_v(X, \mathcal{O}^+_X) = R^+$ and for $i > 0, H^i_v(X, \mathcal{O}_X) = 0$ and $H^i_v(X, \mathcal{O}^+_X)$ is almost zero.

Remark 3.7.4. Recall the structure sheaf is a sheaf for the v-topology on Perfd.

Theorem 3.7.5. ([12] Theorem 8.7, Proposition 8.8). The functors $X \to H^0(X, \mathcal{O}_X)$ and $(X, \mathcal{O}_X)$ and $X \to H^0(X, \mathcal{O}^+_X)$ are sheaves on the v-site. Moreover if X is affinoid then $H^i_v(X, \mathcal{O}_X) = 0$ for $i > 0$, and $H^i_v(X, \mathcal{O}^+_X)$ is almost zero for $i > 0$.

Proof (Sketch [17]). Using pro-étale descent, it is sufficient to check the assertions for totally disconnected spaces. □

Representability follows.

Corollary 3.7.6. ([17] Corollary 17.1.5). Representable presheaves are sheaves on the v-site.

Proof. As in the pro-étale case (cf. Proposition 8.2.8), this follows from Theorem 17.1.3.□

Proposition 3.7.7. (Proposition 11.9). Let Y be a diamond. Then Y is a v-sheaf.

Descent was previously mentioned in Theorem 9.1.3 (i) in the pro-étale case. We have the following descent result.

Proposition 3.7.8. ([12] Proposition 9.3). The functor which assigns to a totally disconnected affinoid perfectoid X the category $\{Y/X$ affinoid perfectoid$\}$ is a stack for the v-topology.

Proof. (Sketch). This follows from Proposition 17.1.4 and faithfully flat descent, tending to subtle issues with rings of integral elements to which faithfully flat descent cannot be applied directly. □

There is an additional descent result on vector bundles.

Lemma 3.7.9. ([12] Lemma 17.1.8). The fibered category sending any $X \in$ Perfd to the category of locally finite free $\mathcal{O}_X$-modules is a stack on the v-site on Perfd.

Proof. Suppose $\tilde{X} \to X$ is a surjective morphism of perfectoid affinoids, with $X = Spa(R, R^+)$ and $\tilde{X} = Spa(\tilde{R}, \tilde{R}^+)$. The idea is to show that the base change functor from finite projective R-modules to finite projective $\tilde{R}$-modules equipped with a descent datum is an equivalence of categories. This is sufficient by Theorem 5.2.8. Full faithfulness follows from the sheaf property of the structure presheaf on the v-site per Theorem 17.1.3. □

Remark 3.7.10. Recall that all diamonds are v-sheaves, sheaves for the v-topology.

Proposition 3.7.11. ([12] Proposition 11.9). Let Y be a diamond. Then Y is a sheaf for the v-topology.

We now discuss small v-sheaves.

3.8 Small v-Sheaves

Definition 3.8.1. ([17] Definition 17.2.1). A v-sheaf $\mathcal{F}$ on Perf is small if there is a surjective map of v-sheaves $X \to \mathcal{F}$ from the sheaf represented by a perfectoid space X.

Any small v-sheaf admits geometric structure.

Proposition 3.8.2. ([17] Proposition 17.2.2 (Sch17, Proposition 12.3). Let $\mathcal{F}$ be a small v-sheaf, and let $X \to \mathcal{F}$ be a surjective map of v-sheaves from a diamond X (e.g., a perfectoid space). Then $R = X \times_{\mathcal{F}} X$ is a diamond, and $\mathcal{F} = X/R$ as v-sheaves.

Proof. (Sketch.) Note that $R \subset X \times X$ is a sub-v-sheaf, where $X \times X$ is a diamond. By Proposition 11.10, any sub-v-sheaf of a diamond is again a diamond. □

Remark 3.8.3. ([17] Remark). So to access v-sheaves takes two steps. First, we analyze diamonds as quotients of perfectoid spaces by representable equivalence relations. Second, then we analyze small v-sheaves as quotients of perfectoid spaces by diamond equivalence relations.

We define the underlying topological space of a small v-sheaf.

Definition 3.8.4. ([17] Definition 17.2.3). Let $\mathcal{F}$ be a small v-sheaf, and let $X \to \mathcal{F}$ be a surjective map of v-sheaves from a diamond X, with $R = X \times_{\mathcal{F}} X$. Then the underlying topological space of $\mathcal{F}$ is $|\mathcal{F}| = |X|/|R|$. This is well-defined and functorial by Proposition 12.7.

Remark 3.8.5. There exists a restricted class of diamonds, called spatial v-sheaves, with $|\mathcal{F}|$ well-behaved. We discuss these spatial diamonds and summarize their main properties.

3.8.1 Spatial v-sheaf

Diamonds for which $|\mathcal{F}|$ is well-behaved are defined as follows.

Definition 3.8.1.1. (Definition 17.3.1). A v-sheaf $\mathcal{F}$ is spatial if

- 1. $\mathcal{F}$ is qcqs (in particular, small), and
- 2. $|\mathcal{F}|$ admits a neighborhood basis consisting of $|\mathcal{G}|$, where $\mathcal{G} \subset \mathcal{F}$ is quasicompact open.

  We say $\mathcal{F}$ is locally spatial if it admits a covering by spatial open subsheaves [17].

Remark 3.8.1.2. ([17] Remark 17.3.2).

- For algebraic spaces, (1) implies (2); however (1) does not imply (2) in the context of small v-sheaves, or even diamonds. See Remark 17.3.6 below.
- If $\mathcal{F}$ is quasicompact, then so is $|\mathcal{F}|$. Indeed, any open cover of $|\mathcal{F}|$ pulls back to a cover of $\mathcal{F}$. However, the converse need not hold true, but it does when $\mathcal{F}$ is locally spatial; cf. [Sch17, Proposition 12.14 (iii)].
- If $\mathcal{F}$ is quasiseparated, then so is any subsheaf of $\mathcal{F}$. Thus if $\mathcal{F}$ is spatial, then so is any quasicompact open subsheaf.

Remark 3.8.1.3 [17].

We demand maps to be representable in locally spatial diamonds so that $Rf_!$ (from the six operations in étale cohomology of diamonds) commutes with all direct sums.

Example 3.8.1.4. ([17] Example 17.3.3). Let K be a perfectoid field in characteristic p, and let $\mathcal{F} = SpaK/Frob^Z$, so that $|\mathcal{F}|$ is one point. Then $\mathcal{F}$ is not quasiseparated. Indeed if $X = Y = SpaK$ (which are quasicompact), then $X \times_{\mathcal{F}} Y$ is a disjoint union of Z copies of $SpaK$. So $X \times_{\mathcal{F}} Y$ is not quasicompact. While $\mathcal{F}$ is not spatial, $\mathcal{F} \times SpaF_p((t^{\frac{1}{p^\infty}})) = (D_K^*/Frob^Z)^\diamond$ is spatial.

Proposition 3.8.1.5. ([17] Proposition 17.3.4) Let $\mathcal{F}$ be a spatial v-sheaf. Then $|\mathcal{F}|$ is a spectral space, and for any perfectoid space X with a map $X \to \mathcal{F}$, the map $|X| \to |\mathcal{F}|$ is a spectral map.

Proof. (Sketch showing $|\mathcal{F}|$ is spectral.) Choose a surjection $X \to \mathcal{F}$ from an affinoid perfectoid space. Let $R = X \times_{\mathcal{F}} X$. This is a qcqs diamond. By Proposition 11.20, it is spatial. Using Lemma 2.9 [12], it suffices to construct many quasicompact open subsets $U \subset |X|$ that are stable under the equivalence relation $|R|$. By Definition 17.3.1 (2), we take the preimages of $|\mathcal{G}|$ for $\mathcal{G} \subset \mathcal{F}$ quasicompact open. $\mathcal{G}$ is quasicompact and $\mathcal{F}$ is quasiseparated. Therefore, $\mathcal{G} \times_{\mathcal{F}} X \subset X$ is still quasicompact. Thus, $|\mathcal{G} \times_{\mathcal{F}} | \subset | \subset X|$ is a quasicompact open subset. □

We use the following proposition to check if a small v-sheaf is spatial.

Proposition 3.8.1.6. ([17] Proposition 17.3.5). Let X be a spectral space, and $R \subset X \times X$ a spectral equivalence relation such that each $R \to X$ is open and spectral. Then X/R is a spectral space, and $X \to X/R$ is spectral.

Remark 3.8.1.7. ([17] Remark 17.3.6). It is important to note that counterexamples to Proposition 17.3.5 exist for the case that $R \to X$ is generalizing but not open. For an example, take X and R are profinite sets. Then one can produce any compact Hausdorff space as X/R. For T any compact Hausdorff space, we can find a surjection $X \to T$ from a profinite set X (e.g., the Stone-Cech compactification of T considered as a discrete set). Then, $R \subset X \times X$ is closed and therefore profinite. If we repeat this construction in the world of diamonds, taking $SpaK \times \underline{X}/SpaK \times \underline{R}$, the result is a qcqs diamond $\mathcal{D}$ with $|\mathcal{D}| = T$.

The corollary is immediate.

Corollary 3.8.1.8. ([17] Corollary 17.3.7). Let $\mathcal{F}$ be a small v-sheaf. Assume there exists a presentation $R \rightrightarrows X \to F$, for R and X spatial v-sheaves (e.g., qcqs perfectoid spaces), and each $R \to X$ is open. Then $\mathcal{F}$ is spatial.

Proof. $\mathcal{F}$ is quasicompact, since X is quasicompact. Since R is quasicompact, $\mathcal{F}$ is quasiseparated. Proposition 17.3.5 shows that $|\mathcal{F}| = |X|/|R|$ is spectral, and $|X| \to |F|$ is spectral. We have that any quasicompact open $U \subset |F|$ defines an open subdiamond $\mathcal{G} \subset \mathcal{F}$ covered by $\mathcal{G} \times_{\mathcal{F}} X \subset X$, which is quasicompact. Therefore, $\mathcal{G}$ is quasicompact. □

Proposition 3.8.1.9. ([17] Proposition 17.3.8). If X is a qcqs analytic adic space over $SpaZ_p$, then $X^\diamond$ is spatial.

Proof. In finding a finite cover of X by affinoid perfectoid spaces, we find that $X^\diamond$ is also quasicompact. Proposition 10.3.7 implies that $|X^\diamond| \simeq X|$ has a basis of opens $|U|$, where $U \subset X$ is quasicompact open. Per Proposition 10.3.6, these correspond to open subdiamonds $U^\diamond \subset X^\diamond$.

We arrive at the culminating theorem, which states that as soon as its points are nice and sufficiently so, a spatial v-sheaf is a diamond.

Theorem 3.8.1.10. ([17] Theorem 17.3.9). Let $\mathcal{F}$ be a spatial v-sheaf. Assume that for all $x \in |\mathcal{F}|$, there is a quasi-pro-étale map $X_x \to \mathcal{F}$ from a perfectoid space X_x such that x lies in the image of $|X_x| \to |\mathcal{F}|$. Then $\mathcal{F}$ is a diamond.

Proof. (Sketch). It is sufficient to find a quasi-pro-étale surjection $X \to \mathcal{F}$ from an affinoid perfectoid space X. We check that all connected components are indeed representable. Let $K \subset \mathcal{F}$ be a connected component. Then K has a unique closed point x. Now let $X_x = Spa(C, C^+) \to \mathcal{F}$ be a quasi-pro-étale morphism such that x lies in the image of $|X_x|$. Let C be an algebraically closed nonarchimedean field and $C^+ \subset C$ is an open and bounded valuation subring. Then the image of $|X_x|$ is exactly K. The map $X_x \to \mathcal{F}$ is an isomoprhism. In summary, every connected component of $\mathcal{F}$ truly is a geometric "point" of the form $Spd(C, C^+)$. Finally, one shows that using Lemma 12.21 [12], the implication is $\mathcal{F}$ itself is representable. □

Additionally we have the following characterization of spatial diamond.

Definition 3.8.1.11. ([12] Definition 1.4). A diamond Y is spatial if it is quasicompact and quasiseparated (qcqs), and $|Y|$ admits a basis for the topology given by $|U|$, where $U \subset Y$ ranges over quasicompact open subdiamonds. More generally, Y is locally spatial if it admits an open cover by spatial diamonds.

Remark 3.8.1.12. ([12] Remark). Any perfectoid space X defines a locally spatial diamond. This diamond is spatial precisely when X is qcqs. We see this as follows. If X is a (locally) spatial diamond, then $|X|$ is a (locally) spectral topological space, and X is quasicompact (resp. quasiseparated) as a v-sheaf precisely when $|X|$ is quasicompact (resp. quasiseparated) as a topological space. Now, if X is a locally spatial diamond, we define $X_{\text{ét}}$ as consisting of (locally separated) étale maps $\mathcal{D} \to X$ from diamonds $\mathcal{D}$ (automatically locally spatial).

Remark 3.8.1.13. The property that a diamond $\mathcal{D}$ is locally spatial guarantees that $|\mathcal{D}|$ is a locally spectral space.

Lemma 3.8.1.14. ([12] Lemma 11.27). Let Y be a spatial diamond. Assume that every connected component of Y is representable by an affinoid perfectoid space. Then Y is representable by an affinoid perfectoid space.

We have two permanence properties.

Corollary 3.8.1.15. ([12] Corollary 11.28). Let Y be a locally spatial diamond, and $Y' \to Y$ a quasi-pro-étale map of pro-étale sheaves. Then $Y'$is a locally spatial diamond.

Corollary 3.8.1.16. ([12] Corollary 11.29). A fibre product of (locally) spatial diamonds is (locally) spatial.

Proposition 3.8.1.17. ([12] Proposition 11.26). Let Y be a spatial diamond. Assume that any surjective étale map $\tilde{Y} \to Y$ that can be written as a composite of quasicompact open immersions and finite étale maps splits. Then Y is a strictly totally disconnected perfectoid space.

Proof (Sketch). Any connected component $Y_0 \subset Y$ admits a pro-étale surjection $Spa(C, C^+) \to Y_0$ for some algebraically closed field C with an open and bounded valuation subring $C^+ \subset C$. □

Lemma 3.8.1.18. ([12] Lemma 11.27). Let Y be a spatial diamond. Assume that every connected component of Y is representable by an affinoid perfectoid space. Then Y is representable by an affinoid perfectoid space.

Fargues suggests the following definition of quasi-pro-étale morphisms.

Proposition 3.8.1.19. ([12] Proposition 13.6). Let $f : Y' \to Y$ be a separated map of v-stacks. Then f is quasi-pro-étale if and only if it is representable in locally spatial diamonds and for all complete algebraically closed fields C with a map $Spa(C, \mathcal{O}_C) \to Y$, the pullback $Y' \times_Y Spa(C, \mathcal{O}_C) \to Spa(C, \mathcal{O}_C)$ is pro-étale.

Spatial morphisms are now discussed.

Definition 3.8.1.20. ([12] Definition 13.1). A map $f : Y' \to Y$ of v-stacks is representable in diamonds if for all diamonds X with a map $X \to Y$, the fibre product $Y' \times_Y X$ is a diamond.

We ensure this notion is well-behaved.

Proposition 3.8.1.21. ([12] Proposition 13.2). Let $f : Y' \to Y$ and $\tilde{Y} \to Y$ be maps of v-stacks, with pullback $\tilde{f} : \tilde{Y}' = Y' \times_Y \tilde{Y} \to \tilde{Y}$.

- If Y is a diamond, then f is representable in diamonds if and only if Y' is a diamond.
- If f is representable in diamonds, then $\tilde{f}$ is representable in diamonds.
- If $\tilde{Y} \to Y$ is surjective as a map of pro-étale stacks and $\tilde{f}$ is representable in diamonds, then f is representable in diamonds.

The definition of locally spatial morphisms follows.

Definition 3.8.1.22. ([12] Definition 13.3). A map $f : Y' \to Y$ of v-stacks is representable in (locally) spatial diamonds if for all (locally) spatial diamonds X with a map $X \to Y$, the fibre product $Y' \times_Y X$ is a (locally) spatial diamond.

We compare the pro-étale and v-cohomology.

Definition 3.8.1.23. ([12] Definition 14.1). Assume that Y is a diamond.

- The quasi-pro-étale site $Y_{\text{qproét}}$ is the site whose objects are (locally separated) quasi-pro-étale maps $Y' \to Y$, with coverings given by families of jointly surjective maps.

  The v-site Y_v is the site whose objects are all maps $Y' \to Y$ from small v-sheaves Y', with coverings given by families of jointly surjective maps.

Remark 3.8.1.24. The topoi are algebraic.

Proposition 3.8.1.25. ([12] Proposition 14.2). The topoi $Y_{\text{ét}}$ respectively, $Y_{\text{qproét}}$ respectively Y_v for a locally spatial diamond respectively diamond respectively small v-stack Y are algebraic. If Y is 0-truncated (i.e., if Y is a small v-sheaf), then an object is quasicompact respectively quasiseparated if and only if it quasicompact respectively quasiseparated as a small v-stack on Perf.

Remark 3.8.1.26. We summarize the main properties and theorems of the category of diamonds before introducing the left-completed derived category $\mathcal{D}_{ét}(X, \Lambda)$ of étale sheaves of Λ-modules on $X_{ét}$, where to X we will associate an étale site $X_{ét}$ and Λ will be a ring such that $n\Lambda = 0$ for some n prime to p.

Proposition 3.8.1.27. ([12] Proposition 11.4). The category of diamonds has all fiber products, cofiltered inverse limits (and even all non-empty limits), but no final object.

3.9 $\mathcal{D}^\diamond$ and the Six Operations

Scholze constructs a six functor formalism on the étale cohomology of diamonds. Our exposition and souscompendium follows [12].

The Grothendieck six operations formalism formalizes the ”refinement” of Poincaré duality to abelian sheaf cohomology [27].

Terminology 3.9.1 [12]. Fix a prime p. Let X be an analytic adic space on which p is toplogically nilpotent. To X was associate an étale site $X_{\acute{e}t}$. Let Λ be a ring such that $n\Lambda = 0$ for some n prime to p. There exists a left-completed derived category $\mathcal{D}_{\acute{e}t}(X, \Lambda)$ of étale sheaves of Λ-modules on $X_{\acute{e}t}$. Let $Perfd$ be the category of perfectoid spaces and $Perf$ be the subcategory of perfectoid spaces of characteristic p. Consider the v-topology on Perf [5].

