Chapter 2. **History and foundations of classical mechanics: the confirmation of the two dichotomies**

*Having clarified that there is a choice of the type of Mathematics in the foundations of Thermodynamics, let us establish whether the same choice applies to the foundations of the prestigious science of Mechanics; at the same time we will recognize another fundamental dichotomy, that of the type of organization.*

2.1 The choice of the type of mathematics in Newton's mechanics

Absolute, true, and mathematical time, of itself, and from its own nature flows equably without regard to anything external … Relative, apparent and common time is a sensible time, the accurate measure (whether accurate or inequable) of duration is obtained by means of motion….

Absolute space, in its own nature, without regard to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies; and which is vulgarly taken for immovable space…

Newton 1687

Scientists, as far as they could, have purged their construction of metaphysics;

but for what they could not, they used it as a key to understanding the whole universe.

E. Burtt, 1924

We have seen in the previous Chapter that Thermodynamics is generally based on a mathematics that is essentially different from that of Carathéodory’s formulation, which is normal in theoretical Physics. We have also found that the pluralism of mathematics also corresponds to a pluralism of essentially different formulations of the same physical theory.

But is this a specific result of the Thermodynamic theory, a theory that many consider "strange" and too different from the others? Or is it an example of a radical theoretical dichotomy which concerns every physical theory?

To find an answer to this question let us look at Mechanics, the theory that most of all represents theoretical physics, and let us ask ourselves if there is the possibility of formulating it on the basis of constructive, rather than classical mathematics. Having found among the various formulations of thermodynamic theory one that uses constructive mathematics, our principal aim will be to study the history of Mechanics in order to find possible alternatives to the dominant formulation.

Before Newton there had been an intense debate concerning the foundations of physics. Suffice it to mention two of the protagonists of this debate: Galilei and Descartes. The extraordinary thing about Galilei's work is that he was fully aware of two dichotomies, the one relating to the type of Mathematics and the one concerning the type of organization; he highlighted them above all in his last two works, the most important. The *Dialogue Concerning the Two Chief World systems* (1632) is not only a discussion about Astronomy, aimed at understanding if it is the Earth that revolves around the Sun or viceversa, but also a discussion about the whole of reality, including the very organization of our way of seeing reality itself. In fact, he was very familiar with the model of the deductive theory, suggested by Aristotle for the organization of a scientific theory; as a young man he had followed lectures on the subject in Rome and had written notes (which are kept in the national edition of all his works). But he never organized his theory deductively; rather his last two works are written in the dialogic manner followed by Plato; or more precisely, in the last work, *The Discourses on Two New Sciences* (1638), deductive passages demonstrating mathematical results are intertwined with investigative passages in which problems are discussed dialogically. This choice is a sign that Galilei wanted to organize his scientific theory in a new way that differed from the dominant mode at that time, that of Euclid. Moreover in this last work he dedicates a whole day (the third) to discussing the decomposition of a segment into infinite rather than finite parts and concludes without having opted for either of the two possibilities. He therefore did not decide between the two alternatives, but showed that he was aware of them and wanted them to form the basis his science.

 Descartes was also aware of these dualisms, but followed resolutely the model of the deductive organization of the theory and deplored the "lack of certain principles" in Galileo's work, that is, a deductive organization. Moreover he (like Huygens) opposed the use of AI, both in Mathematics and in Physics. It is therefore certain that Newton, in constructing his theories with the contributions of previous scientists, was also aware he was making choices that were more or less different from theirs.

Let us then examine the choices Newton made in constructing his physical theory, although he did not declare his choices or discuss them in his writings. Newton clearly chose AI, since he introduced into physics infinitesimal analysis, in which its fundamental element, an infinitesimal, was commonly conceived as the inverse of AI.

Note that recently (1960) it was shown that Newton's mathematics, infinitesimal analysis, was almost modern non-standard Analysis and that the number field adopted by Newton (that of infinitesimals, in modern terminology, hyperreal numbers), was broader than that of the *ε-δ* real numbers of classical mathematics.

Let us see precisely what implications Newton's choice of mathematics had for theoretical physics. We note that the three possible types of mathematics, non-standard analysis, "rigorous" calculation and constructive analysis, present three different interpretations of the basic mathematical concepts. Let us start with the concept of point.

