Conceptus XXVI (1992/93), Nr. 67, S. 3–61

POPPER'S LAWS OF THE EXCESS OF THE PROBABILITY OF THE CONDITIONAL OVER THE CONDITIONAL PROBABILITY

by Georg J.W. Dorn, Salzburg*

Zusammenfassung: Karl Popper erkannte 1938, daß die unbedingte Wahrscheinlichkeit eines materialen Implikationssatzes der Form 'Wenn *A*, dann *B*' normalerweise die bedingte Wahrscheinlichkeit von *B* unter der Bedingung *A* übersteigt. Damit war (ihm wohl als einzigem in jener Zeit) klar, daß bedingte Wahrscheinlichkeit nicht auf unbedingte Wahrscheinlichkeit von materialen Implikationssätzen reduzierbar ist. Ich verfolge zunächst die Entwicklung dieser Erkenntnis in Poppers Schriften und schließe der historischen eine logische Studie an, in der ich Gesetze des Überschusses in der Kolmogorovschen mit denen in der Popperschen Wahrscheinlichkeitssemantik vergleiche.

Summary: Karl Popper discovered in 1938 that the unconditional probability of a conditional of the form 'If *A*, then *B*' normally exceeds the conditional probability of *B* given *A*, provided that 'If *A*, then *B*' is taken to mean the same as 'Not (*A* and not *B*)'. So it was clear (but presumably only to him at that time) that the conditional probability of *B* given *A* cannot be reduced to the unconditional probability of the material conditional 'If *A*, then *B*'. I describe how this insight was developed in Popper's writings and I add to this historical study a logical one, in which I compare laws of excess in Kolmogorov probability semantics with laws of excess in Popper probability semantics.

* * * * *

0 INTRODUCTION

Popper's insight of 1938 that the unconditional probability of a conditional statement of the form 'If *A*, then *B*' is normally different from the conditional probability of *B* given *A*, has become a commonplace among those philosophers who use probability theory for their research. David Lewis, for instance, devotes less than three lines to it before he goes on to develop his triviality results (see LEWIS, Probabilities, p. 298); typically, Popper's laws of excess are mentioned nowhere in Lewis' article. But the difference between unconditional probability of the conditional and conditional probability had not always been as obvious as it is today. To appreciate Popper's insight, let us start by reformulating it more precisely, as follows:

---

* I thank Mag. Monika Feldbacher (Center for Electronic Data Processing, Salzburg University) for valuable criticism of an earlier version of this article.

Normally, $p(A \rightarrow B) > w(B, A)$.

Here A and B are arbitrary statements, and $\rightarrow$ serves as material implication sign; p is an arbitrary unary probability function, i.e., p is a one-place function which maps statements into the reals in accordance with the three standard conditions:

condition 1: $p(A) \geq 0$;
condition 2: if A is a tautology, then $p(A) = 1$;
condition 3: if $A \rightarrow \neg B$ is a tautology, then $p(A \vee B) = p(A) + p(B)$;

and w is the respective binary probability function, i.e., w is a two-place function which maps ordered pairs of statements into the reals in accordance with the standard

condition 4: if $p(A) > 0$, then $w(B, A) = p(B \wedge A)/p(A)$.[1]

Sixty years ago, when probability theory advanced to the status of an axiomatic theory, some philosophers, misled by the ambiguity of natural language, had not even grasped that a statement of the form

The probability of B['s being true], if A [is true], equals r.

can mean at least two different things:

— $p(A \rightarrow B) = r$;
— $w(B, A) = r$.

Confusing unconditional with conditional probability, they could not formulate, far less answer questions about differences and similarities between unconditional probability of the conditional and conditional probability.[2] Others, however, knew from mathematics that probability functions come in two classes: as unary functions (called 'unconditional' or 'absolute probability functions') and as binary functions (called 'conditional' or 'relative probability functions'). They started asking themselves which relationships there are between $p(A \rightarrow B)$ and $w(B, A)$. Among them was Karl Popper, who in the late 1930s was developing an abstract axiomatic probability theory, which was to turn out to be provocatively different from Kolmogorov's famous axiomatization of 1933, with which Popper, incidentally, was not acquainted at that time (see POPPER, LScD, p. 318). According to his own account, in 1938 he had already proved some propositions about the excess of $p(A \rightarrow B)$ over $w(B, A)$. The following proposition, which we shall call 'the main law of excess', must have been one of them:

---

1. $\neg$ serves here as negation, $\wedge$ as conjunction, and $\vee$ as disjunction sign.
2. These times are not quite past, as is illustrated by the case of Richard L. Purtill, who defines on p. 126 of his *Logical Introduction to Philosophy* $p(H \rightarrow G)$ by means of $p(G \wedge H)/p(H)$, adding on p. 127—very misleadingly indeed—that the equations $p(H \rightarrow G) = p(G \wedge H)/p(H)$ and $p(H \wedge G) = p(H) \cdot p(H \rightarrow G)$ "show how the probabilities of conjunctions and conditionals are interrelated". Professor Purtill's *Logical Introduction to Philosophy* was published by Prentice Hall in 1989.

For all A and B: If $0 < p(A) < 1$ and if $w(B, A) < 1$, then $p(A \rightarrow B) > w(B, A)$.

To appreciate the general validity of the main law of excess, remember that here p and w are arbitrary standard probability functions, i.e., nothing more is required of them than fulfilment of the four usual conditions, cited above. So the main law of excess holds for Kolmogorov's, Carnap's, Rényi's and Popper's probability functions. Note also that the numerical difference between $p(A \rightarrow B)$ and $w(B, A)$ is often not negligible at all. For instance, if A is the statement "The next throw of the die will come up a 5", if B is the statement "The next throw of the die will come up an even number", if $p(A) = 1/6$ and if $p(B) = 1/2$, then $p(A \rightarrow B) = 5/6$, whereas $w(B, A) = 0$.

This article has four aims. Firstly, to sketch the history of Popper's laws of excess. Secondly, to find out by means of detailed proofs within an axiomatic framework which laws of excess can be obtained in that version of Kolmogorov probability theory which will be called here 'Kolmogorov probability semantics'. Thirdly, to find out by the same method which of those propositions, presented by Popper as laws of excess, are indeed theorems of that version of his probability theory which will be called here 'Popper probability semantics'. Fourthly, to compare both systems of probability semantics as regards their laws of excess.

Because this article is long, it will be convenient to list its main contents now in order to make its construction explicit and to facilitate access to its subsections:

1 ON THE HISTORY OF POPPER'S LAWS OF EXCESS

1.1 Conjectures and Refutations, 1963

Popper's first published reference to his laws of excess was in chapter 3 *Verisimilitude* in the *Addenda* of the first edition of his *Conjectures and Refutations*, which appeared in 1963. There he wrote on page 396:

> Incidentally, the fact that we have, quite generally,
>
> (21) $Ct_F(a)\text{-}Ct(a \leftarrow a_T) = Ct_F(a)Ct_T(a)$
>
> may appear somewhat surprising. Yet it is an immediate consequence of the following more general formula
>
> (22) $p(a \leftarrow b)\text{-}p(a, b) = Ct(a, b)Ct(b)$,
>
> a formula which I derived many years ago in order to show that the absolute probability of the *one* conditional statement '*a* if *b*' (or of the statement 'if *b* than [!] *a*') exceeds in general the relative probability of some statement *a*, given some other statement *b*.
>
> (Formula (22) thus compares, as it were, the arrow to the left '←' with the comma ',' and calculates the never negative *excess*,
>
> $Exc(a, b) = p(a \leftarrow b)\text{-}p(a, b)$,
>
> of the conditional probability over the relative probability.)

Let us try to state explicitly the definition of excess and the two laws of excess that are hidden in this passage, itself hidden in a very technical addendum which, presumably, has not been studied by any philosopher who is not a philosopher of science. The following terminological remarks should therefore be useful. Firstly, when Popper writes '$a \leftarrow b$', he means $b \rightarrow a$, where '$\rightarrow$' is a material implication sign. Secondly, when Popper writes '$Ct(a, b)$', he means the content of statement a in regard to statement b, i.e. 1-$p(a, b)$, where p is a binary probability function, hence $p(a, b)$ is the relative probability of a in regard to b. Thirdly, when Popper writes '$Ct(b)$', he means the content of b, i.e. 1-$p(b)$, where p is a unary probability function, hence $p(b)$ is the absolute probability of b. (Note that Popper, following common usage, uses 'p' as a sign not only for unary, but also for binary probability functions.) Fourthly, when Popper writes '$Exc(a, b)$', he means neither the excess of statement a over statement b nor the excess of the probability of a over the probability of b, but the excess of the absolute probability of the conditional statement $b \rightarrow a$ over the relative probability of statement a in regard to statement b; hence '$Exc(a, b)$' is an abbreviation of 'the excess of $p(b \rightarrow a)$ over $p(a, b)$'. Since the abbreviative effect of the term '$Exc(a, b)$' is far less than its misleading effect, let us use the more harmless term '$Exc[p(b \rightarrow a), p(a, b)]$' rather than '$Exc(a, b)$'. Finally, when Popper writes 'conditional probability' in the last line of the passage cited above, he means not conditional probability in the sense of 'relative probability', but the absolute probability of the conditional $b \rightarrow a$, thereby deviating from common usage as well as from his own usage elsewhere. So we have as our first exegetic results:

(1.1.1) a definition of excess: $Exc[p(b \rightarrow a), p(a, b)] = p(b \rightarrow a) - p(a, b)$;
(1.1.2) a first law of excess: $Exc[p(b \rightarrow a), p(a, b)] = [1-p(a, b)] \cdot [1-p(b)]$;
(1.1.3) and a second law of excess: $Exc[p(b \rightarrow a), p(a, b)] \geq 0$.

Since no probability (i.e. value of a probability function) can be greater than 1, proposition (1.1.3) is an immediate logical consequence of proposition (1.1.2). Note that neither (1.1.2) nor (1.1.3) contain a condition that the probability of b be greater than 0. Whereas such a condition is very often tacitly to be assumed when one encounters formulations of theorems of Kolmogorov probability theory (according to which $p(a, b)$ may not exist when $p(b) = 0$), it is very seldom tacitly to be assumed when one encounters formulations of theorems of Popper probability theory (according to which $p(a, b)$ exists even when $p(b) = 0$). Since Popper intends propositions (1.1.2) and (1.1.3) to be theorems of *his* probability theory, there is no need here to add "$p(b)>0$" as an if-clause to (1.1.2) or (1.1.3) in order to make an indispensable tacit assumption explicit. Finally, when we have a look at definition (1.1.1), we see that the excess function maps ordered pairs of probabilities into the reals; since probabilities are themselves real numbers, the excess function is a purely numerical function—in contrast to probability functions, which are viewed here as functions which map statements or ordered pairs of statements into the reals.

1.2 Logik der Forschung, 1966

Popper's second published reference to his laws of excess was in the appendix *V *Ableitungen der formalen Wahrscheinlichkeitstheorie* of the second German edition of his *Logik der Forschung*, which appeared in 1966. Here he makes not merely a passing remark about his laws of excess, but goes into details. I shall cite the whole paragraph in which Popper presents his laws of excess. I shall keep, however, not to the second, but to the most recent German edition of his *Logik der Forschung*, and I shall use for easier reading the negation sign '¬' instead of Popper's complement bar '‾':

In ihrer logischen Interpretation (die keineswegs ihre wichtigste ist) kann man die relative Wahrscheinlichkeit also als Verallgemeinerung des Begriffes der Ableitbarkeit auffassen. Es ist jedoch wichtig, die Ableitbarkeit von a aus b *nicht* mit der „materialen Implikation", d.h. mit dem Konditionalsatz „wenn a, dann b" [!] („$b \supset a$"), zu verwechseln, der ein Satz derselben Art ist wie a und b, während „a folgt aus b" und „$p(a, b) = r$" Behauptungen *über* a und b sind. Vor langer Zeit hat Reichenbach vorgeschlagen, $p(a, b)$ als den Grad aufzufassen, in dem $b \supset a$ gilt, mit anderen Worten, $p(a, b) = p(b \supset a)$ zu setzen. Um diesen Vorschlag zu prüfen, berechnete ich 1938 „$Exc(a, b)$", das heißt den „Excess" oder den „Überschuß" von $p(b \supset a)$ über $p(a, b)$. Schon vor der Rechnung sehen wir, daß $-1 \leq Exc \leq +1$ und daß, wenn b widerspruchsvoll ist, $Exc(a, b) = 0$. Wenn b widerspruchsfrei ist, finden wir $Exc(a, b) = p(\neg a, b)p(\neg b)$. In unserem System gilt aber *bedingungslos*: $Exc(a, b) = (1\text{-}p(a, b))p(\neg b) = p(\neg a, b)p(\neg b)(1\text{-}p(\neg b, b)) \geq 0$. Wenn a und b probabilistisch unabhängig sind, dann gilt, falls b widerspruchsfrei ist: $Exc(a, b) = p(\neg a)p(\neg b)$. In diesem Fall ist auch $Exc(a, b) = 1$, wenn $p(a, b) = 0 = p(b)$. Dieser Fall wird verwirklicht durch ein widerspruchsfreies b und jedes beliebige a, wenn $p(b) = 0$ und a entweder von b unabhängig und $p(a) = 0$, oder mit b unvereinbar oder fast unvereinbar ist. (Beispiel: a = „Es existiert ein weißer Rabe"; $b = \neg a$.) Daher ist die Interpretation von $p(a, b)$ durch $p(b \supset a)$ offenbar ganz unzutreffend. (Logik9, pp. 306–307)

Let us study this paragraph sentence by sentence, starting with '$-1 \leq Exc \leq +1$', which we reformulate more fully:

(1.2.1) $-1 \leq Exc[p(b \rightarrow a), p(a, b)] \leq +1$.

(1.2.1) is not surprising in view of (1.1.2). The next sentence is more illuminating. In our reformulation we use the arrow instead of the horse-shoe:

(1.2.2) If b is a contradiction, then $Exc[p(b \rightarrow a), p(a, b)] = 0$.

Note that (1.2.2) has so far been and will remain the only law of excess Popper mentions which gives a condition under which $p(b \rightarrow a)$ does *not* exceed $p(a, b)$ but equals $p(a, b)$. Popper continues with:

(1.2.3) If b is not a contradiction, then $Exc[p(b \rightarrow a), p(a, b)] = p(\neg a, b) \cdot p(\neg b)$.

We now encounter in the cited paragraph a very compactly formulated statement, which we divide into three convenient parts:

(1.2.4) $Exc[p(b \rightarrow a), p(a, b)] = [1\text{-}p(a, b)] \cdot p(\neg b)$
(1.2.5) $Exc[p(b \rightarrow a), p(a, b)] = p(\neg a, b) \cdot p(\neg b) \cdot [1\text{-}p(\neg b, b)]$
(1.2.6) $Exc[p(b \rightarrow a), p(a, b)] \geq 0$

(1.2.5) is new, but (1.2.4) reminds us of (1.1.2), and (1.2.6) we know already as (1.1.3).
The next proposition is about an alleged relationship between probabilistic independence, contradictions, excess and unary probability functions:

(1.2.7) If a and b are probabilistically independent and if b is not a contradiction, then: $Exc[p(b \rightarrow a), p(a, b)] = p(\neg a) \cdot p(\neg b)$.

Having so far experienced no interpretative problems, we have not found it necessary to quote the German sentences one for one in order to dissect their possible meanings. But the next sentence in the cited paragraph is an exception:

(1.2.8') In diesem Fall ist auch $Exc(a, b) = 1$, wenn $p(a, b) = 0 = p(b)$.

It requires special treatment because of its ambiguity. How are we to interpret (1.2.8')? The phrase 'in diesem Fall' suggests that at least one of the two if-clauses in (1.2.7) should be assumed. This suggestion is strengthened by comparison of the second with the ninth German edition of the *Logik der Forschung*. For we read on page 307 of the second edition, instead of (1.2.8'), the following sentence:

(1.2.8'') Auch gilt $Exc(a, b) = 1$ stets, wenn $p(a, b) = 0 = p(b)$.

It seems that Popper saw good reasons after the publication of the second German edition to back away from the straightforward (1.2.8'') and to substitute for it the circumspect (1.2.8'). But nowhere does he say explicitly that (1.2.8'') is false. So we have here four possible laws of excess:

(1.2.8) If $p(a, b) = 0 = p(b)$, then $Exc[p(b \to a), p(a, b)] = 1$.
(1.2.9) If $p(a, b) = 0 = p(b)$ and if a and b are probabilistically independent, then: $Exc[p(b \to a), p(a, b)] = 1$.
(1.2.10) If $p(a, b) = 0 = p(b)$ and if b is not a contradiction, then: $Exc[p(b \to a), p(a, b)] = 1$.
(1.2.11) If $p(a; b) = 0 = p(b)$ and if a and b are probabilistically independent and if b is not a contradiction, then $Exc[p(b \to a), p(a, b)] = 1$.

We now come to the last sentence which is of relevance here:

(1.2.12') Dieser Fall wird verwirklicht durch ein widerspruchsfreies b und jedes beliebige a, wenn $p(b) = 0$ und a entweder von b unabhängig und $p(a) = 0$, oder mit b unvereinbar oder fast unvereinbar ist.

Again, it will be convenient to divide a very condensed statement into three parts:

(1.2.12) If b is not a contradiction and if $p(b) = 0$ and if a and b are probabilistically independent and if $p(a) = 0$, then $Exc[p(b \to a), p(a, b)] = 1$.
(1.2.13) If b is not a contradiction and if $p(b) = 0$ and if a stands in contrary opposition to b, then $Exc[p(b \to a), p(a, b)] = 1$.
(1.2.14) If b is not a contradiction and if $p(b) = 0$ and if a stands almost in contrary opposition to b, then $Exc[p(b \to a), p(a, b)] = 1$:

Thus we have extracted from the cited paragraph 14 propositions that seem to be laws of excess. I should like to stress that the logical question whether each of these propositions is indeed a law of excess, i.e., whether each of these propositions is a theorem of Popper or even Kolmogorov probability theory, has not yet been decided; this question will have to be answered by adducing logical proofs, not by interpreting quotations.

Finally, let us take a look at the cited paragraph from a psychological point of view. Firstly, we observe that again—like the passage cited from *Conjectures and Refutations*—

it is hidden in a very technical appendix, which presumably has not found much attention outside philosophy of science. In addition, it is written in a compact style without any explanation or sketch of a proof. We shall probably not be far off the mark when we suspect that it has not been easily comprehensible to anyone unfamiliar with *Popper* probability theory, although she or he may be well acquainted with *Kolmogorov* probability theory. In fact, especially those philosophers who know Kolmogorov but not Popper probability theory should be most bewildered by propositions like (1.2.2) and (1.2.3) or by the contrast between (1.2.4) and (1.2.5), to mention only three of their possible perplexities. To make matters worse, this very paragraph has been inadvertently omitted in all English editions of the *Logik der Forschung*, although it is the most detailed and comprehensive account Popper has given us of his laws of excess. No wonder, therefore, that philosophers from the English-speaking world do not associate Karl Popper's name with his insight into the excess of the probability of the conditional over the conditional probability. No wonder, either, that the term 'law of excess' has, as far as I know, been neither used nor mentioned in the vast literature on the logic of conditionals, although Popper discovered his laws of excess two generations ago.

1.3 Nature, 1983

Our third encounter with Popper's laws of excess is in the widely-known Popper/Miller anti-induction proof, the first official version of which was published in *Nature* (April 21, 1983) under the title "A Proof of the Impossibility of Inductive Probability" in the section *Letters to Nature*.[3] We shall quote the relevant passage in their letter to *Nature* and then try to extract the laws of excess mentioned there. But first let us preface our citation with a few terminological remarks. We are to understand h as a hypothesis and e as evidence possibly in favour of h. If a and d are any statements whatsoever and if $p(a, d)\text{-}p(a) < 0$, then the degree to which d countersupports a is defined as $p(a, d)\text{-}p(a)$. Now, Popper and Miller inform us in their letter to *Nature* that the degree to which evidence e countersupports the conditional statement $e \rightarrow h$

> equals $Exc(h, e)$, that is the excess of the probability of the conditional over the conditional probability. Theorem 2: Under the same assumptions [as in theorem 1: $p(h, e) \neq 1 \neq p(e)$], $p(h \leftarrow e) - p(h \leftarrow e, e) = p(h \leftarrow e) - p(h, e) = p(-h, e)p(-e) = Exc(h, e) > 0$. (POPPER/MILLER, Proof, p. 688)

This is what we find about the excess of $p(e \rightarrow h)$ over $p(h, e)$ in their letter. To be sure, Popper and Miller introduce a term similar to '$Exc(h, e)$' only seven lines after the end of our citation, but by this new term '$Exc(h, e, b)$' they no longer mean the excess of the absolute probability of a conditional over the respective relative probability, but something

---

3. Another version, published in the same year, but written at least two years earlier, can be found in POPPER (Realism, p. 326). However, no laws of excess are mentioned there.

different, to wit, the excess of the *relative* probability of $e \to h$ in regard to some background knowledge b over the relative probability of $e \to h$ in regard to $e \wedge b$: $Exc(h, e, b) = Excess[p(e \to h, b), p(e \to h, e \wedge b)] = p(e \to h, b) - p(e \to h, e \wedge b)$. A study of the conditions under which, generally, $p(d \to a, c)$ exceeds $p(d \to a, d \wedge c)$ would certainly be instructive, too, but the task of the present study is to amass knowledge about conditions under which $p(d \to a)$ exceeds $p(a, d)$; I shall concentrate on this task.

Interpreting the passage cited above, one immediately asks oneself whether the suggestion that e should be viewed as evidence for a hypothesis h implies pragmatically some tacit assumptions about semantical properties of e, and—if so—which. Evidence, when taken as a conjunction of basic statements or observation reports, will usually be assumed as being not contradictory, indeed as having a probability far above 0. So, when Popper and Miller formulated their theorem 2, did they tacitly assume that e is not a contradiction or even that $p(e)>0$? If so, we would be doing them an injustice if we did not explicitly add to our interpretation their tacit assumptions; if not, we would be doing them an injustice if we added an assumption not tacitly made by them, thereby giving a logically weaker interpretation of theorem 2 than they had intended. Fortunately, even an extremely modest knowledge of probability theory rids us of the problem whether we have to add both assumptions together. We do not, because the proposition "If $p(e)>0$, then e is not a contradiction" holds for every standard probability function p. However, the problem remains whether, if any assumption is to be added at all, it is sufficient to add the condition that e be not a contradiction or whether it is necessary to add the logically stronger condition that the probability of e be greater than 0. As neither text nor context help us to decide this question, I shall list all possible interpretations, on the understanding that if the logically stronger ones turn out to be valid, then it was these Popper and Miller intended; otherwise, they presumably intended the logically weaker ones. By parsing the citation according to the equations in it (omitting the now well-known definition of excess), we obtain first the three propositions (1.3.1), (1.3.2) and (1.3.3). Then by adding separately to each of these three propositions one of the two possible tacit assumptions, we arrive at nine possible laws of excess, all of them new:

(1.3.1) If $p(a, b) \neq 1$ and if $p(b) \neq 1$, then $Exc[p(b \to a), p(a, b)] = p(b \to a) - p(b \to a, b)$.

(1.3.1') If $p(a, b) \neq 1$ and if $p(b) \neq 1$ and if b is not a contradiction, then:
$Exc[p(b \to a), p(a, b)] = p(b \to a) - p(b \to a, b)$.

(1.3.1'') If $p(a, b) \neq 1$ and if $p(b) \neq 1$ and if $p(b)>0$, then:
$Exc[p(b \to a), p(a, b)] = p(b \to a) - p(b \to a, b)$.

(1.3.2) If $p(a, b) \neq 1$ and if $p(b) \neq 1$, then $Exc[p(b \to a), p(a, b)] = p(\neg a, b) \cdot p(\neg b)$.

(1.3.2') If $p(a, b) \neq 1$ and if $p(b) \neq 1$ and if b is not a contradiction, then:
$Exc[p(b \to a), p(a, b)] = p(\neg a, b) \cdot p(\neg b)$.

(1.3.2'') If $p(a, b) \neq 1$ and if $p(b) \neq 1$ and if $p(b)>0$, then:
$Exc[p(b \to a), p(a, b)] = p(\neg a, b) \cdot p(\neg b)$.

(1.3.3) If $p(a, b) \neq 1$ and if $p(b) \neq 1$, then $Exc[p(b \rightarrow a), p(a, b)] > 0$.
(1.3.3') If $p(a, b) \neq 1$ and if $p(b) \neq 1$ and if b is not a contradiction, then: $Exc[p(b \rightarrow a), p(a, b)] > 0$.
(1.3.3'') If $p(a, b) \neq 1$ and if $p(b) \neq 1$ and if $p(b)>0$, then $Exc[p(b \rightarrow a), p(a, b)] > 0$.