Recall the definition of a diamond.

Definition 3.9.2. ([12] Definition 1.3). A diamond is a pro-étale sheaf $\mathcal{D}$ on $Perf$ which can be written as the quotient X/R of a perfectoid space X by a pro-étale equivalence relation $R \subset X \times X$.

Definition 3.9.3. ([12] Definition 1.7). Let X be a small v-stack, and consider the site X_v of all perfectoid spaces over X, with the v-topology. Define the full subcategory $\mathcal{D}_{\acute{e}t}(X, \Lambda) \subset \mathcal{D}(X_v, \Lambda)$ as consisting of all $\mathcal{A} \in \mathcal{D}(X_v, \Lambda$ such that for all (equivalently, one surjective) map $f : Y \to X$ from a locally spatial diamond Y , $f^*\mathcal{A}$ lies in $\hat{\mathcal{D}}(Y_{\acute{e}t}, \Lambda)$.

$\mathcal{D}_{\acute{e}t}(X, \Lambda)$ contains the following six operations.

1. Derived Tensor Product. -$\bigotimes^{\mathbb{L}}_{\Lambda}$ - : $D_{\acute{e}t}(X, \Lambda) \times D_{\acute{e}t}(X, \Lambda) \to D_{\acute{e}t}(X, \Lambda)$.
2. Internal Hom. $R\mathcal{H}om_{\Lambda}(-, -) : D_{\acute{e}t}(X, \Lambda)^{op} \times D_{\acute{e}t}(X, \Lambda) \to D_{\acute{e}t}(X, \Lambda)$.
3. For any map $f : Y \to X$ of small v-stacks, a pullback functor $f^* : D_{\acute{e}t}(X, \Lambda) \to D_{\acute{e}t}(Y, \Lambda)$.
4. For any map $f : Y \to X$ of small v-stacks, a pushforward functor $\mathcal{R}f_* : D_{\acute{e}t}(Y, \Lambda) \to D_{\acute{e}t}(X, \Lambda)$.
5. For any map $f : Y \to X$ of small v-stacks that is compactifiable, representable in locally spatial diamonds, and with dim.trg $f < \infty$ functor $\mathcal{R}f_! : D_{\acute{e}t}(Y, \Lambda) \to D_{\acute{e}t}(X, \Lambda)$.

[5] The v-topology, where a cover $\{f_i : X_i \to X\}$ consists of any maps $X_i \to X$ such that for any quasicompact open subset $U \subset X$, there are finitely many indices i and quasicompact open subsets $U_i \subset X_i$ such that the U_i jointly cover U [12]

6. For any map $f : Y \to X$ of small v-stacks that is compactifiable, representable in locally spatial diamonds, and with dim.trg $f < \infty$, a functor $\mathcal{R}f^! : D_{ét}(X, \Lambda) \to D_{ét}(Y, \Lambda)$.

Remark 3.9.4 [12]. Recall that for any small v-stack Y, we have defined the full subcategory $D_{\text{ét}}(Y, \Lambda) \subset D(Y_v, \Lambda)$.

Lemma 3.9.5. ([12] Lemma 17.1). There is a (natural) presentable stable ∞-category $\mathcal{D}_{\text{ét}}(Y, \Lambda)$ whose homotopy category is $D_{\text{ét}}(Y, \Lambda)$. More precisely, the ∞-derived category $\mathcal{D}(Y_v, \Lambda)$ of Λ-modules on Y_v is a presentable stable ∞-category, and $\mathcal{D}_{\text{ét}}(Y, \Lambda)$ is a full presentable stable ∞-subcategory closed under all colimits.

Proof [12]. $\mathcal{D}(Y_v, \Lambda)$ is a presentable stable ∞-category. This is true for any ringed topos. Next, we check that the full ∞-subcategory $\mathcal{D}_{\text{ét}}(Y, \Lambda)$, with objects those of $D_{\text{ét}}(Y, \Lambda)$, is closed under all colimits in $\mathcal{D}(Y_v, \Lambda)$. This is clear for cones, so we are reduced to filtered colimits. Those commute with canonical truncations, and filtered colimits of étale sheaves are still étale sheaves, as desired. By [Lur09, Proposition 5.5.3.12], it is enough to prove the claim if Y is a disjoint union of strictly totally disconnected perfectoid spaces. In that case, $\mathcal{D}_{\text{ét}}(Y, \Lambda) = \mathcal{D}(Y_{\text{ét}}, \Lambda)$ (as the functor of stable ∞-categories $\mathcal{D}(Y_{\text{ét}}, \Lambda) \to \mathcal{D}(Y_v, \Lambda)$ is fully faithful (as it is on homotopy categories), and has the same objects as $\mathcal{D}_{\text{ét}}(Y, \Lambda)$, which is a presentable ∞-category. □

3.10 Relative Fargues-Fontaine Curve $\mathcal{Y}_{S,E}^{\diamond} = S \times (\mathbf{Spa}\mathcal{O}_E)^{\diamond}$

We now discuss a very important diamond incarnation, on which the conjectured geometrization of the local Langlands correspondence depends, the relative Fargues-Fontaine Curve. Our exposition follows [9].

The set up is the following:

> Let E be an non-archimedean local field, i.e. $E = Fq((t))$ or a finite extension of Q_p. F_q is the residue field of E. $\pi \in O_E$ is a uniformizer. G/E is a reductive group. Fix a choice of algebraic closure $\bar{K}/K$. The Galois group $Gal(\bar{K}/K)$ is denoted $G(K)$, and we write $\mathbf{C}_K$ to denote the completion $\hat{\bar{K}}$ of K endowed with its unique absolute value extending the given absolute value $|\cdot|$ on K [9].

Recall the definition of a v-map.

Definition 3.10.1 [9]. A v-map is a map $f : X \to Y$ of quasi-compact, quasi-separated schemes such that for any map $v : \mathrm{Spec}(V) \to Y$, where V is a valuation ring, there is an extension (of valuation rings) $V \subset W$ and a map Spec W $\to X$ lifting v.

Remark 3.10.2 [9].

> The goals are to study the representation theory of the locally profinite group $G(E)$, to study Spec E and "make Spec E geometric," and to formalize local Langlands as an equivalence of categories [9].

Remark 3.10.3 [9]. Let $E = F_q((t))$ and $\breve{E} = \bar{F}_q((t))$. $Spec\bar{F}_q((t))$ is viewed as a formal punctured open unit disc over $\bar{F}_q$. To make more space, since this only has one point, we pass to an extension C/F_q. This is a complete algebraically closed nonarchimedean field, $C = \hat{\overline{F_q((u))}}$.

Remark 3.10.4 [9].

> The adic spectrum $SpaC \times_{\mathrm{Spa}F_q} \mathrm{Spa}F_q((t))$ is the punctured open unit disk over C, which now has many points.
>
> - $D_C^* = \{x | 0 < |x| < 1\}$.
> - The event of base changing to C passes to a geometric situation.

Definition 3.10.5 (The equal characteristic Fargues-Fontaine Curve). ([9] Definition 2.2). The Fargues-Fontaine Curve for E, C is $X_{C,E} = D_C^* / \phi_C^{\mathbf{Z}}$. This is an adic space over E.

Recall that adic spaces are certain variants of schemes associated to topological rings.

Definition 3.10.6. ([9] Definition 2.15). An adic space is a triple $(X, \mathcal{O}_X, \mathcal{O}_X^+)$ with:

- X is a topological space,
- $\mathcal{O}_X$ is a sheaf of complete topological rings,
- $\mathcal{O}_X^+$is a subsheaf of $\mathcal{O}_X$

that is locally of the form $(\text{Spa}(A, A^+), \mathcal{O}_A, \mathcal{O}_A^+)$, plus the correct analog of "locally ringed."

Remark 3.10.7 [9]. To construct the mixed characteristic Fargues-Fontaine curve, the set up is the following: Let E be a finite extension of Q_p and C/F_q a complete algebraically closed nonarchimedean field.

> Scholze asks what is the meaning of $\text{Spa}E \times_{SpaF_q} \text{Spa}C$, given that there is no map $SpaE \to \text{Spa}F_q$, as E is characteristic 0? The idea is to deform any F_q-algebra R to mixed characteristic via p-typical ramified Witt vectors defined as
>
> - $\widetilde{R} = W_{\mathcal{O}_E}(R) = W(R) \otimes_{W(F_q)} \mathcal{O}_E$.
>
> The analogue of $\text{Spa}F_q((t)) \times_{\text{Spa}F_q} \text{Spa}C$ in mixed characteristic is
>
> - $SpaE \times_{\text{Spa}F_q} \text{Spa}C^m = Y_{C,E} = \{\pi \neq 0, [\iota] \neq 0\} \subset SpaW_{\mathcal{O}_E}(\mathcal{O}_C)$,
>
> still carries an action of ϕ_C that is free and totally discontinuous.

Definition 3.10.8 (The mixed characteristic Fargues-Fontaine curve). ([9] Definition). The Fargues-Fontaine curve is $X_{C,E} = Y_{C,E}/\phi_C^{\mathbf{Z}}$. This is an adic space and lives over $\text{Spa}E$.

What is most important to us is that

> the classical points $X_{C,E}^{cl} \subset |X_{C,E}|$ correspond to untilts C^*/E of C up to $\phi_C^{\mathbf{Z}}$ [9].

Recall the goal is to classify vector bundles on $X_{C,E}$ with the following conjecture (hope).

Conjecture 3.10.9. ([21] Hope 2.3). Bun_G is a "smooth diamond stack."

Remark 3.10.10. The major result is that isocrystals give rise to vector bundles and all vector bundles arise in this way.

Recall, an isocrystal is a pair (V, ϕ) where V is a finite-dimensional $\hat{E}$-vector space, and $\phi_V : V \tilde{\to} V$ is a ϕ_E-linear automorphism. Isocrystals form an E-linear $\otimes$-category $Isoc_E$ [9].

Line bundles and vector bundles semistable of slope 0 on $\mathcal{X}_{C,E}$ are classified. This classification relies on putting a geometric structure on $H^0(X_{C,E}, \mathcal{E})$ for $\mathcal{E} \in VB(X_{C,E})$ and for $H^1(X_{C,E}, \mathcal{E})$. These geometric structures are the same Banach-Colmez spaces, i.e. diamonds, earlier defined (cf. [9] Theorem 15.2.12) [9].

We now discuss the relative Fargues-Fontaine curve $X_{S,E}$ for a perfectoid space S. The set up is the following: Recall that

a perfectoid algebra over F_p is a perfect Tate algebra R over F_p and so it has a topologically nilpotent unit $\bar{\omega}$. Recall, a perfectoid space over F_p is an adic space X/F_p covered by opens $U = \text{Spa}(R, R^+) \subset X$ where R is perfectoid. If $S = \text{Spa}(R, R^+)$ is affinoid perfectoid, we will mimic the earlier construction of the Fargues-Fontaine curve.

Remark 3.10.11 [9]. The construction is the following.

- Replace $\mathcal{O}_C$ by R^+.
- Form $\text{Spa}W_{\mathcal{O}_E}(R^+)$.
- Consider the open subspace $Y_{(R,R^+),E} := \{[\bar{\omega}] \neq 0, \pi \neq 0\} \subset \text{Spa}W_{\mathcal{O}_E}(R^+)$. This is an analytic adic space over E, so it is locally the adic spectrum of a Tate ring.
- Consider $\mathcal{Y}_{(R,R^+),E} = \{[\bar{\omega}] \neq 0\}$. This is an analytic adic space over $\mathcal{O}_E$, so it is locally the adic spectrum of a Tate ring.
- We have the inclusions: $\text{Spa}W_{\mathcal{O}_E}(R^+) \supset \mathcal{Y}_{(R,R^+),E} \supset Y_{(R,R^+),E}$.

Remark 3.10.12 [9]. There exists a radius function rad : $\mathcal{Y}_{(R,R^+),E} \to [0, \infty)$ which sends $y \to \frac{log|\bar{\omega}(\tilde{y}|}{log|\pi(\tilde{y})|}$ for $\tilde{y}$ the maximal rank 1 generalization of y. It sends $\mathcal{Y}_{(R,R^+),E} \to (0, \infty)$.

ϕ_{R^+} is an endomorphism which acts freely and totally discontinuously on $\mathcal{Y}_{(R,R^+),E}$. Taking the quotient by ϕ_R^Z we have the curve.

Definition 3.10.13. ([9] The relative Fargues-Fontaine curve (Definition 5.1). The relative Fargues-Fontaine curve is defined as

- $X_{(R,R^+),E} = Y_{(R,R^+),E}/\phi_R^Z$.

This is an adic space over E.
We now attain the diamond version of $X_{(R,R^+)}, E$ by considering an example.

Example 3.10.14. ([9] Example 5.2). $E = F_q((t))$. $W_{\mathcal{O}_E}(R^+) = R^+[[t]]$. Therefore

- $\mathcal{Y}_{(R,R^+),E} = \mathrm{Spa}(R, R^+) \times_{\mathrm{Spa}F_q} \mathrm{Spa}F_q[[t]] = \mathbb{D}_{\mathrm{Spa}(R,R^+)}$

is the open unit disc which is not quasicompact and is exemplary of fibered products of affinoids not being affinoid.
The subspace

- $Y_{(R,R^+),E} = \mathbb{D}^*_{\mathrm{Spa}(R,R^+)}$

is the punctured open unit disc.

$Y_{(R,R^+),E}$ and $\mathcal{Y}_{(R,R^+),E}$ have a structure map over $\mathrm{Spa}(R, R^+)$, but not after taking the quotient by $\phi_{R^+}^Z$.

The construction glues and there is a diamond equation.

- **Diamond equation 3.10.15**. ([9]). $\mathcal{Y}_{S,E}^{\diamond} = S \times (\mathrm{Spa}\mathcal{O}_E)^{\diamond}$.

The diagram of diamonds glues.

Example 3.10.16. ([9] Example 6.7). Fix a geometric base point $S = \mathrm{Spa}(C, \mathcal{O}_C)$. There is a fully faithful embedding ProFinSet $\hookrightarrow \mathrm{Perf}_S$, constructed by sending $T = \underset{\leftarrow i}{lim} T_i$ to

[6]The colors in the diamond are mathematical impurities reflecting the profinitely many copies of Spa C upon the pullback of the geometric point Spa $C \to \mathcal{D}$ through a quasi-pro-étale cover $X \to \mathcal{D}$, for C an algebraically closed affinoid field and $\mathcal{D}$ a diamond, multiple such covers $X \to \mathcal{D}$ of which are produced.

$$
\begin{array}{ccc}
\mathcal{Y}^{\diamond}_{S,E} & = & S \times (\mathrm{Spa}\,\mathcal{O}_E)^{\diamond} \\
\uparrow & & \uparrow \\
Y^{\diamond}_{S,E} & = & S \times (\mathrm{Spa}\,E)^{\diamond} \\
\downarrow & & \downarrow \\
X^{\diamond}_{S,E} & = & Y^{\diamond}_{S,E}/\phi_S^{\mathbb{Z}}
\end{array}
$$

Figure 4: Gluing Construction for Diamonds[6]

- $\underline{T} \times \mathrm{Spa}(C, \mathcal{O}_C) = \underset{\leftarrow i}{lim}(T_i \times \mathrm{Spa}(C, \mathcal{O}_C)$.

  This could likewise be written as $\mathrm{Spa}(Cont(T, C), Cont(T, \mathcal{O}_C))$ ensuring profinite sets are identified with the affinoid pro-étale spaces over S.

Recall the notion of a locally spatial diamond.

Definition 3.10.17. ([12] Definition 12.12). We say Y is locally spatial if it has an open cover by spatial $U \subset Y$, which implies that $|Y|$ is locally spectral. All relevant diamonds are locally spatial, because Y is spatial if and only if Y is locally spatial and $|Y|$ is qcqs [9].

Remark 3.10.18. ([12] Remark 6.9). We note that the category of locally spatial diamonds has all fiber products and all cofiltered inverse limits with qcqs transition maps. We note that qcqs automatically implies the spectrality, for algebraic spaces, because étale maps are open.

We conclude with some facts about the structure of locally spatial diamonds and the diamond functor.

Proposition 3.10.19. ([12] Proposition 11.19). Let Y be a locally spatial diamond. Then it has an underlying locally spectral space $|Y|$. For every $y \in |Y|$ we have a localization $Y_y \subset Y$ of Y at y, which is $\underset{\leftarrow y \in U}{lim} U$. It comes with a presentation $Y_y = Spa(C, C^+)/\underline{G}$.

Recall, C is a complete algebraically closed nonarchimedean field, $m_{\mathcal{O}_C} \subset C^+ \subset \mathcal{O}_C$ is a valuation subring, and G is a profinite group acting continuously and faithfully on C.

There is a formal similarity between the action of G to C and a Galois action.

Diamond functor 3.10.20. ([9] Definition 6.4). The diamond functor

- $\{\text{analytic adic spaces}/Z_p\} \to \{\text{diamonds}\}$

is constructed by the following proposition, our culminating proposition.

Proposition 3.10.21. ([9] Proposition 6.11). For an analytic adic space X/Z_p, the diamond functor

- $X^\diamond : S \in Perf \to \{S^\#/Z_p \text{ untilts of } S \text{ plus map } S^\# \to X\}$

defines a locally spatial diamond. There are canonical equivalences $|X| \simeq |X^\diamond|$ and $X_{\text{ét}} \simeq X^\diamond_{\text{ét}}$. With X perfectoid, $X^\diamond \simeq X^b$.

Remark 3.10.22 [9].

> If X is an adic space over Q_p, one can define a diamond $X^\diamond$ whose S-valued points are given by an untilt $S^\#$ over Q_p together with a map $S^\# \to X$.

We see this in the diamond version of the relative Fargues-Fontaine curve. Let $S = Spa(R, R^+) \in \text{Perf}/F_q$.

Theorem 3.10.23. ([9] Theorem 6.13). $Y^\diamond_{S,E} = S \times (SpaE)^\diamond$. This means that given a perfectoid T/F_q, an untilt $T^\#/Y_{S,E}$ is the same as an untilt $T^\#/E$ plus a map $T \to S$, where untilts are degree-one Cartier divisors on $Y_{S,E}$.