The point in non-standard analysis is an infinitesimal, that is, "a point that tends to approach point 0"; instead in "rigorous" analysis it is an exact and isolated point (because in this Mathematics it is possible to identify exactly every single point belonging to the continuum); while in constructive analysis it is in general an interval, which however can diminish without limit. As a result, every physical concept, the definition of which derives from the concept of point, can have a different meaning in each of the three mathematics. For example, the first principle of Newton's dynamics implies that one is able to decide when a force in space is exactly zero (because even a force approximately equal to zero could lead to unpredictable results, given that the first principle does not define the effects of a force, which will be known only with the second principle). In constructive mathematics the problem considered above is undecidable (essentially because of the fugitive numbers), while in the other two types of Mathematics the concept of *F* = 0 is legitimate and is represented by a point within the line representing the range of values of force *F*.

Other examples of concepts with different meanings are those defined by *limit properties*. We have already seen that a limit can be a value which can be approximated to any desired level of accuracy, or be an ideal limit, going beyond any possible experimental data and therefore detached from them. For example, Newton attributed to matter the concept, obtained by idealization, of *a perfectly hard body* (so that its form is fixed and therefore does not rebound in a collision). This concept is well known to historians of physics, since the entire development of mechanics in the 18th century was hampered by this ideal concept. Another example is the case of the ether, a concept whose characteristics were obtained as ideal limit properties of the properties of bodies traversed by electromagnetic vibrations. Both of these concepts can be represented in both non-standard and rigorous analysis, but are unacceptable in constructive mathematics, because they are limits located beyond approximations.

Let us now analyze the experimental laws expressed through mathematical functions. Of course, every experimental law is always an approximate equality between approximate values of physical magnitudes. To make the calculations easier each Mathematics represents it through an equality. However, non-standard and rigorous mathematics idealize approximate equality in an exact equality (notice that this idealization does not undermine experimental measurements, which are incapable of appreciating the difference between the idealized value and the experimental result). Constructive mathematics, on the other hand, cannot in general establish whether *f (x)* = 0 and therefore always considers equality to be approximate, yet with an increasingly perfectible approximation. Ultimately, this Mathematics retains all the heritage of experimental laws accumulated so far by physicists. It has, however, the problem, as we have seen in Thermodynamics, of whether the principles are constructive or not, which, being sentences that synthesize myriads of experimental facts (e.g. a body in motion continues its motion until a force changes its motion), are always idealizing propositions regarding them.

The various Mathematics may differ from one another even in relation to some mathematical functions used in theoretical physics. In fact, some functions may exist in rigorous analysis and not exist in constructive analysis. Consider for example the function that Dirichelet defined as equal to 0 on rational numbers and to 1 on irrational numbers (it is approximated by means of the limit of the expression [cos(2*πn!x*)]*m* as *n* and *m* approach ∞).

In constructive mathematics it cannot be decided whether a given number *x* is a rational or an irrational number because the resolution of this problem requires an examination of an actual number of digits and the algorithms to decide this problem (e.g. in the cases of π, √2, etc.), given that each one is composed of a finite number of rules of construction, they in total at most a countable infinity, that is, an infinity less than that of the numbers of the continuum: therefore a function of this type cannot be defined. In the other two Mathematics it is perfectly legitimate.

Are there examples of similar functions in theoretical physics? For example, are the**r**e Newtonian functions based essentially on actual infinity? Clearly, already at the level of variables, absolute space and absolute time have precisely this characteristic, both because they are ideal limits of experimental concepts and because they want to consider every single point, both spatial and temporal, with infinite (actual) precision. At the beginning of the paragraph the magniloquent expressions with which Newton defined these magnitudes as absolute are in contrast with "common" measurable and experimental magnitudes. More than two centuries later, the dramatic crisis of the concept of the ether led physicists to abandon these undecidable (and even non-operational) functions.

Let us now analyze the differential equations of a physical theory. Consider Newton’s differential equation for classical mechanics. Here both non-standard analysis and "rigorous" mathematics constantly search for new techniques to solve the most difficult problems and in the hope that they will all be finally solved. But constructive mathematics shows simply that this differential equation cannot always be solved with a general algorithm. Let us look at the simplest case.

 First note that the differential equation typical of Newtonian dynamics

 

can be translated into *y=dx/dt* and *mdy/dt=f(x,y,t)*

so that it is then possible to proceed with the study of first order differential equations alone.

 In constructive mathematics it is shown that this type of differential equation is solvable in general when *f* is Lipschitzian and is uniformly continuous. In classical mathematics, on the other hand, it is enough that that function is pointwise continuous on the interval of definition. Then with the Heine-Borel theorem it is shown that the function is also uniformly continuous. However that theorem makes use of the Zermelo axiom and therefore it is unacceptable for constructive mathematics.