Note that in (1.3.3'') we encounter for the first time in our interpretation of Popper's writings the main law of excess, although in a formulation which is different from that used in the introduction. But if we take into account that $p(a, b)<1$ iff $p(a, b) \neq 1$, that $p(b)<1$ iff $p(b) \neq 1$, and that $p(b \rightarrow a) > p(a, b)$ iff $Exc[p(b \rightarrow a), p(a, b)] > 0$, then we obtain a reformulation of (1.3.3'') which reveals the identity of proposition (1.3.3'') with the main law of excess: 'If $0 < p(b) < 1$ and if $p(a, b) < 1$, then $p(b \rightarrow a) > p(a, b)$'. Since the main law of excess is logically weaker than the proposition (1.3.3') and since (1.3.3') is again logically weaker than (1.3.3), there would be at least one law of excess which is logically stronger than the main law of excess if proposition (1.3.3') were a theorem of Popper probability theory, and there would be at least two laws of excess stronger than the main law if proposition (1.3.3) were also a theorem.

Propositions (1.3.2') and (1.3.2'') are obvious logical consequences of a proposition which we noted down in subsection 1.2:

(1.2.3) If b is not a contradiction, then $Exc[p(b \rightarrow a), p(a, b)] = p(\neg a, b) \cdot p(\neg b)$.

The logical relationship between (1.3.2) and (1.2.3), however, is an open question. About (1.3.1) we shall have to say more in the next subsection.

1.4 Foundations of Logic and Linguistics, 1985

Karl Popper presented another, more extensive version of the Popper/Miller anti-induction proof at the 7th international congress of logic, methodology and philosophy of science held at Salzburg University in 1983. Two years later, his congress paper, entitled "The Non-Existence of Probabilistic Inductive Support", appeared in the anthology *Foundations of Logic and Linguistics*. Since laws of excess play an essential role in the Popper/Miller anti-induction proof, we find them treated again in this paper. We read in POPPER (Non-Existence, p. 308):

> I shall end this section with an important *Theorem 1*. For every x, y and z, [...]
> 1.16.4 $cs(x \leftarrow y, y, z) = ct(x, yz)ct(y, z) = Exc(x, y, z) \geq 0$ [...]
> 1.16.6 All these formulae remain valid if we erase in each term the second comma (if any) and the variable z.

So let us erase:

1.16.4' For every x and y: $cs(x \leftarrow y, y) = ct(x, y)ct(y) = Exc(x, y) \geq 0$.

That part of 1.16.4' which consists of "$ct(x, y)ct(y) = Exc(x, y) \geq 0$" we know already as propositions (1.1.2) and (1.1.3), taken together:

$Exc[p(b \rightarrow a), p(a, b)] = [1\text{-}p(a, b)] \cdot [1\text{-}p(b)] \geq 0.$

since the functor '*ct*' is here a symbol for 'the content of'. The term '$cs(x \leftarrow y, y)$' is to be read as '$-s(x \leftarrow y, y)$', which denotes the countersupport of the conditional $x \leftarrow y$ by the statement y. Now, $s(x, y)$ is defined as $p(x, y)\text{-}p(x)$. Hence, $s(x \leftarrow y, y) = [p(x \leftarrow y, y) - p(x \leftarrow y)]$, hence $-s(x \leftarrow y, y) = p(x \leftarrow y) - p(x \leftarrow y, y)$. So interpretation of 1.16.4' presents us with the following proposition as a new law of excess:

(1.4.1) $Exc[p(b \rightarrow a), p(a, b)] = p(b \rightarrow a) - p(b \rightarrow a, b)$.

Since $Exc[p(b \rightarrow a), p(a, b)] = p(b \rightarrow a)\text{-}p(a, b)$, the theoremhood of (1.4.1) depends on that of "$p(b \rightarrow a, b) = p(a, b)$". When we look back to (1.3.1), we realize that (1.3.1) is an immediate logical consequence of (1.4.1). Because of the question of tacit assumptions, we were not sure whether proposition (1.3.1) is a theorem of Popper probability theory; now we may be even less sure whether the logically stronger (1.4.1) is. But as there is no indication of tacit assumptions in or around the passage cited above, no logical weakening of (1.4.1) is justified from text or context. On the contrary, two pages later, Popper offers a proof of "$p(b \rightarrow a, b) = p(a, b)$" which proceeds straightforwardly without any assumptions (cf. POPPER, Non-Existence, p. 310):

Proof (1) $p(b \rightarrow a, b) = p[(b \rightarrow a) \wedge b, b] = p(a \wedge b, b) = p(a, b)$.[4]

Obviously, if proof (1) can do without tacit assumptions, so can proposition (1.4.1).

On the previous page, Popper offers a proof of another law of excess. We find in POPPER (Non-Existence, p. 309):

Lemma. $Exc(x, y) = ct(x, y)ct(y)$
Proof: Since $p(x, x) = p(y, y) = 1$, [...] we have
$ct(x, y)ct(y) = ct(y) + p(xy)\text{-}p(x, y)$ [...]
$= p(x \leftarrow y)\text{-}p(x, y)$ [...]
$= Exc(x, y)$.

In this second citation we encounter a further possible law of excess. To see this clearly, let us first rewrite the proof in our terminology:

(1) $p(a, a) = p(b, b) = 1$
(2) $[1\text{-}p(a, b)] \cdot [1\text{-}p(b)] = 1\text{-}p(b) + p(a \wedge b)\text{-}p(a, b)$
(3) $= p(b \rightarrow a)\text{-}p(a, b)$

4. I have experienced that proof (1) can look so strange to people who are used to working within the Kolmogorovian framework that they are sure its formulation contains a misprint, distorting the meaning, and must be corrected, as follows: $p(b \rightarrow a, b) = p[(b \rightarrow a) \wedge b]/p(b) = p(a \wedge b, b) = p(a, b)$. Since stating anything about the quotient $p[(b \rightarrow a) \wedge b]/p(b)$ means presupposing that $p(b)>0$, they go on to point out—and rightly so—that their "corrected" proof will never work for (1.4.1), but only for the weaker proposition: "If $p(b)>0$, then $Exc[p(b \rightarrow a), p(a, b)] = p(b \rightarrow a) - p(b \rightarrow a, b)$". As one of the editors of POPPER (Non-Existence), I should like to stress, however, that there is no misprint in the formulation of proof (1) in POPPER (Non-Existence, p. 310). Proof (1) was intended as formulated.

(4) $= Exc[p(b \to a), p(a, b)]$.

From step (2) and (4) we obtain:

(1.4.2) $Exc[p(b \to a), p(a, b)] = 1\text{-}p(b)+p(a \wedge b)\text{-}p(a, b)$.

Here the theoremhood of (1.4.2) will obviously depend on the theoremhood of

$p(b \to a) = 1\text{-}p(b)+p(a \wedge b)$,

which is fortunately well established.

1.5 Philosophical Transactions of the Royal Society of London, 1987

Karl Popper and David Miller elaborate their anti-induction proof still further and defend it impressively against manifold criticisms in their paper "Why Probabilistic Support Is Not Inductive", published in the *Philosophical Transactions of the Royal Society of London* in 1987. This paper has been the last one so far, to the best of my knowledge, in which Popper says something about the excess of $p(b \to a)$ over $p(a, b)$. He stresses again that $p(a, b)$ never exceeds $p(b \to a)$, he alludes again to proposition (1.4.1), and he offers again a proof of the proposition "$p(b \to a, b) = p(a, b)$", on the theoremhood of which (as we know from subsection 1.4) the theoremhood of (1.4.1) wholly depends. As this new proof differs slightly but instructively from proof (1) above, we note it also, using our own terminology (cf. POPPER/MILLER, Support, p. 576):

Proof (2) Because $p(a, b) = p(a \wedge b, b)$ and because $(b \to a) \wedge b$ is logically equivalent to $a \wedge b$, it follows that $p(b \to a, b) = p(a \wedge b, b) = p(a, b)$.

Note that proof (2) proceeds, like proof (1), without the assumption that $p(b)>0$.

1.6 The Discoveries of 1938

From the point of the history of philosophy, it is noteworthy that Popper tells us in POPPER/MILLER (Support, p. 576) for the second time the year of his discovery of laws of excess: 1938. He gave this information first in POPPER (LdF2, p. 307) (see subsection 1.2 above) and he has repeated it in correspondence (e.g. in a letter to the present author, dated August 25, 1987). As he published no references to his laws of excess until 1963, we have to rely here on his words. Fortunately, Popper's claim of having discovered laws of excess as early as 1938 is a wholly plausible one, considering his great logical achievement of 1937: his first axiomatization of probability theory. In a note to *Mind*, dated November 20, 1937, he let the philosophical community know of his axiomatization. His note was published six months later in *Mind* 47 (1938), pp. 275–277, under the title "A

Set of Independent Axioms for Probability". Anyone who could construct in 1937 a consistent and independent set of axioms strong enough to allow the deduction of mathematical probability theory as known at that time, could easily derive in 1938 some illuminating theorems concerning relationships between $p(b \to a)$ and $p(a, b)$, especially when his note introduces at the very beginning *two* probability functors—'$pa(x_1)$' for the absolute probability of x_1, '$p(x_1, x_2)$' for the relative probability of x_1 in regard to x_2—where the arguments x_1 and x_2 need not be sets (as in Kolmogorov's axiomatization), but may, for instance, be statements or formulae. In retrospect, Popper's axiomatization appears as the most appropriate logical framework in the 1930s for developing insights, examining conjectures and deriving theorems about relationships between absolute and relative probabilities of statements. Having created an innovative axiomatic probabilistic system of his own, Popper certainly had the means at that time to test the conjecture "$p(b \to a) = p(a, b)$" thoroughly within his system and, finding the conjecture false, to prove the existence of other relationships between $p(b \to a)$ and $p(a, b)$.

But exactly which relationships between $p(b \to a)$ and $p(a, b)$ did Popper discover and prove in 1938? To this question, I have no full answer, but a partial one. The only thing we learn from POPPER (LdF2, p. 307) and POPPER/MILLER (Support, p. 576) is that he calculated the excess of $p(b \to a)$ over $p(a, b)$ in 1938 and found that it was never negative. So it may appear at first sight that he must have known and proved at that time the proposition "$Exc[p(b \to a), p(a, b)] \geq 0$", which he mentions in *Conjectures and Refutations* and later in the second German edition of his *Logik der Forschung*. But—speaking strictly and with a tinge of pedantry—it could not, in my opinion, have been this proposition which he proved in 1938; it must have been the following, logically weaker one:

If $p(b)>0$, then $Exc[p(b \to a), p(a, b)] \geq 0$.

The probabilistic system in which he seems to have worked at that time—he called it 'system S_2' in his note to *Mind*—would have been logically too weak to allow a proof of the former proposition, but was strong enough to allow a proof of the latter one. It is true that already his system S_2 differs essentially from Kolmogorov's axiomatization, for S_2 *leads* to theorems which Kolmogorov's axiomatization *presupposes*; but S_2 still introduces—as does Kolmogorov's axiomatization—the two-place probability functor '$p(a, b)$' (*our* symbols now) by the standard condition "If $p(b)>0$, then $p(a, b) = p(a \wedge b)/p(b)$". Hence, if it was the system S_2 in which Popper constructed his proofs in 1938, he—like Kolmogorov or any other probability theorist of the time—could say nothing about $p(a, b)$ when $p(b) = 0$; hence he could find out something about the excess of $p(b \to a)$ over $p(a, b)$, only when he assumed that $p(b)>0$. Now, we know from POPPER (LdF9, p. 260; LScD, p. 319) that he worked from 1937 up to the early 1950s within the system S_2, for which he developed more and more elegant axiomatic bases over the years. Hence Popper could only have proved in 1938 such laws of excess the if-clause of which guarantees that $p(b)>0$. Hardly any of the propositions which we have listed in this section as possible laws of excess have such an if-clause. So when Popper informs us that he discovered and proved in 1938 the never negative excess of $p(b \to a)$ over $p(a, b)$, then I think we are to understand that he discovered and proved in 1938 at least the following three propositions (p is a unary or binary standard probability function according to context):

(1.6.1) If $p(b)>0$, then $Exc[p(b \to a), p(a, b)] = [1-p(a, b)] \cdot [1-p(b)]$.
(1.6.2) If $p(b)>0$, then $Exc[p(b \to a), p(a, b)] \geq 0$.
(1.6.3) If $0 < p(b) < 1$ and if $p(a, b) < 1$, then $Exc[p(b \to a), p(a, b)] > 0$.

(1.6.1) must have been the central insight, (1.6.2) and (1.6.3) were welcome corollaries to it. Note that (1.6.1) is a weakened version of (1.1.2), and (1.6.2) a weakened version of (1.1.3) or (1.2.6). (1.6.3) is the main law of excess—without which (1.6.2) would be pointless, because (1.6.2) could be true even when $p(b \to a)$ always equals $p(a, b)$.

Having detailed what I consider to be the smallest set of laws of excess which were discovered and proved by Popper in 1938, I should like to illustrate by means of three typical examples which propositions could not have been proved in system S_2. Because context in POPPER (LdF2, p. 307) might misleadingly suggest that, in 1938, Popper proved every proposition listed in subsection 1.2, I shall take these examples from there:

(1.2.2) If b is a contradiction, then $Exc[p(b \to a), p(a, b)] = 0$.
(1.2.3) If b is not a contradiction, then $Exc[p(b \to a), p(a, b)] = p(\neg a, b) \cdot p(\neg b)$.
(1.2.6) $Exc[p(b \to a), p(a, b)] \geq 0$

Since S_2 does not allow proofs of propositions about $p(a, b)$ without a guarantee that $p(b)>0$, neither (1.2.2) nor (1.2.3) nor (1.2.6) is provable in S_2. (1.2.6) is the most transparent case, because it simply has no if-clause. (1.2.2) is the most blatant case, because its if-clause guarantees that $p(b) = 0$. (1.2.3) makes us ask the question whether not being a contradiction is a guarantee for having a probability greater than 0. In probability theories like those of Kolmogorov and Popper (from S_2 up to the most recent ones) the answer is no. Indeed, in the opinion of many a philosopher of science, especially Popper's (see Ldf9, pp. 313–328), universal hypotheses are very good examples for statements which are not contradictions but have probability 0. It was eventually Popper himself who became so dissatisfied with probability theory for its inability to tell us anything about $p(a, b)$ when $p(b) = 0$ that he began, in the 1950s, to develop probability theories which do tell us something about $p(a, b)$ even when $p(b) = 0$. It is this family of new probability theories, published in the late 1950s and the following years, which is now summarizingly called 'Popper probability theory', and it is Popper probability theory of this new kind—and no longer S_2—which is the theoretical frame of every passage we have cited from Popper's writings. So, if propositions like (1.2.2), (1.2.3) and (1.2.6) are not theorems of S_2, then this should not be taken to mean that these propositions are not theorems of Popper probability theory.

This ends our historical study of those propositions in Popper's writings which seem to be laws of excess. We have found 27 of them. Let us now turn to their logical examination.

2 LAWS OF EXCESS WITHIN KOLMOGOROV PROBABILITY SEMANTICS

2.1 Elementary Kolmogorov Probability Semantics

According to Kolmogorov's axiomatization of probability theory, unary probability functions are real-valued, non-negative, additive set-functions normalized to 1, the sets being elements of a field $\mathfrak{F}$ of subsets of some non-empty set E (cf. Kolmogoroff, Grundbegriffe, p. 2; Kolmogorov, Foundations, p. 2). So, central terms like '$P(A)$', '$P(A+B)$', and '$P(E)$' are to be read as 'the probability of set A', 'the probability of the union of sets A and B', and 'the probability of the universal set', respectively. Binary probability functions are also non-negative, additive functions, mapping some or all elements of $\mathfrak{F}\times\mathfrak{F}$ into the reals according to the standard condition "If $P(A)>0$, then $P_A(B) = P(AB)/P(A)$", where 'AB' stands for the intersection of the sets A and B, and '$P_A(B)$' for $p(B, A)$. However, it is possible to consider A and B not as sets, but as formulae of a logical language $\mathfrak{L}$, to use the set intersection and set union signs as conjunction and disjunction signs, and to let 'E' stand for an arbitrary tautology of $\mathfrak{L}$. Under this reading, '$P(A+B)$', for instance, means the probability of the disjunction of the formulae A and B. Let us call 'Kolmogorov probability semantics' that version of Kolmogorov's original axiomatic probability theory which deals with those probability functions that attach probabilities not to sets of a field $\mathfrak{F}$, nor to ordered pairs of $\mathfrak{F}\times\mathfrak{F}$, but to formulae of a logical language $\mathfrak{L}$ or to ordered pairs of $\mathfrak{L}\times\mathfrak{L}$.

Let $\mathfrak{L}$ be our object-language with $\{\neg, \wedge, \vee, \rightarrow\}$ as its set of logical constants and let the set of formulae of $\mathfrak{L}$ be inductively defined as usual (whence every formula of $\mathfrak{L}$ is of finite length). All we need from classical semantics for developing Kolmogorov probability semantics is the set of theorems of the following seven familiar definitions (let us call this set of theorems 'the classical theory of truth value functions and truth-functional attributes'):

(1) f is a classical truth value function defined on $\mathfrak{L}$ iff f is a function from $\mathfrak{L}$ into $\{0, 1\}$ such that for all formulae x and y of $\mathfrak{L}$: $f(\neg x) = 1$ iff $f(x) = 0$; $f(x\wedge y) = 1$ iff $f(x) = 1 = f(y)$; $f(x\vee y) = 1$ iff $f(x) = 1$ or $f(y) = 1$; and $f(x\rightarrow y) = 1$ iff $f(x) = 0$ or $f(y) = 1$.

(2) If x is a formula of $\mathfrak{L}$, then x is a tautology$_K$ [in $\mathfrak{L}$] iff for all classical truth value functions f defined on $\mathfrak{L}$: $f(x) = 1$.

(3) If x is a formula of $\mathfrak{L}$, then x is a contradiction$_K$ [in $\mathfrak{L}$] iff for all classical truth value functions f defined on $\mathfrak{L}$: $f(x) = 0$.

(4) If x and y are formulae of $\mathfrak{L}$, then x truth-functionally follows$_K$ from y [in $\mathfrak{L}$] iff for all classical truth value functions f defined on $\mathfrak{L}$: if $f(x) = 0$, then $f(y) = 0$.

(5) If x and y are formulae of $\mathfrak{L}$, then x is truth-functionally equivalent$_K$ to y [in $\mathfrak{L}$] iff for all classical truth value functions f defined on $\mathfrak{L}$: $f(x) = f(y)$.
(6) If x and y are formulae of $\mathfrak{L}$, then x stands in truth-functional contrary opposition$_K$ to y [in $\mathfrak{L}$] iff for all classical truth value functions f defined on $\mathfrak{L}$:
if $f(x) = 1$, then $f(y) = 0$.
(7) If x and y are formulae of $\mathfrak{L}$, then x stands in truth-functional contradictory opposition$_K$ to y [in $\mathfrak{L}$] iff for all classical truth value functions f defined on $\mathfrak{L}$: $f(x) \neq f(y)$.

We extend our classical theory of truth value functions and truth-functional attributes to [the] Kolmogorov probability semantics [of $\mathfrak{L}$] by adding appropriate definitions of unary and of binary Kolmogorov probability functions and of a ternary relation of probabilistic independence; these definitions will serve as the three specific axioms K1, K2 and K3 of elementary Kolmogorov probability semantics (elementary, because K1 and K2 say nothing about probability functions that map infinitely long formulae into the reals):

Axiom K1: For all f and $\mathfrak{L}$: f is a unary (or absolute or unconditional) Kolmogorov probability function defined on $\mathfrak{L}$ iff f is a function from $\mathfrak{L}$ into the reals such that for all formulae x and y of $\mathfrak{L}$: $f(x) \geq 0$; if x is a tautology$_K$, then $f(x) = 1$; and if x stands in truth-functional contrary opposition$_K$ to y, then $f(x \vee y) = f(x)+f(y)$.
Axiom K2: For all f, g and $\mathfrak{L}$: f is a g-based binary (or relative or conditional) Kolmogorov probability function defined on $\mathfrak{L}$ iff g is a unary Kolmogorov probability function defined on $\mathfrak{L}$ and f is a function from a non-empty subset of $\mathfrak{L} \times \mathfrak{L}$ into the reals such that for all formulae x and y of $\mathfrak{L}$: if $g(x) > 0$, then $f(y, x) = g(y \wedge x)/g(x)$.
Axiom K3: For all x, y, $\mathfrak{L}$ and f: if x and y are formulae of $\mathfrak{L}$, then: x is f-independent$_K$ of y [in $\mathfrak{L}$] iff, firstly, f is a unary Kolmogorov probability function defined on $\mathfrak{L}$, and, secondly, $f(x \wedge y) = f(x) \cdot f(y)$.

As the conditions in the definiens of K1, K2 and K3 are the usual ones and correspond in an obvious way to Kolmogorov's familiar probability axioms and independence definition, I will not here elaborate on them, but only point to the instructive fact that every classical truth value function defined on $\mathfrak{L}$ is a unary Kolmogorov probability function defined on $\mathfrak{L}$, hence the term 'semantics' in 'Kolmogorov probability semantics' is a highly appropriate one.[5]

To save ink, paper and time, let us agree to use the first six capital letters of the Latin alphabet as variables for formulae of $\mathfrak{L}$, the subindexed letter 'p_K' as a variable for unary Kolmogorov probability functions defined on $\mathfrak{L}$, and the subindexed letter 'w_K' as a variable for p_K-based binary Kolmogorov probability functions defined on $\mathfrak{L}$. So, instead of beginning each formulation of a theorem of our Kolmogorov probability semantics with the cumbersome preamble:

5. For more informations on probability semantics in general see LEBLANC (Alternatives, pp. 225–274).

For all $\mathfrak{L}$, x, y, f and g: If x and y are formulae of $\mathfrak{L}$ and if f is a g-based binary Kolmogorov probability function defined on $\mathfrak{L}$, then...

we shall simply write:

For all A, B, p_k and w_k...

where w_k is a p_k-based binary Kolmogorov probability function. For instance, we shall prefer to the following long formulation of, say,

Theorem K20: For all $\mathfrak{L}$, x, y, f and g: If x and y are formulae of $\mathfrak{L}$ and if f is a g-based binary Kolmogorov probability function defined on $\mathfrak{L}$, then:
if $g(x)>0$, then $f(y, x) = 0$ iff $g(x \wedge y) = 0$.

this short formulation of

Theorem K20: For all A, B, p_k and w_k: If $p_k(A)>0$, then:
$w_k(B, A) = 0$ iff $p_k(A \wedge B) = 0$.

We shall need not only theorem K20 for the later development of the theory of excess of $p_k(A \rightarrow B)$ over $w_k(B, A)$, but also 17 other simple theorems of elementary Kolmogorov probability semantics. I have listed these theorems in appendix 1 for easy reference. Since they are well established, they are stated without proof.

2.2 *Extension of Elementary Kolmogorov Probability Semantics to the Theory Ke about the Excess of $p_k(A \rightarrow B)$ over $w_k(B, A)$*

2.2.1 *The Specific Axiom of Ke*

Let us first observe that we cannot extend elementary Kolmogorov probability semantics to a theory of excess (call it 'Ke') without adding to:

excess$[p_k(A \rightarrow B), w_k(B, A)] = p_k(A \rightarrow B) - w_k(B, A)$

the constraint that $p_k(A)>0$. For suppose, $p_k(A) = 0$. Then K2 does not guarantee that there is a real number r such that $w_k(B, A) = r$. But if we have no guarantee that $w_k(B, A)$ exists, then we have no guarantee that excess is a function from ordered pairs of probabilities into the reals. But as we want excess to be such a function, we must make sure that there is an r such that $w_k(B, A) = r$. This we do by requiring that the probability of A be greater than 0, for if $p_k(A)>0$, then, by K2, $w_k(B, A) = p_k(B \wedge A)/p_k(A)$, hence there is an r, the quotient $p_k(B \wedge A)/p_k(A)$, such that $w_k(B, A) = r$. Therefore the specific axiom of our theory Ke will have to be formulated more carefully, as follows:

Axiom Ke1:
For all A, B, p_k and w_k: If $p_k(A)>0$, then:
excess$[p_k(A \rightarrow B), w_k(B, A)] = p_k(A \rightarrow B) - w_k(B, A)$.