We have a classification of G-bundles on the Fargues-Fontaine curve. Per the usual notation we choose $\bar{F}_q$ to give $\breve{E} = W_{\mathcal{O}_E}(\bar{F}_q)[\frac{1}{\pi}]$.

Definition 3.10.24. ([9] Definition 11.20). Let $S \in \text{Perf}_{\bar{F}_q}$. A G-torsor on X_S is an exact $\otimes$-functor $\mathcal{E} : \text{Rep}_E(G) \to VB(X_S)$.

Definition 3.10.25. ([9] Definition 11.21). We define the stack of G-bundles on the Fargues-Fontaine curve Bun_G to be the v-stack on $\text{Perf}_{\bar{F}_q}$ sending $S \to \{G\text{-bundles on } X_S\}$. This is a v-stack, which follows from v-descent for vector bundles.

A warning follows.

Warning 3.10.26. ([9] Warning 11.22). There is no such thing as "the" Fargues-Fontaine curve, as the construction of $X_{C,E}$ depends on some input field C. But the "stack of G-bundles on the Fargues-Fontaine curve" is well-defined.

We arrive at our two culminating theorems.

Theorem 3.10.27. ([9] Theorem 11.23 (Fargues if E/Q_p, Anschütz in general)). If $S = Spa(C, C^+)$ where C is a complete algebraically closed field, then the functor

- $G \to Isoc \to Bun_G(S)$

sending a G-torsor on $Spa\breve{E}/\sigma^Z$ to its pullback to $Y_S/\phi^Z = X_S$ induces a bijection on isomorphism classes $Bun_G(S)/\sim \tilde{\to} B(G)$ which is even a homeomorphism $|Bun_G(S)| \simeq B(G)$.

Let G/E be a reductive group. The moduli stack Bun_G of G-bundles on the Fargues-Fontaine curve represents the functor taking

- $S \in \text{Perf}\bar{F}_q \to \{G\text{-bundles on } X_S\}$.

Theorem 3.10.28. ([9] Theorem 16.1).

- 1. Bun_G is an Artin v-stack, cohomologically smooth of dimension 0.
- 2. The map $|Bun_G| \to B(G)$ is a continuous bijection.
- 3. For any $b \in B(G)$, we get a locally closed stratum $|Bun_G^b| \subset Bun_G$. It has the form $Bun_G^b = [*/\mathcal{G}_b]$ where $\mathcal{G}_b$ fits into a short exact sequence
- $1 \to \underbrace{\text{unipotent group diamond}}_{\text{iterated extension of positive Banach-Colmez spaces}} \to \mathcal{G}_b \to G_b(E) \to 1.$

3.11 Geometrization of the Local Langlands Correspondence

We now give a brief summary of Scholze and Fargues' geometrization of the local Langlands correspondence. Our exposition follows [9]. For complete details, see [21] and [9].

We fix a discrete Langlands parameter $\phi : W_E \to {}^L G$ where ${}^L G$ is the ℓ-adic Langlands dual of G, $\ell \neq p$ and W_E is the Weil group of E. Let S be an F_q-perfectoid space. The geometrization conjecture associates to ϕ an S_ϕ-equivariant Hecke eigensheaf $\mathcal{F}_\phi$ on the stack of G-bundles over the Fargues-Fontaine curve [21].

Remark 3.11.1. To define Hecke correspondences, we recall the following [21].

- $\mathcal{F}_\phi$ has the *character sheaf property.*
- $\mathcal{F}_\phi$ is a Weil-perverse sheaf on $Bun_G \otimes \bar{F}_q$.
- $Spa(E)^\diamond = Spa(E)$ when $E = F_q((\pi))$;
- $Spa(E)^\diamond$ is the sheaf of untilts when $E|Q_p$.
- Any morphism $S \to Spa(E)^\diamond$ defines a Cartier divisor $S^\# \hookrightarrow X_S$ where $S^\# = S$ if $E = F_q((\pi))$.
- $S^\#$ is the corresponding untilt of S when $E|Q_p$.
- When $S = Spa(R, R^+)$, the formal completion of X_S along this Cartier divisor is the formal spectrum of Fontaine's ring $B^+_{dR}(R^\#)$.

The goals of the geometrization of the local Langlands correspondence are to:

> study the representation theory of the locally profinite group $G(E) = Gal(\bar{E}/E)$; give a construction of the map $\pi \to \phi_\pi$ from irreducible representations to L-parameters, which is purely local and works uniformly for any reductive group G; and formulate a more structured form of the local Langlands correspondence as an equivalence of categories using perfectoid geometry.

The set up is the following: E is a non-archimedean local field such as $F_q((t))$ or a finite extension of Q_p); F_q is the residue field of E; $\pi \in \mathcal{O}_E$ is a uniformizer; and G/E is a reductive group.

Recall the following definition.

Definition 3.11.2. ([9] Definition 1.1). Let Γ be a locally profinite group. A smooth representation of Γ over a field L is an L-vector space V plus a map $\Gamma \to GL(V)$ such that for all $v \in V$, $\text{Stab}(v) \subset \Gamma$ is open.

Recall the local Langlands correspondence.

Conjecture 3.11.3. (Local Langlands correspondence ([9] Conjecture 1.6)). Consider representations over $L = \mathbf{C}$. There exists a natural map

- $\text{Irrep}(G, E))/\sim \to \text{Hom}(W_E, \hat{G}(\mathbf{C}))/G(\hat{\mathbf{C}})$

where $\hat{G}$ is the Langlands dual group, W_E is the Weil group of E which is surjective with finite fibers (called L-packets) defined as the pre-image of $Z \subset \hat{Z}$ under the surjection $Gal(\bar{E}/E) \to \bar{Z}$ corresponding to the maximal unramified extension of E.

Remark 3.11.4. ([9] Remark 1.7). There are conjectural improvements which describe the whole category $\text{Rep}(G(E))$ in terms of coherent sheaves on the Artin stack $\text{Hom}(W_E, \hat{G}(\mathbf{C})/\hat{G}(\mathbf{C})$.

An example of the local Langlands correspondence is the following.

Example 3.11.5. ([9] Example 1.4.1). Let $G = G_m$. Then $G(E) = E^\times$ is abelian, so

- $\text{Irrep}(E^\times) = \{\chi : E^\times \to \mathbf{C}^x\}$

is the set of characters. The dual group is $\hat{G} = G_m$. Then

- $\text{Hom}(W_E, \hat{G}(\mathbf{C})) = \text{Hom}(W_E, \mathbf{C}^x) = \text{Hom}(W_E^{ab}, \mathbf{C}^x)$,

recalling that local class field theory identifies

- $E^\times \xrightarrow{\sim} W_E^{ab}$.

Remark 3.11.6. ([9] Section 1.7). To study $\text{Rep}(G(E))$, we investigate Spec E via either its étale site where SpecE is controlled by $\pi_1^{\prime et}(\text{SpecE}) = Gal(\bar{E}/E)$ or via coherent sheaves which are E-vector spaces, and "make SpecE more geometric."

Remark 3.11.7. ([9] Section 1.7). The Zariski site of Spec E is a point and the étale site of E is the category of finite sets with continuous $Gal(\bar{E}/E)$-action. Neither are very geometric.

In general we consider the groupoid of G-torsors.

- $[\mathrm{pt}/G](SpecE) = \amalg_{\alpha \in H^1(E,G)} [\mathrm{pt}/G_\alpha(E)]$

  where $G_\alpha(E)$ is the inner form corresponding to the torsor α, $[\mathrm{pt}/G(E)]$ is an open and closed substack, and

- $\mathrm{Rep}(G(E)) = \mathrm{Shv}([\mathrm{pt}/G(E)])$

  is embedded in a natural way into the larger category

- $\mathrm{Shv}([\mathrm{pt}/G](\text{Spec E}))$ [9].

Remark 3.11.8. ([9] Section 1.7). A series of modifications occurs. $Gal(\bar{E}/E)$ is changed to W_E. For a scheme X/F_q, we replace X by $X_{\bar{F}_q}$ its Frob_q-action. Then

- $\pi_1(X_{\bar{F}_q}/\mathrm{Frob}_q) = \pi_1(X_{\bar{F}_q}) \rtimes Z$

is the Weil form of the fundamental group. Next, we replace $\mathrm{Spec}F_q((t))$ by $\mathrm{Spec}\bar{F}_q((t))/Frob_q$ and Spec E by $\mathrm{Spec}\hat{E}/\mathrm{Frob}_q$ where $\hat{E}$ is the completion of the maximal unramified extension of E.

- $\pi_1(\mathrm{Spec}\hat{E}/\mathrm{Frob}_q) = W_E$

controls the étale site while, on the coherent side, vector bundles are by descent the category of isocrystals.

- $\mathrm{Isoc}_E = \{\hat{E} - \text{vector spaces} + \text{Frob-semilinear}\phi : V \tilde{\rightarrow} V\}$.

Remark 3.11.9. ([9] Section 1.7). These structures are much richer than E-vector spaces. By the Dieudonne-Manin Theorem, Isoc_E decomposes into

- $\bigoplus_{\lambda \in \mathbf{Q}} \mathrm{Isoc}_E^\lambda$

where Isoc_E^λ consists of isocrystals which are of pure slope λ. The category $Isoc_E^\lambda$ is identified with D_λ-modules, where D_λ/E is the central division algebra of an invariant $|\lambda| \in Q/Z$.

Remark 3.11.10. ([9] Section 1.7). In general, we consider the category of G-torsors in Isoc_E. This was classified by Kottwitz and he showed that the groupoid of G-isocrystals is

- $\amalg_{b \in B(E,G)}[pt/G_b(E)]$

for G_b an inner form of a Levi subgroup of G. There exists an injection $H^1(E,G) \hookrightarrow B(E,G)$ and it is advantageous to consider all the G_b simultaneously.

Remark 3.11.11. ([9] Section 1.8). The idea is to upgrade

- $[pt/G](SpecE) = \amalg_{\alpha \in H^1(E,G)}[pt/G_\alpha(E)]$

to something more geometric, a stack of G-isocrystals. There is a way to do this using perfectoid rings, but we first discuss perfect algebras.

Remark 3.11.12. ([9] Section 1.8.2). The idea is the following. Replace $\bar{F}_q$ by any perfect $\bar{F}_q$-algebra R, which means

- replace $Spec\bar{F}_q((t))/Frob_q$ with $SpecR((t))/Frob_q$ and
- replace $Spec\breve{E}/Frob_q$ with $Spec(W(R)) \otimes_{W(\bar{F}_q)} E)/Frob_q$.

Definition 3.11.13. ([9] Section 1.8.2). We then define a stack, called G-Isoc, on perfect $\bar{F}_q$-algebras.

- $R \rightarrow \{$G-torsors on Spec$R((t))/Frob_q$ or Spec$(W(R) \otimes_{W(\bar{F}_q)} E)/Frob_q\}$.

Theorem 3.11.14. ([9] Theorem 1.14 (Rapoport-Richartz, Caraiani-Scholze, Ivanov, Anschütz)).

- G-Isoc is a stack for the v-topology.
- For any $b \in B(E,G)$, there is a locally closed substack $G - Isoc_b \subset G - Isoc$ where the isocrystal is isomorphic to b, and $G - Isoc_b \simeq [pt/G_b(E)]$.

Remark 3.11.15. ([9] Section 1.8.3). We now use perfectoid rings instead of perfect algebras. The set up is the following: to recall that the relation to the Langlands dual group $\hat{G}$ is through Hecke operators; and $R((t))$ is a small punctured disk. We consider correspondences on the space of G-torsors.

The construction is the following. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two G-torsors over (Spec$R((t))/Frob_q$). We consider the space parametrizing $\mathcal{E}_1$ and $\mathcal{E}_2$ and an isomorphism $\mathcal{E}_1 \simeq \mathcal{E}_2$ not near a

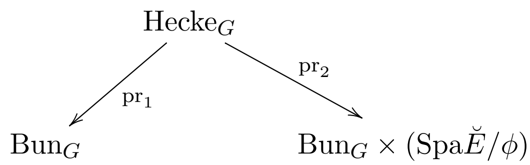


Figure 5: *Geometric Satake equivalence* in the geometrization of the local Langlands correspondence where Bun_G is an Artin diamond v-stack, cohomologically smooth of dimension 0 (cf. [9]).[8]

divisor $D \subset \text{Spec}R((t))$, which requires a section $\text{Spec}R \to \text{Spec}R((t))$ (which does not exist if R is a discrete ring), which requires R to be a Banach algebra. That is the motivation for bringing in perfectoid geometry.

We associate the Fargues-Fontaine curve

- $\mathbb{D}^*_{\text{Spa}(R,R^+)}/\text{Frob}_q$

(respectively a similar object in mixed characteristic) to any perfectoid affinoid algebra (R, R^+) over $\bar{F}_q$. We then consider the moduli space of G-torsors on $\mathbb{D}^*_{\text{Spa}(R,R^+)}/\text{Frob}_q$.

This analytic approach is advantageous because the punctured disk is more geometric, which allows for consideration of the Hecke operators. [7]

[7]([28] Section 4.3) Let $M_k(N)$ (resp., $S_k(N)$) denote the space of modular forms (resp., cusp forms) of weight k (automorphicity condition) and level N. Then $M_k(N)$ and $S_k(N)$ are finite-dimensional vector spaces over $\mathbb{C}$. For each prime p not dividing N the Hecke operators T_p and $< p >$ are endomorphisms of $M_k(N)$ defined by

- $(T_p g)(\tau) = p^{k-1} g(pr) + \frac{1}{g}\sum_{j=0}^{p-1} g(\frac{\tau+j}{p})$.
- $(< p > g)(\tau) = g(\gamma\tau)(c\tau + d)^{-k}$.
- $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is any element of $SL_2(\mathbb{Z})$ with $c \equiv 0(modN)$ and $d \equiv p(modN)$.
- A modular form is an eigenform if it is an eigenvector for all the Hecke operators.

In Figure 5,

> $\mathrm{Spa}(\breve{E}/\phi)$ parametrizes sections of $\mathbb{D}^*_{\mathrm{Spa}(R,R^+)}/\mathrm{Frob}_q$. $Hecke_G$ is infinite-dimensional, while it has finite-dimensional strata $Hecke^{\mu}_G$ [9].

Definition 3.11.16. ([9] Section 1.8.3). Now define the operators T_μ.

- $T_\mu := pr_{2!}pr_1^* : D(Bun_G, \bar{Q}_\ell) \to D(Bun_G \times Spa\breve{E}/\phi; \bar{Q}_\ell) = D(Bun_G, \bar{Q}_\ell)^{W_E}$.

The *Geometric Satake equivalence* says that T_μ are enumerated by $Rep(\hat{G})$.

Remark 3.11.17. (Section 1.8.3). Recall that $D(Bun_G, \bar{Q}_\ell)$ is an aggregate of

- $Rep(G_b(E); \bar{Q}_\ell)$ for all b.

This categorical structure is exactly what is needed to define L-parameters attached to $Rep(G_b(E); \bar{Q}_\ell)$.

Recall, we have a classification of G-bundles on the Fargues-Fontaine curve. Per the usual notation choose $\bar{F}_q$ to give $\breve{E} = W_{\mathcal{O}_E}(\bar{F}_q)[\frac{1}{\pi}]$.

Definition 3.11.18. ([9] Definition 11.20). Let $S \in \mathrm{Perf}_{\bar{F}_q}$. A G-torsor on X_S is an exact $\otimes$-functor $\mathcal{E}$:$\mathrm{Rep}_E(G) \to VB(X_S)$.

Definition 3.11.19. ([9] Definition 11.21). We define the stack of G-bundles on the Fargues-Fontaine curve Bun_G to be the v-stack on $\mathrm{Perf}_{\bar{F}_q}$ sending $S \to \{G\text{−bundles on } X_S\}$. This is a v-stack, which follows from v-descent for vector bundles.

Recall, we have the two culminating theorems [9].

Theorem 3.11.20. (Theorem 11.23 (Fargues if E/Q_p, Anschütz in general)). If $S = Spa(C, C^+)$ where C is a complete algebraically closed field, then the functor $G \to Isoc \to$

[8] For any $b \in B(G)$, we get a locally closed stratum $|Bun_G^b| \subset Bun_G$. It has the form $Bun_G^b = [*/\mathcal{G}_b]$ where $\mathcal{G}_b$ fits into a short exact sequence

- $1 \to$ "unipotent group diamond" (it. ext of positive Banach-Colmez spaces) $\to \mathcal{G}_b \to G_b(E) \to 1$.

$Bun_G(S)$ sending a G-torsor on $Spa\hat{E}/\sigma^Z$ to its pullback to $Y_S/\phi^Z = X_S$ induces a bijection on isomorphism classes $Bun_G(S)/\sim \tilde{\rightarrow} B(G)$ which is even a homeomorphism $|Bun_G(S)| \simeq B(G)$.

Let G/E be a reductive group. The moduli stack Bun_G of G-bundles on the Fargues-Fontaine curve represents the functor taking $S \in \text{Perf}\bar{F}_q \rightarrow \{G\text{-bundles on } X_S\}$.

Theorem 3.11.21. ([9] Theorem 16.1).

- 1. Bun_G is an Artin v-stack, cohomologically smooth of dimension 0.
- 2. The map $|\text{Bun}_G| \rightarrow B(G)$ is a continuous bijection.
- 3. For any $b \in B(G)$, we get a locally closed stratum $|\text{Bun}_G^b| \subset \text{Bun}_G$. It has the form $\text{Bun}_G^b = [*/\mathcal{G}_b]$ where $\mathcal{G}_b$ fits into a short exact sequence.
- $1 \rightarrow$ "unipotent group diamond" (iterated extension of positive Banach-Colmez spaces) $\rightarrow \mathcal{G}_b \rightarrow G_b(E) \rightarrow 1$.

Because of the gluing of the diamonds and Fargues' hope for the diamond stack incarnation of Bun_G, we use the diamond relative Fargues-Fontaine curve

- $\mathcal{Y}_{S,E}^{\diamond} = S \times (\text{Spa}\mathcal{O}_E)^{\diamond}$

in the localization sequence to investigate the K^{Efimov}-groups.