The case of pointwise continuous but not uniformly continuous functions is therefore incompatible with the use of constructive mathematics. In Mechanics we have such functions when we consider force functions that are not limited: for example, central forces (such as gravity), which for *r* = 0 give infinite force. Then there are discontinuous forces, those that occur in instantaneous impacts. Thus, when constructive mathematics is adopted, the differential equation of the second principle of Newton's dynamics is in some cases undecidable, i.e. it cannot be solved with the same general algorithm holding for all other cases. In the cases just mentioned of unlimited or discontinuous forces, the equation can still be used, but only as a heuristic principle. In conclusion, it is capable of suggesting the resolution of many mechanical problems, but not all problems.

It follows that this differential equation can no longer represent a principle-axiom, which by its nature should be universal, that is, cover all real physical cases. Because of this fact, constructive mathematics does not accept Newton's formulation, which furthermore is based on an incompatible inertia principle since it requires *f* = 0 with absolute precision. Actually, the metaphysical character of the Newtonian formulation had already been pointed out as soon as it appeared by famous scholars: Leibniz, Berkeley, and then, subsequently, by D'Alembert, L. Carnot, Lagrange, Mach, Poincaré.

If we then examine some of the best known of the other formulations of Mechanics, we find a seemingly discouraging result. The variational formulations (Maupertuis’ minimal action approach, the various minimum principles, Gauss’s least squares method) require solving an equation of the type

 *Adt*=0.

But we know that *x* = 0 exactly is undecidable, equality of a function with 0 more so and that of a functional even more so. All these formulations are therefore unacceptable for constructive mathematics. The Lagrange formulation based on a differential equation that is even more complex than *F = ma*, certainly repeats the problems of the preceding differential equation.

In conclusion, the differences presented by the three different types of Mathematics lead to different ways of formulating the basic principles and concepts a physical theory, so that it may happen that a particular formulation of a physical theory agrees with only teo types of Mathematics: for example, Newtonian Mechanics is only compatible with classical mathematics and non-classical analysis. And it can be the case, as in classical mechanics, that there is apparently no acceptable formulation in constructive mathematics. In the following we will see however that a more accurate inspection of history of mechanics will suggest a formulation of classical mechanics whose mathematics is constructive.

*2.2 Leibniz's search for an alternative to Newton's mechanics*

The principles of Newtonian Mechanics were almost exclusively formulated by Newton. They were formulated almost three centuries ago and for two centuries remained the foundation of a science, Mechanics, which in the 1700s included more and more fields of phenomena in its theory, so that for a long time it was thought that physical theory was simply the theory of mechanics. One consequence was that Newton was seen as the Aristotle of the modern era; no one more than he had such a lasting influence on the mentality of modern man.

But constructive mathematics cannot accept Newton's formulation, based on infinitesimals and differential equations; and usually in Mechanics (unlike Thermodynamics) no one is aware of any other formulation that is so simple mathematically as to be directly recognized as being based on constructive mathematics. For this reason it is worth carrying out a supplementary investigation into the history of physics to see if there have been attempts, and with what scope, to formulate a mechanical theory different from Newton’s.

For a long time it was extremely difficult to evaluate the scope prefisco extent of the thought of Newton’s contemporary, Leibniz (1646-1716) on the subject of dynamics. His scientific contribution was certainly wide and profound. This contribution includes the mathematical theory of infinitesimal analysis (of which he was perhaps the sole inventor); moreover, he laid the foundations of modern mathematical logic. But his theory of mechanics seemed either too naïve (because based on very elementary examples of mechanical motions) o metaphysical (because including principles and considerations of this nature) or very sophisticated (in the applications of infinitesimal analysis to some problems).

The problem of an accurate appraisal of his contribution to mechanics stems from various factors. Leibniz was enormously productive and the material to be taken into consideration by an historian is very broad, with manuscripts still to be published. Furthermore, Leibniz's thought evolved: he was first a Cartesian, then a follower of Huygens, and finally his thought underwent an independent development. In addition, Leibniz’s dynamics, compared with the contemporary Newtonian theory of 1687, is clearly different, but incomplete, so that it requires a specific analysis. Moreover, Leibniz’s whole scientific programme remained incomplete, since he believed that infinitesimal analysis constituted merely one part of that very general mode of reasoning by symbols that he had envisioned. Finally, let us add the fact that Leibniz was considered by most commentators to be the great metaphysician (he is usually remembered for saying that everything is conceivable with the metaphysical "monads" and finally that this world is "the best of all possible worlds", etc.). The consequence was that a proper study of his scientific texts in order to find a coherence in his thought was not carried out. A sign of this lack of interest is the fact that several of his mechanical works long remained untranslated into other languages (e.g. Italian).