2.2.2 *Laws of Excess of* $p_k(A \rightarrow B)$ over $w_k(B, A)$ *in Ke*

A corollary to Ke1 is:

Theorem Ke2:
For all A, B, p_k and w_k: If $p_k(A)>0$, then:
$\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = p_k(A \rightarrow B) - w_k(A \rightarrow B, A)$; and
$\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = 1 - p_k(A) + p_k(B \wedge A) - w_k(B, A)$.

Proof of theorem Ke2:

(1)	$p_k(A)>0$	assumption
(2)	$w_k(A \rightarrow B, A) = p_k[(A \rightarrow B) \wedge A]/p(A)$	(1), K2
(3)	$p_k[(A \rightarrow B) \wedge A] = p_k(B \wedge A)$	K5
(4)	$w_k(A \rightarrow B, A) = p_k(B \wedge A)/p_k(A)$	(2), (3)
(5)	$w_k(B, A) = p_k(B \wedge A)/p_k(A)$	(1), K2
(6)	$w_k(A \rightarrow B, A) = w_k(B, A)$	(4), (5)
(7)	$\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = p_k(A \rightarrow B) - w_k(A \rightarrow B, A)$	(1), Ke1, (6)
(8)	$p_k(A \rightarrow B) = p_k(\neg A) + p_k(B \wedge A) = 1 - p_k(A) + p_k(B \wedge A)$	K11, K4
(9)	$\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = 1 - p_k(A) + p_k(B \wedge A) - w_k(B, A)$	(1), Ke1, (8)

Our next theorem shows that the difference between the absolute probability of $A \rightarrow B$ and the relative probability of B in regard to A is equal to a product of absolute and relative probabilities, provided that $p_k(A)>0$:

Theorem Ke3:
For all A, B, p_k and w_k:
If $p_k(A)>0$, then $\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = w_k(\neg B, A) \cdot p_k(\neg A)$.[6]

Proof of theorem Ke3:

(1)	$p_k(A)>0$	assumption
(2)	$p_k(A \rightarrow B) = p_k(\neg A) + p_k(B \wedge A)$	K11
(3)	$p_k(A \rightarrow B) = p_k(\neg A) + [w_k(B, A) \cdot p_k(A)]$	(2), (1), K2
(4)	$p_k(A \rightarrow B) - w_k(B, A) = p_k(\neg A) + [w_k(B, A) \cdot p_k(A)] - w_k(B, A)$	(3)
(5)	$p_k(A \rightarrow B) - w_k(B, A) = p_k(\neg A) + [w_k(B, A) \cdot (p_k(A) - 1)]$	(4)
(6)	$p_k(A \rightarrow B) - w_k(B, A) = p_k(\neg A) \cdot [1 - w_k(B, A)]$	(5), K4
(7)	$p_k(A \rightarrow B) - w_k(B, A) = p_k(\neg A) \cdot w_k(\neg B, A) = w_k(\neg B, A) \cdot p_k(\neg A)$	(1), K12, (6)
(8)	$\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = w_k(\neg B, A) \cdot p_k(\neg A)$	(1), Ke1, (7)

This leads at once to the following two theorems:

Theorem Ke4:
For all A, B, p_k and w_k: If $p_k(A)>0$, then $\text{excess}[p_k(A \rightarrow B), w_k(B, A)] \geq 0$.

Proof of theorem Ke4:

(1)	$p_k(A)>0$	assumption
(2)	$\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = w_k(\neg B, A) \cdot p_k(\neg A)$	(1), Ke3

---

6. Note that David Lewis uses the condition "$p_k(\neg A) \cdot [p_k(\neg B \wedge A)/p_k(A)]$", hence "$p_k(\neg A) \cdot w_k(\neg B, A)$", hence "$w_k(\neg B, A) \cdot p_k(\neg A)$", hence the excess of $p_k(A \rightarrow B)$ over $w_k(B, A)$ as a measure of the diminution of the assertability of the truth-functional conditional $A \rightarrow B$ (cf. LEWIS, Probabilities, p. 306).

(3) $p_k(\neg A) \geq 0$ and $w_k(\neg B, A) \geq 0$. K6, (1), K13
(4) $\text{excess}[p_k(A \rightarrow B), w_k(B, A)] \geq 0$ (2), (3)

Theorem Ke5:
For all A, B, p_k and w_k: If $p_k(A)>0$, then:
$\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = [1-w_k(B, A)] \cdot [1-p_k(A)]$; and
$\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = [1-w_k(B, A)] \cdot p_k(\neg A)$; and
$\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = w_k(\neg B, A) \cdot p_k(\neg A) \cdot [1-w_k(\neg A, A)]$.

Proof of theorem Ke5:
(1) $p_k(A)>0$ assumption
(2) $\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = w_k(\neg B, A) \cdot p_k(\neg A)$ (1), Ke3
(3) $\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = [1-w_k(B, A)] \cdot [1-p_k(A)]$ (1), (2), K12, K4
(4) $\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = [1-w_k(B, A)] \cdot p_k(\neg A)$ (2), (3)
(5) $w_k(\neg A, A) = 0$ (1), K12, K18
(6) $\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = w_k(\neg B, A) \cdot p_k(\neg A) \cdot [1-w_k(\neg A, A)]$ (2), (5)

A more informative corollary to theorem Ke3 is:

Theorem Ke6:
For all A, B, p_k and w_k: If $0<p_k(A)<1$, then:
if $p_k(B) = 0$ or B is a contradiction$_k$, then $\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = p_k(\neg A) > 0$.

Proof of theorem Ke6:
(1) $0<p_k(A)<1$ assumption
(2) If B is a contradiction$_k$, then $p_k(B) = 0$. K10
(3) If $p_k(B) = 0$, then $w_k(B, A) = 0$. (1), K16
(4) If $w_k(B, A) = 0$, then $w_k(\neg B, A) = 1$. (1), K12
(5) If $w_k(\neg B, A) = 1$, then $\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = p_k(\neg A)$. (1), Ke3, (4)
(6) $p_k(\neg A)>0$ (1), K9
(7) If $p_k(B) = 0$ or B is a contradiction$_k$, then: (2), (3), (4), (5), (6)
$\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = p_k(\neg A) > 0$.

Another is the main law of excess:

Theorem Ke7:
For all A, B, p_k and w_k:
If $0<p_k(A)<1$ and if $w_k(B, A)<1$, then $\text{excess}[p_k(A \rightarrow B), w_k(B, A)] > 0$.

Proof of theorem Ke7:
(1) $0<p_k(A)<1$ assumption
(2) $w_k(B, A)<1$ assumption
(3) $\text{excess}[p_k(A \rightarrow B), w_k(B, A)] = p_k(\neg A) \cdot w_k(\neg B, A)$ (1), Ke3
(4) $p_k(\neg A)>0$ (1), K9
(5) $w_k(\neg B, A)>0$ (1), (2), K17
(6) $\text{excess}[p_k(A \rightarrow B), w_k(B, A)] > 0$ (3), (4), (5)

And a third one is our next theorem, which describes a general relationship between probabilistic independence$_k$ and excess:

Theorem Ke8:

For all A, B, p_k and w_k: If $p_k(A)>0$ and if B is p_k-independent$_k$ of A, then: excess$[p_k(A\rightarrow B), w_k(B, A)] = p_k(\neg B)\cdot p_k(\neg A)$.

Proof of theorem Ke8:

(1)	$p_k(A)>0$	assumption
(2)	B is p_k-independent$_k$ of A.	assumption
(3)	excess$[p_k(A\rightarrow B), w_k(B, A)] = p_k(\neg A)\cdot w_k(\neg B, A)$	(1), Ke3
(4)	$w_k(\neg B, A) = p_k(\neg B)$	(1), (2), K21
(5)	excess$[p_k(A\rightarrow B), w_k(B, A)] = p_k(\neg A)\cdot p_k(\neg B) = p_k(\neg B)\cdot p_k(\neg A)$	(3), (4)

Now to the range of the excess function. We know already from Ke4 that the excess of $p_k(A\rightarrow B)$ over $w_k(B, A)$ is never less than 0, we prove now that it never reaches 1, but does reach 0 (if $p_k(A)>0$). Hence the range of the excess function is the half-open interval [0, 1), provided that its arguments are values of *Kolmogorov* probability functions (as should be stressed since things will be different when we calculate the range of the excess function under the assumption that its arguments are values of Popper probability functions; see subsection 3.3).

Theorem Ke9:

For all A, B, p_k and w_k: If $p_k(A)>0$, then excess$[p_k(A\rightarrow B), w_k(B, A)] < 1$.

Proof of theorem Ke9:

(1)	$p_k(A)>0$	assumption
(2)	If $[p_k(A\rightarrow B)\text{-}w_k(B, A)] >1$, then: if $w_k(B, A)\geq 0$, then $p_k(A\rightarrow B)>1$.	arithmetics
(3)	$w_k(B, A)\geq 0$	(1), K13
(4)	If $[p_k(A\rightarrow B)\text{-}w_k(B, A)] >1$, then $p_k(A\rightarrow B)>1$.	(2), (3)
(5)	It is not the case that $p_k(A\rightarrow B)>1$.	K6
(6)	$[p_k(A\rightarrow B)\text{-}w_k(B, A)] \leq 1$	(4), (5)
(7)	If $[p_k(A\rightarrow B)\text{-}w_k(B, A)] = 1$, then $p_k(A\rightarrow B) = 1$ and $w_k(B, A) = 0$.	(3), (6)
(8)	If $p_k(A\rightarrow B) = 1$, then $p_k(A\wedge B) \neq 0$.	K8, (1)
(9)	If $w_k(B, A) = 0$, then $p_k(A\wedge B) = 0$.	(1), K20
(10)	It is not the case that both $p_k(A\rightarrow B) = 1$ and $w_k(B, A) = 0$.	(8), (9)
(11)	$[p_k(A\rightarrow B)\text{-}w_k(B, A)] \neq 1$	(7), (10)
(12)	$[p_k(A\rightarrow B)\text{-}w_k(B, A)] < 1$	(6), (11)
(13)	excess$[p_k(A\rightarrow B), w_k(B, A)] < 1$	(1), Ke1, (12)

Now to the conditions under which $p_k(A\rightarrow B)$ equals $w_k(B, A)$. The following two theorems summarize the most important among them.

Theorem Ke10:

For all A, B, p_k and w_k: If $p_k(A)>0$ and if $p_k(B) = 1$ or if $w_k(B, A) = 1$ or if B is a tautology$_k$ or if B truth-functionally follows$_k$ from A or if A is truth-functionally equivalent$_k$ to B, then: excess$[p_k(A\rightarrow B), w_k(B, A)] = 0$.

Proof of theorem Ke10:

(1)	$p_k(A)>0$	assumption
(2)	If $p_k(B) = 1$, then $w_k(B, A) = 1$.	(1), K15

(3) If $w_k(B, A) = 1$, then $p_k(A \to B) = 1$. (1), K14

(4) If B is a tautology$_k$, then $w_k(B, A) = 1$. K1, (2)

(5) If B truth-functionally follows$_k$ from A or if A is truth-functionally equivalent$_k$ to B, then $w_k(B, A) = 1$. (1), K18

(6) If $p_k(B) = 1$ or if $w_k(B, A) = 1$ or if B is a tautology$_k$ or if B truth-functionally follows$_k$ from A or if A is truth-functionally equivalent$_k$ to B, then: $w_k(B, A) = p_k(A \to B)$. (2), (3), (4), (5)

(7) If $w_k(B, A) = p_k(A \to B)$, then $p_k(A \to B) - w_k(B, A) = 0 = \text{excess}[p_k(A \to B), w_k(B, A)]$ (1), Ke1

(8) If $p_k(B) = 1$ or if $w_k(B, A) = 1$ or if B is a tautology$_k$ or if B truth-functionally follows$_k$ from A or if A is truth-functionally equivalent$_k$ to B, then $\text{excess}[p_k(A \to B), w_k(B, A)] = 0$. (6), (7)

Theorem Ke11:

For all A, B, p_k and w_k:

If $p_k(A) = 1$ or if A is a tautology$_k$, then $\text{excess}[p_k(A \to B), w_k(B, A)] = 0$.

Proof of theorem Ke11:

(1) If $p_k(A) = 1$, then $w_k(B, A) = p_k(B) = p_k(A \to B)$. K19, K7

(2) If $p_k(A) = 1$, then $p_k(A \to B) - w_k(B, A) = 0 = \text{excess}[p_k(A \to B), w_k(B, A)]$. (1), Ke1

(3) If A is a tautology$_k$, then $\text{excess}[p_k(A \to B), w_k(B, A)] = 0$. K1, (2)

(4) If $p_k(A) = 1$ or if A is a tautology$_k$, then $\text{excess}[p_k(A \to B), w_k(B, A)] = 0$. (2), (3)

Finally, let us calculate the excess of $p_k(A \to B)$ over $w_k(B, A)$, when A stands in truth-functional contrary or contradictory opposition$_k$ to B. (Note that we cannot calculate the excess of $p_k(A \to B)$ over $w_k(B, A)$, when $p_k(A) = 0$ or A is a contradiction$_k$.)

Theorem Ke12:

For all A, B, p_k and w_k: If $p_k(A) > 0$ and if A stands in truth-functional contrary opposition$_k$ to B, then $\text{excess}[p_k(A \to B), w_k(B, A)] = p_k(\neg A)$.

Proof of theorem Ke12:

(1)	$p_k(A) > 0$	assumption
(2)	A stands in truth-functional contrary opposition$_k$ to B.	assumption
(3)	$p_k(B \wedge A) = 0$	(2), K10
(4)	$w_k(B, A) = 0$	(1), K20, (3)
(5)	$p_k(A \to B) = p_k(\neg A) + p_k(B \wedge A) = p_k(\neg A)$	K11, (3)
(6)	$p_k(A \to B) - w_k(B, A) = p_k(\neg A) - 0 = p_k(\neg A)$	(5), (4)
(7)	$\text{excess}[p_k(A \to B), w_k(B, A)] = p_k(\neg A)$	(1), Ke1, (6)

Theorem Ke13:

For all A, B, p_k and w_k: If $p_k(A) > 0$ and if A stands in truth-functional contradictory opposition$_k$ to B, then $\text{excess}[p_k(A \to B), w_k(B, A)] = p_k(B)$.

Proof of theorem Ke13:

(1)	$p_k(A) > 0$	assumption
(2)	A stands in truth-functional contradictory opposition$_k$ to B.	assumption
(3)	A stands in truth-functional contrary opposition$_k$ to B.	(2)
(4)	$\text{excess}[p_k(A \to B), w_k(B, A)] = p_k(\neg A)$	(1), (3), Ke12
(5)	$\neg A$ is truth-functionally equivalent$_k$ to B.	(2)
(6)	$p_k(\neg A) = p_k(B)$	(5), K5
(7)	$\text{excess}[p_k(A \to B), w_k(B, A)] = p_k(B)$	(4), (6)

3.1 Basic Popper Probability Semantics

In Popper's first publication of 1938 on the axiomatization of probability theory, axioms for a theory of absolute probability functions were given; this theory was then extended to a theory of relative probability functions by means of the standard condition "If $p(b)>0$, then $p(a, b) = p(a \bullet b)/p(b)$", where '•' is a two-place operator (the axioms are understood to take care that • is boolean meet; see POPPER, Axioms). This axiomatization was called 'S_2'. In the same publication, Popper presented a further idea on how to axiomatize probability theory: to give first axioms for a theory of relative probability functions and then to extend it to a theory of absolute probability functions by means of the condition "$p(a) = p[a, (a \bullet a^*)^*]$", where '*' is a one-place operator (* may be understood as boolean complement). Such an axiomatization was called 'S_1', but no axioms were given. In Popper's second publication on the axiomatization of probability theory—dated 1955—two axiomatic bases were presented. The first one was a refined version of S_2: again axioms for a theory of absolute probability functions were given, which was again extended to a theory of relative probability functions, this time by means of three conditions and in such a way that the values of relative probability functions were defined even for those b with absolute probability 0 (see POPPER, Axiom Systems, pp. 53–55). The second one was a worked-out version of S_1 (see POPPER, Axiom Systems, pp. 56–57). This innovative kind of axiomatizations in the S_1-style has become known as 'Popper probability theory'. Popper has constructed and discussed several such "axiomatic systems for relative probability", as he calls them; we shall keep to the axiomatic system developed in appendix *V of the *Logik der Forschung*, for it is in appendix *V that Popper develops his new S_1-style axiomatic probability theory in the most detail, and it is in appendix *V that Popper gives the most extensive account of his laws of excess, which—so the context suggests—are to be viewed as theorems of precisely this axiomatic probability theory, suitably extended to a theory of the excess of $p(A \rightarrow B)$ over $w(B, A)$.[7]

---

7. S_1-systems can be found in POPPER (Philosophy, appendix) [1957]; POPPER (LScD, pp. 326–358) [1959]; POPPER (Conjectures, pp. 59–60, pp. 388–389) [1963]; POPPER (Definitions, p. 169) [1963]; POPPER (LdF2, pp. 268–308) [1966]; POPPER (Non-Existence, pp. 317–318) [1985]. Although Popper prefers S_1-systems to S_2-systems, he has not stopped to develop axiomatic bases for S_2-systems; the most recent one (so far I am aware) was published in POPPER (LdF7, pp. 419–424) [1982].—For secondary literature on Popper probability theory see the probabilistic work of Leblanc and his collaborators, mainly: LEBLANC (Requirements), LEBLANC/VAN FRAASSEN (Functions), LEBLANC (Autonomy), LEBLANC (Contributions), LEBLANC/ROEPER (Probabilities), ROEPER/LEBLANC (Indiscernibility) and LEBLANC/ROEPER (Functions); these articles contain sophisticated elaborations and comparisons of Carnap's, Kolmogorov's, Popper's and Rényi's probability theories. I have also found the following article helpful: HARPER (Belief, pp. 84–112), which contains not only a

Relative Popper probability functions are functions from $S \times S$ into the reals—S being a non-empty set, closed under an arbitrary unary operation * and under an arbitrary binary operation •; absolute Popper probability functions are functions from S into the reals. We note that every sentential-logical language $\mathfrak{L}$ with $\{\neg, \wedge, \vee, \rightarrow\}$ as its set of connectives behaves as is required of S: $\mathfrak{L}$ is always non-empty and closed under negation and conjunction (or disjunction). Let us call '[the] Popper probability semantics [of $\mathfrak{L}$]' that version of Popper probability theory which deals with those Popper probability functions that attach probabilities not to the elements of an *arbitrary* set S of the kind described above, nor to ordered pairs of $S \times S$, but to formulae of a logical language $\mathfrak{L}$ or to ordered pairs of them. So, as in Kolmogorov probability semantics, any sentential-logical language $\mathfrak{L}$ is our object-language. But whereas Kolmogorov probability semantics is an extension of the classical theory of truth value functions and truth-functional attributes, Popper probability semantics is not. Hence we embed our Popper probability semantics *not* in our theory of truth value functions and truth-functional attributes, but simply in informal set theory and arithmetic by adding to the set of their theorems appropriate definitions of binary and of unary Popper probability functions and of a ternary relation of probabilistic independence; these definitions will serve as the three specific axioms Pb1, Pb2 and Pb3 of basic Popper probability semantics (basic, because the set {Pb1, Pb2, Pb3} is the basis on which we shall build a theory of some important semantical properties and relations and, after that, a theory of the excess of the probability of the conditional over the conditional probability):

Axiom Pb1: For all f and $\mathfrak{L}$: f is a binary (or relative or conditional) Popper probability function defined on $\mathfrak{L}$ iff f is a function from $\mathfrak{L} \times \mathfrak{L}$ into the reals such that the following seven conditions are fulfilled:
Popper's axiom A1: There is at least one formula x of $\mathfrak{L}$ and at least one formula y of $\mathfrak{L}$ such that $f(x, x) \neq f(x, y)$.
Popper's axiom A2: For all formulae y and z of $\mathfrak{L}$: if there is at least one formula x of $\mathfrak{L}$ such that $f(x, y) \neq f(x, z)$, then there is at least one formula u of $\mathfrak{L}$ such that $f(y, u) \neq f(z, u)$.
Popper's axiom A3: For all formulae x and y of $\mathfrak{L}$: $f(x, x) \leq f(y, y)$.
Popper's axiom B1: For all formulae x, y and z of $\mathfrak{L}$: $f(x \wedge y, z) \leq f(x, z)$.
Popper's axiom B2: For all formulae x, y and z of $\mathfrak{L}$: $f(x \wedge y, z) = f(x, y \wedge z) \cdot f(y, z)$.
Popper's axiom C: For all formulae x, y and z of $\mathfrak{L}$:
if $f(x, x) \neq f(y, z)$, then $f(x, x) = f(x, z) + f(\neg x, z)$.
Condition D: For all formulae x, y and z of $\mathfrak{L}$:
$f(x \vee y, z) = f[\neg(\neg x \wedge \neg y), z)$ and $f(x \rightarrow y, z) = f(\neg x \vee y, z)$.

Axiom Pb2: For all f, g and $\mathfrak{L}$: g is an f-based unary (or absolute or unconditional)

succinct comparison of Popper probability theory with traditional probability theory, but also an application of Popper probability theory to the problem of rational belief change.

Popper probability function defined on $\mathfrak{L}$ iff, firstly, f is a binary Popper probability function defined on $\mathfrak{L}$ and, secondly, for all x and y: if x and y are formulae of $\mathfrak{L}$, then: $g(x) = f[x, \neg(y\wedge\neg y)]$.

Axiom Pb3: For all x, y, $\mathfrak{L}$ and f: if x and y are formulae of $\mathfrak{L}$, then: x is f-independent$_p$ of y [in $\mathfrak{L}$] iff, firstly, f is a binary Popper probability function defined on $\mathfrak{L}$, and, secondly, for some formula z of $\mathfrak{L}$: $f(x, y) = f[x, \neg(z\wedge\neg z)]$.

As regards axiom Pb1, cf. POPPER (LdF9, p. 298) and POPPER (LScD, p. 349). Note that axiom A1 in (LScD, p.349) is logically stronger than axiom A1 in (LdF9, p. 298). Although the formulations of the other axioms of Popper probability theory in (Logic, p. 349) are slightly different from the respective formulations in (LdF9, p. 298), they are logically equivalent to each other. (Some of Popper's axioms in Pb1 may be better understandable if '$f(x, x)$' is read as '1'.) As regards axiom Pb2, cf. POPPER (LdF9, p. 275, p. 284 and p. 302) and POPPER (LScD, p. 333, p. 337 and p. 353). The main message of Pb2 is simply that the absolute probability of x equals the relative probability of x in regard to a tautology $\neg(y\wedge\neg y)$. As regards axiom Pb3, cf. POPPER (LdF9, p. 422), from which an independence definition such as Pb3 can be gathered (unfortunately, there is no definition of probabilistic independence to be found in the appendix *V of the *Logik der Forschung*). Pb3 was chosen here instead of the following definition Pb3*, which would also have been possible:

Pb3* x is f-independent* of y [in $\mathfrak{L}$] iff, firstly, f is a binary Popper probability function defined on $\mathfrak{L}$, and, secondly, for some formula z of $\mathfrak{L}$: $f[x\wedge y, \neg(z\wedge\neg z)] = f[x, \neg(z\wedge\neg z)]\cdot f[y, \neg(z\wedge\neg z)]$.

Pb3* has the advantage of closely corresponding to definition K3 in Kolmogorov probability semantics, but it is certainly not an adequate definition of probabilistic independence in *Popper* probability semantics. For instance, Pb3* would declare y to be f-independent* of x if $f[y, \neg(z\wedge\neg z)] = 0$—even when y truth-functionally follows$_k$ from x. Furthermore, Popper's formulation of his proposition:

(1.2.12) If b is not a contradiction and if $p(b) = 0$ and if a and b are probabilistically independent and if $p(a) = 0$, then $Exc[p(b\rightarrow a), p(a, b)] = 1$.

strongly suggests that he did not have something like Pb3* in his mind when he wrote down (1.2.12), since a and b would be probabilistically [p-]independent* anyway when $p(a) = 0$ or $p(b) = 0$; so why require that a and b be probabilistically independent *and* $p(a) = 0$ *and* $p(b) = 0$? In addition, not even that weakened version of (1.2.12) which is provable in Pe (see subsection 3.3.2) could be proved if Pb3* had here been used instead of Pb3. This is not to say that Pb3 is a satisfying definition of probabilistic independence in Popper probability semantics; Pb3 yields its own stock of counterintuitive theorems. It is just to say that using Pb3* would have been worse than using Pb3. Surely, Popper probability semantics is still in need of an adequate definition of probabilistic independence.