3.11.1 $\mathcal{G}$-torsors

We review $\mathcal{G}$-torsors for our exposition of the geometrization of local Langlands. Our brief review follows ([12] (Appendix to Lecture 19)).

The set up is the following. $\mathcal{G}$ denotes the adic space $\text{Spa}(R, R^+)$ if $\mathcal{G} = \text{Spec}R$ and $R^+ \subset R$ is the integral closure of Z_p. For every adic space S over Z_p, one has

- $\mathcal{G}(S) = \mathcal{G}(\mathcal{O}_S(S))$.

Definition 3.11.1.1.([12] Discussion Appendix to Lecture 19). There are three possible definitions of $\mathcal{G}$-torsors.

- A geometric $\mathcal{G}$-torsor is an adic space $\mathcal{P} \to X$ over X with an action of $\mathcal{G}$ over X such that étale locally on X, there is a $\mathcal{G}$-equivariant isomorphism $\mathcal{P} \simeq \mathcal{G} \times X$.
- (Cohomological) A cohomological $\mathcal{G}$-torsor is an étale sheaf $\mathcal{Q}$ on X with an action of $\mathcal{G}$ such that étale locally on X, there is a $\mathcal{G}$-equivariant isomorphism $\mathcal{Q} \simeq \mathcal{G}$.
- (Tannakian) A Tannakian $\mathcal{G}$-torsor is an exact $\otimes$-functor $P : \mathrm{Rep}\ \mathcal{G} \to Bun(X)$, where $Bun(X)$ is the category of vector bundles on X.

Theorem 3.11.1.2. ([12] Theorem 19.5.2). The categories of geometric, cohomological, and Tannakian $\mathcal{G}$-torsors on X are canonically equivalent.

We conclude with the descent of $\mathcal{G}$-torsors on open subsets of $S \dot{\times} \mathrm{Spa} Z_p$.

Proposition 3.11.1.3. ([12] Proposition 19.5.3). Let $S \in \mathrm{Perf}$ be a perfectoid space of characteristic p and let $U \subset S \dot{\times} \mathrm{Spa} Z_p$ be an open subset. The functor on Perf_S sending any $S' \to S$ to the groupoid of $\mathcal{G}$-torsors on

- $U \times_{S \dot{\times} \mathrm{Spa} Z_p} S' \dot{\times} \mathrm{Spa} Z_p$

is a v-stack.

3.11.2 Étale Locus

To study moduli spaces of shtukas, we quickly review vector bundles on the relative Fargues-Fontaine curve $\mathcal{X}_{FF,S}$,

which are also studied via ϕ-modules on subspaces of $\mathcal{Y}_{(0,\infty)}(S)$ [17].

Two foundational theorems about vector bundles on $\mathcal{X}_{FF,S}$ are proved by Kedlaya-Liu:

the first concerns the semicontinuity of the Newton polygon; the second the open locus where the Newton polygon is 0.

The set up is the following: Let S be a perfectoid space of characteristic p, and let $\mathcal{E}$ be a vector bundle on $\mathcal{X}_{FF,S}$. Passing to an open and closed cover of S, we assume that the rank of $\mathcal{E}$ is constant. For any $s \in S$, we choose a geometric point

- $\bar{s} = \mathrm{Spa}(C, C^+) \to S$

whose closed point maps to s, and pullback $\mathcal{E}$ to a vector bundle $\mathcal{E}_{\bar{s}}$ on $\mathcal{X}_{FF,C}$ [17].

Theorem 3.11.2.1. ([17] Theorem 22.2.1 [KL15]). The map $\nu_{\mathcal{E}}$ is upper semicontinuous.

To construct examples of the open locus where the Newton polygon is 0, we take a pro-étale $\underline{Q_p}$-local system $\mathbb{L}$ on S and look at

- $\mathcal{E} = \mathbb{L} \otimes_{\underline{Q_p}} \mathcal{O}_{\mathcal{X}_{FF,S}}$.

Using the pro-étale or even v-descent of vector bundles on $\mathcal{Y}_{(0,\infty)}(S)$, and thus on $\mathcal{X}_{FF,S}$ ([17] Proposition 19.5.3), we have a vector bundle on $\mathcal{X}_{FF,S}$, upon descending to the case where $\mathbb{L}$ is trivial [17].

We now have the second theorem of Kedlaya-Liu.

Theorem 3.11.2.2. ([17] Theorem 22.3.1 [KL15). This construction defines an equivalence between the category of pro-étale Q_p-local systems on S and the category of vector bundles $\mathcal{E}$ on $\mathcal{X}_{FF,S}$ which are trivial at every geometric point of S (which is to say, the function $\nu_{\mathcal{E}}$ is identically 0).

For the link to isocrystals and further discussion, see [17] Corollary 22.3.3, Definition 22.4.1, and Remark 22.4.2.

We now present the formalism of Efimov K-theory and extend Efimov K-theory to $\mathcal{D}^{\diamond}$, therein newly introducing the Efimov K-theory of diamonds.

4 Efimov K-theory of Diamonds

Efimov K-theory is K-theory for large stable ∞-categories. Efimov's idea is to weaken to dualizable the compactly generated condition carefully ensuring K-theory remains defined. Dualizability is a nice condition because $\mathcal{C}$ dualizable implies that $\mathcal{C}$ fits into a localization sequence $\mathcal{C} \to \mathcal{S} \to \mathcal{X}$ with $\mathcal{S}$ and $\mathcal{X}$ compactly generated. Then the Efimov K-theory should be the fiber of the K-theory in $\mathcal{C} \to \mathcal{S} \to \mathcal{X}$ [16].

Efimov formalizes this idea in the following theorem.

Theorem 4.1 ([43] Theorem [Efimov]). Let $\mathcal{T}$ be a category. The functor

- $Fun(Pr_{St}^{dual}, \mathcal{T}) \longrightarrow Fun(Cat_{St}^{idem}, \mathcal{T})$, $F \mapsto F \circ Ind$,

restricts to an isomorphism between the full subcategories of localizing invariants, with inverse $F \mapsto F_{cont}$. In particular,

- if $\mathcal{C} \in Pr_{St}^{cg}$, then $F_{cont}(\mathcal{C}) = F(C^{\omega})$.

We now explain this theorem and its proof.

4.1 Efimov K-theory

Our terminology, exposition, and souscompendium follows Hoyois [43].

- ∞-categories are called categories.
- Let $\mathcal{P}r$ denote the category of presentable categories and colimit-preserving functors.
- Let $\mathcal{P}r^{dual} \subset \mathcal{P}r$ denote the subcategory of dualizable objects and right-adjointable morphisms (with respect to the symmetric monoidal and 2-categorical structures of $\mathcal{P}r$).

  Let $\mathcal{P}r^{cg} \subset \mathcal{P}r$ be the subcategory of compactly generated categories and compact functors. (Compact functors are functors whose right adjoints preserve filtered colimits).
- Let $\mathcal{P}r_{St}^{\star}$ denote the corresponding full subcategories consisting of stable categories.
- **Definition 4.1.1** [43]. A localization sequence in any such category is a cofiber sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ where $\mathcal{A} \to \mathcal{B}$ is fully faithful.

Proposition 4.1.2 [43]. $\mathcal{P}r_{St}^{cg}$ is a full subcategory of $\mathcal{P}r_{St}^{dual}$.

Theorem 4.1.3. ([43] Theorem 1 (Lurie)). For $\mathcal{C}$ a stable presentable category, the following are equivalent:

- $\mathcal{C}$ is dualizable in $\mathcal{P}r$.
- $\mathcal{C}$ is a retract in a $\mathcal{P}r$ of a compactly generated category.
- The colimit functor colim: $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $y\hat{}: \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.

Remark 4.1.4 [43]. $y\hat{}$ is fully faithful. We have that, for $\mathcal{C}$ compactly generated, $y\hat{}$ is the Ind-extension of the inclusion $\mathcal{C}^w \subset \mathcal{C}$.

We now define the Calkin category.

Definition 4.1.5. ([43] Definition 2). For $\mathcal{C}$ a presentable stable category, the Calkin category $\mathrm{Calk}(\mathcal{C}) \subset \mathrm{Ind}(\mathcal{C})$ is the kernel of the colimit functor colim: $\mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$.

Proposition 4.1.6. ([43] Proposition 3). Suppose $\mathcal{C} \in \mathcal{P}r$ is stable and dualizable. Then:

- $\mathrm{Calk}(\mathcal{C}) = \mathrm{Ind}(\mathrm{Calk}(\mathcal{C})^w)$.
- The inclusion $\mathrm{Calk}(\mathcal{C}) \subset \mathrm{Ind}(\mathcal{C})$ admits a left adjoint $\Phi : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Calk}(\mathcal{C})$ which preserves compact objects.
- There is a localization sequence of cocomplete stable categories
  - $\mathcal{C} \underset{\mathrm{colim}}{\overset{\hat{y}}{\rightleftarrows}} \mathrm{Ind}(\mathcal{C}) \overset{\Phi}{\rightleftarrows} \mathrm{Calk}(\mathcal{C})$.

Proof. See [43] Proposition 3.

Remark 4.1.7 [43]. $\mathrm{Calk}(\mathcal{C}^w)$ is a locally small but large category. This means that $\mathrm{Calk}(\mathcal{C})$ is not presentable unless $\mathcal{C} = 0$.

We now define continuous K-theory.

Definition 4.1.8. ([43] Definition 4). Let $C \in \mathcal{P}r$ be stable and dualizable. We define the continuous K-theory of $\mathcal{C}$ as the space

- $\mathrm{K_{cont}}(\mathcal{C}) = \Omega\ \mathrm{K}(\mathrm{Calk}(C)^{\omega})$.

Lemma 4.1.9. ([43] Lemma 5). If $\mathcal{C}$ is compactly generated, then $\mathrm{K}_{cont}(\mathcal{C}) = \mathrm{K}(\mathcal{C}^w)$.

Proof. The localization sequence of Proposition 3 is Ind of the sequence

- $\mathcal{C}^w \hookrightarrow \mathcal{C} \to \mathrm{Calk}(\mathcal{C})^w$.

As $\mathrm{K}(\mathcal{C}) = 0$, the result follows from the localization theorem in K-theory. □

> Efimov explains that we can define a continuous version of any localizing invariant and we used that K-theory is defined on the large category $\mathrm{Calk}(\mathcal{C})^w$. Efimov wishes to generalize to any abstrat localizing invariant and so needs to use a presentable version of $\mathrm{Calk}(\mathcal{C})$ depending on the choice of a large enough regular cardinal [43].

Definition 4.1.10. ([43] Definition 6). Let $\mathcal{C}$ be a presentable stable category and κ a regular cardinal. The κ-Calkin category $\mathrm{Calk}_\kappa(\mathcal{C}) \subset \mathrm{Ind}(\mathcal{C}^\kappa)$ is the kernel of the colimit functor colim: $\mathrm{Ind}(\mathcal{C}^\kappa) \to \mathcal{C}$.

Definition 4.1.11 [43]. If $\mathcal{C} \in \mathcal{P}r$ is dualizable, there exists a regular cardinal κ such that the fully faithful functor $\hat{y} : \mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ lands in $\mathrm{Ind}(\mathcal{C}^\kappa)$. $\mathcal{C}$ is then said to be κ-dualizable. Let $\mathcal{P}r^{\kappa-dual} \subset \mathcal{P}r^{dual}$ be the full subcategory spanned by the κ-dualizable categories. Then

- $\mathcal{P}r^{dual} = \bigcup_\kappa \mathcal{P}r^{\kappa-dual}$ and $\mathcal{P}r^{\kappa-dual} \subset \mathcal{P}r^{\kappa-cg}$.

This means a stable presentable category is w-dualizable if and only if it is compactly generated. Given $\mathcal{C} \in \mathcal{P}r^{\kappa-dual}_{St}$, we have a localization sequence

- $\mathcal{C} \underset{\text{colim}}{\overset{\hat{y}}{\rightleftarrows}} \mathrm{Ind}(\mathcal{C}^\kappa) \overset{\Phi_\kappa}{\rightleftarrows} \mathrm{Calk}_\kappa(\mathcal{C})$. $(\star)$

in $\mathcal{P}r$, where Φ_κ is the restriction of Φ.

Remark 4.1.12 [43]. We see that this diagram is functorial in $\mathcal{C}$.

Remark 4.1.13 [43]. We have that $\mathrm{Calk}_\kappa : \mathcal{P}r_{St}^{\kappa-dual} \to \mathcal{P}r_{St}^{cg}$ preserves localization sequences.

Remark 4.1.14 [43]. Let Cat_{St}^{idem} be the category of small stable idempotent complete categories and exact functors. Ind-completion induces an isomorphism $\mathrm{Cat}_{St}^{idem} \simeq \mathcal{P}r_{St}^{cg}$.

We now define a localizing invariant.

Definition 4.1.15 [43]. If $\mathcal{T}$ is a category, we say that a functor F: $\mathrm{Cat}_{St}^{idem} \longrightarrow \mathcal{T}$ is a localizing invariant if it preserves final objects and sends localization sequences to fiber sequences.

Lemma 4.1.16. ([43] Lemma 8). Let $\mathcal{T}$ be a category and $F : \mathrm{Cat}_{St}^{idem} \to \mathcal{T}$ a localizing invariant. Let $\mathcal{C} \in \mathcal{P}r_{St}$ be dualizable and let $\kappa \leq \lambda$ be uncountable regular cardinals such that $\hat{y}(\mathcal{C}) \subset \mathrm{Ind}(\mathcal{C}^\kappa)$. Then the inclusion $\mathrm{Calk}_\kappa(\mathcal{C}) \subset \mathrm{Calk}_\lambda(\mathcal{C})$ induces an isomorphism

- $\mathrm{F}(\mathrm{Calk}_\kappa(\mathcal{C})^w) \simeq \mathrm{F}(\mathrm{Calk}_\lambda(\mathcal{C})^w)$.

Proof. Applying Calk_κ to $(\star)$ for λ, gives a localization sequence

- $\mathrm{Calk}_\kappa(\mathcal{C}) \to \mathrm{Calk}_\kappa(\mathrm{Ind}(\mathcal{C}^\lambda)) \to \mathrm{Calk}_\kappa(\mathrm{Calk}_\lambda(\mathcal{C}))$.

Also, applying $(\star)$ to $\mathrm{Calk}_\lambda(\mathcal{C})$ gives a localization sequence

- $\mathrm{Calk}_\lambda(\mathcal{C}) \to \mathrm{Ind}(\mathrm{Calk}_\lambda(\mathcal{C})^\kappa) \to \mathrm{Calk}_\kappa(\mathrm{Calk}_\lambda(\mathcal{C}))$.

Therefore, we obtain isomorphisms

- $F(\mathrm{Calk}_\lambda(\mathcal{C})^\omega) \simeq \Omega F(\mathrm{Calk}_\kappa(\mathrm{Calk}_\lambda(\mathcal{C}))^\omega) \simeq F(\mathrm{Calk}_\lambda(\mathcal{C})^\omega)$,

natural in λ, which proves the claim. □

Definition 4.1.17. ([43] Definition 9). Let $\mathcal{T}$ be a category and $F : \mathrm{Cat}_{St}^{idem} \to \mathcal{T}$ a localizing invariant. The continuous extension of F is the functor $F_{cont} : \mathcal{P}r_{St}^{dual} \to \mathcal{T}$ defined by

- $F_{cont}(\mathcal{C}) = \Omega\mathrm{F}(\mathrm{Calk}_\kappa(\mathcal{C})^w)$, for κ any uncountable regular cardinal such that $\hat{y}(\mathcal{C}) \subset \mathrm{Ind}(\mathcal{C}^{\kappa})$.

Remark 4.1.18 [43]. A more formal construction is to consider $(\mathrm{Calk}_{\kappa})_{\kappa}$ as a functor

- $\mathcal{P}r^{dual}_{St} \to \widehat{Ind}(\mathcal{P}r^{cg}_{St}) \simeq \widehat{Ind}(Cat^{idem}_{St})$

and compose it with the Ind-extension of ΩF. The result lands in the subcategory $\mathcal{T} \subset \widehat{Ind}(\mathcal{T})$ by Lemma 4.1.16.

> Recall, K-theory commutes with filtered colimits. So, in the case of K-theory, there is agreement between Definition 4.1.17 and Definition 4.1.8 since $\mathrm{Calk}(\mathcal{C})^{w}$ $= \mathrm{colim}_{\kappa}\mathrm{Calk}_{\kappa}(\mathcal{C})^{w}$ [43].

Definition 4.1.19 [43]. A functor $F : \mathcal{P}r^{dual}_{St} \to \mathcal{T}$ is called a localizing invariant if it preserves final objects and sends localization sequences to fiber sequences.

Theorem 4.1.20. ([43] Theorem 10). (Efimov). Let $\mathcal{T}$ be a category. The functor

- $Fun(Pr^{dual}_{St}, \mathcal{T}) \longrightarrow Fun(Cat^{idem}_{St}, \mathcal{T})$, $F \mapsto F \circ Ind$,

restricts to an isomorphism between the full subcategories of localizing invariants, with inverse $F \mapsto F_{cont}$. In particular,

- if $\mathcal{C} \in Pr^{cg}_{St}$, then $F_{cont}(\mathcal{C}) = F(C^{\omega})$.

Proof. Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a localization sequence in $\mathcal{P}r^{dual}_{St}$. Then for large enough κ we have an induced localization sequence

- $\mathrm{Calk}_{\kappa}(\mathcal{A}) \to \mathrm{Calk}_{\kappa}(\mathcal{B}) \to \mathrm{Calk}_{\kappa}(\mathcal{C})$.

It follows that F_{cont} is a localizing invariant. By Lemma we have $F_{cont}\circ$ Ind $\simeq F$, functorially in F. To $\mathcal{C} \in \mathcal{P}r^{dual}_{St}$ we can associate the filtered diagram of localization sequences

- $\mathcal{C} \to \mathrm{Ind}(\mathcal{C}^{\kappa}) \to \mathrm{Calk}_{\kappa}(\mathcal{C})$ for $\kappa >> 0$. This gives a functorial isomorphism $F \simeq (F\circ \mathrm{Ind})_{cont}$. □

Recall, our conjecture extends this formalism to diamonds.