 It should be noted, however, that several times after his death, Leibniz was partially re-evaluated, acknowledging his anticipation of many subsequent scientific ideas. For example, in the nineteenth century his anticipation of the energy conservation law was recognized; and in the XX century his anticipation of modern mathematical logic, of Relativity and then of the foundation of theoretical physics on symmetries became manifest.

Leibniz was always clearly opposed to Newtonian Mechanics and even to the very name mechanics (it is he who by way of opposition invented the word "dynamics"), as he was to the concept of attraction at a distance (accused of being a kind of Aristotelian "occult quality", given that Newton left it unexplained) and to the whole logical structure of Newton’s theory. Already in response to Descartes regarding the correct mathematical notion of *vis viva* he had launched the programme of a "dynamic reform", which he pursued with even more determination after the publication of Newton's *Principia*, which he never accepted. Thus Leibniz wrote to Honoratus Fabri:

We should have inquired whether complex natural phenomena could not be derived from other known and studied phenomena. It is in fact useless to assume possible causes in the place of true causes, when the true and certain causes are before our eyes. So I believe that with my example more intelligent things could be proposed for research; to treat natural philosophy [read: theoretical physics] in the future without imaginary hypotheses and presuppose such causes whose actual [read: experimental] reality is well established in nature. No one, as far as I know, has so far tried to explain phenomena starting from phenomena, to explain *the congery* *of particularities by moving from a few general phenomena*, which is the real demonstrative procedure of physics.[[1]](#footnote-1)

The following will be a reconstruction of his mechanics, obtained by extracting the most significant parts from his last works, those written between 1690 and 1698.

*2.3 Leibniz’s principles*

Leibniz stated strongly that there are two general principles of human knowledge, the first is the *principle of non-contradiction* (it is absurd that both *A* and *not A* at the same time); the second is *the principle of sufficient reason*, which he stated as follows:

nothing is without reason, that is, every truth may be demonstrated [by means of an] a priori [method, i.e. it may be], deduced from the concepts of its terms, although it is not always in our power to arrive at this analysis....[[2]](#footnote-2)

Note that Leibniz’s last sentence explains clearly that we are forced to state the first sentence, which has two negations, because it is not always possible to establish with certainty the derivations we are looking for (this is the meaning of the second sentence).

A doubly negated sentence is used in this quotation. This occurs in many areas of science, but has so far gone unnoticed, one reason being that in natural languages there are ambiguities in the use of a double negation. A careful examination is therefore needed to understand when a doubly negated sentence coincides with a single, but reinforced, negation ("I do not want go no further "), or it is a merely emphatic affirmative expression ("I don’t have more than one euro"), or, it is a hidden double negation because one negation is understood (for example: who would ever object ?[= nobody objects] It is only apparently false that ... [= it is not true that it is false])

Let us begin to clarify the meaning of the statements for which ¬¬A ** A does not hold by asking ourselves what their domain of definition is. It is a fuzzy set. In such a case ~~then~~ its double complementation does not bring us back to the initial set, because we do not know if the points of the blurred contour of ¬¬A are the same as those of the blurred contour of A. We see here another difficulty that scientists have encountered in recognizing the above sentences as scientific: imprecision. Such imprecision seemed foreign to mathematics and rigorous science, which in general idealize reality with clear concepts, even at the cost ~~perhaps~~ of detaching themselves from reality. Until a few decades ago mathematicians stated with Hilbert: "In Mathematics there is no such thing as the sentence <We will not know even in the future>"; sooner or later everything will definitely be decided and therefore any indecision and imprecision will come to an end. However, 1900 mathematicians discovered infinite undecidable problems in the foundations of all mathematics).

At the level of logic, the statement sentence proposition ¬¬*A* which does not imply *A*, concerns a set or a physical reality that we do not know how to decide completely: we cannot state *A*; but, on the other hand, it is absurd that it is not *A*. In science there are many situations of this kind. Traditionally, the theorist has tried to avoid them, looking for the certainty of a precise statement, as if every statement of the theory could be decided experimentally either positively or negatively, in all possible cases. In fact in theory very often some concepts and some situations are idealized (we have already seen some principles used in the various formulations of Thermodynamics; in mechanics Newton idealized space as absolute and time as absolute, force as a cause, etc.).