Let us again use the letters 'A' to 'F' as variables for formulae of $\mathfrak{L}$. Let the subindexed letter 'w_p' be a variable for binary Popper probability functions defined on $\mathfrak{L}$, and the subindexed letter 'p_p' a variable for w_p-based unary Popper probability functions defined on $\mathfrak{L}$. So, instead of beginning each formulation of a theorem of our Popper probability semantics with the cumbersome preamble:

For all $\mathfrak{L}$, x, y, f and g: If x and y are formulae of $\mathfrak{L}$ and if f is a g-based unary Popper probability function defined on $\mathfrak{L}$, then...

we shall simply write:

For all A, B, p_p and w_p...

where p_p is a w_p-based unary Popper probability function. For instance, we shall prefer to the following long formulation of:

Popper's theorem (96): For all $\mathfrak{L}$, x, y, f and g: If x and y are formulae of $\mathfrak{L}$ and if f is a g-based unary Popper probability function defined on $\mathfrak{L}$, then: $g(y, x) \cdot f(x) = f(y \wedge x)$.

this short formulation of:

Popper's theorem (96): For all A, B, p_p and w_p: $w_p(B, A) \cdot p_p(A) = p_p(B \wedge A)$.

We shall need not only Popper's theorem (96) for the later development of the theory of the excess of $p_p(A \rightarrow B)$ over $w_p(B, A)$, but also 34 other theorems of basic Popper probability semantics. 19 of them can be taken over from Popper's own development of his probability theory in his (LdF9, pp. 298–304) or (LScD, pp. 349–355); there he derives 100 theorems from his axioms in a fascinating step-by-step process. I have listed these 19 theorems in appendix 2 for easy reference. The remaining 16 theorems from basic Popper probability semantics (numbered Pb101–Pb116) will be formulated and proved in the text when they are required for proofs of further theorems.

Finally, in order to facilitate comparisons between Kolmogorov and Popper probability semantics, let us say that a proposition y is the Kolmogorov analogue to a theorem x of Popper probability semantics if the formulation of y results from replacing each occurrence of the subindex '$_p$' in the formulation of x by an occurrence of the subindex '$_k$'. For instance, when we replace in the formulation of Popper's theorem (96):

'For all A, B, p_p and w_p: $w_p(B, A) \cdot p_p(A) = p_p(B \wedge A)$.'

each occurrence of the subindex '$_p$' by an occurrence of '$_k$', then we obtain the following sentence:

'For all A, B, p_k and w_k: $w_k(B, A) \cdot p_k(A) = p_k(B \wedge A)$.'

and the proposition which is expressed by this sentence is the Kolmogorov analogue to Popper's theorem (96). Significantly, the Kolmogorov analogue to Popper's theorem (96) is *not* a theorem of Kolmogorov probability semantics. An immediate arithmetical consequence of Popper's theorem (96) is Popper's theorem (97):

For all A, B, p_p and w_p: If $p_p(A) \neq 0$, then $w_p(B, A) = p_p(B \wedge A)/p_p(A)$.

The Kolmogorov analogue to Popper's theorem (97):

For all A, B, p_k and w_k: If $p_k(A) \neq 0$, then $w_k(B, A) = p_k(B \wedge A)/p_k(A)$.

is a theorem of Kolmogorov probability semantics. Likewise, we shall also speak of the Popper analogue to a theorem of Kolmogorov probability semantics.

3.2 Extension of Basic Popper Probability Semantics to the Theory Ps of some Fundamental Semantical Properties and Relations

3.2.1 The Specific Axioms of Ps

We extend basic Popper probability semantics to a probabilistic theory of some fundamental semantical properties and relations (call it 'Ps') by adding to the axioms Pb1, Pb2 and Pb3 of basic Popper probability semantics the following seven definitions Ps1–Ps7, which will serve as the specific axioms of Ps:

Axiom Ps1: For all A and B: B truth-functionally follows$_p$ from A iff for all w_p and C: $w_p(B \wedge A, C) = w_p(A, C)$.
Axiom Ps2: For all A and B: A is truth-functionally equivalent$_p$ to B iff B truth-functionally follows$_p$ from A, and A truth-functionally follows$_p$ from B.
Axiom Ps3: For all A and B: A stands in truth-functional contrary opposition$_p$ to B iff $\neg B$ truth-functionally follows$_p$ from A.
Axiom Ps4: For all A and B: A stands in truth-functional contradictory opposition$_p$ to B iff A is truth-functionally equivalent$_p$ to $\neg B$.
Axiom Ps5: For all A: A is a contradiction$_p$ iff A stands in truth-functional contrary opposition$_p$ to A.
Axiom Ps6: For all A: A is a tautology$_p$ iff $\neg A$ is a contradiction$_p$.
Axiom Ps7: For all A and w_p: A is w_p-absurd iff for all B: $w_p(B, A) = 1$.

Axiom Ps1 is modelled after (but not identical with) that definition of "B (truth-functionally) follows$_p$ from A" which Popper gives in appendix *V (cf. definition (D3) on p. 305 in POPPER, LdF9; and the last paragraph on p. 356 in POPPER, LScD). Of course, Ps1 is chosen in such a way that the following theorem which connects our classical theory of truth value functions and truth-functional attributes with Popper probability semantics be valid:

For all A and B: B truth-functionally follows$_k$ from A iff B truth-functionally follows$_p$ from A.

However, I shall not have recourse to this bridge theorem. All theorems of Popper probability semantics which will here be required will be neatly proved *within* Popper probability semantics, be they deemed to be trivial or not. No (specific) theorems of Kolmogorov probability semantics and no (specific) theorems of the classical theory of truth value functions and truth-functional attributes will be used in any of these proofs. Only in this way can it be seen how Popper probability semantics works.

3.2.2 Theorems concerning "A Truth-functionally follows$_p$ from B" and "A is Truth-functionally equivalent$_p$ to B"

Popper's own step-by-step development of his probability theory ends on page 304 of the *Logik der Forschung* with theorem (100). On page 307 he lists his laws of excess. In between, on page 306, he informs us that there are two noteworthy theorems in his (extended) probability theory which are only theorems of his, but not theorems of Kolmogorov probability theory, namely (in *our* terminology, universal quantification over w_p added):

B truth-functionally follows$_p$ from A iff for all w_p: $w_p(B, \neg B \wedge A) \neq 0$; and
B truth-functionally follows$_p$ from A iff for all w_p: $w_p(B, \neg B \wedge A) = 1$.

To build a bridge of theorems from the last proved theorem (100) on page 304 to the first law of excess on page 307, it will be convenient to start with a proof of these two important propositions. Call the first one 'Ps10' and the second one 'Ps11'. By working our way to Ps10 and Ps11, we shall collect a lot of theorems, all of them helpful for the later proofs of Popper's laws of excess.—To prove Ps10, two lemmata will suffice:

Theorem Ps8: [lemma 1 for Ps10; also lemma for Ps15 and Pe33]

For all A, B and w_p: If B truth-functionally follows$_p$ from A, then $w_p(B, A) = 1$.

Proof of theorem Ps8:

(1)	B truth-functionally follows$_p$ from A.	assumption
(2)	For all C: $w_p(B \wedge A, C) = w_p(A, C)$.	Ps1, (1)
(3)	$w_p(B \wedge A, A) = w_p(A, A)$	(2)
(4)	$w_p(B \wedge A, A) = w_p(A \wedge B, A) = w_p(B, A)$	Popper's theorems (40) and (29)
(5)	$w_p(A, A) = 1$	Popper's theorem (23)
(6)	$w_p(B, A) = 1$	(4), (3), (5)

Theorem Ps9: [lemma 2 for Ps10]

For all A, B, C: If B truth-functionally follows$_p$ from A, then B truth-functionally follows$_p$ from $A \wedge C$.

Proof of theorem Ps9:

(1)	B truth-functionally follows$_p$ from A.	assumption
(2)	For all C: $w_p(B \wedge A, C) = w_p(A, C)$.	Ps1, (1)
(3)	$w_p(B \wedge A, C \wedge D) = w_p(A, C \wedge D)$	(2)
(4)	$w_p[(B \wedge A) \wedge C, D] = w_p(B \wedge A, C \wedge D) \cdot w_p(C, D)$	Popper's axiom B2

(5)	$w_p[(B\wedge A)\wedge C, D] = w_p(A, C\wedge D)\cdot w_p(C, D)$	(3), (4)
(6)	$w_p[(B\wedge A)\wedge C, D] = w_p[B\wedge(A\wedge C), D]$	Popper's theorem (62)
(7)	$w_p[B\wedge(A\wedge C), D] = w_p(A, C\wedge D)\cdot w_p(C, D)$	(5), (6)
(8)	$w_p(A\wedge C, D) = w_p(A, C\wedge D)\cdot w_p(C, D)$	Popper's axiom B2
(9)	$w_p[B\wedge(A\wedge C), D] = w_p(A\wedge C, D)$	(7), (8)
(10)	For all D and w_p: $w_p[B\wedge(A\wedge C), D] = w_p(A\wedge C, D)$.	(9), for neither D nor w_p is free in (1) or (10)
(11)	B truth-functionally follows$_p$ from $A\wedge C$.	Ps1, (10)

Theorem Ps10: [lemma for Ps20]
For all A and B: If B truth-functionally follows$_p$ from A, then for all w_p: $w_p(B, \neg B\wedge A) = 1 \neq 0$.

Proof of theorem Ps10:

(1)	B truth-functionally follows$_p$ from A.	assumption
(2)	B truth-functionally follows$_p$ from $A\wedge\neg B$.	(1), Ps9
(3)	$w_p(B, \neg B\wedge A) = w_p(B, A\wedge\neg B) = 1 \neq 0$	Popper's theorem (40), (2), Ps8
(4)	For all w_p: $w_p(B, \neg B\wedge A) = 1 \neq 0$.	(3), for w_p is neither free in (1) nor in (4)

Note that the Kolmogorov analogue to Ps9 is an unspecific theorem of Kolmogorov probability semantics, whereas the Kolmogorov analogues to Ps8 and Ps10 are not theorems of it—an obvious shortcoming of Kolmogorov probability semantics.

To prove Ps11, we shall need a sublemma (Pb101) and five lemmata (Pb102–Pb106). None of the Kolmogorov analogues to Pb101–Pb106 is a theorem of Kolmogorov probability semantics.

Theorem Pb101: [lemma for Pb106; also lemma for Pb108, Ps11, and Ps24]
For all A, B, C and w_p: $w_p(B, A) = w_p(\neg\neg B, A)$ and $w_p(A\wedge\neg\neg B, C) = w_p(A\wedge B, C)$ and $w_p(B, A) = w_p(B, \neg\neg A)$ and $w_p(B, A) = w_p(B, \neg\neg A\wedge A)$.

Proof of theorem Pb101:

(1)	$w_p(\neg\neg B, A) = 1\text{-}w_p(\neg B, A)+w_p(\neg A, A)$	Popper's theorem (64)
(2)	$w_p(\neg B, A) = 1\text{-}w_p(B, A)+w_p(\neg A, A)$	Popper's theorem (64)
(3)	$w_p(\neg\neg B, A) = 1\text{-}[1\text{-}w_p(B, A)+w_p(\neg A, A)]+w_p(\neg A, A) = w_p(B, A)$	(1), (2)
(4)	For all A and B: $w_p(B, A) = w_p(\neg\neg B, A)$.	(3), for neither A nor B is free in (4)
(5)	$w_p(\neg\neg B\wedge A, C) = w_p(\neg\neg B, A\wedge C)\cdot w_p(A, C)$	Popper's axiom B2
(6)	$w_p(B, A\wedge C) = w_p(\neg\neg B, A\wedge C)$	(4)
(7)	$w_p(\neg\neg B\wedge A, C) = w_p(B, A\wedge C)\cdot w_p(A, C)$	(5), (6)
(8)	$w_p(B, A\wedge C)\cdot w_p(A, C) = w_p(B\wedge A, C)$	Popper's axiom B2
(9)	$w_p(\neg\neg B\wedge A, C) = w_p(B\wedge A, C)$	(7), (8)
(10)	$w_p(A\wedge\neg\neg B, C) = w_p(A\wedge B, C)$	(9), Popper's theorem (40)
(11)	$w_p(B, A) = w_p(\neg\neg B, A)$ and $w_p(A\wedge\neg\neg B, C) = w_p(A\wedge B, C)$.	(3), (10)
(12)	For all D and E: If for all C: $w_p(D, C) = w_p(E, C)$, then for all C: $w_p(C, D) = w_p(C, E)$.	Popper's axiom A2
(13)	For all C: $w_p(A, C) = w_p(\neg\neg A, C)$	(4), for C is not free in (13)
(14)	If for all C: $w_p(A, C) = w_p(\neg\neg A\ C)$, then for all C: $w_p(C, A) = w_p(C, \neg\neg A)$.	(12)
(15)	$w_p(B, A) = w_p(B, \neg\neg A)$	(13), (14)
(16)	For all B: $w_p(\neg\neg B\wedge A, C) = w_p(B\wedge A, C)$.	(9), for B is not free in (16)
(17)	$w_p(\neg\neg A\wedge A, C) = w_p(A\wedge A, C) = w_p(A, C)$	(16), Popper's theorem (32)

(18)	For all C: $w_p(A, C) = w_p(\neg\neg A \wedge A, C)$.	(17), for C is not free in (18)
(19)	For all C: $w_p(C, A) = w_p(C, \neg\neg A \wedge A)$.	(12), (18)
(20)	$w_p(B, A) = w_p(B, \neg\neg A \wedge A)$	(19)

Theorem Pb102: [lemma 1 for Ps11]

For all A, B and w_p: If $w_p(B, \neg B \wedge A) \neq 0$, then for all C: $w_p(C, \neg B \wedge A) = 1$.

Proof of theorem Pb102:

(1)	$w_p(B, \neg B \wedge A) \neq 0$	assumption
(2)	$w_p(\neg B, A \wedge \neg B) = 1 = w_p(\neg B, \neg B \wedge A)$	Popper's theorems (28) and (40)
(3)	$1 = 1 - w_p(B, \neg B \wedge A) + w_p[\neg(\neg B \wedge A), \neg B \wedge A]$	Popper's theorem (64), (2)
(4)	$w_p(B, \neg B \wedge A) = w_p[\neg(\neg B \wedge A), \neg B \wedge A] \neq 0$	(3), (1)
(5)	For all C: $w_p(C, \neg B \wedge A) = 1$.	Popper's theorem (63), (4)

Theorem Pb103: [lemma 2 for Ps11; also lemma for Pb104]

For all A, B and w_p: If $w_p(B \wedge \neg B, A) = 1$, then $w_p(A, B) = w_p(\neg B, B)$.

Proof of theorem Pb103:

(1)	$w_p(B \wedge \neg B, A) = 1$	assumption
(2)	$w_p(\neg B \wedge A, B) = w_p(\neg B, A \wedge B) \cdot w_p(A, B)$	Popper's axiom B2
(3)	$w_p(B \wedge \neg B, A) = w_p(\neg B \wedge B, A)$	Popper's theorem (40)
(4)	$1 = w_p(\neg B \wedge B, A) = w_p(\neg B, B \wedge A) \cdot w_p(B, A)$	(1), (3), Popper's axiom B2
(5)	$w_p(\neg B, A \wedge B) = w_p(\neg B, B \wedge A)$	Popper's theorem (40)
(6)	$w_p(\neg B, A \wedge B) \cdot w_p(B, A) = 1$	(5), (4)
(7)	$w_p(B, A) \leq 1$ and $w_p(\neg B, A \wedge B) \leq 1$.	Popper's theorem (16)
(8)	$w_p(\neg B, A \wedge B) = 1$	(6), (7)
(9)	$w_p(\neg B \wedge A, B) = 1 \cdot w_p(A, B) = w_p(A, B)$	(2), (8)
(10)	$w_p(\neg B \wedge A, B) = w_p(\neg B, B)$	Popper's theorem (69)
(11)	$w_p(A, B) = w_p(\neg B, B)$	(9), (10)

Theorem Pb104: [lemma 3 for Ps11; also lemma for Ps17]

For all A and w_p: for all B: $w_p(B, A) = 1$ iff for all B: $w_p(\neg A, B) = 1$.

Proof of theorem Pb104:

(1)	For all B: $w_p(B, A) = 1$.	assumption →
(2)	$w_p(B \wedge \neg B, A) = 1$	(1)
(3)	$w_p(A, B) = w_p(\neg B, B)$	(2), Pb103
(4)	$w_p(\neg A, B) = 1 - w_p(A, B) + w_p(\neg B, B) = 1$	Popper's theorem (64), (3)
(5)	For all B: $w_p(\neg A, B) = 1$.	(4), for B is neither free in (1) nor in (5)
(6)	For all B: $w_p(\neg A, B) = 1$.	assumption ←
(7)	$w_p(\neg A, A) = 1$	(6)
(8)	For all B: $w_p(B, A) = 1$.	Popper's theorem (63), (7)

Theorem Pb105: [lemma 4 for Ps11; also lemma for Ps19]

For all A, B and w_p: $w_p(B \wedge \neg B, A) = w_p(\neg A, A) = w_p(\neg B \wedge B, A)$.

Proof of theorem Pb105:

(1)	$w_p(B \wedge \neg B, A) = 1 - w_p[\neg(B \wedge \neg B), A] + w_p(\neg A, A) =$ $= 1 - 1 + w_p(\neg A, A) = w_p(\neg A, A)$	Popper's theorems (64) and (74)
(2)	$w_p(B \wedge \neg B, A) = w_p(\neg B \wedge B, A)$	Popper's theorem (40)
(3)	$w_p(B \wedge \neg B, A) = w_p(\neg A, A) = w_p(\neg B \wedge B, A)$	(1), (2)

Theorem Pb106: [lemma 5 for Ps11; also lemma for Ps18, Pb113, and Ps26]

For all A, B, C and w_p:

$w_p(A \to B, C) = w_p[\lnot(A \land \lnot B), C] = w_p(B \land A, C) + w_p(\lnot A, C) - w_p(A \land \lnot A, C)$.

Proof of theorem Pb106:

(1)	$w_p(A \to B, C) = w_p(\lnot A \lor B, C) = w_p[\lnot(\lnot\lnot A \land \lnot B), C]$	Pb1
(2)	$w_p[\lnot(\lnot\lnot A \land \lnot B), C] = 1 - w_p(\lnot\lnot A \land \lnot B, C) + w_p(\lnot C, C)$	Popper's theorem (64)
(3)	$w_p(A \to B, C) = 1 - w_p(\lnot\lnot A \land \lnot B, C) + w_p(\lnot C, C)$	(1), (2)
(4)	$w_p[\lnot(\lnot(A \land B) \land \lnot\lnot A), C] = 1 - w_p[\lnot(A \land B) \land \lnot\lnot A, C] + w_p(\lnot C, C)$	Popper's theorem (64)
(5)	$w_p[\lnot(A \land B) \land \lnot\lnot A, C] = w_p[\lnot(A \land B), \lnot\lnot A \land C] \cdot w_p(\lnot\lnot A, C)$	Popper's axiom B2
(6)	$w_p[(\lnot(A \land B), \lnot\lnot A \land C] = 1 - w_p(A \land B, \lnot\lnot A \land C) + w_p[\lnot(\lnot\lnot A \land C), \lnot\lnot A \land C]$	Popper's theorem (64)
(7)	$w_p(A \land B, \lnot\lnot A \land C) = w_p[A, B \land (\lnot\lnot A \land C)] \cdot w_p(B, \lnot\lnot A \land C)$	Popper's axiom B2
(8)	$w_p[A, B \land (\lnot\lnot A \land C)] = w_p[A, (\lnot\lnot A \land C) \land B] = w_p[A, \lnot\lnot A \land (C \land B)]$	Popper's theorems (40), (62)
(9)	$w_p[\lnot\lnot A, (C \land B) \land \lnot\lnot A] = 1 = w_p[\lnot\lnot A, \lnot\lnot A \land (C \land B)] =$ $= w_p[A, \lnot\lnot A \land (C \land B)]$	Popper's theorems (28) and (40), Pb101
(10)	$w_p(A \land B, \lnot\lnot A \land C) = w_p(B, \lnot\lnot A \land C)$	(7), (8), (9)
(11)	$w_p(\lnot B, \lnot\lnot A \land C) = 1 - w_p(B, \lnot\lnot A \land C) + w_p[\lnot(\lnot\lnot A \land C), \lnot\lnot A \land C]$	Popper's theorem (64)
(12)	$w_p(\lnot B, \lnot\lnot A \land C) = w_p[\lnot(A \land B), \lnot\lnot A \land C]$	(11), (10), (6)
(13)	$w_p[\lnot(A \land B) \land \lnot\lnot A, C] = w_p(\lnot B, \lnot\lnot A \land C) \cdot w_p(\lnot\lnot A, C)$	(12), (5)
(14)	$w_p(\lnot\lnot A \land \lnot B, C) = w_p(\lnot B \land \lnot\lnot A, C)$	Popper's theorem (40)
(15)	$w_p(\lnot B \land \lnot\lnot A, C) = w_p(\lnot B, \lnot\lnot A \land C) \cdot w_p(\lnot\lnot A, C)$	Popper's axiom B2
(16)	$w_p(\lnot\lnot A \land \lnot B, C) = w_p[\lnot(A \land B) \land \lnot\lnot A, C]$	(14), (15), (13)
(17)	$w_p(A \to B, C) = w_p[\lnot(\lnot(A \land B) \land \lnot\lnot A), C]$	(16); (3), (4)
(18)	$w_p[\lnot(\lnot(A \land B) \land \lnot\lnot A), C] = w_p(A \land B, C) + w_p(\lnot A, C) - w_p[(A \land B) \land \lnot A, C]$	Popper's theorem (79)
(19)	$w_p[(A \land B) \land \lnot A, C] = w_p[A \land (B \land \lnot A), C]$	Popper's theorem (62)
(20)	$w_p[A \land (B \land \lnot A), C] = w_p[(B \land \lnot A) \land A, C]$	Popper's theorem (40)
(21)	$w_p[(B \land \lnot A) \land A, C] = w_p[B \land (\lnot A \land A), C]$	Popper's theorem (62)
(22)	$w_p[B \land (\lnot A \land A), C] = w_p[B, (\lnot A \land A) \land C] \cdot w_p(\lnot A \land A, C)$	Popper's axiom B2
(23)	$w_p(B \land C, \lnot A \land A) = w_p[B, C \land (\lnot A \land A)] \cdot w_p(C, \lnot A \land A)$	Popper's axiom B2
(24)	$w_p(B \land C, \lnot A \land A) = 1 = w_p(C, \lnot A \land A)$	Popper's theorem (33')
(25)	$w_p[B, C \land (\lnot A \land A)] = 1 = w_p[B, (\lnot A \land A) \land C]$	(23), (24), Popper's theorem (40)
(26)	$w_p[B \land (\lnot A \land A), C] = w_p(\lnot A \land A, C) = w_p(A \land \lnot A, C)$	(22), (25), Popper's theorem (40)
(27)	$w_p[(A \land B) \land \lnot A, C] = w_p(A \land \lnot A, C)$	(19), (20), (21), (26)
(28)	$w_p[\lnot(\lnot(A \land B) \land \lnot\lnot A), C] = w_p(A \land B, C) + w_p(\lnot A, C) - w_p(A \land \lnot A, C)$	(18), (27)
(29)	$w_p(A \land B, C) = w_p(B \land A, C)$	Popper's theorem (40)
(30)	$w_p(A \to B, C) = w_p(B \land A, C) + w_p(\lnot A, C) - w_p(A \land \lnot A, C)$	(17), (28), (29)
(31)	$w_p(\lnot\lnot A \land \lnot B, C) = w_p(\lnot\lnot A, \lnot B \land C) \cdot w_p(\lnot B, C)$	Popper's axiom B2
(32)	$w_p(A, \lnot B \land C) = w_p(\lnot\lnot A, \lnot B \land C)$	Pb101
(33)	$w_p(\lnot\lnot A \land \lnot B, C) = w_p(A, \lnot B \land C) \cdot w_p(\lnot B, C)$	(31), (32)
(34)	$w_p(A \land \lnot B, C) = w_p(A, \lnot B \land C) \cdot w_p(\lnot B, C)$	Popper's axiom B2
(35)	$w_p(A \land \lnot B, C) = w_p(\lnot\lnot A \land \lnot B, C)$	(34), (33)
(36)	$1 - w_p(A \land \lnot B, C) + w_p(\lnot C, C) = 1 - w_p(\lnot\lnot A \land \lnot B, C) + w_p(\lnot C, C)$	(35)
(37)	$w_p[\lnot(A \land \lnot B), C] = w_p[\lnot(\lnot\lnot A \land \lnot B), C]$	Popper's theorem (64), (36)
(38)	$w_p(A \to B, C) = w_p[\lnot(A \land \lnot B), C]$	(1), (37)
(39)	$w_p(A \to B, C) = w_p[\lnot(A \land \lnot B), C] = w_p(B \land A, C) + w_p(\lnot A, C) - w_p(A \land \lnot A, C)$	(38), (30)

Theorem Ps11: [lemma for Ps20]

For all A and B:

If for all w_p: $w_p(B, \lnot B \land A) \neq 0$, then B truth-functionally follows$_p$ from A.