Conjecture 2.4. Let $\mathcal{D}_\diamond$ be the complex of v-stacks of locally spatial diamonds. Let $\mathcal{D}^\diamond$ be the $(\infty,1)$-category of diamonds. Let $\mathcal{Y}_{(R,R^+),E} = \mathrm{Spa}(R,R^+)\times_{\mathrm{Spa}F_q} \mathrm{Spa}F_q[[t]]$ be the relative Fargues-Fontaine curve. Let $(\mathcal{Y}^\diamond_{S,E})$ be the diamond relative Fargues-Fontaine curve. There exists a localization sequence

- $K(\mathcal{D}_\diamond) \to K^{\mathrm{Efimov}}(\mathcal{Y}^\diamond_{S,E}) \to K^{\mathrm{Efimov}}(\mathcal{Y}_{(R,R^+),E})$.

Example 4.1.21. ([43] Example 11). An example of a localizing invariant $\mathcal{P}r^{dual}_{St} \to \mathrm{Sp}$ is the functor sending a dualizable category to its Euler characteristic. This functor is the continuous extension of

- THH: $\mathrm{Cat}^{idem}_{St} \to \mathrm{Sp}$

as THH($\mathcal{C}$) is the Euler characteristic of Ind($\mathcal{C}$) in $\mathcal{P}r_{St}$.

Example 4.1.22. ([43] Example 12). Efimov's Theorem implies the following:

- THH_{cont} : $\mathcal{P}r^{dual}_{St} \to$ Sp factors through the stable category CycSp of cyclotomic spectra.
- $\mathrm{TP}_{cont}(\mathcal{C}) \simeq \mathrm{THH}_{cont}(\mathcal{C})^{t\mathbb{T}}$ and $\mathrm{TC}_{cont}(\mathcal{C})$ is the mapping spectrum $\mathrm{Map}_{\mathrm{CycSp}}(1, THH_{\mathrm{cont}}(\mathcal{C}))$.
- $\Omega^\infty(\mathbb{K}_{cont}) \simeq \mathrm{K}_{cont}$, where K is nonconnective K-theory.
- The cyclotomic trace $\mathbb{K} \to$ TC extends uniquely to a transformation $\mathbb{K}_{cont} \to TC_{\mathrm{cont}}$.

Lemma 4.1.22. ([43] Lemma 14). Let $\mathcal{T}$ be a category, F : $\mathrm{Cat}^{idem}_{St} \to \mathcal{T}$ a localizing invariant, and $\mathcal{K}$ a small category. Then F preserves $\mathcal{K}$-indexed colimits if and only if F_{cont} preserves $\mathcal{K}$-indexed colimits.

Proof. This is immediate from ($\star$) and from the fact that Ind:$Cat^{idem}_{St} \to \mathcal{P}r_{St}$ preserves colimts. □

Theorem 4.1.22. ([43] Theorem 15). Let X be a locally compact Hausdorff topological space, $\mathcal{C}$ a stable dualizable presentable category, and R a sheaf of E_1-ring spectra on X.

- (Lurie) $\mathrm{Mod}_R(Shv(X,\mathcal{C})$ is dualizable in $\mathcal{P}r$.

2. (Efimov) Suppose that $Shv(X)$ is hypercomplete (i.e., X is a topological manifold). Let $\mathcal{T}$ be a stable compactly generated category and F: $\mathrm{Cat}_{St}^{idem} \to \mathcal{T}$ a localizing invariant that preserves filtered colimits. Then

- $F_{cont}(Mod_R(Shv(X,\mathcal{C}) \simeq \Gamma_c(X, F_{cont}(Mod_R(\mathcal{C})))$.

In particular,

- $F_{cont}(Shv(\mathbb{R}^n,\mathcal{C})) \simeq \Omega^n F_{cont}(\mathcal{C})$.

Proof. See [43] Theorem 15.

We modify this theorem to diamonds. Recall our Conjecture 2.2.

Conjecture 2.2. Let S be a perfectoid space, $\mathcal{D}^\diamond$ a stable dualizable presentable $(\infty,1)$-category, and R a sheaf of E_1-ring spectra on S. Let $\mathcal{T}$ be a stable compactly generated $(\infty,1)$-category and F: $\mathrm{Cat}_{St}^{idem} \to \mathcal{T}$ a localizing invariant that preserves filtered colimits. Then

- $F_{cont}(Shv(\mathbb{S}^n,\mathcal{D}^\diamond)) \simeq \Omega^n F_{cont}(\mathcal{D}^\diamond)$.

4.2 Efimov K-theory of Diamonds

4.2.1 Waldhausen S-Construction

The main goal is to develop a Waldhausen S-construction on $\mathcal{D}^\diamond$ to obtain the K-theory diamond spectrum $K\mathcal{D}^\diamond$.

Recall the following definition of the Waldhausen category.

Definition 4.2.1.1. ([26] Definition). The Waldhausen category is a category C with zero object 0 equipped with a choice of two classes of maps cof of the cofibrations and w.e. of weak equivalences such that

- (C1) all isomorphisms are cofibrations

- (C2) there is a zero object 0 and for any object a the unique morphism 0→a is a cofibration
- (C3) if $a * b$ is a cofibration and $a \to c$ any morphism then the pushout $c \to b \cup_a c$ is a cofibration
- (W1) all isomorphisms are weak equivalences
- (W2) weak equivalences are closed under composition (make a subcategory)
- (W3) "glueing for weak equivalences".

  For any cofibration $A \hookrightarrow B$ there is a cofibration sequence

  - $A \hookrightarrow B \to B/A$ where B/A is the choice of the cokernel $B \cup_A 0$.

Recall the Waldhausen S-Construction formalism.

Definition 4.2.1.2. ([26] Definition). To a Waldhausen category C whose weak equivalence classes form a set, one defines $K_0(C)$ as an abelian group whose elements are the weak equivalence classes modulo the relation $[A]+[B/A] = [B]$ for any cofibration sequence

- $A \hookrightarrow B \to B/A$ where B/A.

The Waldhausen S-construction $C \to S.C$ is devised from Waldhausen categories to simplicial categories with cofibrations and weak equivalences. The K-theory space of a Waldhausen S-construction is given by formula

- $\Omega\text{hocolim}_{\Delta^{op}}([n] \to N \cdot (w.e.(S_n C)))$

where Ω is the loop space functor, N is the simplicial nerve, w.e. takes the (simplicial) subcategory of weak equivalence and $[n] \in \Delta$.

Remark 4.2.1.3.

> This can be improved using iterated Waldhausen S-construction to the K-theory Ω-spectrum of C, wherein the K-theory space is the one-space of the K-theory spectrum.
>
> Therefore, the K-groups are the homotopy groups of the K-theory space [26].

4.2.2 $(\infty, 1)$-site and Topological Localization

In parallel with the S-Construction, to construct a topology on $\mathcal{D}^\diamond$, we must first construct the $(\infty, 1)$-site on $\mathcal{D}^\diamond$.

We recall the definition of an $(\infty, 1)$-site.

Definition 4.2.2.1. ([8] Definition). The $(\infty, 1)$-site on an $(\infty, 1)$-category $\mathcal{C}$ is the data encoding an $(\infty, 1)$-category of $(\infty, 1)$-sheaves $Sh(\mathcal{C}) \hookrightarrow PSh(\mathcal{C})$ inside the $(\infty, 1)$-category of $(\infty, 1)$-presheaves on $\mathcal{C}$.

Remark 4.2.2.2. We start with a κ-small construction. The following three categories are endowed with the structure of a site:

- Perfd, the category of perfectoid spaces,
- Perf $\subset$ Perfd, the full subcategory of perfectoid spaces in characteristic p,
- and $X_{\text{proét}}$, the category of perfectoid spaces pro-étale over X, where X is any perfectoid space.

by establishing that a collection of morphisms $\{f_i : Y_i \to Y\}$ is a covering (a pro-étale cover) if the f_i are pro-étale, and if for all quasi-compact open $U \subset Y$, there exists a finite subset $I_U \subset I$, and quasicompact open subsets $U_i \subset Y_i$ for $i \in I_U$, such that

- $U = \bigcup_{i \in I_U} f_i(U_i)$,

requiring stricter conditions than the f_i merely being a topological cover [17].

Remark 4.2.2.3. It is conjectured that $\mathcal{D}^\diamond$ admits a topological localization [7]. Extending a perfectoid version of topological localization is not straightforward as the topology of the diamond $\mathcal{D}$ can be highly pathological and not even T_0.

Consider the following example exhibiting a high pathology.

Example 4.2.2.4. ([17] Discussion of Remark 10.3.2 [17]). Consider the quotient of the constant perfectoid space Z_p over a perfectoid field by the equivalence relation "congruence modulo Z." This yields a diamond with underlying topological space

- Z_p/Z.

Restricting to a special class of well-behaved diamonds, called qcqs, resolves these issues.

Proposition 4.2.2.5. ([17] Proposition). Let $\mathcal{D}$ be a quasicompact quasiseparated diamond. Then

- $\mathcal{D}$ is T_0.

We now recall the following definition of topological localization.

Definition 4.2.2.6. ([35] Definition). A topological localization is a left exact localization of an $(\infty, 1)$-category in the sense of passing to a reflective sub-$(\infty, 1)$-category at a collection of morphisms that are monomorphisms. A reflective sub-$(\infty, 1)$-category

- $\mathcal{D} \overset{L}{\underset{\hookrightarrow}{\leftarrow}} \mathcal{C}$

is obtained by localizing at a collection S of morphisms of C.

We recall the definition of a reflective subcategory.

Definition 4.2.2.7. ([29] Definition). A reflective subcategory is a full subcategory $\mathcal{C} \hookrightarrow \mathcal{D}$ such that objects d and morphisms $f : d \to d'$ in $\mathcal{D}$ have "reflections" Td and $Tf : Td \to Td'$ in $\mathcal{C}$. Every object in $\mathcal{D}$ looks at its own reflection via a morphism $d \to Td$ and the reflection of an object $c \in \mathcal{C}$ is equipped with an isomorphism $Tc \simeq c$. The inclusion creates all limits of $\mathcal{D}$ and $\mathcal{C}$ has all colimits which $\mathcal{D}$ admits.

Remark 4.2.2.8. The construction of the topological localization seems appropriately meta, in its construction of a reflection of what is already profinitely many copies and v-stacks.

Recall there is a bijection between equivalence classes of topological localizations and Grothendieck topologies on $(\infty, 1)$-categories C. Topological localizations are appropos to our formalism because in passing to the full reflective sub-$(\infty, 1)$-category, objects and morphisms have reflections in the category, just as geometric points have reflections in the profinitely many copies of $Spa(\mathcal{C})$.

Recall, the category of sheaves on a (small) site is a Grothendieck topos [8]. Lurie discusses the structure needed for our construction. Recall the following definition of an $(\infty, 1)$-category of $(\infty, 1)$-sheaves.

Definition 4.2.2.8. ([8] Definition 6.2.2.6). An $(\infty, 1)$-category of $(\infty, 1)$-sheaves is a reflective sub-$(\infty, 1)$-category

- $Sh(C) \overset{L}{\underset{\hookrightarrow}{\leftarrow}} PSh(C)$

of an $(\infty, 1)$-category of $(\infty, 1)$-presheaves such that the following equivalent conditions hold:

1. L is a topological localization.
2. There is the structure of an $(\infty, 1)$-site on C such that the objects of $Sh(C)$ are precisely those $(\infty, 1)$-presheaves A that are local objects with respect to the covering monomorphisms

   - $p : U \to j(c)$ in $PSh(C)$ in that
   - $A(c) \simeq PSh(j(c), A) \xrightarrow{\text{PSh(p,A)}} PSh(U, A)$

   is an $(\infty, 1)$-equivalence in ∞Grpd.
3. The $(\infty, 1)$-equivalence is the descent condition and the presheaves satisfying it are the $(\infty, 1)$-sheaves.

As a Grothendieck topology is a collection of morphisms designated as covers [8], we take Scholze's six operations as our basic building blocks in constructing the covers, in the following sense. Scholze's six operations live in the derived categories of sheaves and derived categories are the ∞-categorical localization of the category of chain complexes at the class of quasi-isomorphisms [31].

We must construct the localizing invariants and extend to $\mathcal{D}^\diamond$ the Waldhausen S-Construction.

4.2.3 Diamond Chromatic Tower

We wish to construct a chromatic filtration on $K\mathcal{D}^\diamond$ We take L_n as the topological localization of $K\mathcal{D}^\diamond$ and the Lubin-Tate formal $\mathcal{O}_F$-module law [22] as a perfectoid version of formal group law.

We propose the following:

Conjecture 4.2.3.1. The diamond spectrum is an object representing the étale cohomology of diamonds.

Conjecture 4.2.3.2. The Waldhausen S-construction on $\mathcal{D}^\diamond$ generates the K-theory spectrum of $\mathcal{D}^\diamond$, herein denoted $K(\mathcal{D}^\diamond)$.

Recall that in chromatic homotopy theory, the moduli stack of formal groups admits a stratification where the open strata are indexed by the height of formal groups [8]. We know that by the Landweber exact functor theorem, complex oriented cohomology theories can be constructed from these formal groups and inherit the height filtration, therein inducing a chromatic filtration [37].

Recall also the Bousfield localization procedure.

Definition 4.2.3.3. ([34] Definition). Bousfield localization of spectra is a localization of the stable $(\infty, 1)$-category of spectra at the collection of morphisms which become equivalences under the smash product with a given spectrum E. Any such E represents a generalized homology theory. So this may also be thought of as E-homology localization. If the stable $(\infty, 1)$-category of spectra is presented by a (stable) model category, then the ∞-categorical localization can be presented by the operation of Bousfield localization of model categories [34].

We want to supervene formal group laws and Bousfield localization. Recall, the following theorem.

Theorem 4.2.3.4.([14] Theorem 4.1). The Lubin-Tate tower at infinite level defined as

- $\mathcal{M}_{LT,\infty} = \tilde{U}_x \times^{GL_2(\mathbb{Q}_P)_1} GL_2(\mathbb{Q}_P) \cong \bigsqcup_Z \tilde{U}_x$. $\mathcal{M}_{LT,\infty}$

is a perfectoid space.

Remark 4.2.3.5. Also recall that $E(k, \Gamma)$, the Lubin-Tate spectrum associated to the universal deformation of a formal group law Γ over k, is a Morava E-theory, E_n [30].

Now recall the definition of the Lubin-Tate formal $\mathcal{O}_F$-module law.

Definition 4.2.3.6. ([22] [17] Section 4.5). Every finite extension F/Q_p of degree n has a corresponding connected finite étale n-fold cover of

- $\tilde{\mathcal{D}}^*_C/\underline{Q_p^x}$.

Let $\bar{\omega} \in \mathcal{O}_F$ be a uniformizer. Let $LT/\mathcal{O}_F$ be a Lubin-Tate formal $\mathcal{O}_F$-module law. This is a formal scheme isomorphic to Spf$\mathcal{O}_F[[T]]$. It has an $\mathcal{O}_F$-module structure such that multiplication by $\bar{\omega}$ sends T to a power series congruent to T^q modulo $\bar{\omega}$ for $q = \#\mathcal{O}_F/\bar{\omega}$.

Remark 4.2.3.7. Importantly, we have that $LT_{\mathcal{O}_F}$ is a p-divisible group of height n and dimension one and thus, in prestidigitorium, more connections arise. P-divisible formal group laws are great sources of arithmetically profinite extensions (APF), any completion of which is a perfectoid space [17].

Recall the definition of an APF extension.

Definition 4.2.3.8. ([42] Definition [25]). Let p be a prime and K a finite extension of Q_p with residue field k and valuation v_K normalized so that $v_K(K^x) = \mathbb{Z}$. Fix an algebraic closure $\bar{K}$ of K, and for any subfield E of $\bar{K}$ containing K write $G_E := Gal(\bar{K}/E)$. Recall that an infinite, totally wildly ramified extension L/K is said to be arithmetically profinite (APF) if the upper numbering ramification groups $G^u_K G_L$ are open in G_K for all $u \geq 0$.

Altogether, we have the following:

Conjecture 4.2.3.8. We have a topological localization of spectra $L_n; = L_{E_n}$ where E_n is the Lubin-Tate spectrum *nth* Morava E-theory, the *nth* chromatic localization.

Remark 4.2.3.9. Recall, the generalized Eilenberg-Steenrod cohomology is defined as the intrinsic cohomology of the $(\infty, 1)$-category of spectra [30]. It is known that every generalized cohomology is aptly represented by mapping spectra into a spectrum, per the Brown Representability Theorem. Recall,

> generalized Eilenberg-Steenrod cohomology is cohomology $E(X) = H(X, E)$ with coefficients in E a spectrum object [36].

We take the E spectrum object to be our diamond K-theory spectrum $K(\mathcal{D}^\diamond)$.

Our conjecture is immediate.

Conjecture 4.2.3.10. The étale cohomology of diamonds generalizes to a generalized Eilenberg-Steenrod cohomology and inherits the height filtration, therein inducing a chromatic filtration.

We now state our conjectured chromatic tower.

Conjecture 4.2.3.11. There exists a chromatic tower of the diamond K-theory spectrum

- $K(\mathcal{D}^\diamond) \to ... \to L_n K(\mathcal{D}^\diamond) \to L_{n-1} K(\mathcal{D}^\diamond) \to ... \to L_0 K(\mathcal{D}^\diamond)$,

where the open strata are labeled by the heights of the $LT/\mathcal{O}_F$.

If we can get $K\mathcal{D}^\diamond$ to be an Ω-spectrum, then our tower follows accordingly. There exists a chromatic tower of the diamond K-theory spectrum

- $\Omega K(\mathcal{D}^\diamond) \to ... \to L_n \Omega K(\mathcal{D}^\diamond) \to L_{n-1} K(\mathcal{D}^\diamond) \to ... \to L_0 \Omega K(\mathcal{D}^\diamond)$.

We now briefly review Rognes' Redshift Conjecture and its possible extension to large stable ∞-categories.

4.3 Efimov Redshift Conjecture

Recall Rognes' Redshift Conjecture.

Conjecture 4.3.1. ([24] Conjecture). Algebraic K-theory increases chromatic complexity. Let $K(R)$ be the K-theory spectrum of an E_∞ ring R. Then $K(R)$ has chromatic level one higher than R.