 Nothing strange then if a scientific theory, using idealized concepts, reduces the propositions ¬¬*A* to *A*, or to something similar to *A*. For example in the collision of the bodies the time of impact is very short compared to the duration of the trajectories. Therefore the statement "It is not true that *t* is not zero" holds. The search for certainty here suggests reducing the sentence to *t = 0*, which in itself is an absurdity; but if we then assume that in practice we must take into account that *t = 0* is always approximable with very small *t* values, all is correct. (This ‘all is correct’, even if through a restriction that remains implied, is actually taken by theoretical physicists as proof that idealization is unrestrictedly valid!).

Now, let us look at the question from the point of view of the type of logic. In classical logic the law ¬¬*A* = *A* holds. In what logic are we if this law is no longer valid, as in our case? Extensive studies by mathematical logicians, aimed at reducing as far as possible the distance between classical and non-classical logic, have, however, had to conclude that this distance is irreducible, essentially by virtue of the aforementioned logical law, which either is or is not valid. It has been shown that the non-validity of this law is typical of (almost) all non-classical logics (from intuitionist to minimal).[[3]](#footnote-3)

Here we see another difficulty in recognizing that a proposition ¬¬A does not imply *A*. In this case it belongs to non-classical logic, that is to a type of logic that has always been accused of being abstruse, or unrealistic, or deviant with respect to the ideal rationality of classical logic. Yet the opposite is true. To the conscious activity of the I, classical logic seems a source of certainty because all questions are decided through it with a clear distinction between "true" or "false". But it should be noted that in doing so, such a logic remains abstracted from reality and from life, where we almost always encounter situations that are complex, ambiguous and elusive, and therefore far removed from absolute precision; e.g. will it rain or not? Does he love or not love me? (this also applies to computers, based essentially on the approximation of real numbers and therefore unable to decide with a calculation whether *π* has infinite digits or not). But science, from Newton onwards, has always been presented as certain and without doubt about what has already derived from hard experimenta data. As a result, non-classical logics have always been considered not strictly scientific.

Returning to the principle of sufficient reason, its double negation introduces us to a logic that has no absolute certainties (or *A* or ¬*A*), but rather to an inductive, heuristic logic, which has not cut itself off from creativity: ¬¬*A* represents e.g. a future possibility. With words typical of the language of Leibniz: experimental science is not expressed by "necessary" propositions (that is, necessary to our mind, which is a direct creation of the Divine), as are mathematical truths (which, according to the tradition of that time, are the expression of Truth itself). Experimental science, on the other hand, is instead expressed by "contingent" truths, that is, they depend on imperfect external reality, as Leibniz says with precision: each of them is such that its contrary proposition does not imply contradiction (once again note the double negation, which cannot be substituted by the corresponding positive statement; because the latter would promote contingent truths to priori, i.e. necessary, truths). Today we express this concept by saying: every physical law is valid within certain limits (those of experimental errors); its contrary, entering the domain of values not yet established experimentally, merges the false with the possible and is therefore not absurd. In this sense it is contingent.

In particular, physics cannot claim to be based on perfect equalities and absolute identities, but rather on the search for the "indiscernible" (another double negation, when it is noted that "discerning" means distinguishing, declaring "unequal"), i.e. an approximate equality. That which is stated above eliminates at the root the preconception that traditional theoretical physics has maintained for centuries i.e. that there is only the principle of non-contradiction and therefore only classical logic. It is possible to reason through non-classical logic and this can also be done in Physics.

These aspects of mechanical theory, which belong to theoretical physics, may seem to involve philosophy. If we look for a principle of Leibniz’s that is only physical in nature and which is linked to these ideas, we find that general principle, which Stevino and Huygens had already enunciated and applied fruitfully to establish laws of Physics: the impossibility of perpetual motion. We note that the statement of this principle is still a double negation (since "perpetual" means "without end", as Stevino rightly pointed out). Therefore the statement of the impossibility of perpetual motion cannot be an a priori proposition, nor is it evident to reason, but is the product of an enormous number of common experiences, and is proved by a *reductio ad absurdum* ("If it were possible, then the whole world would be different from the way it is; for example, work would be done without expenditure of energy"). Therefore this principle is ~~perfectly~~ connected with the philosophical principle of sufficient reason, because both follow the same non-classical logic.