Proof of theorem Ps11:

(1)	For all w_p: $w_p(B, \neg B\wedge A) \neq 0$.	assumption
(2)	For all C: $w_p(C, \neg B\wedge A) = 1$.	(1), Pb102
(3)	$w_p(C, \neg B\wedge A) = 1$	(2)
(4)	$w_p(C, A\wedge\neg B) = 1$	(3), Popper's theorem (40)
(5)	For all E: $w_p(A, E) = w_p(\neg\neg A, E)$.	Pb101
(6)	$w_p(C, A\wedge\neg B) = w_p(C, \neg\neg A\wedge\neg B)$	(4), Popper's theorem (99)
(7)	$w_p(C, \neg\neg A\wedge\neg B) = 1$	(6), (4)
(8)	For all C: $w_p(C, \neg\neg A\wedge\neg B) = 1$.	(7), for C is neither free in (1) nor in (8)
(9)	For all C: $w_p[\neg(\neg\neg A\wedge\neg B), C] = 1$.	(8), Pb104
(10)	$w_p[\neg(\neg\neg A\wedge\neg B), C] = 1 = w_p(B\wedge A, C)+w_p(\neg A, C)-w_p(A\wedge\neg A, C)$	(9), Pb106, Pb101
(11)	$w_p(\neg A, C) = 1-w_p(A, C)+w_p(\neg C, C)$	Popper's theorem (64)
(12)	$w_p(A\wedge\neg A, C) = w_p(\neg C, C)$	Pb105
(13)	$1 = w_p(B\wedge A, C)+1-w_p(A, C)+w_p(\neg C, C)-w_p(\neg C, C)$	(10), (11), (12)
(14)	$w_p(B\wedge A, C) = w_p(A, C)$	(13)
(15)	For all w_p and C: $w_p(B\wedge A, C) = w_p(A, C)$.	(14), for neither w_p nor C is free in (1) or (15)
(16)	B truth-functionally follows$_p$ from A.	Ps1, (15)

So the first half of the bridge—spanning the distance from Popper's theorem (100) on page 304 to theorems Ps10 and Ps11 on page 306 of the *Logik der Forschung*—is completed. We have to construct now the second half of the bridge from theorems Ps10 and Ps11 to Popper's laws of excess on page 307; this will require proving further 18 theorems. Some of them (though not their proofs) will look familiar from Kolmogorov probability semantics. We continue with three theorems concerning the relation of truth-functional equivalence$_p$: Ps12, Ps13 and Ps14. The Kolmogorov analogue to Ps12 is not a theorem of Kolmogorov probability semantics—which is another one of its shortcomings. The Kolmogorov analogue to Ps13 is theorem K5; and that to Ps14 is a well-known theorem of the classical theory of truth-value functions and truth-functional attributes.

Theorem Ps12: [lemma for Ps13, Ps14, and Pe13]
For all A and B:
A is truth-functionally equivalent$_p$ to B iff for all w_p and C: $w_p(A, C) = w_p(B, C)$.

Proof of theorem Ps12:

(1)	A is truth-functionally equivalent$_p$ to B.	assumption →
(2)	B truth-functionally follows$_p$ from A, and A truth-functionally follows$_p$ from B.	Ps2, (1)
(3)	$w_p(B\wedge A, C) = w_p(A, C)$	Ps1, (2)
(4)	$w_p(A\wedge B, C) = w_p(B, C)$	Ps1, (2)
(5)	$w_p(A\wedge B, C) = w_p(B\wedge A, C)$	Popper's theorem (40)
(6)	$w_p(A, C) = w_p(B, C)$	(3), (4), (5)
(7)	For all w_p and C: $w_p(A, C) = w_p(B, C)$.	(6), for neither w_p nor C is free in (1) or (7)
(8)	For all w_p and C: $w_p(A, C) = w_p(B, C)$.	assumption ←
(9)	$w_p(A, A\wedge C) = w_p(B, A\wedge C)$	(8)
(10)	$w_p(A, C\wedge A) = 1 = w_p(A, A\wedge C)$	Popper's theorems (28) and (40)
(11)	$w_p(B, A\wedge C) = 1$	(9), (10)
(12)	$w_p(B\wedge A, C) = w_p(B, A\wedge C)\cdot w_p(A, C) = w_p(A, C)$	Popper's axiom B2, (11)
(13)	For all w_p and C: $w_p(B\wedge A, C) = w_p(A, C)$.	(12), for neither w_p nor C is free in (8) or (13)
(14)	B truth-functionally follows$_p$ from A.	Ps1, (13)

(15)	$w_p(A, B\wedge C) = w_p(B, B\wedge C)$	(8)
(16)	$w_p(B, C\wedge B) = 1 = w_p(B, B\wedge C)$	Popper's theorems (28) and (40)
(17)	$w_p(A, B\wedge C) = 1$	(15), (16)
(18)	$w_p(A\wedge B, C) = w_p(A, B\wedge C)\cdot w_p(B, C) = w_p(B, C)$	Popper's axiom B2, (17)
(19)	For all w_p and C: $w_p(A\wedge B, C) = w_p(B, C)$.	(18), for neither w_p nor C is free in (8) or (19)
(20)	A truth-functionally follows$_p$ from B.	Ps1, (19)
(21)	A is truth-functionally equivalent$_p$ to B.	Ps2, (14), (20)

Theorem Ps13: [lemma for Ps16]

For all A, B and p_p: If A is truth-functionally equivalent$_p$ to B, then $p_p(A) = p_p(B)$.

Proof of theorem Ps13:

(1)	A is truth-functionally equivalent$_p$ to B.	assumption
(2)	For all C: $w_p(A, C) = w_p(B, C)$.	Ps12, (1)
(3)	$w_p[A, \neg(C\wedge\neg C)] = w_p[B, \neg(C\wedge\neg C)]$	(2)
(4)	$p_p(A) = p_p(B)$	(3), Popper's theorem (75)

Theorem Ps14: [lemma for Pe13]

For all A and B: $(A\to B)\wedge A$ is truth-functionally equivalent$_p$ to $B\wedge A$.

Proof of theorem Ps14:

(1)	$w_p[(A\to B)\wedge A, C] = w_p(A\to B, A\wedge C)\cdot w_p(A, C)$	Popper's axiom B2
(2)	$w_p(A\to B, A\wedge C) = w_p[\neg(A\wedge\neg B), A\wedge C]$	Pb106
(3)	$w_p[\neg(A\wedge\neg B), A\wedge C] = 1\text{-}w_p(A\wedge\neg B, A\wedge C)+w_p[\neg(A\wedge C), A\wedge C]$	Popper's theorem (64)
(4)	$w_p(A\wedge\neg B, A\wedge C) = w_p(\neg B\wedge A, A\wedge C)$	Popper's theorem (40)
(5)	$w_p(\neg B\wedge A, A\wedge C) = w_p[\neg B, A\wedge(A\wedge C)]\cdot w_p(A, A\wedge C)$	Popper's axiom B2
(6)	$w_p(A, C\wedge A) = 1 = w_p(A, A\wedge C)$	Popper's theorems (28) and (40)
(7)	$w_p(A\wedge\neg B, A\wedge C) = w_p[\neg B, A\wedge(A\wedge C)]$	(4), (5), (6)
(8)	$w_p[\neg B, A\wedge(A\wedge C)] = w_p[\neg B, (A\wedge A)\wedge C]$	Popper's theorem (62)
(9)	For all E: $w_p(A\wedge A, E) = w_p(A, E)$.	Popper's theorem (32)
(10)	$w_p[\neg B, (A\wedge A)\wedge C] = w_p(\neg B, A\wedge C)$	(9), Popper's theorem (99)
(11)	$w_p[\neg B, A\wedge(A\wedge C)] = w_p(\neg B, A\wedge C)$	(8), (10)
(12)	$w_p(\neg B, A\wedge C) = 1\text{-}w_p(B, A\wedge C)+w_p[\neg(A\wedge C), A\wedge C]$	Popper's theorem (64)
(13)	$w_p(A\wedge\neg B, A\wedge C) = 1\text{-}w_p(B, A\wedge C)+w_p[\neg(A\wedge C), A\wedge C]$	(7), (11), (12)
(14)	$w_p[\neg(A\wedge\neg B), A\wedge C] = 1\text{-}[1\text{-}w_p(B, A\wedge C)+w_p[\neg(A\wedge C), A\wedge C]]+w_p[\neg(A\wedge C), A\wedge C]$	(3), (13)
(15)	$w_p[\neg(A\wedge\neg B), A\wedge C] = w_p(B, A\wedge C)$	(14)
(16)	$w_p(A\to B, A\wedge C) = w_p(B, A\wedge C)$	(2), (15)
(17)	$w_p[(A\to B)\wedge A, C] = w_p(B, A\wedge C)\cdot w_p(A, C)$	(1), (16)
(18)	$w_p(B\wedge A, C) = w_p(B, A\wedge C)\cdot w_p(A, C)$	Popper's axiom B2
(19)	$w_p[(A\to B)\wedge A, C] = w_p(B\wedge A, C)$	(17), (18)
(20)	For all w_p and C: $w_p[(A\to B)\wedge A, C] = w_p(B\wedge A, C)$.	(19), for neither w_p nor C is free in (20)
(21)	$(A\to B)\wedge A$ is truth-functionally equivalent$_p$ to $B\wedge A$.	Ps12, (20)

3.2.3 Theorems concerning "A stands in Truth-functional Contrary Opposition$_p$ to B" and "A stands in Truth-functional Contradictory Opposition$_p$ to B"

We shall need only two of them for the time being: Ps15 and Ps16.

Theorem Ps15: [lemma for Pe11 and Pe36]

For all A, B and w_p: If A stands in truth-functional contrary opposition$_p$ to B, then $w_p(\neg B, A) = 1$.

Proof of theorem Ps15: immediately from Ps3 and Ps8.

For the proof of theorem Ps16 (and of many theorems to come), we need the negation theorem for unary Popper probability functions, i.e., $p_p(\neg A) = 1\text{-}p_p(A)$, which will be our theorem Pb110. To make the proof of Pb110 transparent, we prove three lemmata first: Pb107 as a starter, which is only a convenient lemma for the important theorem Pb108, to which Pb109 is a corollary, from which, finally, Pb110 follows at once. Note that the Komogorov analogues to Pb108 and Pb109 are not theorems of Kolmogorov probability semantics.

Theorem Pb107: [lemma for Pb108; also lemma for Pb111 and Pb115]

For all A, B and p_p: $p_p(A \wedge B) = p_p(B \wedge A)$.

Proof of theorem Pb107:

(1)	$p_p(A \wedge B) = w_p[A \wedge B, \neg(C \wedge \neg C)]$	Popper's theorem (75)
(2)	$p_p(B \wedge A) = w_p[B \wedge A, \neg(C \wedge \neg C)]$	Popper's theorem (75)
(3)	$w_p[A \wedge B, \neg(C \wedge \neg C)] = w_p[B \wedge A, \neg(C \wedge \neg C)]$	Popper's theorem (40)
(4)	$p_p(A \wedge B) = p_p(B \wedge A)$	(1), (2), (3)

Theorem Pb108: [lemma for Pb109]

For all A and w_p: If $w_p(\neg A, A) \neq 0$, then $w_p(A, \neg A) = 0$.

Proof of theorem Pb108:

(1)	$w_p(\neg A, A) \neq 0$ and $w_p(A, \neg A) \neq 0$.	assumption [for indirect proof]
(2)	$w_p(B, A) = 1$	(1), Popper's theorem (63)
(3)	$w_p(A, \neg A) = w_p(\neg\neg A, \neg A)$	Pb101
(4)	$w_p(A, \neg A) = 1$	(1), (3), Popper's theorem (63)
(5)	If $w_p(\neg B, B) = 0$, then $w_p(A \wedge A, B) + w_p(\neg A \wedge A, B) = w_p(A, B)$.	Popper's theorem (70)
(6)	If $w_p(\neg B, B) = 0$, then $w_p(A, B) + w_p(\neg A \wedge A, B) = w_p(A, B)$.	(5), Popper's theorem (32)
(7)	If $w_p(\neg B, B) = 0$, then $w_p(\neg A \wedge A, B) = 0 = w_p(A \wedge \neg A, B)$.	(6), Popper's theorem (40)
(8)	$w_p(A \wedge \neg A, B) = w_p(A, \neg A \wedge B) \cdot w_p(\neg A, B)$	Popper's axiom B2
(9)	$w_p(B \wedge A, \neg A) =$ $= w_p(B, A \wedge \neg A) \cdot w_p(A, \neg A) = w_p(B, \neg A \wedge A) \cdot w_p(A, \neg A) = 1 \cdot 1 = 1$	Popper's axiom B2, Popper's theorems (40) and (33'), (4)
(10)	$w_p(B \wedge A, \neg A) = w_p(A \wedge B, \neg A) =$ $= w_p(A, B \wedge \neg A) \cdot w_p(B, \neg A) = w_p(A, \neg A \wedge B) \cdot w_p(B, \neg A)$	Popper's axiom B2, Popper's theorem (40)
(11)	$w_p(A, \neg A \wedge B) \cdot w_p(B, \neg A) = 1$	(10), (9)
(12)	$0 \leq w_p(A, \neg A \wedge B) \leq 1$ and $0 \leq w_p(B, \neg A) \leq 1$.	Popper's theorem (16)
(13)	$w_p(A, \neg A \wedge B) = 1$	(11), (12)
(14)	$w_p(A \wedge \neg A, B) = w_p(\neg A, B)$	(8), (13)
(15)	If $w_p(\neg B, B) = 0$, then $w_p(\neg A, B) = 0$.	(7), (14)
(16)	If $w_p(\neg B, B) = 0$, then $0 = 1 - w_p(A, B) + 0$.	Popper's theorem (64), (15)
(17)	If $w_p(\neg B, B) = 0$, then $w_p(A, B) = 1$.	(16)
(18)	If $w_p(\neg B, B) \neq 0$, then $w_p(A, B) = 1$.	Popper's theorem (63)
(19)	$w_p(A, B) = 1$	(17), (18)
(20)	$p_p(B) = p_p(A \wedge B)$	(19), Popper's theorem (96)

(21)	$p_p(A) = p_p(B \wedge A)$	(2), Popper's theorem (96)
(22)	$p_p(A) = p_p(B)$	(21), (20), Pb107
(23)	For all B: $p_p(A) = p_p(B)$.	(22), for B is neither free in (1) nor in (23)
(24)	$p_p(A) = p_p(C) = p_p(\neg C) = p_p[\neg\neg(B\wedge\neg B)] = p_p(C\wedge\neg C)$	(23)
(25)	$w_p[C, \neg(B\wedge\neg B)] = w_p[\neg C, \neg(B\wedge\neg B)] = w_p[\neg\neg(B\wedge\neg B), \neg(B\wedge\neg B)]$	(24), Popper's theorem (75)
(26)	$w_p[\neg C, \neg(B\wedge\neg B)] = 1\text{-}w_p[C, \neg(B\wedge\neg B)]+w_p[\neg\neg(B\wedge\neg B), \neg(B\wedge\neg B)]$	Popper's theorem (64)
(27)	$w_p[\neg C, \neg(B\wedge\neg B)] = 1$	(26), (25)
(28)	$p_p(\neg C) = 1 = p_p(C\wedge\neg C)$	(27), Popper's theorem (75), (24)
(29)	$w_p(C, \neg C)\cdot p_p(\neg C) = p_p(C\wedge\neg C)$	Popper's theorem (96)
(30)	$w_p(C, \neg C) = 1$	(29), (28)
(31)	For all C: $w_p(C, \neg C) = 1$.	(30), for C is neither free in (1) nor in (31)
(32)	$w_p(\neg C, \neg\neg C) = 1$	(31)
(33)	$w_p(\neg C, C) = 1$	(32), Pb101
(34)	For all C: $w_p(\neg C, C) = 1$.	(33), for C is neither free in (1) nor in (34)
(35)	There is at least one C such that $w_p(\neg C, C) \neq 1$.	Popper's theorem (25)
(36)	There is at least one C such that $w_p(\neg C, C) \neq 1$, and there is no C such that $w_p(\neg C, C) \neq 1$.	(35), (34)

Theorem Pb109: [lemma for Pb110; also lemma for Pb113, Pb115 and Ps26]

For all A, B, w_p and p_p: $w_p[B\wedge\neg B, \neg(A\wedge\neg A)] = 0 = p_p(B\wedge\neg B)$.

Proof of theorem Pb109:

(1)	$w_p[\neg(B\wedge\neg B), B\wedge\neg B] = w_p[\neg(B\wedge\neg B), \neg B\wedge B] \neq 0$	Popper's theorems (40) and (33')
(2)	$w_p[B\wedge\neg B, \neg(B\wedge\neg B)] = 0$	(1), Pb108
(3)	$p_p(B\wedge\neg B) = w_p[B\wedge\neg B, \neg(B\wedge\neg B)]$	Popper's theorem (75)
(4)	$p_p(B\wedge\neg B) = w_p[B\wedge\neg B, \neg(A\wedge\neg A)]$	Popper's theorem (75)
(5)	$w_p[B\wedge\neg B, \neg(A\wedge\neg A)] = 0 = p_p(B\wedge\neg B)$	(4), (3), (2)

Theorem Pb110: [lemma for Ps16; also lemma for 14 further theorems]

For all A and p_p: $p_p(\neg A) = 1\text{-}p_p(A)$.

Proof of theorem Pb110:

(1)	$w_p[\neg A, \neg(B\wedge\neg B)] = 1\text{-}w_p[A, \neg(B\wedge\neg B)]+w_p[\neg\neg(B\wedge\neg B), \neg(B\wedge\neg B)]$	Popper's theorem (64)
(2)	$w_p[B\wedge\neg B, \neg(B\wedge\neg B)] = 0 = w_p[\neg\neg(B\wedge\neg B), \neg(B\wedge\neg B)] = 0$	Pb109, Pb101
(3)	$w_p[\neg A, \neg(B\wedge\neg B)] = 1\text{-}w_p[A, \neg(B\wedge\neg B)]$	(1), (2)
(4)	$p_p(\neg A) = 1\text{-}p_p(A)$	(3), Popper's theorem (75)

Theorem Ps16: [lemma for Pe37 and Pe38]

For all A, B and p_p: If A stands in truth-functional contradictory opposition$_p$ to B, then: A stands in truth-functional contrary opposition$_p$ to B; and

$p_p(A) = p_p(\neg B)$ and $p_p(\neg A) = p_p(B)$.

Proof of theorem Ps16:

(1)	A stands in truth-functional contradictory opposition$_p$ to B.	assumption
(2)	A is truth-functionally equivalent$_p$ to $\neg B$.	Ps4, (1)
(3)	$\neg B$ truth-functionally follows$_p$ from A.	Ps2, (2)
(4)	A stands in truth-functional contrary opposition$_p$ to B.	Ps3, (3)
(5)	$p_p(A) = p_p(\neg B)$	(2), Ps13
(6)	$p_p(\neg A) = 1\text{-}p_p(A) = 1\text{-}p_p(\neg B) = 1\text{-}[1\text{-}p_p(B)] = 1\text{-}1+p_p(B) = p_p(B)$	Pb110, (5)
(7)	$p_p(A) = p_p(\neg B)$ and $p_p(\neg A) = p_p(B)$.	(5), (6)

3.2.4 *Theorems concerning w_p-Absurdities, Contradictions$_p$ and Tautologies$_p$*

We need also some theorems concerning w_p-absurdities, contradictions$_p$ and tautologies$_p$. We start with two central theorems concerning w_p-absurdities: Ps17 and Ps18.

Theorem Ps17: [lemma for Ps18, Ps20, Pe3–Pe6, Pe8, Pe9, Pe16, Pe29, and Pe36]
For all A, B and w_p: A is w_p-absurd iff
$w_p(\neg A, A) \neq 0$; or iff
$w_p(\neg B, A) \neq 1\text{-}w_p(B, A)$; or iff
for all C: $w_p(\neg A, C) = 1$.

Proof of theorem Ps17:

(1)	A is w_p-absurd iff for all B: $w_p(B, A) = 1$.	Ps7
(2)	$w_p(\neg A, A) \neq 0$ iff for all B: $w_p(B, A) = 1$.	Popper's theorem (63)
(3)	A is w_p-absurd iff $w_p(\neg A, A) \neq 0$.	(1), (2)
(4)	$w_p(\neg B, A) = 1\text{-}w_p(B, A)+w_p(\neg A, A)$	Popper's theorem (64)
(5)	$w_p(\neg B, A) \neq 1\text{-}w_p(B, A)$ iff $w_p(\neg A, A) \neq 0$.	(4)
(6)	A is w_p-absurd iff $w_p(\neg B, A) \neq 1\text{-}w_p(B, A)$.	(3), (5)
(7)	For all B: $w_p(B, A) = 1$ iff for all C: $w_p(\neg A, C) = 1$.	Pb104
(8)	A is w_p-absurd iff for all C: $w_p(\neg A, C) = 1$.	(1), (7)
(9)	A is w_p-absurd iff $w_p(\neg A, A) \neq 0$ or iff $w_p(\neg B, A) \neq 1\text{-}w_p(B, A)$ or iff for all C: $w_p(\neg A, C) = 1$.	(3), (6), (8)

To learn also something about general relationships between w_p-absurdities and *unary* Popper probability functions, another theorem from basic Popper probability semantics is required:

Theorem Pb111: [lemma for Ps18, Pb114, Pe22, Pb116, and Pe25]
For all A, B and p_p: If $p_p(A) = 0$, then $p_p(A\wedge B) = 0 = p_p(B\wedge A)$.

Proof of theorem Pb111:

(1)	$p_p(A) = 0$	assumption
(2)	$0 = w_p[A, \neg(C\wedge\neg C)]$	Popper's theorem (75), (1)
(3)	$w_p[B\wedge A, \neg(C\wedge\neg C)] = w_p[B, A\wedge\neg(C\wedge\neg C)]\cdot w_p[A, \neg(C\wedge\neg C)] = 0$	Popper's axiom B2, (2)
(4)	$p_p(B\wedge A) = w_p[B\wedge A, \neg(C\wedge\neg C)] = 0$	Popper's theorem (75), (3)
(5)	$p_p(A\wedge B) = 0 = p_p(B\wedge A)$	Pb107, (4)

Theorem Ps18: [lemma for Pe3, Pe8, Pe28, and Pe40]
For all A, B, p_p and w_p: If A is w_p-absurd, then:
$p_p(A) = 0 = p_p(A\wedge B) = p_p(B\wedge A)$ and $p_p(\neg A) = 1 = p_p(A\rightarrow B)$.

Proof of theorem Ps18:

(1)	A is w_p-absurd.	assumption
(2)	$w_p[\neg A, \neg(C\wedge\neg C)] = 1 = p_p(\neg A)$	(1), Ps17, Popper's theorem (75)
(3)	$p_p(A) = 1\text{-}p_p(\neg A) = 1\text{-}1 = 0$	Pb110, (2)
(4)	$p_p(A\wedge B) = 0 = p_p(B\wedge A)$	(3), Pb111
(5)	For all B: $p_p(A\wedge B) = 0$.	(4), for B is neither free in (1) nor in (5)
(6)	$p_p(A\wedge\neg B) = 0$	(5)
(7)	$p_p(A\rightarrow B) = w_p[A\rightarrow B, \neg(C\wedge\neg C)] = w_p[\neg(A\wedge\neg B), \neg(C\wedge\neg C)]$	Popper's theorem (75), Pb106

(8)	$w_p[\neg(A\wedge\neg B); \neg(C\wedge\neg C)] = p_p[\neg(A\wedge\neg B)]$	Popper's theorem (75)
(9)	$p_p[\neg(A\wedge\neg B)] = 1\text{-}p_p(A\wedge\neg B) = 1\text{-}0 = 1$	Pb110, (6)
(10)	$p_p(A\rightarrow B) = 1$	(7), (8), (9)
(11)	$p_p(A) = 0 = p_p(A\wedge B) = p_p(B\wedge A)$ and $p_p(\neg A) = 1 = p_p(A\rightarrow B)$.	(3), (4), (2), (10)

We continue with four theorems concerning contradictions$_p$. First a proof of the proposition that the paragons of contradictions$_k$ are contradictions$_p$, then a proof of the central proposition that contradictions$_p$ are precisely those formulae of $\mathfrak{L}$ which are absurd in regard to *every* binary Popper probability function:

Theorem Ps19: [lemma for Pb125]

For all A: $A\wedge\neg A$ is a contradiction$_p$.