> Rognes's Redshift Conjecture states that the chromatic level of the algebraic K-theory spectrum $K(R)$ of an E-∞ ring R is one higher than R [24].

Question 4.3.2. What is the Efimov version of this Redshift Conjecture on the diamond spectrum?

Recall, Blumberg and Mandell construct a localization sequence of spectra

- $K(Z) \to K(ku) \to K(KU) \to \sigma K(Z)$

for the algebraic K-theory of topological K-theory, as a consequence of the following dévissage theorem [1].

Theorem 4.3.3. ([1] D'évissage Theorem). Let R be a connective S-algebra (A_∞ ring spectrum) with $\pi_0 R$ left Noetherian. Then there is a natural isomorphism in the stable category $K'(\pi_0 R) \to K'(R)$, where $K'(\pi_0 R)$ is Quillen's K-theory of the exact category of finitely generated left $\pi_0 R$-modules, and $K'(R)$ is the Waldhausen K-theory of the category of finitely generated finite stage Postnikov towers of left R-modules, $\mathcal{P}_R$.

We will use a similar formalism for the S-construction in the diamond setting. We now commence our perfectoid spaces presentation.

5 Perfectoid Spaces

Our exposition and souscompendium follow [17].

Our main definition is the following.

Definition 5.1. ([17] Definition 7.1.2). A perfectoid space is an adic space covered by affinoid adic spaces of the form $\mathrm{Spa}(R, R^+)$ with R a perfectoid ring.

We recall the definition of a perfectoid ring. Let p be a fixed prime.

Definition 5.2. ([Fon13] [17] Definition 6.1.1). A complete Tate ring R is perfectoid if R is uniform and there exists a pseudo-uniformizer $\bar{\omega} \in R$ such that $\bar{\omega}^p | p$ holds in R°, and such that the pth power Frobenius map

- $\Phi : R^\circ/\bar{\omega} \to R^\circ/\bar{\omega}^p$.

is an isomorphism.

We recall the highly refined structure of a totally disconnected perfectoid space.

Definition 5.3. ([12] Definition 7.1). A perfectoid space X is totally disconnected if it is qcqs and every open cover splits. A perfectoid space X is strictly totally disconnected if it is qcqs and every étale cover splits. Any totally disconnected perfectoid space is affinoid.

Moreover, we have the following.

Lemma 5.4. ([12] Lemma 7.3). Let X be a totally disconnected perfectoid space.

- There is a continuous projection $\pi : X \to \pi_0(X)$ to the profinite set $\pi_0(X)$ of connected components.
- All fibres of π are of the form $\mathrm{Spa}(K, K^+)$ for some perfectoid field K with an open and bounded valuation subring $K^+ \subset K$.

5.1 Tilting

We recall the tilting procedure.

Definition 5.1.1. ([17] Definition 6.2.1). Let R be a perfectoid Tate ring. The tilt of R is

- $R^b = \underset{\substack{\leftarrow \\ x \to x^p}}{lim} R$

given the inverse limit topology. A priori this is just a topological multiplicative monoid. It is given a ring structure with addition law

- $(x^{(0)}, x^{(1)}, \cdots) + (y^{(0)}, y^{(1)}, \cdots) = (z^{(0)}, z^{(1)}, \cdots)$

where

- $z^{(i)} = \lim_{n \to \infty} (x^{(i+n)} + y^{(i+n)})^{p^n} \in R.$

Theorem 5.1.2. ([17] Theorem 7.1.4). For any perfectoid space X with tilt X^b , the functor $Y \to Y^b$ induces an equivalence between the categories of perfectoid spaces over X, respectively X^b.

We now discuss the corresponding inverse functor, the untilt. The motivating question is the following:

Question 5.1.3 [17]. Given a perfectoid algebra R in characteristic p, what are all the untilts $R^\#$ of R?

We start with a pair (R, R^+).

Lemma 5.1.3. ([17] Lemma 6.2.8). Let $(R^\#, R^{\#+})$ be an untilt of (R, R^+), i.e. a perfectoid Tate ring $R^\#$ together with an isomorphism $\iota : R^{\#b} \to R$, such that $R^{\#+}$ and R^+ are identified under Lemma 6.2.5.

- **Lemma 5.1.4**. ([17] Lemma 6.2.5). The set of rings of integral elements $R^+ \subset R^\circ$ is in bijection with the set of rings of integral elements $R^{b+} \subset R^{b\circ}$, via $R^{b+} = \underset{\leftarrow x \to x^p}{lim} R^+$. Also, $R^{b+}/\bar{\omega}^b = R^+/\bar{\omega}$.

5.2 Adic Spaces and Formal Schemes

We review adic spaces as developed by Huber, formal schemes, rigid analytic varieties, and perfectoid rings, beginning with formal schemes and rigid-analytic geometries, two categories of geometric objects from nonarchimedean geometry.

Definition 5.2.1. ([17] Discussion Section 2.1). An adic ring is a topological ring carrying the I-adic topology for an ideal $I \subset A$, called an ideal of definition.

Examples 5.2.2 [17]. Examples are the following:

- $A = Z_p$ and $I = pZ_p$;
- $A = Z_p[[T]]$ and $I = (p, T)$;
- $A =$ an arbitrary discrete ring and $I = (0)$.

Definition 5.2.3. ([17] Discussion Section 2.1). If A is an adic ring, Spf$A$ is the set of open prime ideals of A. This agrees with SpecA/I for any ideal of definition I. Spf$A$ is given a topology and a sheaf of topological rings.

Definition 5.2.4. ([17] Discussion Section 2.1). For any $f \in A$, the nonvanishing locus is definied as $D(f) \subset$ Spf$A$ with the declaration that the $D(f)$ generate the topology of Spf$A$. We define the structure sheaf $\mathcal{O}_{\mathrm{Spf}A}$ by setting

- $\mathcal{O}_{\mathrm{Spf}A}(D(f))$

to be the I-adic completion of $A[f^{-1}]$.

Definition 5.2.5. ([17] Discussion Section 2.1). A formal scheme is a topologically ringed space which is locally of the form Spf$A$ for an adic ring A. Considering A with its discrete topology, the category of formal schemes contains the category of schemes as a full subcategory, via the functor which carries Spec A to Spf$A$.

Example 5.2.6 [17]. An example of a formal scheme is

- $X = \mathrm{Spf} Z_p[[T]]$.

  This is the formal open unit disc over Z_p. For any adic Z_p-algebra R, we have $X(R) = R^{\circ\circ}$, the ideal of topologically nilpotent elements of R.

Definition 5.2.7. ([17] Discussion Section 2.1). Let K be a nonarchimedean field. This is a field complete with respect to a nontrivial nonarchimedean absolute value $||$. For each $n \geq 0$ we have the Tate K-algebra

- $K < T_1, \cdots, T_n >$,

which is the completion of the polynomial ring $K[T_1, ..., T_n]$ under the Gauss norm. Equivalently, $K < T_1, ..., T_n >$ is the ring of formal power series in $T_1, ..., T_n$ with coefficients in K tending to 0. A K-affinoid algebra is a topological K-algebra A which is isomorphic to a quotient of some $K < T_1, , T_n >$.

Definition 5.2.8. ([17] Discussion Section 2.1). A rigid-analytic space over K is a G-topologized space equipped with a sheaf of K-algebras, which is locally isomorphic to a K-affinoid space. $X = \mathrm{MaxSpec} Q_p < T >$, the rigid closed unit disc over Q_p is a standard example.

Remark 5.2.9. ([17] Discussion Section 2.1). The category of formal schemes is linked to the category of rigid analytic varieties.

> There is a generic fiber functor $X \to X_\eta$ from a certain class of formal schemes over SpfZ_p, locally formally of finite type, to rigid-analytic spaces over Q_p.

Example 5.2.10. ([17] Discussion Section 2.1). As an example, the image of the formal open disc Spf$Z_p[[T]]$ over Z_p under this functor is the rigid open disc over Q_p. This is not a generic fiber since SpfZ_p has only one point so there is no generic point η with residue field Q_p.

Remark 5.2.11 [17]. The goal is to build a a category of adic spaces with both formal schemes and rigid-analytic spaces realized as full subcategories.

- $X \to X^{\mathrm{ad}}$

  denotes the functor from formal schemes to adic spaces. Objects are topologically ringed spaces, but now $(\mathrm{Spf}Z_p)^{\mathrm{ad}}$ has two points: a generic point η and a special point s. This makes $(\mathrm{Spf}Z_p)^{\mathrm{ad}}$ as a topological space the same as SpecZ_p.

We recall the definition of the adic generic fiber.

Definition 5.2.12. ([17] Discussion Section 2.1). For X a formal scheme over SpfZ_p, X^{ad} is fibered over $(\mathrm{Spf}Zp)^{\mathrm{ad}}$. The adic generic fiber of X is defined by

- $X^{\mathrm{ad}}_{\eta} = X^{ad} \times_{(\mathrm{Spf}Z_p)^{\mathrm{ad}}} \{\eta\}$.

If X is locally formally of finite type, then X^{ad}_{η} agrees with the adic space attached to Berthelot's X_{η}.

Remark 5.2.13. ([17] Discussion Section 2.1). There is a nice symmetry in the construction of adic spaces that follows that of formal and rigid analytic varieties. Specifically,

- as formal schemes are built from affine formal schemes associated to adic rings,
- and rigid analytic spaces from afifnoid spaces associated to affinoid algebras,
- adic spaces are built from affinoid adic spaces, which are associated to pairs of topological rings (A, A^+).

5.3 The Adic Spectrum Spa(A, A^+)

We recall the definition of an adic spectrum.

Definition 5.3.1. ([17] Discussion Section 2.1). The affinoid adic space associated to such a pair (A, A^+) is denoted

- $\mathrm{Spa}(A, A^+)$.

$\mathrm{Spa}(A, A^+)$ is called the adic spectrum.

We recall the definition of the adic spectrum $\mathrm{Spa}(A, A^+)$.

Definition 5.3.2. ([17] Definition 2.3.12). The adic spectrum $\mathrm{Spa}(A, A^+)$ is the set of equivalence classes of continuous valuations $|\cdot|$ on A such that $|A^+| \leq 1$. For $x \in \mathrm{Spa}(A, A^+)$ write $g \to |g(x)|$ for a choice of corresponding valuation.

The topology on $\mathrm{Spa}(A, A^+)$ is generated by open subsets of the form

- $\{x||f(x)| \leq |g(x)| \neq 0\}$

with $f, g \in A$.

Definition 5.3.3. ([17] Definition 2.2.1). A Huber ring is a topological ring A that admits an open subring $A^\circ \subset A$ which is adic with a finitely generated ideal of definition. A Huber pair is a pair (A, A^+), where A is a Huber ring and $A+ \subset A^\circ$ is an open and integrally closed subring.

> We can construct a spectral topological space $X = \mathrm{Spa}(A, A^+)$ consisting of equivalence classes of continuous valuations $| \circ |$ on A such that $|A^+| \leq 1$ [17].

Theorem 5.3.4. ([17] Theorem 2.3.3 [Hub93] Theorem 3.5 (i)]). The topological space $\mathrm{Spa}(A, A^+)$ is spectral.

Definition 5.3.5. ([17] Definition 2.2.5). A Huber ring R is Tate if it contains a topologically nilpotent unit (pseudo-uniformizers). An affinoid (Tate) ring is a pair (A, A^+) where A is a Tate ring and A^+ is a ring of integral elements of A.

Definition 5.3.6. ([17] Definition 6.1.1). A complete Tate ring R is perfectoid if R is uniform (the set of power-bounded elements $R^\circ \subset R$ is bounded) and there exists a pseudouniformizer $\bar{\omega} \in R$ such that $\bar{\omega}^p|p$ holds in R° ,and such that the pth power Frobenius map $\Phi : R^\circ/\bar{\omega} \to R^\circ/\bar{\omega}^p$ is an isomorphism.

Definition 5.3.7. ([17] Definition 6.1.7). A perfectoid field is a perfectoid Tate ring R which is a nonarchimedean field. The space of all maximal ideals, $\mathrm{Spm}(A)$ of a Tate algebra A is called an affinoid space.

Theorem 5.3.8. ([17] Theorem 6.1.10). Let (R, R^+) be a Huber pair such that R is perfectoid. Then for all rational subsets $U \subset X = \mathrm{Spa}(R, R^+)$, $\mathcal{O}_X(U)$ is again perfectoid. In particular, (R, R^+) is stably uniform, hence sheafy.

5.4 Sousperfectoid

Definition 5.4.1. ([17]Definition 6.3.1). Let R be a complete Tate-Z_p-algebra. Then R is sousperfectoid if there exists a perfectoid Tate ring $\tilde{R}$ with an injection $R \hookrightarrow \tilde{R}$ that splits as topological R-modules.

Example 5.4.2. ([17] Example 6.3.2). Two examples are the following.

- Any perfectoid ring is sousperfectoid.
- A Tate algebra $R = Q_p < T >$ is sousperfectoid, by taking
  - $\tilde{R} = Q_p^{cycl} < T^{\frac{1}{p^\infty}} >$.

Definition 5.4.3. ([17] Definition 7.1.2). A perfectoid space is an adic space covered by affinoid adic spaces of the form $\mathrm{Spa}(R, R^+)$ where R is a perfectoid ring.

Lemma 5.4.4. ([17] Lemma 8.3.5). The (absolute) product of two perfectoid spaces of characteristic p is again a perfectoid space.

Remark 5.4.5. ([17] Remark 8.3.6). Perf does not contain a final object. This means that an absolute product is not a fiber product.

5.5 Examples of Perfectoid Spaces

Examples of perfectoid spaces are the following:.

- **Example 5.5.1** [17]. The perfectoid Shimura variety
  - $S_{K^p} \sim \varprojlim_{K^p}(S_{K^pK_p} \bigotimes_E E_p)^{ad}$ [18].
- **Example 5.5.2** [17]. Any completion of an arithmetically profinite extension (APF) extension, in the sense of Fontaine and Wintenberger, is perfectoid.
  - [17] A source of APF extensions is p-divisible formal group laws.
- **Example 5.5.3** [17]. If K is a perfectoid field and $K+ \subset K$ is a ring of integral elements, then $\text{Spa}(K, K+)$ is a perfectoid space.
- **Example 5.5.4** [17]. Zariski closed subsets of an affinoid perfectoid space support a unique perfectoid structure.
- **Example 5.5.5** [17]. Any locally Zariski closed subset of a perfectoid space is a perfectoid space.
- **Example 5.5.6** [17]. (Non-example). The nonarchimedean field Q_p is not perfectoid as there is no topologically nilpotent element $\xi \in Z_p$ whose pth power divides p.

We have the following equivalence of categories.

Theorem 5.5.7. ([17] Theorem 9.4.4). The following categories are equivalent.

- Perfectoid spaces over Q_p.
- Perfectoid spaces X of characteristic p equipped with a "structure morphism"
  - $X \to \text{Spd}Q_p$.

Remark 5.5.8.

Both categories are fibered over Perf. For the category of perfectoid spaces X of characteristic p equipped with a "structure morphism" $X \to \text{Spd}Q_p$, this is obvious. For the category of perfectoid spaces over Q_p, the morphism to Perf is $X \to X^b$ [17].

Theorem 7.1.4 gives the existence of pullbacks. So we have that if $X \to Y$ is a morphism in Perf and $Y^{\#}$ is an untilt of Y, then there exists a unique morphism of perfectoid spaces

- $X^{\#} \to Y^{\#}$ whose tilt is $X \to Y$.

Remark 5.5.9. What is most important is that

the two fibered categories correspond to two presheaves of groupoids on Perf:

- $X \to \text{Untilt}_{Q_p}(X) = \{(X^{\#}, \iota)\}$,

where $X^{\#}$ is a perfectoid space over Q_p and $\iota : X^{\#b} \simeq X$ is an isomorphism.

- $SpdQ_p$ [17].

Proof. Sections of both presheaves have no nontrivial automorphisms. Thus, we can think of these as presheaves of sets. This means that illustrating an isomorphism between the fibered categories is the same as illustrating an isomorphism between the presheaves.

By definition, SpdQ_p is a sheaf for the pro-étale topology on Perf. It is shown that UntiltQ_p is a sheaf also. Let Untilt be the presheaf on Perf which assigns to X the set of pairs $(X^{\#}, \iota)$, where $X^{\#}$ is a perfectoid space (of whatever characteristic), and $\iota : X^{\#b} \simeq X$ is an isomorphism. The following lemma is immediate.

Lemma 5.5.10. ([17] Lemma 9.4.5 [Lemma 15.1 [12]]). Untilt is a sheaf on Perf.
Since the invertibility of p can be checked locally, it follows that Untilt$_{Q_p}$ is a sheaf as well. Now since both Untilt$_{Q_p}$ and SpdQ_p are sheaves on Perf, the constructions above globalize to give the desired isomorphism. □

5.6 Perfectoid Shimura Variety

A perfectoid Shimura variety investigates the torsion coefficients in the cohomology of locally symmetric spaces. The setting is the following:

Working adelically, we have the Shimura variety S_K, which is a quasiprojective scheme over the reflex field E, which is a finite extension of Q_p and K is any compact open subgroup $K \subset G(\mathbb{A}_f)$. Compact open subgroups are defined as:

- $K = K^p \times K_p \subset G(\mathbb{A}_f^p) \times G(\mathbb{Q}_p$

where K^p and K_p are compact open.

Now fix any prime p, and let $\mathbf{p}|p$ be a place of the reflex field E. Denote by S_K the rigid analytic variety, or adic space, corresponding to $S_K \otimes_E E_p$ and similarly for $F\ell_{\mathcal{G},\mu}$ [18].

The following theorem is a refinement of the theory of the Hodge-Tate period map from [17], which is a p-adic version of the antiholomorphic Borel embedding.

Theorem 5.6.1 ([18] Theorem 1.10). Assume that the Shimura datum is of Hodge type. Then for any sufficiently small compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$, there is a perfectoid space S_{K^p} over E_p such that

- $S_{K^p} \sim \varprojlim_{K^p} (S_{K^pK_p} \bigotimes_E E_p)^{\mathrm{ad}}$.