Proof of theorem Ps19:

(1)	$w_p(\neg B, B) = w_p[\neg(A\wedge\neg A)\wedge(A\wedge\neg A), B]$	Pb105
(2)	$w_p(\neg B, B) = w_p(A\wedge\neg A, B) = w_p[\neg(A\wedge\neg A)\wedge(A\wedge\neg A), B]$	Pb105
(3)	$w_p[\neg(A\wedge\neg A)\wedge(A\wedge\neg A), B] = w_p(A\wedge\neg A, B)$	(1), (2)
(4)	For all w_p, B: $w_p[\neg(A\wedge\neg A)\wedge(A\wedge\neg A), B] = w_p(A\wedge\neg A, B)$.	(3), for neither w_p nor B is free in (4)
(5)	$\neg(A\wedge\neg A)$ truth-functionally follows$_p$ from $A\wedge\neg A$.	Ps1, (4)
(6)	$A\wedge\neg A$ stands in truth-functional contrary opposition$_p$ to $A\wedge\neg A$.	Ps3, (5)
(7)	$A\wedge\neg A$ is a contradiction$_p$.	Ps5, (6)

Theorem Ps20: [lemma for Ps21 and Ps22]

For all A: A is a contradiction$_p$ iff for all w_p: A is w_p-absurd.

Proof of theorem Ps20:

(1)	A is a contradiction$_p$ iff $\neg A$ truth-functionally follows$_p$ from A.	Ps5, Ps3
(2)	$\neg A$ truth-functionally follows$_p$ from A iff for all w_p: $w_p(\neg A, \neg\neg A\wedge A) \neq 0$.	Ps10, Ps11
(3)	For all w_p: $w_p(\neg A, A) = w_p(\neg A, \neg\neg A\wedge A)$.	Pb101
(4)	A is a contradiction$_p$ iff for all w_p: $w_p(\neg A, A) \neq 0$.	(1), (2), (3)
(5)	For all w_p: A is w_p-absurd iff $w_p(\neg A, A) \neq 0$.	Ps17
(6)	For all w_p: A is w_p-absurd iff for all w_p: $w_p(\neg A, A) \neq 0$.	(5)
(7)	A is a contradiction$_p$ iff for all w_p: A is w_p-absurd.	(4), (6)

We add to Ps20 two convenient corollaries (note that the Kolmogorov analogue to Ps21 is not a theorem of Kolmogorov probability semantics):

Theorem Ps21: [lemma for Ps24]

For all A: If A is a contradiction$_p$, then for all B and w_p: $w_p(B, A) = 1 = w_p(\neg A, B)$.

Proof of theorem Ps21: immediately from Ps20, Ps7 and Ps17.

Theorem Ps22: [lemma for Ps23]

For all A: If A is a contradiction$_p$, then for all B and p_p:

$p_p(A) = 0 = p_p(A\wedge B) = p_p(B\wedge A)$ and $p_p(\neg A) = 1 = p_p(A\rightarrow B)$.

Proof of theorem Ps22: immediately from Ps20 and Ps18.

So much for contradictions$_p$. For the purpose of proving laws of excess in Popper probability semantics, two theorems concerning tautologies$_p$ will also be required. The first one will certainly not come as a surprise, but the second one might, because its Kolmogorov analogue is not a theorem of Kolmogorov probability semantics (which is still another shortcoming of Kolmogorov probability semantics):

Theorem Ps23: [lemma for Pe31 and Pe32]

For all A: If A is a tautology$_p$, then for all p_p: $p_p(A) = 1$.

Proof ot theorem Ps23:

(1)	A is a tautology$_p$.	assumption
(2)	$\neg A$ is a contradiction$_p$.	Ps6, (1)
(3)	$p_p(\neg\neg A) = 1 = 1\text{-}p_p(\neg A) = 1\text{-}[1\text{-}p_p(A)] = 1\text{-}1\text{+}p_p(A) = p_p(A)$	(2), Ps22, Pb110
(4)	For all p_p: $p_p(A) = 1$.	(3), for p_p is neither free in (1) nor in (4)

Theorem Ps24: [lemma for Pe32]

For all A: If A is a tautology$_p$, then for all B and w_p: $w_p(A, B) = 1$.

Proof of theorem Ps24:

(1)	A is a tautology$_p$.	assumption
(2)	$\neg A$ is a contradiction$_p$.	Ps6, (1)
(3)	$w_p(\neg\neg A, B) = 1 = w_p(A, B)$	(2), Ps21, Pb101
(4)	For all B and w_p: $w_p(A, B) = 1$.	(3), for w_p is neither free in (1) nor in (4)

With theorem Ps24 the second half of the bridge between theorems Ps10 and Ps11 on page 306 and Popper's laws of excess on page 307 of the *Logik der Forschung* is completed. It should now be easy to arrive at those of Popper's laws of excess which are provable in Popper probability semantics. In the next section, we shall extend theory Ps non-creatively to the theory Pe of the excess of $p_p(A\rightarrow B)$ over $w_p(B, A)$ by adding to the axiomatic basis {Pb1–Pb3, Ps1–Ps7} a further and last axiom which corresponds to Popper's definition of the excess of $p(b\rightarrow a)$ over $p(a, b)$.

3.3 Extension of Ps to the Theory Pe about the Excess of $p_p(A\rightarrow B)$ over $w_p(B, A)$

3.3.1 The Specific Axiom of Pe

According to axiom Pb1, every binary Popper probability function is a function from $\mathfrak{L}\times\mathfrak{L}$ into the reals, hence it holds for all A, B and w_p that there is exactly one real number r such that $w_p(B, A) = r$. Hence in Popper probability semantics—in contrast to Kolmogorov probability semantics—$w_p(B, A)$ always exists, whatever the absolute probability of A. Hence we can take over the gist of Popper's definition of excess without further ado as the specific axiom of Pe:

Axiom Pe1:

For all A, B, p_p and w_p: excess$[p_p(A\rightarrow B), w_p(B, A)] = p_p(A\rightarrow B)\text{-}w_p(B, A)$.

The only difference, if any, to Popper's original definition of the excess of $p(b\rightarrow a)$ over $p(a, b)$ is that, in Pe1, A and B are formulae, whereas, in Popper's definition, a and b are, presumably, statements.

3.3.2 *Popper's Laws of the Excess of $p_p(A \rightarrow B)$ over $w_p(B, A)$*

Let us follow closely Popper's presentation of laws of excess in POPPER (LdF9, p. 307), from which we gained in subsection 1.2 fourteen propositions as possible laws of excess. As it is our aim to prove these propositions and since, strictly speaking, all propositions which we have listed as possible laws of excess say something about statements expressed by sentences in some natural language rather than about meaningless formulae of some logical language, we have to presume here that each of the following theorems which is true for an arbitrary formula A is also true for every statement a that correponds to formula A. We start with the proof of proposition (1.2.1), which states that the values of the excess function range at most from -1 to +1. To prove proposition (1.2.1), the following theorem of basic Popper probability semantics is required:

Theorem Pb112: [lemma for Pe2, Pe7, Pe16, Pe19, and Pe40]
For all A and p_p: $0 \leq p_p(A) \leq 1$.

Proof of theorem Pb112:

(1)	$0 \leq w_p[A, \neg(B \wedge \neg B)] \leq 1$	Popper's theorem (16)
(2)	$p_p(A) = w_p[A, \neg(B \wedge \neg B)]$	Popper's theorem (75)
(3)	$0 \leq p_p(A) \leq 1$	(1), (2)

Theorem Pe2: [cf. proposition (1.2.1)]
For all A, B, p_p and w_p: $-1 \leq \text{excess}[p_p(A \rightarrow B), w_p(B, A)] \leq +1$.

Proof of theorem Pe2:

(1)	If $p_p(A \rightarrow B) \geq 0$ and $w_p(B, A) \leq 1$, then $\text{excess}[p_p(A \rightarrow B), w_p(B, A)] \geq -1$.	Pe1
(2)	If $p_p(A \rightarrow B) \leq 1$ and $w_p(B, A) \geq 0$, then $\text{excess}[p_p(A \rightarrow B), w_p(B, A)] \leq +1$.	Pe1
(3)	$0 \leq p_p(A \rightarrow B) \leq 1$ and $0 \leq w_p(B, A) \leq 1$.	Pb112, Popper's theorem (16)
(4)	$-1 \leq \text{excess}[p_p(A \rightarrow B), w_p(B, A)] \leq +1$	(1), (2), (3)

We continue with the only one of Popper's laws of excess that states a condition (i.e. A's being a contradiction$_p$) under which $p_p(A \rightarrow B)$ does not exceed $w_p(B, A)$. We prove this law by proving a logically stronger one:

Theorem Pe3: [cf. proposition (1.2.2)]
For all A, B, p_p and w_p: If A is w_p-absurd, then $\text{excess}[p_p(A \rightarrow B), w_p(B, A)] = 0$.

Proof of theorem Pe3:

(1)	A is w_p-absurd.	assumption
(2)	$p_p(A \rightarrow B) = 1 = w_p(B, A)$	Ps18, Ps7
(3)	$p_p(A \rightarrow B) - w_p(B, A) = 0$	(2)
(4)	$\text{excess}[p_p(A \rightarrow B), w_p(B, A)] = 0$	Pe1, (3)

We come to Popper's central proposition (1.2.3), formulated in our terminology:

For all A, B, p_p and w_p: If A is not a contradiction$_p$, then:
$\text{excess}[p_p(A \rightarrow B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A)$..

Unfortunately, proposition (1.2.3) is not provable in Pe. In order to prove (1.2.3), we would have to prove (as will be illustrated by the proof of Pe4 below):

(i) If A is not a contradiction$_p$, then $w_p(\neg B, A) = 1\text{-}w_p(B, A)$.

Now, if (i) were a theorem of Popper probability semantics, then this would also be one:

(ii) A is a contradiction$_p$ iff $w_p(\neg B, A) \neq 1\text{-}w_p(B, A)$.

But (ii) is logically equivalent to the non-theorem:

(iii) A is a contradiction$_p$ iff (there is at least one w_p such that) A is w_p-absurd.

Popper seems to take (i) for granted (as I gather from correspondence with him), and so it is completely understandable that he considers proposition (1.2.3) to be a theorem of his (extended) probability theory. But taking (i) for granted means equating w_p-absurdities with contradictions$_p$, as is illustrated by (iii). However, as Leblanc and his collaborators found out in the 1970s[8], it is precisely *Popper* probability semantics in which the fine distinction between w_p-absurdities and contradictions$_p$ can be made and has to be made, if one wants all contradictions$_p$ to be contradictions$_k$: for whereas every contradiction$_k$ is indeed w_p-absurd, not every w_p-absurdity is a contradiction$_k$. So not (i), but only this weaker version of (i) is a theorem of Ps:

(iv) If A is not a contradiction$_p$, then there is at least one w_p such that $w_p(\neg B, A) = 1\text{-}w_p(B, A)$.

Theorem (iv) (which is a corollary to Ps20 and Ps17) is not strong enough to yield (1.2.3).[9] However, if we take w_p-absurdities instead of contradictions$_p$, then we can prove a proposition which is, admittedly, logically weaker than (1.2.3), but which is nevertheless illuminating and, in addition, still useful for proving further laws of excess: this proposition will be theorem Pe4. First two convenient theorems from basic Popper probability semantics:

Theorem Pb113: [lemma for Pb114, Pe4, Pe20, and Ps25]
For all A, B and p_p: $p_p(A \rightarrow B) = 1\text{-}p_p(A)+p_p(B \wedge A)$.

Proof of theorem Pb113:

(1)	$w_p[A \rightarrow B, \neg(A \wedge \neg A)] = w_p[B \wedge A, \neg(A \wedge \neg A)]+w_p[\neg A, \neg(A \wedge \neg A)]\text{-}w_p[A \wedge \neg A, \neg(A \wedge \neg A)]$	Pb106
(2)	$p_p(A \rightarrow B) = p_p(\neg A)+p_p(B \wedge A)\text{-}p_p(A \wedge \neg A)$	Popper's theorem (75), (1)
(3)	$p_p(A \wedge \neg A) = 0$	Pb109
(4)	$p_p(A \rightarrow B) = 1\text{-}p_p(A)+p_p(B \wedge A)$	(2), (3), Pb110

8. This discovery was triggered off by a remark in STALNAKER (Probability, p. 70) [1970] to the effect that A need not be a tautology$_k$ if A is w_p-valid, i.e., if $\neg A$ is w_p-absurd. Stalnaker's remark was elaborated in HARPER (Belief, p. 109) [1976], and the problem was taken care of in LEBLANC/VAN FRAASSEN (Functions) [1979]. The 1980s and the early 1990s brought an abundancy of refinements of these early results; see especially ROEPER/LEBLANC (Indiscernibility) [1991].

9. Ironically, if p were taken to be a Carnap probability function, then proposition (1.2.3) could be proved in Carnap probability semantics (the same holds true for propositions (1.2.7), (1.2.12), and (1.2.13)). But Carnap probability semantics is known to turn a controversial proposition into a theorem which Popper has never accepted, to wit: "If $p(A) = 0$, then A is a contradiction$_k$".

Theorem Pb114: [lemma for Pe9]

For all A, B and p_p: If $p_p(A) = 0$, then $p_p(A \rightarrow B) = 1$.

Proof of theorem Pb114:

(1)	$p_p(A) = 0$	assumption
(2)	$p_p(B \wedge A) = 0$	(1), Pb111
(3)	$p_p(A \rightarrow B) = 1 - p_p(A) + p_p(B \wedge A) = 1 - 0 + 0 = 1$	Pb113, (1), (2)

Now to the proof of the weakened version of proposition (1.2.3):

Theorem Pe4: [weakened version of proposition (1.2.3)]

For all A, B, p_p and w_p: If A is not w_p-absurd, then:
excess$[p_p(A \rightarrow B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A)$.

Proof of theorem Pe4:

(1)	A is not w_p-absurd.	assumption
(2)	$w_p(\neg B, A) = 1 - w_p(B, A)$	Ps17, (1)
(3)	$p_p(\neg A) = 1 - p_p(A)$	Pb110
(4)	$w_p(\neg B, A) \cdot p_p(\neg A) = [1 - w_p(B, A)] \cdot [1 - p_p(A)]$	(2), (3)
(5)	$w_p(\neg B, A) \cdot p_p(\neg A) = 1 - p_p(A) - w_p(B, A) + [w_p(B, A) \cdot p_p(A)]$	(4)
(6)	$w_p(B, A) \cdot p_p(A) = p_p(B \wedge A)$	Popper's theorem (96)
(7)	$w_p(\neg B, A) \cdot p_p(\neg A) = 1 - p_p(A) - w_p(B, A) + p_p(B \wedge A)$	(5), (6)
(8)	$p_p(A \rightarrow B) = 1 - p_p(A) + p_p(B \wedge A)$	Pb113
(9)	$w_p(\neg B, A) \cdot p_p(\neg A) = p_p(A \rightarrow B) - w_p(B, A)$	(7), (8)
(10)	excess$[p_p(A \rightarrow B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A)$	Pe1, (9)

A nice side result is this: since, according to Ps7, A is not w_p-absurd if $w_p(B, A) \neq 1$, theorem Pe4 guarantees at once the theoremhood of propositions (1.3.2), (1.3.2'), and (1.3.2'') (compare subsection 1.3.).

We continue with the proofs of Popper's laws of excess (1.2.4) and (1.2.5):

Theorem Pe5: [cf. proposition (1.2.4)]

For all A, B, p_p and w_p: excess$[p_p(A \rightarrow B), w_p(B, A)] = [1 - w_p(B, A)] \cdot p_p(\neg A)$.

Proof of theorem Pe5:

(1)	If A is w_p-absurd, then $w_p(B, A) = 1$.	Ps7
(2)	If A is w_p-absurd, then $[1 - w_p(B, A)] \cdot p_p(\neg A) = 0$.	(1)
(3)	If A is w_p-absurd, then excess$[p_p(A \rightarrow B), w_p(B, A)] = [1 - w_p(B, A)] \cdot p_p(\neg A)$.	Pe3, (2)
(4)	If A is not w_p-absurd, then excess$[p_p(A \rightarrow B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A)$.	Pe4
(5)	If A is not w_p-absurd, then $w_p(\neg B, A) = 1 - w_p(B, A)$.	Ps17
(6)	If A is not w_p-absurd, then excess$[p_p(A \rightarrow B), w_p(B, A)] = [1 - w_p(B, A)] \cdot p_p(\neg A)$.	(4), (5)
(7)	excess$[p_p(A \rightarrow B), w_p(B, A)] = [1 - w_p(B, A)] \cdot p_p(\neg A)$	(3), (6)

Theorem Pe6: [cf. proposition (1.2.5)]

For all A, B, p_p and w_p:
excess$[p_p(A \rightarrow B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A) \cdot [1 - w_p(\neg A, A)]$.

Proof of theorem Pe6:

(1)	If A is w_p-absurd, then $w_p(\neg A, A) = 1$.	Ps17
(2)	If A is w_p-absurd, then $w_p(\neg B, A) \cdot p_p(\neg A) \cdot [1 - w_p(\neg A, A)] = 0$.	(1)
(3)	If A is w_p-absurd, then excess$[p_p(A \rightarrow B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A) \cdot [1 - w_p(\neg A, A)]$.	Pe3, (2)

(4) If A is not w_p-absurd, then $w_p(\neg A, A) = 0$. Ps17

(5) If A is not w_p-absurd, then $w_p(\neg B, A) \cdot p_p(\neg A) \cdot [1 - w_p(\neg A, A)] = w_p(\neg B, A) \cdot p_p(\neg A)$. (4)

(6) If A is not w_p-absurd, then: Pe4, (5)
excess$[p_p(A \to B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A) \cdot [1 - w_p(\neg A, A)]$.

(7) excess$[p_p(A \to B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A) \cdot [1 - w_p(\neg A, A)]$ (3), (6)

Now comes the proof of proposition (1.2.6), i.e. Popper's law of the never negative excess of $p_p(A \to B)$ over $w_p(B, A)$:

Theorem Pe7: [cf. proposition (1.2.6)]
For all A, B, p_p and w_p: excess$[p_p(A \to B), w_p(B, A)] \geq 0$.

Proof of theorem Pe7:

(1) excess$[p_p(A \to B), w_p(B, A)] = [1 - w_p(B, A)] \cdot p_p(\neg A)$ Pe5

(2) $w_p(B, A) \leq 1$ and $p_p(\neg A) \geq 0$. Popper's theorem (16), Pb112

(3) excess$[p_p(A \to B), w_p(B, A)] \geq 0$ (1), (2)

Proposition (1.2.7) would be the next one to require proof, but (see non-theorem (i) above again and step (4) in the proof of Pe8 below) that proposition which *can* be proved is the logically weaker:

Theorem Pe8: [weakened version of proposition (1.2.7)]
For all A, B, p_p and w_p: If B is w_p-independent$_p$ of A and if A is not w_p-absurd, then excess$[p_p(A \to B), w_p(B, A)] = p_p(\neg B) \cdot p_p(\neg A)$.

Proof of theorem Pe8:

(1) B is w_p-independent$_p$ of A. assumption

(2) A is not w_p-absurd. assumption

(3) $w_p(B, A) = w_p[B, \neg(C \wedge \neg C)] = p_p(B)$ Pb3, (1), Popper's theorem (75)

(4) $w_p(\neg B, A) = 1 - w_p(B, A) = 1 - p_p(B) = p_p(\neg B)$ (2), Ps17, (3), Pb110

(5) excess$[p_p(A \to B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A) = p_p(\neg B) \cdot p_p(\neg A)$ (2), Pe4, (4)

We turn now to the propositions (1.2.8)–(1.2.14), which state conditions under which excess$[p_p(A \to B), w_p(B, A)] = 1$. First we discern that Popper's original formulation 'Auch gilt $Exc(a, b) = 1$ stets, wenn $p(a, b) = 0 = p(b)$' of proposition (1.2.8) is alright after all:

Theorem Pe9: [cf. proposition (1.2.8)]
For all A, B, p_p and w_p:
If $w_p(B, A) = 0 = p_p(A)$, then excess$[p_p(A \to B), w_p(B, A)] = 1$.

Proof of theorem Pe9: immediately from Pb114 and Pe1.

This result guarantees also the theoremhood of propositions (1.2.9), (1.2.10) and (1.2.11), which are weakened versions of proposition (1.2.8).

Propositions (1.2.12) and (1.2.13) cannot be proved in Popper probability semantics for the old reason: not being a contradiction$_p$ is not enough for not being w_p-absurd. But the *weakened* versions of propositions (1.2.12), in which "not being a contradiction$_p$" is replaced by "not being w_p-absurd", can be proved:

Theorem Pe10 [weakened proposition (1.2.12)]

For all A, B, p_p and w_p: If A is not w_p-absurd and if $p_p(A) = 0$ and if B is w_p-independent$_p$ of A and if $p_p(B) = 0$, then excess$[p_p(A \rightarrow B), w_p(B, A)] = 1$.

Proof of theorem Pe10:

(1)	A is not w_p-absurd.	assumption
(2)	$p_p(A) = 0$	assumption
(3)	B is w_p-independent$_p$ of A.	assumption
(4)	$p_p(B) = 0$	assumption
(5)	excess$[p_p(A \rightarrow B), w_p(B, A)] = p_p(\neg B) \cdot p_p(\neg A)$	(3), (1), Pe8
(6)	$p_p(\neg A) = 1$ and $p_p(\neg B) = 1$.	Pb110, (2), (4)
(7)	excess$[p_p(A \rightarrow B), w_p(B, A)] = 1 \cdot 1 = 1$	(5), (6)

Theorem Pe11: [weakened proposition (1.2.13)]

For all A, B, p_p and w_p: If A is not w_p-absurd and if $p_p(A) = 0$ and if A stands in truth-functional contrary opposition$_p$ to B, then excess$[p_p(A \rightarrow B), w_p(B, A)] = 1$.

Proof of theorem Pe11:

(1)	A is not w_p-absurd.	assumption
(2)	$p_p(A) = 0$	assumption
(3)	A stands in truth-functional contrary opposition$_p$ to B.	assumption
(4)	excess$[p_p(A \rightarrow B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A)$	(1), Pe4
(5)	$w_p(\neg B, A) = 1$	(3), Ps15
(6)	$p_p(\neg A) = 1$	Pb110, (2)
(7)	excess$[p_p(A \rightarrow B), w_p(B, A)] = 1 \cdot 1 = 1$	(4), (5), (6)

Finally, we observe that proposition (1.2.14):

For all A, B, p_p and w_p: If A is not a contradiction$_p$, if $p_p(A) = 0$ and if A stands almost in truth-functional contrary opposition$_p$ to B, then excess$[p_p(A \rightarrow B), w_p(B, A)] = 1$.

is not provable in Pe, because Ps, in which Pe is embedded, simply does not contain an axiom which says something about the relation of almost contrary opposition. As I have found no indications in Popper's writings at how such an axiom could look like, I thought it best to let the axiomatic basis of Ps stand as it is. Its extension in the direction of almost truth-functional entailment and opposition might be fruitful, but would certainly go beyond Popper probability semantics as known so far.

So 9 out of 14 propositions listed in subsection 1.2 as possible laws of excess have turned out to be theorems of Pe. Let us now go through the other subsections of section 1 and find out which of the further propositions listed there are theorems of Pe.

In subsection 1.1, we interpreted a citation from POPPER (Conjectures, p. 396) and found, besides the definition of excess, two pertinent propositions: proposition (1.1.3) being Popper's law of the never negative excess of $p_p(A \rightarrow B)$ over $w_p(B, A)$—already proved above as theorem Pe7—as well as proposition (1.1.2), which is a corollary to theorem Pe5:

Theorem Pe12: [cf. proposition (1.1.2)]

For all A, B, p_p and w_p: excess$[p_p(A \rightarrow B), w_p(B, A)] = [1-w_p(B, A)] \cdot [1-p_p(A)]$.

Proof of theorem Pe12: immediately from Pe5 and Pb110.

We come to subsection 1.3, in which we interpreted a small citation from the 1983 Popper/Miller letter to *Nature* and arrived, because of the question of tacit assumptions, at the high number of nine propositions which could have been intended as laws of excess. These were the propositions:

(1.3.1) If $p(a, b) \neq 1$ and $p(b) \neq 1$, then $Exc[p(b \rightarrow a), p(a, b)] = p(b \rightarrow a) - p(b \rightarrow a, b)$.
(1.3.2) If $p(a, b) \neq 1$ and $p(b) \neq 1$, then $Exc[p(b \rightarrow a), p(a, b)] = p(\neg a, b) \cdot p(\neg b)$.
(1.3.3) If $p(a, b) \neq 1$ and $p(b) \neq 1$, then $Exc[p(b \rightarrow a), p(a, b)] > 0$.

plus two weakened versions of each of them. We prove propositions (1.3.1) and (1.3.2) as well as their weakened versions by proving two propositions which are logically stronger than (1.3.1) and (1.3.2), respectively. We know already from subsection 1.4 that proposition (1.3.1) is an immediate logical consequence of the last but one proposition which we encountered in Popper's writings as a possible law of excess, i.e. proposition:

(1.4.1) $Exc[p(b \rightarrow a), p(a, b)] = p(b \rightarrow a) - p(b \rightarrow a, b)$.

So if we prove proposition (1.4.1), we have also proved proposition (1.3.1) and its two weakened versions. In addition, as our proof follows Popper's proofs (1) and (2) of proposition (1.4.1) (see subsections 1.4 and 1.5), its correctness confirms theirs.

Theorem Pe13: [cf. proposition (1.4.1)]
For all A, B, p_p and w_p: $\text{excess}[p_p(A \rightarrow B), w_p(B, A)] = p_p(A \rightarrow B) - w_p(A \rightarrow B, A)$.

Proof of theorem Pe13:

(1)	$w_p[(A \rightarrow B) \wedge A, A] = w_p(B \wedge A, A)$	Ps14, Ps12
(2)	$w_p(B \wedge A, A) = w_p(A \wedge B, A) = w_p(B, A)$	Popper's theorems (40) and (29)
(3)	$w_p[(A \rightarrow B) \wedge A, A] = w_p(B, A)$	(1), (2)
(4)	$w_p[(A \rightarrow B) \wedge A, A] = w_p[A \wedge (A \rightarrow B), A] = w_p(A \rightarrow B, A)$	Popper's theorems (40) and (29)
(5)	$w_p(B, A) = w_p(A \rightarrow B, A)$	(3), (4)
(6)	$p_p(A \rightarrow B) - w_p(B, A) = p_p(A \rightarrow B) - w_p(A \rightarrow B, A)$	(5)
(7)	$\text{excess}[p_p(A \rightarrow B), w_p(B, A)] = p_p(A \rightarrow B) - w_p(A \rightarrow B, A)$	Pe1, (6)

The next theorem guarantees the theoremhood of proposition (1.3.2) and (once again after theorem Pe4) that of its two weakened versions:

Theorem Pe14: [covers propositions (1.3.2), (1.3.2') and (1.3.2'')]
For all A, B, p_p and w_p: If $w_p(B, A) \neq 1$, then:
$\text{excess}[p_p(A \rightarrow B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A)$.

Proof of theorem Pe14:

(1)	$w_p(B, A) \neq 1$	assumption
(2)	A is not w_p-absurd.	Ps7, (1)
(3)	$\text{excess}[p_p(A \rightarrow B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A)$	(2), Pe4

Step (3) illustrates that Pe14 logically follows from Pe4 in Pe,.but not vice versa, because the proposition "If A is not w_p-absurd, then $w_p(B, A) \neq 1$" is not a theorem of Pe. This answers the question, put in subsection 1.3, which logical relationship holds between proposition (1.2.3), represented by theorem Pe4, and proposition (1.3.2), represented by:

Theorem Pe15: [cf. proposition (1.3.2)]
For all A, B, p_p and w_p: If $w_p(B, A) \neq 1$ and if $p_p(A) \neq 1$, then:
excess$[p_p(A \rightarrow B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A)$.

Proof of theorem Pe15: immediately from Pe14.

Since the condition "$w_p(B, A) \neq 1$" guarantees that A is not w_p-absurd, Pe15 is still logically stronger than Pe4, hence proposition (1.3.2) is logically stronger than proposition (1.2.3).—Now to the proof of proposition (1.3.3):

Theorem Pe16: [cf. proposition (1.3.3)]
For all A, B, p_p and w_p:
If $w_p(B, A) \neq 1$ and if $p_p(A) \neq 1$, then excess$[p_p(A \rightarrow B), w_p(B, A)] > 0$.

Proof of theorem Pe16:

(1)	$w_p(B, A) \neq 1$	assumption
(2)	$p_p(A) \neq 1$	assumption
(3)	A is not w_p-absurd.	Ps7, (1)
(4)	excess$[p_p(A \rightarrow B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A)$	(3), Pe4
(5)	$w_p(B, A) < 1$	Popper's theorem (16), (1)
(6)	$w_p(\neg B, A) = 1 - w_p(B, A)$	Ps17, (3)
(7)	$w_p(\neg B, A) > 0$	(6), (5)
(8)	$p_p(A) < 1$	Pb112, (2)
(9)	$p_p(\neg A) > 0$	Pb110, (8)
(10)	$[w_p(\neg B, A) \cdot p_p(\neg A)] > 0$	(7), (9)
(11)	excess$[p_p(A \rightarrow B), w_p(B, A)] > 0$	(4), (10)

Hence the two weakened versions of proposition (1.3.3) are also theorems of Pe:

Theorem Pe17: [cf. proposition (1.3.3')]
For all A, B, p_p and w_p: If $w_p(B, A) \neq 1$ and if $p_p(A) \neq 1$ and if A is not a contradiction$_p$, then excess$[p_p(A \rightarrow B), w_p(B, A)] > 0$.

Proof of theorem Pe17: immediately from Pe16.

Theorem Pe18: [cf. proposition (1.3.3'')]
For all A, B, p_p and w_p: If $w_p(B, A) \neq 1$ and if $p_p(A) \neq 1$ and if $p_p(A) > 0$, then:
excess$[p_p(A \rightarrow B), w_p(B, A)] > 0$.

Proof of theorem Pe18: immediately from Pe16.

Note that Pe18 is not only logically weaker than Pe16, but also logically weaker than Pe17, for the condition "$p_p(A) > 0$" guarantees that A is not a contradiction$_p$, whereas the condition "A is not a contradiction$_p$" does not guarantee that $p_p(A) > 0$. Since Pe18 is—because of Popper's theorem (16) and Pb112—logically equivalent to:

Theorem Pe19: [the main law of excess]
For all A, B, p_p and w_p:

If $0<p_p(A)<1$ and if $w_p(B, A)<1$, then excess$[p_p(A\rightarrow B), w_p(B, A)] > 0$.

this means that there are in Pe at least two laws of excess which are logically stronger than the main law of excess, to wit: Pe16 and Pe17.

We still have to deal with proposition (1.4.2), which is a by-product of Pb113:

Theorem Pe20: [cf. proposition (1.4.2)]
For all A, B, p_p and w_p: excess$[p_p(A\rightarrow B), w_p(B, A)] = 1-p_p(A)+p_p(B\wedge A)-w_p(B, A)$.
Proof of theorem Pe20: immediately from Pb113 and Pe1.

This completes our chain of proofs of those laws of excess which have been mentioned or alluded to in Popper's writings.

In the next subsection we prove some further laws of excess in preparation for the comparison of Pe with Ke in section 4.

3.3.3 Further Laws of the Excess of $p_p(A\rightarrow B)$ over $w_p(B, A)$

Firstly, three theorems concerning the excess of $p_p(A\rightarrow B)$ over $w_p(B, A)$, when $p_p(A) = 0$ or $p_p(B) = 0$.

Theorem Pe21:
For all A, B, p_p and w_p:
If $p_p(A) = 0$, then excess$[p_p(A\rightarrow B), w_p(B, A)] = 1-w_p(B, A)$.
Proof of theorem Pe21: immediately from Pe12.

Note that Popper's law of excess Pe9 is an immediate logical consequence of Pe21.

Theorem Pe22:
For all A, B, p_p and w_p:
If $p_p(B) = 0$, then excess$[p_p(A\rightarrow B), w_p(B, A)] = p_p(\neg A)-w_p(B, A)$.
Proof of theorem Pe22:

(1)	$p_p(B) = 0$	assumption
(2)	$p_p(B\wedge A) = 0$	(1), Pb111
(3)	excess$[p_p(A\rightarrow B), w_p(B, A)] = 1-p_p(A)+p_p(B\wedge A)-w_p(B, A)$	Pe20
(4)	excess$[p_p(A\rightarrow B), w_p(B, A)] = 1-p_p(A)-w_p(B, A) = p_p(\neg A)-w_p(B, A)$	(3), (2), Pb110

Secondly, two theorems concerning the excess of $p_p(A\rightarrow B)$ over $w_p(B, A)$, when $p_p(A) = 1$ or $p_p(B) = 1$. First, we need two further theorems from basic Popper probability semantics:

Theorem Pb115: [lemma for Pb116]
For all A, B and p_p: $p_p(B) = p_p(A\wedge B)+p_p(\neg A\wedge B) = p_p(B\wedge A)+p_p(B\wedge\neg A)$.
Proof of theorem Pb115:

(1)	$w_p[A\wedge B, \neg(C\wedge\neg C)]+w_p[\neg A\wedge B, \neg(C\wedge\neg C)] =$ $= w_p[B, \neg(C\wedge\neg C)]+w_p[\neg\neg(C\wedge\neg C), \neg(C\wedge\neg C)]$	Popper's theorem (70)

(2) $w_p[C\wedge\neg C, \neg(C\wedge\neg C)] = 0 = w_p[\neg\neg(C\wedge\neg C), \neg(C\wedge\neg C)]$ — Pb109, Pb101

(3) $w_p[B, \neg(C\wedge\neg C)] = w_p[A\wedge B, \neg(C\wedge\neg C)]+w_p[\neg A\wedge B, \neg(C\wedge\neg C)]$ — (1), (2)

(4) $p_p(B) = p_p(A\wedge B)+p_p(\neg A\wedge B)$ — Popper's theorem (75), (3)

(5) $p_p(A\wedge B) = p_p(B\wedge A)$ and $p_p(\neg A\wedge B) = p_p(B\wedge\neg A)$. — Pb107

(6) $p_p(B) = p_p(A\wedge B)+p_p(\neg A\wedge B) = p_p(B\wedge A)+p_p(B\wedge\neg A)$ — (4), (5)

Theorem Pb116: [lemma for Pe23 and Pe24]

For all A, B and p_p: If $p_p(A) = 1$, then $p_p(A\wedge B) = p_p(B) = p_p(B\wedge A)$.

Proof of theorem Pb116:

(1) $p_p(A) = 1$ — assumption

(2) $p_p(\neg A) = 0$ — Pb110, (1)

(3) $p_p(\neg A\wedge B) = 0 = p_p(B\wedge\neg A)$ — (2), Pb111

(4) $p_p(A\wedge B) = p_p(B) = p_p(B\wedge A)$ — Pb115, (3)

Theorem Pe23:

For all A, B, p_p and w_p: If $p_p(A) = 1$, then excess$[p_p(A\to B), w_p(B, A)] = 0$.

Proof of theorem Pe23:

(1) $p_p(A) = 1$ — assumption

(2) $p_p(B\wedge A) = p_p(B)$ — (1), Pb116

(3) $p_p(A\to B) = 1\text{-}p_p(A)+p_p(B\wedge A) = 1\text{-}1+p_p(B) = p_p(B)$ — Pb113, (1), (2)

(4) $w_p(B, A) = p_p(B\wedge A) = p_p(B)$ — Popper's theorem (96), (1), (2)

(5) excess$[p_p(A\to B), w_p(B, A)] = p_p(A\to B)\text{-}w_p(B, A) = p_p(B)\text{-}p_p(B) = 0$ — Pe1, (3), (4)

Theorem Pe24:

For all A, B, p_p and w_p:

If $p_p(B) = 1$, then excess$[p_p(A\to B), w_p(B, A)] = 1\text{-}w_p(B, A)$.

Proof of theorem Pe24:

(1) $p_p(B) = 1$ — assumption

(2) $p_p(B\wedge A) = p_p(A)$ — (1), Pb116

(3) $p_p(A\to B) = 1\text{-}p_p(A)+p_p(B\wedge A) = 1\text{-}p_p(A)+p_p(A) = 1$ — Pb113, (2)

(4) excess$[p_p(A\to B), w_p(B, A)] = 1\text{-}w_p(B, A)$ — Pe1, (3)

Thirdly, three theorems concerning the excess of $p_p(A\to B)$ over $w_p(B, A)$, when $p_p(A)>0$, and $p_p(B) = 0$ or $p_p(B) = 1$.

Theorem Pe25:

For all A, B, p_p and w_p:

If $p_p(A)>0$ and if $p_p(B) = 0$, then excess$[p_p(A\to B), w_p(B, A)] = p_p(\neg A)$.

Proof of theorem Pe25:

(1) $p_p(A)>0$ — assumption

(2) $p_p(B) = 0$ — assumption

(3) $w_p(B, A) = p_p(B\wedge A)/p_p(A)$ — (1), Popper's theorem (97)

(4) $p_p(B\wedge A) = 0$ — (2), Pb111

(5) $w_p(B, A) = 0$ — (3), (4)

(6) excess$[p_p(A\to B), w_p(B, A)] = p_p(\neg A)\text{-}w_p(B, A) = p_p(\neg A)$ — (2), Pe22, (5)

Theorem Pe26:

For all A, B, p_p and w_p:

If $0<p_p(A)<1$ and if $p_p(B) = 0$, then excess$[p_p(A \to B), w_p(B, A)] > 0$.

Proof of theorem Pe26: immediately from Pe25 and Pb110.

Theorem Pe27:

For all A, B, p_p and w_p:

If $p_p(A)>0$ and if $p_p(B) = 1$, then excess$[p_p(A \to B), w_p(B, A)] = 0$.

Proof of theorem Pe27:

(1)	$p_p(A)>0$	assumption
(2)	$p_p(B) = 1$	assumption
(3)	$p_p(B \wedge A) = p_p(A)$	(2), Pb116
(4)	$w_p(B, A) = p_p(B \wedge A)/p_p(A) = p_p(A)/p_p(A) = 1$	(1), Popper's theorem (97), (3)
(5)	excess$[p_p(A \to B), w_p(B, A)] = 1 - w_p(B, A) = 1-1 = 0$	(2), Pe24, (4)

Fourthly, five theorems concerning the excess of $p_p(A \to B)$ over $w_p(B, A)$, when A or B are known to be w_p-absurdities or tautologies$_p$ or not w_p-absurdities or not tautologies$_p$. The excess of $p_p(A \to B)$ over $w_p(B, A)$ equals 0, when A is a w_p-absurdity (see Pe3). No such informative result can be obtained in regard to B. When we know only that B is not a w_p-absurdity, then we can calculate neither $p_p(A \to B)$ nor $w_p(B, A)$. And when we know only that B is a w_p-absurdity, then we get no more than a weakened version of Pe22:

Theorem Pe28:

For all A, B, p_p and w_p:

If B is w_p-absurd, then excess$[p_p(A \to B), w_p(B, A)] = p_p(\neg A) - w_p(B, A)$.

Proof of theorem Pe28: immediately from Pe22 and Ps18.

However, when we also know that A is not a w_p-absurdity, then we can obtain:

Theorem Pe29:

For all A, B, p_p and w_p: If B is w_p-absurd and if A is not w_p-absurd, then:
excess$[p_p(A \to B), w_p(B, A)] = p_p(\neg A)$.

Proof of theorem Pe29:

(1)	B is w_p-absurd.	assumption
(2)	A is not w_p-absurd.	assumption
(3)	$w_p(\neg B, A) = 1$	(1), Ps17
(4)	excess$[p_p(A \to B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A) = p_p(\neg A)$	(2), Pe4, (3)

Theorem Pe30:

For all A, B, p_p and w_p: If B is w_p-absurd and if A is not w_p-absurd and if $p_p(A) = 0$, then excess$[p_p(A \to B), w_p(B, A)] = 1$.

Proof of theorem Pe30: immedatiately from Pe29 and Pb110.

When we know only that A is not a tautology$_p$ or that B is not a tautology$_p$, then nothing new can be proved. But when we know that A is a tautology$_p$ or that B is one, then we immediately have:

Theorem Pe31:

For all A, B, p_p and w_p: If A is a tautology$_p$, then excess$[p_p(A \to B), w_p(B, A)] = 0$.

Proof of theorem Pe31: immediately from Ps23 and Pe23.

Theorem Pe32:

For all A, B, p_p and w_p: If B is a tautology$_p$, then excess$[p_p(A \rightarrow B), w_p(B, A)] = 0$.

Proof of theorem Pe32: immediately from Ps23, Pe24 and Ps24.

Fifthly, two theorems concerning the excess of $p_p(A \rightarrow B)$ over $w_p(B, A)$, when truth-functional entailment or equivalence is known to exist between A and B. First a theorem from Ps:

Theorem Ps25: [lemma for Pe33]

For all A, B and p_p: If B truth-functionally follows$_p$ from A, then $p_p(A \rightarrow B) = 1$.

Proof of theorem Ps25:

(1)	B truth-functionally follows$_p$ from A.	assumption
(2)	$w_p[B \wedge A, \neg(C \wedge \neg C)] = w_p[A, \neg(C \wedge \neg C)] = p_p(B \wedge A) = p_p(A)$	Ps1, (1), Popper's theorem (75)
(3)	$p_p(A \rightarrow B) = 1 - p_p(A) + p_p(B \wedge A) = 1$	Pb113, (2)

Theorem Pe33:

For all A, B, p_p and w_p:

If B truth-functionally follows$_p$ from A, then excess$[p_p(A \rightarrow B), w_p(B, A)] = 0$.

Proof of theorem Pe33: immediately from Ps25, Ps8 and Pe1.

Theorem Pe34:

For all A, B, p_p and w_p:

If A is truth-functionally equivalent$_p$ to B, then excess$[p_p(A \rightarrow B), w_p(B, A)] = 0$.

Proof of theorem Pe34: immediately from Ps2 and Pe33.

Sixthly, four theorems concerning the excess of $p_p(A \rightarrow B)$ over $w_p(B, A)$, when contrary or contradictory opposition$_p$ is known to exist between A and B. First we need once more a theorem from Ps:

Theorem Ps26: [lemma for Pe35]

For all A and B:

If A stands in truth-functional contrary opposition$_p$ to B, then for all p_p: $p_p(B \wedge A) = 0$.

Proof of theorem Ps26:

(1)	A stands in truth-functional contrary opposition$_p$ to B.	assumption
(2)	$\neg B$ truth-functionally follows$_p$ from A.	Ps3, (1)
(3)	$w_p(\neg B \wedge A, C) = w_p(A, C)$	Ps1, (2)
(4)	$w_p(A \wedge \neg B, C) = w_p(\neg B \wedge A, C) = w_p(A, C)$	Popper's theorem (40), (3)
(5)	$w_p[\neg(A \wedge \neg B), C] = w_p(\neg A, C)$	Popper's theorem (64), (4)
(6)	$w_p(\neg A, C) = w_p(B \wedge A, C) + w_p(\neg A, C) - w_p(A \wedge \neg A, C)$	Pb106, (5)
(7)	$w_p(B \wedge A, C) = w_p(A \wedge \neg A, C)$	(6)
(8)	For all C: $w_p(B \wedge A, C) = w_p(A \wedge \neg A, C)$.	(7), for C is neither free in (1) nor in (8)
(9)	$w_p[B \wedge A, \neg(C \wedge \neg C)] = w_p[A \wedge \neg A, \neg(C \wedge \neg C)]$	(8)
(10)	$p_p(B \wedge A) = p_p(A \wedge \neg A) = 0$	Popper's theorem (75), (9), Pb109
(11)	For all p_p: $p_p(B \wedge A) = 0$.	(10), for p_p is neither free in (1) nor in (11)

Note that A's standing in truth-functional contrary opposition$_p$ to B has the same effect on $p_p(B \wedge A)$ as B's having probability 0; so, not surprisingly, theorem Pe22 and the following theorem share the same then-clause:

Theorem Pe35:

For all A, B, p_p and w_p: If A stands in truth-functional contrary opposition$_p$ to B, then: excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] $=$ $p_p(\neg A) - w_p(B, A)$.

Proof of theorem Pe35: like that of Pe22, just use Ps26 instead of Pb111.

A more informative result emerges, when we extend the if-clause:

Theorem Pe36:

For all A, B, p_p and w_p: If A stands in truth-functional contrary opposition$_p$ to B and if A is not w_p-absurd, then excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] $= p_p(\neg A)$.

Proof of theorem Pe36:

(1)	A stands in truth-functional contrary opposition$_p$ to B.	assumption
(2)	A is not w_p-absurd.	assumption
(3)	excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] $= p_p(\neg A) - w_p(B, A)$	(1), Pe35
(4)	$w_p(\neg B, A) = 1 = 1 - w_p(B, A)$	(1), Ps15, (2), Ps17
(5)	$w_p(B, A) = 0$	(4)
(6)	excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] $= p_p(\neg A)$	(3), (5)

Similarly with truth-functional contradictory opposition$_p$ and excess:

Theorem Pe37:

For all A, B, p_p and w_p: If A stands in truth-functional contradictory opposition$_p$ to B, then excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] $= p_p(B) - w_p(B, A)$.

Proof of theorem Pe37: immediately from Ps16 and Pe35.

Theorem Pe38:

For all A, B, p_p and w_p: If A stands in truth-functional contradictory opposition$_p$ to B and if A is not a contradiction$_p$, then excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] $= p_p(B)$.

Proof of theorem Pe38: immediately from Ps16 and Pe36.

Seventhly, two theorems concerning the range of the excess function, when its arguments are values of Popper probability functions:

Theorem Pe39:

For all A, B, p_p and w_p: $0 \leq$ excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] ≤ 1.

Proof of theorem Pe39: immediately from Pe2 and Pe7.

Theorem Pe40:

For all A, B, p_p and w_p: If $p_p(A)>0$, then $0 \leq$ excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] < 1.

Proof of theorem Pe40:

(1)	$p_p(A)>0$	assumption
(2)	A is not w_p-absurd.	(1), Ps18
(3)	excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] $= w_p(\neg B, A) \cdot p_p(\neg A)$	(2), Pe4

(4) $0 \le w_p(\neg B, A) \le 1$ and $0 \le p_p(\neg A) < 1$. Popper's theorem (16), Pb112, (1), Pb110
(5) $[w_p(\neg B, A) \cdot p_p(\neg A)] < 1$ (4)
(6) $0 \le \text{excess}[p_p(A \to B), w_p(B, A)] < 1$ Pe7, (3), (5)

Finally, let us summarize conditions under which excess$[p_p(A\to B), w_p(B, A)] = 0$, and conditions under which excess$[p_p(A\to B), w_p(B, A)] = 1$.

Theorem Pe41:
For all A, B, p_p and w_p: excess$[p_p(A\to B), w_p(B, A)] = 0$,
if $p_p(A) = 1$ [Pe23]; or
if $w_p(B, A) = 1$ [Pe12]; or
if $p_p(A) > 0$ and $p_p(B) = 1$ [Pe27]; or
if A is a tautology$_p$ [Pe31]; or
if A is w_p-absurd [Pe3]; or
if A is a contradiction$_p$ [Ps20, Pe3]; or
if B is a tautology$_p$ [Pe32]; or
if B truth-functionally follows$_p$ from A [Pe33]; or
if A is truth-functionally equivalent$_p$ to B [Pe34].

Theorem Pe42:
For all A, B, p_p and w_p: excess$[p_p(A\to B), w_p(B, A)] = 1$,
if $w_p(B, A) = 0 = p_p(A)$ [Pe9]; or
if $p_p(B) = 1$ and $w_p(B, A) = 0$ [Pe24]; or
if A is not w_p-absurd and $p_p(A) = 0$ and B is w_p-independent$_p$ of A and $p_p(B) = 0$ [Pe10]; or
if A is not w_p-absurd and $p_p(A) = 0$ and A stands in truth-functional contrary opposition$_p$ to B [Pe11]; or
if A is not w_p-absurd and $p_p(A) = 0$ and B is w_p-absurd [Pe30]; or
if A is not w_p-absurd and $p_p(B) = 1$ and A stands in truth-functional contradictory opposition$_p$ to B [Pe38].