Moreover, there is a Hodge-Tate period map

- $\pi_{HT} : S_{K^p} \to F\ell_{G,\mu}$

which agrees with the Hodge-Tate period map constructed in [17] for the Siegel case, and is functorial in the Shimura datum.

Remark 5.6.2 [18]. The geometry of this map is curious as the fiber dimension "jumps." Specifically, the map π_{HT} is a profinite étale cover when restricted to Drinfeld's upper half-plane $\hat{\Omega}^2 = \hat{\mathbb{P}}^1 n \mathbb{P}^1(\mathbb{Q}_p)$. However, fibres over points in $\mathbb{P}^1(\mathbb{Q}_p)$ are Igusa curves, which are coarse moduli spaces of elliptic curves in characteristic p with a level structure that gives a finite covering [14].

We conclude our discussion with the culminating-following:

Question 5.6.3. What, conjecturally, could the higher $\mathrm{K}^{\mathrm{Efimov}}$-groups of S_K, $\mathcal{M}_{LT,\infty}$, and APF extensions, respectively, compute?

6 Global Langlands over Number Fields

We now give an exposition of Drinfeld's lemma for diamonds via shtukas in the global Langlands correspondence over function fields, illuminating how

> the diamond formalism transports finite étale maps between different presentations of a diamond as the diamond of an analytic adic space [17].

Our main goal will be establishing a connection between moduli spaces of mixed-characteristic local G-shtukas, which are representable by locally spatial diamonds, and higher $\mathrm{K}^{\mathrm{Efimov}}$-groups of diamonds.

Our exposition and souscompendium follows [17] and [12].

To motivate the function field case, we quickly review the global Langlands Correspondence over number fields concluding with Scholze's construction of p-adic Galois representations and work on perfectoid shimura varieties. All constructions and definitions follow [13].

Recall the Global Langlands (Clozel-Fontaine-Mazur) Conjecture.

Conjecture 6.1. ([13] Global Langlands (Clozel-Fontaine-Mazur) Conjecture I.1). Let F be a number field, p some rational prime, and fix an isomorphism $\mathbb{C} \simeq \mathbb{Q}_p$. Then for any $n \geq 1$ there is a unique bijection between the set of L-algebraic cuspidal automorphic representations of $GL_n(\mathbb{A}_F)$, and the set of (isomorphism classes of) irreducible continuous representations $Gal(\bar{F}/F) \to GL_n(\mathbb{Q}_l)$ which are almost everywhere unramified, and de Rham at places dividing p, such that the bijection matches Satake parameters with eigenvalues of Frobenius elements.

> For the L-algebraic automorphic representation, the (normalized) infinitesimal character of π_v is integral for all infinite places v of F. Also, $\mathbb{A}_F = \prod_v{}' F_v$ denotes the adeles of F, which is the restricted product of the completions F_v of F at all (finite or infinite places) of F [13].

Remark 6.2. The main point is the existence of p-adic Galois representations associated to L-algebraic automorphic representations for a general connected reductive group over a number field. To construct these representations, the idea is to work adelically. So for any compact open subgroup $K \subset G(\mathbb{A}_f)$, we have the Shimura variety S_K which is a quasiprojective scheme over the reflex field E, which is a finite extension of Q_p, and where K denotes compact open subgroups

- $K = K^p \times K_p \subset G(\mathbb{A}_f^p) \times G(\mathbb{Q}_p)$

where K^p and K_p are compact open [13].

Recall the following theorems.

Theorem 6.3. ([13] Theorem V.4.1). Every cuspidal regular algebraic automorphic representation of GL_n over a totally real or CM field F has an associated Galois representation. For the special case $F = Q$ we have

Theorem 6.4. ([13] Corollary V.1.7 (Scholze)). Let g be a Hecke eigenclass in the singular cohomology $H^j(\Gamma/\mathscr{H}_n, C)$, and let $a_{,i}(g)$ be the eigenvalue of $T_{,i}$ on g for $\nmid N$ prime and $i = 1, ..., n$. There exists a continuous semi-simple p-adic Galois representation $\rho : Gal(\bar{Q}/Q) \rightarrow GL_n(\bar{Q_p})$ which is associated to g in the sense that for all primes $\nmid Np$ ρ is unramified at and the characteristic polynomial of $\rho(Frob)$ is $x^n + \Sigma(-1)^{kk(k-1)/2}\alpha_{,k}(g)x^{n-k}$.

Theorem 6.5. ([13] Corollary V.4.2). Let π be a cuspidal algebraic automorphic representation of GL_n. Let p be a prime, and let $\iota : C \rightarrow \bar{Q_p}$ be a field isomorphism. Then there exists an irreducible p-adic Galois representation $\rho : Gal(\bar{Q}/Q) \rightarrow GL_n(\bar{Q_p})$, such that for almost all primes , ρ is unramified at and the roots of the characteristic polynomial of $\rho(Frob_l)$ are the images under ι of the Satake parameters $\alpha_{,1}, * * * \alpha_{,n}$ of π. Furthermore, ρ is geometric.

Remark 6.6.

The converse is the Fontaine-Mazur conjecture. So, there is a conjectural bijection between algebraic cuspidal automorphic representations of GL_n and a certain class of n-dimensional p-adic representations of $Gal(\bar{Q}/Q)$ [13].

This concludes our review of global Langlands for number fields, where Langlands functoriality and perfectoid spaces make a prominent appearance.

We now begin our discussion of global Langlands for function fields. Our exposition follows [17].

6.1 Global Langlands over Function Fields

Remark 6.1.1. The connection lies in developing the geometric Langlands program over the Fargues-Fontaine curve, using the geometry of perfectoid spaces and diamonds. Diamonds contain such rich geometric structure and, as Scholze says,

> diamonds are to perfectoid spaces as algebraic spaces are to schemes [17].

6.2 Shtukas

To introduce Drinfeld's lemma for diamonds, we begin with Drinfeld's notion of an X-shtuka, for X/F_p a smooth projective curve with function field K. Our exposition and souscompendium follow [17].

Definition 6.2.1. ([17] Definition 1.1.1). A shtuka of rank n over an F_p-scheme S is a pair $(\mathcal{E}, \phi_E)$, where $\mathcal{E}$ is a rank n vector bundle over $S \times_{F_p} X$ and

- $\phi_E : Frob_S^* \mathcal{E} \dashrightarrow \mathcal{E}$

is a meromorphic isomorphism which is defined on an open subset $U \subset S \times_{F_p} X$ that is fiberwise dense in X. Here,

- $Frob_S : S \times_{F_p} X \to S \times_{F_p} X$

refers to the product of the pth power Frobenius map on S with the identity on X.

Remark 6.2.2. ([17] Discussion Section 1.1). We obtain the Langlands correspondence for X by studying moduli spaces of X-shtukas. The idea is the following:

> To a shtuka $(\mathcal{E}, \phi_{\mathcal{E}})$ of rank n over $S = Spec\ k$, for k algebraically closed, we attach these discrete data:

- 1. The collection of points $x_1, \cdots, x_m \in X(k)$ where $\phi_{\mathcal{E}}$ is undefined. These points are called the legs of the shtuka.
- 2. For each $i = 1, \cdots, m$ a conjugacy class μ_i of cocharacters $G_m \to GL_n$, encoding the behaviour of ϕ_E near x_i.

By fixing these data we define a moduli space $Sht_{GL_n,\{\mu_1,\cdots,\mu_m\}}$ whose k-points classify the following data:

- 1. An m-tuple of points $(x_1, \cdots, x_m)$ of $X(k)$.
- 2. A shtuka $(\mathcal{E}, \phi_E)$ of rank n with legs $x_1, \cdots, x_m$, for which the relative position of $\mathcal{E}\hat{x_i}$ and $(Frob_S^*\mathcal{E})\hat{x_i}$ is bounded by the cocharacter μ_i for all $i = 1, \cdots, m$ [17].

Remark 6.2.3. ([17] Discussion Section 1.1).

- $Sht_{GL_n,\{\mu_1,\cdots,\mu_m\}}$ is a Deligne-Mumford stack.

Let

- $f : Sht_{GL_n,\{\mu_1,\cdots,\mu_m\}} \to X^m$

map a shtuka onto its m-tuple legs.

Remark 6.2.4. ([17] Discussion Section 1.1). $Sht_{GL_n,\{\mu_1,\cdots,\mu_m\}}$ takes the form of an equal-characteristic analogue of Shimura varieties fibered over $SpecZ$.

What makes the function field more complete is that Shimura varieties are not fibered over anything of the form

- $SpecZ \times_{F_1} SpecZ$.

Remark 6.2.5. ([17] Discussion Section 1.1).

To obtain a family of stacks $Sht_{GL_n,\{\mu_1,\cdots,\mu_m\},N}$ level structures are added parametrized by finite closed subschemes $N \subset X$. These N are effective divisors. A level N structure on $(\mathcal{E}, \phi_{\mathcal{E}})$ is a trivialization of the pullback of $\mathcal{E}$ to N in a way compatible with $\phi_{\mathcal{E}}$. Along with the family of stacks $Sht_{GL_n,\{\mu_1,\cdots,\mu_m\},N}$ we get morphisms

- $f_N : Sht_{GL_n,\{\mu_1,\cdots,\mu_m\},N} \to (X/N)^m$.

$Sht_{GL_n,\{\mu_1,\cdots,\mu_m\},N}$ carries an action of $GL_n(\mathcal{O}_N)$, by altering the trivialization of $\mathcal{E}$ on N.

Linking us back to Langlands, we have the inverse limit

- $\underset{\leftarrow N}{lim} Sht_{GL_n,\{\mu_1,\cdots,\mu_m\},N}$

admits an action of $GL_n(\mathbb{A}_K)$, via Hecke correspondences.

Remark 6.2.6. Recall, $\mathbb{A}_K = \prod_v' F_v$ denotes the adeles of F. (finite or infinite places) of F.

Remark 6.2.7. ([17] Discussion Section 1.1). To connect cuspidal automorphic representations of $GL_n(\mathbb{A}_K)$ with ℓ-adic representations of $Gal(\bar{K}/K)$, Scholze considers the middle cohomology of the stack $Sht_{GL_n,\{\mu_1,\cdots,\mu_m\},N}$. Scholze considers the cohomology

- $R^d(f_N)_!\bar{Q}_\ell$,

an étale $\bar{Q}_\ell$-sheaf on X^m, taking d the relative dimension of f.

Example 6.2.8. ([17] Example Section 1.1). We start with a simpler example. Take a $\bar{Q}_\ell$ -sheaf $\mathbb{L}$ on X^m. This becomes lisse upon restriction to U^m for some dense open subset $U \subset X$. $\mathbb{L}$ is a representation of the étale fundamental group $\pi_1(U^m)$ on an $\bar{Q}_\ell$ -vector space. This should connect with $\pi_1(U)$, because this is a quotient of $Gal(\bar{K}/K)$. We consider the natural homomorphism

- $\pi_1(U^m) \to \pi_1(U) \times \cdots \times \pi_1(U)$ (m copies).

At this point "partial Frobenii" are introduced to deal with injectivity issues. These take the following form.

Definition 6.2.9. ([17] Definition Section 1.1). For $i = 1, \cdots, m$, we have a partial Frobenius map

- $F_i : X^m \to X^m$.

This is $Frob_X$ on the ith factor, and the identity on each other factor. For an étale morphism $V \to X^m$, a system of partial Frobenii on V is called a commuting collection of isomorphisms $F_i^* V \simeq V$ over X^m. Their product is the relative Frobenius of $V \to X^m$. Finite étale covers of U^m equipped with partial Frobenii form a Galois category. Therefore, they are classified by continuous actions of a profinite group

- $\pi_1(U^m/\text{partial Frob.})$

on a finite set.

We are now ready to state Drinfeld's lemma, the diamond version of which follows.

Lemma 6.2.10. ([17] Lemma 1.1.2 Drinfeld's Lemma [Dri80, Theorem 2.1]). The natural map

- $\pi_1(U^m/\text{partial Frob.}) \to \pi_1(U) \times \cdots \times \pi_1(U)$ (m copies)

is an isomorphism.

Remark 6.2.11. This lemma has nice consequences. One is that

> if $\mathbb{L}$ is a $\bar{Q}_\ell$ -local system on U^m, which comes with commuting isomorphisms $F_i^*\mathbb{L} \simeq \mathbb{L}$ then $\mathbb{L}$ determines a representation of $\pi_1(U)^m$ on a $\bar{Q}_\ell$-vector space...The moduli space of shtukas admits partial Frobenii morphisms lying over the F_i. Therefore, its cohomology does as well [17].

We now summarize Drinfeld, L. Lafforgue, and V. Lafforgue's Langlands work.

> Taking the limit over N results in a big representation of
>
> - $Gl_n(\mathbb{A}_K) \times Gal(\bar{K}/K) \times \cdots \times Gal(\bar{K}/K)$ on $\underset{\to N}{lim} R^d(f_N)_! \bar{Q}_\ell$.
>
> The expectation is that the cuspidal part of this space decomposes as
>
> - $\underset{\to N}{lim} R^d(f_N)_! \bar{Q}_\ell = \bigoplus_\pi \pi \otimes (r_1 \circ \sigma(\pi)) \otimes \cdots \otimes (r_m \circ \sigma(\pi))$.

where

- π runs over cuspidal automorphic representations of $GL_n(\mathbb{A}_K)$.
- $\sigma(\pi) : Gal(\bar{K}/K) \to GL_n(\bar{Q}_\ell)$ is the corresponding $\mathbb{L}$-parameter, and
- $r_i : GL_n \to GL_{n_i}$ is an algebraic representation corresponding to μ_i [17].

Remark 6.2.12. In summary, Drinfeld (for n = 2) and L. Lafforgue (for general n) consider $m = 2$, with cocharacters μ_1 the n-tuple (1, 0, . . . , 0) and μ_2 the n-tuple (0, . . . , 0, -1).

> They correspond to the tautological representation $r_1 : GL_n \to GL_n$ and r_2 its dual. They prove the above decomposition and constructed the Langlands correspondence $\pi \to \sigma(\pi)$.
>
> In parallel, V. Lafforgue considers, in place of GL_n, general reductive groups G and uses G-shtukas, shtukas with vector bundles replaced by G-bundles. V. Lafforgue produces a correspondence $\pi \to \sigma(\pi)$ from cuspidal automorphic representations of G to L-parameters using moduli of G-shtukas $Sht_{G,\{\mu_1,\cdots,\mu_m\}}$ with arbitrary legs and cocharacters [17].

6.3 $\mathbf{Spa}Q_p \times \mathbf{Spa}Q_p$

Frobenius appears critically in the global Langlands program. Frobenius does not appear in geometric Langlands, which studies the (hopeful) diamond stack Bun_G on X. "Arithmetic Langlands" studies Bun_G together with its Frobenius map.

We now turn our attention to the use of diamonds in Arithmetic Langlands. Our exposition follows [17] Section 1.2. The goal is to have moduli spaces of shtukas over number fields. One issue to be resolved is that these moduli spaces would exist over an object of the form

- $\mathrm{Spec}Z \times \mathrm{Spec}Z$

where the product is over F_1 [17].

Question. 6.3.1. ([17] Discussion Section 1.2). What if, instead of working with $\mathrm{Spec}Z \times \mathrm{Spec}Z$, we work with

- $SpecZ_p \times SpecZ_p$,

which is the completion of Spec$Z$ $\times$ Spec$Z$ at (p, p).

Remark. 6.3.2. ([17] Discussion Section 1.2). SpecZ_p $\times$ SpecZ_p is a creature of nonarchimedean analytic geometry. Thus it is more properly called

- SpaZ_p $\times$ SpaZ_p,

where Spa denotes the adic spectrum. SpaZ_p $\times$ SpaZ_p should contain SpaQ_p $\times$ SpaQ_p as a dense open subset.

Remark. 6.3.3. ([17] Discussion Section 1.2). Motivated by Lafforgue and Drinfeld's use of shtukas to prove the global langlands correspondence for our $n = 2$, Scholze defines mixed charactersitic local shtukas, constructs moduli spaces of shtukas in mixed characteristic, and proves the analogue of Drinfeld's lemma for the product $SpaQ_p \times SpaQ_p$.

SpaQ_p $\times$ SpaQ_p is described as follows.

Definition. 6.3.4. ([17] Discussion Section 1.2). Consider the open unit disc

- $D_{Q_p} = \{x||x| \leq 1\}$

as a subgroup of the adic version of cocharacter G_m via $x \to (1+x)^p - 1$. Consider

- $\tilde{D}_{Q_p} = \underset{x \to (1+x)^p - 1}{\underset{\leftarrow}{lim}} D_{Q_p}$.

Upon base extending to a perfectoid field, this is a perfectoid space. It carries the structure of a Q_p-vector space and its punctured form is

- $\tilde{D}^*_{Q_p} = \tilde{D}_{Q_p} \backslash \{0\}$.

This has an action of Q_p^x. The quotient

- $\tilde{D}^*_{Q_p} / Z_p^x$,

which does not exist in the category of adic spaces, is considered.

We finally have the definition of SpaQ_p $\times$ SpaQ_p.

Definition 6.3.5. ([17] Definition 1.2.1). Let

- $\mathrm{Spa}Q_p \times \mathrm{Spa}Q_p = \tilde{D}^*_{Q_p}/Z_p^x$,

the quotient being taken formally.

Remark 6.3.6. ([17] Discussion 1.2.1). Let

  - $X = (\tilde{D}^*_{Q_p})/\phi^Z$.

  A finite étale cover of X is a Q_p^x-equivariant finite étale cover of $\tilde{D}^*_{Q_p}$ and the corresponding profinite group $\pi_1(X)$ classifies the Q_p^x-equivariant finite étale covers.

The following theorem, our culminating theorem, is a local version of Drinfeld's lemma in the case $m = 2$.

Theorem 6.3.7.([17] Drinfelds lemma $m = 2$). We have

- $\pi_1(X) \simeq Gal(\bar{Q_p}/Q_p) \times Gal(\bar{Q}p/Q_p)$.

For $K = F_p((t))$, the quotient $X = D^*K/\phi^Z$ is an adic space. This makes

- $\pi_1(X) \simeq Gal(\bar{K}/K) \times Gal(\bar{K}/K)$.