4 COMPARATIVE TABLES OF LAWS OF EXCESS IN PE AND KE

In order to get an overview of the laws of excess we have obtained, and in order to see at a glance which of the Kolmogorov analogues to theorems of Pe are also theorems of Ke, we group all our laws of excess systematically in three tables. The left-hand column of each table lists the theorems of Pe, the right the respective theorems of Ke—if they exist. Note that some of the theorems which appear in the left-hand column of the tables have not been explicitly mentioned in section 3, for they are, in an obvious way, merely weakenings of theorems of Pe, proved above. In the first table we collect those laws of excess which are equations but which attach no definite numerical value to the excess of $p_p(A\to B)$ over

$w_p(B, A)$ or of $p_k(A \to B)$ over $w_k(B, A)$. In the second table, we collect those laws of excess which are equations and which do attach a definite numerical value to the excess of $p_p(A \to B)$ over $w_p(B, A)$ or of $p_k(A \to B)$ over $w_k(B, A)$. And in the third table, we collect those laws of excess which are inequations.

Table 1: Equations without Definite Numerical Values

Theorems of Pe	*Respective Theorems of Ke (if any)*
excess$[p_p(A \to B), w_p(B, A)] =$ $= p_p(A \to B) - w_p(A \to B, A)$ [Pe13] $= 1 - p_p(A) + p_p(B \wedge A) - w_p(B, A)$ [Pe20] $= [1 - w_p(B, A)] \cdot [1 - p_p(A)]$ [Pe12] $= [1 - w_p(B, A)] \cdot p_p(\neg A)$ [Pe5] $= w_p(\neg B, A) \cdot p_p(\neg A) \cdot [1 - w_p(\neg A, A)]$ [Pe6]	
If A is not w_p-absurd, then: [Pe4] excess$[p_p(A \to B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A)$.	
If $w_p(B, A) \neq 1$, then: [Pe14] excess$[p_p(A \to B), w_p(B, A)] = w_p(\neg B, A) \cdot p_p(\neg A)$.	
If $p_p(A)>0$, then excess$[p_p(A \to B), w_p(B, A)] =$ $= p_p(A \to B) - p_p(A \to B, A)$ $= 1 - p_p(A) + p_p(B \wedge A) - p_p(B, A)$ $= w_p(\neg B, A) \cdot p_p(\neg A)$ [Pe4, Ps18] $= [1 - w_p(B, A)] \cdot [1 - p_p(A)]$ $= [1 - w_p(B, A)] \cdot p_p(\neg A)$ $= w_p(\neg B, A) \cdot p_p(\neg A) \cdot [1 - w_p(\neg A, A)]$.	If $p_k(A)>0$, then excess$[p_k(A \to B), w_k(B, A)] =$ $= p_k(A \to B) - w_k(A \to B, A)$ [Ke2] $= 1 - p_k(A) + p_k(B \wedge A) - w_k(B, A)$ [Ke2] $= w_k(\neg B, A) \cdot p_k(\neg A)$ [Ke3] $= [1 - w_k(B, A)] \cdot [1 - p_k(A)]$ [Ke5] $= [1 - w_k(B, A)] \cdot p_k(\neg A)$ [Ke5] $= w_k(\neg B, A) \cdot p_k(\neg A) \cdot [1 - w_k(\neg A, A)]$. [Ke5]
If B is w_p-independent$_p$ of A and if A is not w_p-absurd, then: [Pe8] excess$[p_p(A \to B), w_p(B, A)] = p_p(\neg B) \cdot p_p(\neg A)$.	
If $p_p(A)>0$ and if B is w_p-independent$_p$ of A, then: excess$[p_p(A \to B), w_p(B, A)] = p_p(\neg B) \cdot p_p(\neg A)$. [Pe8, Ps18]	If $p_k(A)>0$ and if B is p_k-independent$_k$ of A, then: excess$[p_k(A \to B), w_k(B, A)] = p_k(\neg B) \cdot p_k(\neg A)$. [Ke8]
If $p_p(A) = 0$ or if $p_p(B) = 1$, then: [Pe21, Pe24] excess$[p_p(A \to B), w_p(B, A)] = 1 - w_p(B, A)$.	
If $p_p(B) = 0$ or B is w_p-absurd, then: [Pe22, Pe28] excess$[p_p(A \to B), w_p(B, A)] = p_p(\neg A) - w_p(B, A)$.	
If A is not w_p-absurd and B is w_p-absurd, then: excess$[p_p(A \to B), w_p(B, A)] = p_p(\neg A)$. [Pe29]	
If $p_p(A)>0$ and if $p_p(B) = 0$ or B is a contradiction$_p$, then excess$[p_p(A \to B), w_p(B, A)] = p_p(\neg A)$. [Pe25, Ps22]	If $p_k(A)>0$ and if $p_k(B) = 0$ or B is a contradiction$_k$, then excess$[p_k(A \to B), w_k(B, A)] = p_k(\neg A)$. [Ke6]

If A stands in truth-functional contrary opposition$_p$ to B, then: [Pe35] excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] = $p_p(\neg A)$-$w_p(B, A)$.	
If A is not w_p-absurd and if A stands in truth-functional contrary opposition$_p$ to B, then: excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] = $p_p(\neg A)$. [Pe36]	
If $p_p(A)>0$ and if A stands in truth-functional contrary opposition$_p$ to B, then: [Pe36, Ps18] excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] = $p_p(\neg A)$.	If $p_k(A)>0$ and if A stands in truth-functional contrary opposition$_k$ to B, then: [Ke12] excess[$p_k(A \rightarrow B)$, $w_k(B, A)$] = $p_k(\neg A)$.
If A stands in truth-functional contradictory opposition$_p$ to B, then: [Pe37] excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] = $p_p(B)$-$w_p(B, A)$.	
If A is not w_p-absurd and if A stands in truth-functional contradictory opposition$_p$ to B, then: excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] = $p_p(B)$. [Pe38]	
If $p_p(A)>0$ and if A stands in truth-functional contradictory opposition$_p$ to B, then: [Pe38, Ps18] excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] = $p_p(B)$.	If $p_k(A)>0$ and if A stands in truth-functional contradictory opposition$_k$ to B, then: [Ke13] excess[$p_k(A \rightarrow B)$, $w_k(B, A)$] = $p_k(B)$.

Table 2: Equations with Definite Numerical Values

Theorems of Pe	*Respective Theorems of Ke (if any)*
excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] = 0, if: $p_p(A)$ = 1 or A is a tautology$_p$. [Pe41]	excess[$p_k(A \rightarrow B)$, $w_k(B, A)$] = 0, if: $p_k(A)$ = 1 or A is a tautology$_k$. [Ke11]
excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] = 0, if: [Pe41] $w_p(B, A)$ = 1; or A is w_p-absurd; or A is a contradiction$_p$; or B is a tautology$_p$; or B truth-functionally follows$_p$ from A; or A is truth-functionally equivalent$_p$ to B.	
excess[$p_p(A \rightarrow B)$, $w_p(B, A)$] = 0, if $p_p(A)>0$ and: [Pe41] $p_p(B)$ = 1; or $w_p(B, A)$ = 1; or B is a tautology$_p$; or B truth-functionally follows$_p$ from A; or A is truth-functionally equivalent$_p$ to B.	excess[$p_k(A \rightarrow B)$, $w_k(B, A)$] = 0, if $p_k(A)>0$ and: $p_k(B)$ = 1; or $w_k(B, A)$ = 1; or B is a tautology$_k$; or B truth-functionally follows$_k$ from A; or A is truth-functionally equivalent$_k$ to B. [Ke10]

excess$[p_p(A \to B), w_p(B, A)] = 1$, if: [Pe42] $w_p(B, A) = 0 = p_p(A)$; or $p_p(B) = 1$ and $w_p(B, A) = 0$; or A is not w_p-absurd and $p_p(A) = 0$ and B is w_p-independent$_p$ of A and $p_p(B) = 0$; or A is not w_p-absurd and $p_p(A) = 0$ and A stands in truthfunctional contrary opposition$_p$ to B; or A is not w_p-absurd and $p_p(A) = 0$ and B is w_p-absurd; or A is not w_p-absurd, $p_p(B) = 1$ and A stands in truth-functional contradictory opposition$_p$ to B.	

Table 3: Inequations

Theorems of Pe	*Respective Theorems of Ke (if any)*
excess$[p_p(A \to B), w_p(B, A)] \geq 0$ [Pe7]	
If $p_p(A)>0$, then excess$[p_p(A \to B), w_p(B, A)] \geq 0$.	If $p_k(A)>0$, then excess$[p_k(A \to B), w_k(B, A)] \geq 0$. [Ke4]
excess$[p_p(A \to B), w_p(B, A)] \leq 1$ [Pe39]	
If $p_p(A)>0$, then excess$[p_p(A \to B), w_p(B, A)] < 1$. [Pe40]	If $p_k(A)>0$, then excess$[p_k(A \to B), w_k(B, A)] < 1$. [Ke9]
If $p_p(A)<1$ and $w_p(B, A)<1$, then: excess$[p_p(A \to B), w_p(B, A)] > 0$. [Pe16]	
If $0<p_p(A)<1$ and $w_p(B, A)<1$, then: excess$[p_p(A \to B), w_p(B, A)] > 0$.	If $0<p_k(A)<1$ and $w_k(B, A)<1$, then: excess$[p_k(A \to B), w_k(B, A)] > 0$. [Ke7]
If $0<p_p(A)<1$ and if $p_p(B) = 0$ or B is a contradiction$_p$, then: excess$[p_p(A \to B), w_p(B, A)] > 0$. [Pe26, Ps22]	If $0<p_k(A)<1$ and if $p_k(B) = 0$ or B is a contradiction$_k$, then: excess$[p_k(A \to B), w_k(B, A)] > 0$. [Ke6]

We close the paper with two lists of those specific theorems of Kolmogorov and of Popper probability semantics which have been used, but not proved in the text.

APPENDIX 1:

THEOREMS OF ELEMENTARY KOLMOGOROV PROBABILITY SEMANTICS WHICH HAVE BEEN USED IN THE TEXT[10]

Theorem K4: [lemma for Ke2 and Ke5]
For all A and p_k: $p_k(\neg A) = 1\text{-}p_k(A)$.
Theorem K5: [lemma for Ke2 and Ke13]
For all A, B and p_k: If A is truth-functionally equivalent$_k$ to B, then $p_k(A) = p_k(B)$.
Theorem K6: [lemma for Ke4 and Ke9]
For all A and p_k: $0 \leq p_k(A) \leq 1$.
Theorem K7: [lemma for Ke11]
For all A, B and p_k: If $p_k(A) = 1$, then $p_k(A \rightarrow B) = p_k(B)$.
Theorem K8: [lemma for Ke9]
For all A, B and p_k: If $p_k(A \rightarrow B) = 1$, then $p_k(A) = p_k(A \wedge B)$.
Theorem K9: [lemma for Ke6 and Ke7]
For all A, B and p_k: If $p_k(A) < 1$, then $p_k(\neg A) > 0$.
Theorem K10: [lemma for Ke6 and Ke12]
For all A and p_k: If A is a contradiction$_k$, then $p_k(A) = 0$.
Theorem K11: [lemma for Ke2, Ke3 and Ke12]
For all A, B and p_k: $p_k(A \rightarrow B) = p_k(\neg A) + p_k(B \wedge A)$.
Theorem K12: [lemma for Ke3, Ke5, and Ke6]
For all A, B, p_k and w_k: If $p_k(A) > 0$, then $w_k(\neg B, A) = 1\text{-}w_k(B, A)$.
Theorem K13: [lemma for Ke4 and Ke9]
For all A, B, p_k and w_k: If $p_k(A) > 0$, then $0 \leq w_k(B, A) \leq 1$.
Theorem K14: [lemma for Ke10]
For all A, B, p_k and w_k: If $p_k(A) > 0$ and if $w_k(B, A) = 1$, then $p_k(A \rightarrow B) = 1$.
Theorem K15: [lemma for Ke10]
For all B, A, p_k and w_k: If $p_k(A) > 0$ and if $p_k(B) = 1$, then $w_k(B, A) = 1$.
Theorem K16: [lemma for Ke6]
For all A, B, p_k and w_k: If $p_k(A) > 0$ and if $p_k(B) = 0$, then $w_k(B, A) = 0$.

---

10. Note that all these theorems are *specific* theorems of elementary Kolmogorov probability semantics, i.e., they follow from the axiomatic basis {K1, K2, K3} and, in addition, they do not follow from any consistent set S of propositions if none of the sets {K1}, {K2}, {K3} is a subset of S. Elementary Kolmogorov probability semantics contains also *un*specific theorems, these are the theorems of that theory in which elementary Kolmogorov probability theory is embedded, i.e. the classical theory of truth value functions and truth-functional attributes. Those of the unspecific theorems of Kolmogorov probability semantics which have been used in the text are so well-known that no special mention has been made of them.—Note, finally, that each of the Popper analogues to the theorems in appendix 1 is a theorem of Popper probability semantics.

Theorem K17: [lemma for Ke7]
For all A, B, p_k and w_k: If $p_k(A)>0$ and if $w_k(B, A)<1$, then $w_k(\neg B, A)>0$.
Theorem K18: [lemma for Ke5 and Ke10]
For all B, A, p_k and w_k: If $p_k(A)>0$ and
if B truth-functionally follows$_k$ from A or A is truth-functionally equivalent$_k$ to B, then $w_k(B, A) = 1$.
Theorem K19: [lemma for Ke11]
If $p_k(A) = 1$, then $w_k(B, A) = p_k(B)$.
Theorem K20: [lemma for Ke9 and Ke12]
For all A, B, p_k and w_k: If $p_k(A)>0$, then $w_k(B, A) = 0$ iff $p_k(A \wedge B) = 0$.
Theorem K21: [lemma for Ke8]
For all A, B, p_k and w_k: If $p_k(A)>0$, then:
B is p_k-independent$_k$ of A iff $w_k(\neg B, A) = p_k(\neg B)$.

APPENDIX 2: THEOREMS OF BASIC POPPER PROBABILITY SEMANTICS WHICH HAVE BEEN USED IN THE TEXT[11]

Popper's theorem (16): [lemma for Pb103, Pb108, Pb112, Pe2, Pe7, Pe40, Pe8, Pe13 and Pe16]
For all A, B and w_p: $0 \leq w_p(B, A) \leq 1$.
Popper's theorem (23): [lemma for Ps8, Pb108, Pb110, Pb110 and Pb115]
For all A and w_p: $w_p(A, A) = 1$.
Popper's theorem (25): [lemma for Pb108]
For every w_p there is at least one A such that $w_p(\neg A, A) = 0$.
Popper's theorem (28): [lemma for Pb102, Pb106, Ps12 and Ps14]
For all A, B and w_p: $w_p(B, A \wedge B) = 1$.
Popper's theorem (29): [lemma for Ps8 and Pe13]
For all A, B and w_p: $w_p(A \wedge B, A) = w_p(B, A)$.
Popper's theorem (32): [lemma for Ps14 and Pb108]
For all A, B and w_p: $w_p(B \wedge B, A) = w_p(B, A)$.
Popper's theorem (33'): [lemma for Pb106, Pb108 and Pb115]
For all A, B and w_p: $w_p(B, \neg A \wedge A) = 1$.

11. The numbering of these theorems is identical with Popper's numbering of his theorems in his own development of his probability theory in (LdF9, pp. 298–304). You may also consult POPPER (LScD, pp. 349–355), but note that the numbering of theorems in (LScD) is not identical with the numbering in (LdF9) up to theorem (32). Incidentally, none of the Kolmogorov analogues to the theorems in appendix 2 is a theorem of Kolmogorov probability semantics.

Popper's theorem (40): [lemma for Ps8, Ps9, Pb102, Pb103, Pb105, Pb106, Pb107, Pb108, Pb111, Ps11, Ps12, Ps14, Pe13, Pb115, Pb116]

For all A, B, C and w_p: $w_p(A \wedge B, C) = w_p(B \wedge A, C)$ and $w_p(C, A \wedge B) = w_p(C, B \wedge A)$.

Popper's theorem (62): [lemma for Ps9, Pb106 and Ps14]

For all A, B, C, D and w_p:

$w_p[(A \wedge B) \wedge C, D] = w_p[A \wedge (B \wedge C), D]$ and $w_p[D, (A \wedge B) \wedge C] = w_p[D, A \wedge (B \wedge C)]$.

Popper's theorem (63): [lemma for Pb102, Pb104, Pb108, Ps17, Ps23]

For all A and w_p: $w_p(\neg A, A) \neq 0$ iff for all B: $w_p(B, A) = 1$.

Popper's theorem (64): [lemma for Pb101, Pb102, Pb104, Pb106, Ps11, Ps14, Pb108, Pb110, Ps17, Pe4, and Pe5]

For all A, B and w_p: $w_p(\neg B, A) = 1 - w_p(B, A) + w_p(\neg A, A)$.

Popper's theorem (69): [lemma for Pb103]

For all A, B and w_p: $w_p(\neg A \wedge B, A) = w_p(\neg A, A)$.

Popper's theorem (70): [lemma for Pb108 and Pb115]

For all A, B, C and w_p: $w_p(A \wedge B, C) + w_p(\neg A \wedge B, C) = w_p(B, C) + w_p(\neg C, C)$.

Popper's theorem (74): [lemma for Pb105]

For all A, B and w_p: $w_p[\neg(\neg B \wedge B), A] = 1 = w_p[\neg(B \wedge \neg B), A]$.

Popper's theorem (75): [lemma for Pb101, Pb107, Pb108, Pb110, Pb112, Pb111, Ps18, Ps23, Pb115, Pb116, Ps25, and Ps26]

For all A, B, p_p and w_p: $p_p(A) = w_p[A, \neg(B \wedge \neg B)]$.

Popper's theorem (79): [lemma for Pb106]

For all A, B, C and w_p: $w_p[\neg(\neg A \wedge \neg B), C] = w_p(A, C) + w_p(B, C) - w_p(A \wedge B, C)$.

Popper's theorem (96): [lemma for Pb108, Pe4, Pe8, and Pe23]

For all A, B, p_p and w_p: $w_p(B, A) \cdot p_p(A) = p_p(B \wedge A)$.

Popper's theorem (97): [lemma for Pe25, Pe27 and Pb120]

For all A, B, p_p and w_p: If $p_p(A) \neq 0$, then $w_p(B, A) = p_p(B \wedge A) / p_p(A)$.

Popper's theorem (99): [lemma for Ps11 and Ps14]

For all A, B, C, D and w_p:

If for all E: $w_p(A, E) = w_p(B, E)$, then $w_p(C, A \wedge D) = w_p(C, B \wedge D)$.

References

DORN/WEINGARTNER (Foundations)

Georg Dorn and Paul Weingartner (eds.): *Foundations of Logic and Linguistics. Problems and Their Solutions.*—London: Plenum, 1985.

DUNN/GUPTA (Truth)

J. Michael Dunn and Anil Gupta (eds.): *Truth or Consequences. Essays in Honor of Nuel Belnap.*—Dordrecht: Kluwer, 1990.

GABBAY/GUENTHNER (Handbook)
D. Gabbay and F. Guenthner (eds.): *Handbook of Philosophical Logic. Volume I: Elements of Classical Logic.*—Dordrecht: Reidel, 1983.
HARPER (Belief)
William L. Harper: "Rational Belief Change, Popper Functions and Counterfactuals".—In: HARPER/HOOKER (Foundations, pp. 73–112).
HARPER/HOOKER (Foundations)
William L. Harper and Clifford A. Hooker (eds.): *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science. Proceedings of an International Research Colloquium held at the University of Western Ontario, London, Canada, 10–13 May 1973.* Volume 1: *Foundations and Philosophy of Peistemic Applications of Probability Theory.*—Dordrecht: Reidel, 1976.
KOLMOGOROFF (Grundbegriffe)
Andrej Nikolajevič Kolmogoroff: *Grundbegriffe der Wahrscheinlichkeitsrechnung.*—Berlin: Verlag von Julius Springer, 1933.
KOLMOGOROV (Foundations)
Andrej Nikolajevič Kolmogorov: *Foundations of the Theory of Probability.*—New York: Chelsea Publishing Company, [2]1956.
LEBLANC (Alternatives)
Hugues Leblanc: "Alternatives to Standard First-Order Semantics".—In: GABBAY/ GUENTHNER (Handbook), pp. 189–274.
LEBLANC (Autonomy)
Hugues Leblanc: "The Autonomy of Probability Theory (Notes on Kolmogorov, Rényi, and Popper)".—In: *The British Journal for the Philosophy of Science* 39 (1988), pp. 167–181.
LEBLANC (Contributions)
Hugues Leblanc: "Popper's Formal Contributions to Probability Theory".—In: NOTTURNO/BRILL (Perspectives), pp. 341–367.
LEBLANC (Requirements)
Hugues Leblanc: "On Requirements for Conditional Probability Functions".—In: *The Journal of Symbolic Logic* 1960 (25), pp. 238–242.
LEBLANC/VAN FRAASSEN (Functions)
Hugues Leblanc and Bas C. Van Fraassen: On Carnap and Popper Probability Functions.—In: *The Journal of Symbolic Logic* 44 (1979), pp. 369–373.
LEBLANC/ROEPER (Functions)
Hugues Leblanc and Peter Roeper: Probability Functions: "The Matter of their Recursive Definability".—In: *Philosophy of Science* 59 (1992), pp. 372–388.
LEBLANC/ROEPER (Probabilities)
Hugues Leblanc and Peter Roeper: "What are Absolute Probabilities a Function of?"—In: DUNN/ GUPTA (Truth), pp. 307–325.

LEWIS (Probabilities)
David Lewis: "Probabilities of Conditionals and Conditional Probabilities".—In: *The Philosophical Review* 85 (1976), pp. 297–315.
MACE (Philosophy)
C. A. Mace (ed.): *British Philosophy in the Mid-Century: A Cambridge Symposium.*—London: George Allen and Unwin, 1957.
NOTTURNO/BRILL (Perspectives)
M. A. Notturno and E. J. Brill (eds.): *Perspectives on Psychologism.*—Leiden: Martinus Nijhoff, 1989.
POPPER (Axioms)
Karl R. Popper: "A Set of Independent Axioms for Probability".—In: *Mind* 47 (1938), pp. 275–277. [Errata, *ibidem*, p. 415, p. 552]
POPPER (Axiom Systems)
Karl R. Popper: "Two Autonomous Axiom Systems for the Calculus of Probabilities".—In: *The British Journal for the Philosophy of Science* 6 (1955/56), pp. 51–57. [Errata, *ibidem*, p. 176, p. 351]
POPPER (Conjectures)
Karl R. Popper: *Conjectures and Refutations. The Growth of Scientific Knowledge.*—London: Routledge and Kegan Paul, ^{4}1972.
POPPER (Definitions)
Karl R. Popper: "Creative and Non-Creative Definitions in the Calculus of Probability".—In: *Synthese* 15 (1963), pp. 167–186. [Correction: *Synthese* 21 (1970), p. 107]
POPPER (LdF2)
Karl R. Popper: *Logik der Forschung.* Zweite, erweiterte Auflage.—Tübingen: J. C. B. Mohr (Paul Siebeck), 1966.
POPPER (LdF7)
Karl R. Popper: *Logik der Forschung.* Siebente, verbesserte und durch sechs Anhänge vermehrte Auflage.—Tübingen: J. C. B. Mohr (Paul Siebeck), 1982.
POPPER (LdF9)
Karl R. Popper: *Logik der Forschung.* Neunte, verbesserte Auflage.—Tübingen: J. C. B. Mohr (Paul Siebeck), 1989.
POPPER (LScD)
Karl R. Popper: *The Logic of Scientific Discovery.*—London: Hutchinson, ^{10}1980.
POPPER (Non-Existence)
Karl R. Popper: "The Non-Existence of Probabilistic Inductive Support".—In: DORN/-WEINGARTNER (Foundations, pp. 303–318).
POPPER (Philosophy)
Karl R. Popper: "Philosophy of Science: A Personal Report".—In: MACE (Philosophy, pp. 155–191).

POPPER (Realism)
Karl R. Popper: *Realism and the Aim of Science.* [Volume 1 of *Postscript to the Logic of Scientific Discovery.* Edited by W. W. Bartley, III].—London: Hutchinson, 1983.
POPPER/MILLER (Proof)
Karl R. Popper and David W. Miller: "A Proof of the Impossibility of Inductive Probability".—In: *Nature* 302 (April 21, 1983), pp. 687–688.
POPPER/MILLER (Support)
Karl R. Popper and David W. Miller: "Why Probabilistic Support Is Not Inductive".—In: *Philosophical Transactions of the Royal Society of London* 321 (1987), pp. 569–591.
ROEPER/LEBLANC (Indiscernibility)
Peter Roeper and Hugues Leblanc: "Indiscernibility and Identity in Probability Theory".—In: *Notre Dame Journal of Formal Logic* 32 (1991), pp. 1–46.
STALNAKER (Probability)
Robert Stalnaker: "Probability and Conditionals", *Philosophy of Science* 37 (1970), pp. 64–80.