Remark 6.3.8. ([17] Discussion 1.2.2). This remarkable theorem suggests that if a moduli space of Q_p-shtukas

  fibered over products such as $\mathrm{Spa}Q_p \times \mathrm{Spa}Q_p$ could be defined, then its cohomology would produce representations of

  - $Gal(\bar{Q_p}/Q_p) \times Gal(\bar{Q_p}/Q_p)$.

Remark 6.3.9. ([17] Discussion 1.2.2).

  A Q_p-shtuka over S would be a vector bundle $\mathcal{E}$ over

  - $\mathrm{Spa}Q_p \times S$,

  with a meromorphic isomorphism

- $Frob_S^*\mathcal{E} \dashrightarrow \mathcal{E}$.

We have to make Spa$Q_p \times S$ and $Frob_S$ geometric, similar to giving Spa$Q_p \times$ SpaQ_p geometric meaning.

Remark 6.3.10. ([9] Discussion 1.2.2). Working in the equal characteristic setting is hopeful.

> Let K be a local field of characteristic p, $K = F_p((t))$. Take a topologically finite type adic space S over SpaF_p. The product
>
> - Spa$K \times_{\mathrm{Spa}F_p} S$
>
> is again an adic space. It is the punctured open unit disc over S, so G-bundles can be defined here, as well as shtukas and moduli spaces of K-shtukas.
>
> For $K = Q_p$, meaning is given to
>
> - $SpaQ_p \times S$

for S a perfectoid space of characteristic p. Then moduli spaces of p-adic shtukas can be built.

Remark 6.3.11. This is our main point. Moduli spaces of p-adic shtukas are diamonds, which are quotients of perfectoid spaces by pro-étale equivalence relations. Specifically, this moduli space of shtukas is a diamond fibered over the m-fold product

- $SpaQ_p \times SpaQ_p \times \cdots_m \times SpaQ_p$.

We want to look at the Efimov K-theory of these moduli spaces.

The intent is to construct isomorphism classes of diamonds and build a K-theory framework to recover global information from the higher $\mathrm{K}^{\mathrm{Efimov}}$-groups of these specific diamonds:

- $K^{\mathrm{Efimov}}(\mathcal{Y}_{S,E}^{\diamond})$ for $\mathcal{Y}_{S,E}^{\diamond} = S \times (Spa\mathcal{O}_E)^{\diamond}$ and
- Spd$Q_p \times_{\diamond}$ SpdQ_p, the diamond self-product.

Remark 6.3.12. Extensive work has already been done examining G-torsors on the relative Fargues-Fontaine curve $\mathcal{X}_{FF,S}$. Let $S = Spa(R, R^+)$ be affinoid with R a perfectoid ring. We have the following equivalence of categories:

Theorem 6.3.13. ([17] Theorem 9.4.4). The following categories are equivalent:

- Pro-étale $\underline{G(Q_p)}$-torsors on S, and
- G-torsors on $\mathcal{X}_{FF,S}$, which are trivial at every geometric point of S.

The moduli spaces of shtukas in mixed characteristic live in the category of diamonds. Studying the isomorphism classes of moduli spaces of shtukas as the higher K-groups of diamonds could link diamnods and global Langlands over function fields.

Let us formalize the connection between shtukas and diamonds. Our exposition and sous-compendium summarizes the main constructions in [17].

6.4 Mixed-characteristic local G-shtukas and locally spatial diamonds

Varshavsky constructed moduli spaces of global equal-characteristic shtukas. Moduli spaces of mixed-characteristic local G-shtukas are the local analogues. The set up is a smooth projective geometrically connected curve X defined over a finite field F_q and a reductive group G/F_q.

Definition 6.4.1. ([17] Discussion Lecture 23). The moduli space of G-shtukas with m legs is a stack $Sht_{G,m}$, which is not locally of finite type, equipped with a morphism to X^m. For an F_q-scheme S, the S-points of $Sht_{G,m}$ classify triples consisting of a G-torsor $\mathcal{P}$ on $S \times_{F_q} X$, an m-tuple $x_1, \cdots, x_m \in X(S)$, and an isomorphism

- $\phi_{\mathcal{P}} : (Frob_S^* \mathcal{P})|_{(S \times_{F_q} X) \setminus \bigcup_{i=1}^m \Gamma_{x_i}} \tilde{\rightarrow} \mathcal{P}|_{(S \times_{F_q} X) \setminus \bigcup_{i=1}^m \Gamma_{x_i}}$.

Given an m-tuple of conjugacy classes of cocharacters $\mu_1, \cdots, \mu_m$ of G, we can define a closed substack $Sht_{G,\{\mu_i\}}$ of $Sht_{G,m}$ classifying shtukas where $\phi^{\mathcal{P}}$ is bounded by μ_i.

Then $Sht_{G,\{\mu_i\}}$ is a Deligne-Mumford stack, which is locally of finite type over X^m.

Definition 6.4.2. ([17] Discussion Lecture 23).

We can then add level structures to these spaces of shtukas and obtain a tower of moduli spaces which admits an action of the adelic group $G(\mathbb{A}_F)$, where F is the function field of X. V. Lafforgue uses the cohomology of these towers of moduli spaces as the primary means to construct the "automorphic to Galois" direction of the Langlands correspondence for G over F.

We now define the space of local mixed-characteristic G-shtukas and the main theorem.

The set up is the following:

Let G be a reductive group over Q_p. G does not live over Z_p in the mixed characteristic setting. So we choose a smooth group scheme $\mathcal{G}$ over Z_p with generic fiber G and connected special fiber. Now let $S = Spa(R, R^+)$ be an affinoid perfectoid space of characteristic p, with pseudouniformizer $\hat{\omega}$. Take k a discrete algebraically closed field, and $L = W(k)[1/p]$ [17].

Definition 6.4.3. ([17] Definition 23.1.1). Let $(\mathcal{G}, b, \{\mu_i\})$ be a triple consisting of a smooth group scheme $\mathcal{G}$ with reductive generic fiber G and connected special fiber, an element $b \in G(L)$, and a collection $\mu_1, \cdots, \mu_m$ of conjugacy classes of cocharacters $G_m \to G_{\bar{Q_p}}$. For $i = 1, \cdots, m$, let E_i/Q_p be the field of definition of μ_i , and let $\hat{E_i} = E_i \cdot L$. The moduli space

- $Sht_{\mathcal{G},\lfloor,\{\mu_\rangle} \to Spd\hat{E}_1 \times_{Spdk} \cdots \times_{Spdk} Spd\hat{E}_m$

of shtukas associated with $(\mathcal{G}, b, \{\mu_i\})$ is the presheaf on $Perf_k$ sending $S = Spa(R, R^+)$ to the set of quadruples $(\mathcal{P}, \{S_i^{\#}\}, \phi_{\mathcal{P}}, \iota_r\}$ where:

- $\mathcal{P}$ is a G-torsor on $S \dot{\times} SpaZ_p$,
- $S_i^{\#}$ is an untilt of S to $\hat{E}_i$ for $i = 1, \cdots, m$,
- $\phi^{\mathcal{P}}$ is an isomorphism $\phi_{\mathcal{P}} : (Frob_S^* \mathcal{P})|_{(S \times_{F_q} X) \setminus \bigcup_{i=1}^m \Gamma_{x_i}} \tilde{\to} \mathcal{P}|_{(S \times_{F_q} X) \setminus \bigcup_{i=1}^m \Gamma_{x_i}}$ and finally
- ι_r is an isomorphism $\iota_r : \mathcal{P}|_{\mathcal{Y}_{[r,\infty)}}(S) \tilde{\to} G \times \mathcal{Y}_{[r,\infty)}(S)$ for large enough r, under which $\phi^{\mathcal{P}}$ gets identified with $b \times Frob_S$.

Recall the open subset $\mathcal{Y}_{[r,\infty)}(S) = \{|\bar{\omega}| \leq |\bar{p}^r| \neq 0\} \subset S \dot{\times} SpaZ_p$.

The main theorem is the following.

Theorem 6.4.4. ([17] Theorem 23.1.4). The moduli space $Sht_{G,b,\{\mu_i\}}$ is a locally spatial diamond.

Remark 6.4.5. We note that descent results imply that $Sht_{G,b,\{\mu.\}}$ is a v-sheaf and we have already seen that all diamonds are v-sheaves. The goal is that, through our Efimov K-theory, the higher $\mathrm{K}^{\mathrm{Efimov}}$-groups of the diamond Fargues-Fontaine curve will return the mixed-characteristic shtuka datum and or any property of a Banach-Colemz space.

A diamond version of Drinfeld's lemma is constructed to analyze the moduli space of shtukas, which is a diamond. Before we state the theorem we summarize the construction and the definition of the Fargues-Fontaine curve through the category of Banach-Colmez spaces, which is an important class of diamonds. Our exposition and souscompendium follows [17]. We commence with the following:

Definition 6.4.6. ([17] Definition 15.2.6). The Fargues-Fontaine curve is the quotient

- $\mathcal{X}_{FF,S} = \mathcal{Y}_{(0,\infty)}(S)/\phi^Z$.

This is a sousperfectoid [9] adic space over Q_p. For any perfectoid field K/Q_p, the base change

- $X_{(FF,S)} \times_{SpaQ_p} SpaK$

is a perfectoid space. There is a diamond version.

Proposition 6.4.7. ([17] Proposition 15.2.7). There is a natural isomorphism

- $\mathcal{X}^{\diamond}_{FF,S} \simeq S/\phi^Z \times SpdQ_p$.

We now explain the $\mathcal{Y}_{(0,\infty)}$ term in the Fargues-Fontaine Curve.

Definition 6.4.8. ([17] Proposition 11.2.1). The fiber product

[9] cf. [17] Definition 6.3.1.

- $SpaC_p^b \dot{\times} SpaZ_P$

is defined as the analytic adic space

- $\mathcal{Y}_{(0,\infty)} = SpaA_{inf} \backslash \{[p]^b\} = 0$

with $A_{inf} = W(\mathcal{O}_{C_p^b})$ and its associated diamond $SpaC_p^b \times SpdZ_p$. Invert p to get

- $SpaC_p^b \times SpdQ_p$

as the diamond associated to

- $\mathcal{Y}_{(0,\infty)} = SpaA_{inf} \backslash \{p[p]^b\} = 0.$

From this construction we have

- $\mathcal{Y}^{\diamond}_{(0,\infty)/G_{Q_p}} = SpdQ_p \times SpdQ_p.$

Remark 6.4.9. Recall that the Fargues-Fontaine curve was defined as

- $\mathcal{X}_{FF} = \mathcal{Y}_{(0,\infty)}/\phi^Z.$

  We have $\pi_1(\mathcal{X}_{FF}) = G_{Q_p}$ by Theorem 13.5.7. Additionally we have

  - $\pi_1(\mathcal{X}_{FF}) = \pi_1(\mathcal{X}^{\diamond}_{FF}) = \pi_1((SpdC_p \times SpdQ_p)/Frob^Z_{SpdQ_p}) = \pi_1((\tilde{\mathcal{D}}^*_{C_p})^{\diamond}/\underline{Q_p^x}).$

  So the $n = 2$ case of Drinfeld's Lemma for diamonds follows [17].

Theorem 6.4.10. ([17] Theorem 15.1.1 Drinfeld's Lemma for diamonds).

- $\pi_1((\tilde{\mathcal{D}}^*_{Q_p})^{\diamond}/\underline{Q_p^{\times}} \simeq G_{Q_p} \times G_{Q_p}.$

6.5 Banach-Colmez Spaces

We now introduce the category of Banach-Colmez spaces and state the general theorem of Drinfeld's lemma for diamonds.

Definition 6.5.1. ([17] Definition 15.2.1). Fix an algebraically closed nonarchimedean field C/Q_p. The category of Banach-Colmez spaces over C is the thick abelian subcategory of the category of pro-étale sheaves of $\underline{Q_p}$-modules on $Perfd_C$ generated by $\underline{Q_p}$ and $G^{\diamond}_{a,C}$.

Definition 6.5.2. ([17] Definition 15.2.8). Let $\mathcal{E}$ be a coherent sheaf on X_{FF} whose torsion-free quotient has only nonnegative slopes. The Banach-Colmez space $\mathcal{BC}(\mathcal{E})$ associated with $\mathcal{E}$ is the pro-´etale sheaf

- $\mathcal{BC}(\mathcal{E}) :\in Perfd_C \to H^0(\mathcal{X}_{\mathcal{FF},\mathcal{S}}, \mathcal{E}|_{\mathcal{X}_{\mathcal{FF},\mathcal{S}}})$.

The grand conclusion is the main theorem of Le Bras which describes the category of Banach-Colmez spaces via coherent sheaves on the Fargues-Fontaine curve.

> We need to consider vector bundles $\mathcal{E}$ all of whose Harder-Narasimhan slopes are negative and whose global sections of $\mathcal{E}$ vanish.

Definition 6.5.3. ([17] Definition 15.2.11). Let $\mathcal{E}$ be a vector bundle on $X\mathcal{X}_{FF}$ all of whose Harder-Narasimhan slopes are negative. The Banach-Colmez space $\mathcal{BC}(\mathcal{E}[1])$ associated with $\mathcal{E}$ is the pro-étale sheaf

- $\mathcal{BC}(\mathcal{E}[1]) :\in Perfd_C \to H^1(\mathcal{X}_{\mathcal{FF},\mathcal{S}}, \mathcal{E}|_{\mathcal{X}_{\mathcal{FF},\mathcal{S}}})$.

We conclude this section with Le Bras' theorem.

Theorem 6.5.4. ([17] Theorem 15.2.12 (Le Bras [LB18])). All Banach-Colmez spaces are diamonds. The category of Banach-Colmez spaces is equivalent to the full subcategory of the derived category of coherent sheaves on $\mathcal{X}_{FF}$ of objects of the form

- $\mathcal{E}_{\geq 0} \oplus \mathcal{E}_{<0}[1]$,

where the torsion-free quotient of $\mathcal{E}_{\geq 0}$ has only nonnegative Harder-Narasimhan slopes, and $\mathcal{E}_{<0}$ is a vector bundle all of whose Harder-Narasimhan slopes are negative. In particular, the category of Banach-Colmez space depends on C only through its tilt C^b.

6.6 Drinfeld's Lemma for Diamonds

We will now state the diamond version of Drinfeld's lemma and show another important example of

> the diamond formalism transporting finite étale maps between different presentations of a diamond as the diamond of an analytic adic space [17].

Our discussion follows [17] Section 16.3.

Definition 6.6.1. ([17] Discussion Section 16.3). A diamond $\mathcal{D}$ is defined to be connected if its underlying topological space $|\mathcal{D}|$ is connected.

Definition 6.6.2. ([17] Discussion Section 16.3). Finite étale covers of $\mathcal{D}$ of a connected diamond $\mathcal{D}$ form a Galois category. For a geometric point $x \in \mathcal{D}(C, \mathcal{O}_C)$ Scholze defines a profinite group

- $\pi_1(\mathcal{D}, x)$,

such that finite $\pi_1(\mathcal{D}, x)$-sets are equivalent to finite étale covers $\mathcal{E} \to \mathcal{D}$, for vector bundle $\mathcal{E}$.

All the connected schemes X_i appearing in Drinfeld's lemma are replaced with diamond $SpdQ_p$. While Q_p has characteristic 0, its diamond $SpdQ_p$ admits an absolute Frobenius

- $F : SpdQ_p \to SpdQ_p$.

Remark 6.6.3. ([17] Discussion Section 16.3). Recall,

> there is an absolute Frobenius defined on perfectoid affinoids in characteristic p and $SpdQ_p$ is a sheaf on the category Perf of perfectoid affinoids in characteristic p.
>
> Let
>
> - $(SpdQ_p \times \cdots \times SpdQ_p/p.Fr.)_{\text{fét}}$
>
> be the category of finite étale covers $E \tilde{\to} (SpdQ_p)^n$ equipped with commuting isomorphisms $\beta_i : E \tilde{\to} F_i^* E$ (where F_i is the ith partial Frobenius), $i = 1, \cdots, n$ such that

- $\prod_i \beta_i = F_{E/(SpdQ_p)^n} : E \tilde{\rightarrow} F^*E$.

This is equivalent to the category of finite étale covers $E \to (SpdQ_p)^n$ equipped with commuting isomorphisms $\beta_1, \cdots, \beta_{n-1}$. What is new is the action of

- $F_1^Z \times \cdots \times F_{n-1}^Z$ on $|(SpdQ_p)^n|$

is free and totally discontinuous, making the quotient

- $(SpdQ_p)^n/(F_1^Z \times \cdots \times F_{n-1}^Z)$

a diamond. Thus, $((SpdQ_p)^n/p.Fr.)_{fet}$ is the category of finite étale covers of $(SpdQ_p)^n/(F_1^Z \times \cdots \times F_{n-1}^Z)$ [17].

We now state the version of Drinfeld's lemma needed to analyze the cohomology of moduli spaces of shtukas.

For $SpdQ_p$ a diamond, we have the following.

Theorem 6.6.4. ([17] Theorem 16.3.1 (Drinfeld's Lemma for Diamonds)).

- $\pi_1((SpdQ_p)^n/p.Fr.) \simeq G_{Q_p}^n$.

  Now define $X = SpdQ_p/F^Z$ and consider the base change X_C for any algebraically closed nonarchimedean field C/F_p. The next lemma is equivalent to the simple connectivity of the Fargues-Fontaine curve [17].

Lemma 6.6.5. ([17] Lemma 16.3.2). For any algebraically closed nonarchimedean field C/F_p, one has $\pi_1(X_C) \simeq G_{Q_p}$.
By Proposition (15.2.7), we have

- $(X_C)_{f'et} = ((SpdQ_p/F^Z) \times SpdC)_{f'et} = (SpdQ_p \times (SpdC/F_C^Z))_{f'et} = (\mathcal{X}_{FF}^{\diamond})_{f'et}$.

The result follows from the Fargues-Fontaine where

- $(\mathcal{X}_{FF})_{f'et} \simeq (X_{FF})_{f'et} \simeq (Q_p)f'et$.

To recap, the moduli spaces of shtukas in mixed characteristic live in the category of diamonds. Studying the isomorphism classes of moduli spaces of shtukas as the higher K$^{\text{Efimov}}$-groups of diamonds could link diamonds and global Langlands over function fields.

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