*2.4 Fundamentals of a dynamic of interaction*

The impossibility of perpetual motion unifies the principle of sufficient reason with the essentially heuristic method of subsequent Leibnizian physical theory. Precisely because perpetual motion is impossible, the causes of an observed motion have to be sought. Thus, in his physics the concept of cause is not metaphysical, but is derived from a real impossibility. This theoretical orientation, therefore, does not accept the concept of force-cause presented as metaphysical by Newton, who considered gravitational force as the only force in the world because it was the expression of God's intervention in all places. Moreover in *f = ma* the force is impossible to define without implying an arbitrary assumption: the static force is *e*qual to the dynamic force; for example, if the wire of a dynamometer that supports a body breaks, the force that is exerted on the free body is the same as that acting on the body when stationary. This is an assumption because we have no means of comparing the static force, defined through other static forces, with the corresponding dynamic force, which is defined with the dynamic magnitude, acceleration Instead, according to the development of Newton’s theory it is precisely the concept of force that should lead us to explain precisely the transition from the static to the dynamic of in the theory of Mechanics.

The concept of force is present also in Leibniz’s writings, but his is an abuse of language (which other authors will continue to repeat until the end of 1870), because he applied this word to the concept of kinetic energy, or other types of energy. Moreover in his theory the specific concept of force is useful only if it has an important specification: theoretical reversibility in the transition from force to effects: "*Causa aequat effectum*" (not as in Newtonian force-cause, which, wanting to represent God's intervention in the world, makes the transition from a metaphysical cause to a physical effect); from the effects it must be possible to completely reconstruct the cause.

Much less did Leibniz want to link his concept of force to the mathematics of differential analysis, knowing well the artificiality and idealized nature of the infinitesimal (he called them "beings of reason"), in contrast to the contingent-experimental character of physics. Instead he made extensive use of combinatorics, algebra, and the idea, still not formalized, of vector quantity; it was on this last idea that he based his polemic regarding the theory of impact of Descartes, who believed that the magnitude of the total momentum, considered as a scalar *mv*, was conserved, whereas Leibniz correctly pointed out that the actual conserved quantity is the total . considered as a vectorial quantity.

Moreover his contingentist conception, based on imprecision, does not ascribe primary importance to Newtonian forces which bring together in an idealizing fashion all the various influences of the external world on the body considered as uniform but rather to the interactions of bodies with one another ("everything is connected"); and in these relationships, the principle of action and reaction. Thus the scientific method of Leibniz focusses on bodies together with their interactions that are objectively given by their reciprocal collisions; in other words, its method is not analytical (analysis of a complete system by breaking it down into its elementary bodies and then reconstituting the system with those elements), but globalistic.

In this perspective (considering the system as a whole) it is natural that the principle of inertia also has a different version from that of Newton; it can indeed be enunciated, precisely according to the double negation of the principle of sufficient reason, through the doubly negated phrase: "The in-difference of bodies to rest or to motion". Which, at least according to Koyré, is the true theoretical content of this principle (and not Newton’s animistic "persevering").

This perspective, not anchored to absolute truths and essentially heuristic, is also devoid of all the other Newtonian certainties, or rather it is in contrast with them: for example. *space and time are not absolute*. Leibniz's controversy with Clarke (representing Newton’s position) regarding these concepts is famous. Space for Leibniz is not the "sensory [ear] of God", it does not exist a priori, it is a relative concept, the order of relations between things themselves and it exists only in so far as things exist. Likewise, time, which is simply an ordering between events, is also relative.

If there is no longer absolute space, the concept of space is relative to the observer and it then becomes natural to consider the composition of different movements as a basic theoretical fact. The composition of movements is clear to Leibniz when one observer is in motion with respect to another. We can recall the classic example of the boat in uniform motion, on whose deck the movements take place as on land, while from land their values are obtained by adding the speed of the boat to the speeds as measured on the boat. Furthermore, the vectorial nature of velocity is clear and therefore the principle of composition of velocities, the so-called parallelogram, is also clear.

*2.5 Theory of impact of bodies*

In particular, the field of phenomena that for Leibniz is most attractive and stimulating is that of the *phenomena of impact*. Impact is a terrestrial phenomenon that is extraneous to the abstractions of astronomy (science without experiments repeatable by the scientist). Moreover, the phenomenon of impact is not comprehensible within the Newtonian schema, both because the equation *f=ma* is not applicable (since in this case the forces are instantaneous, i.e. they represent singularities of differential equations), and because Newton had suggested a schematization that Leibniz did not share: according to Newton, God created the world through perfectly hard bodies, such that they are invariable in form even in the most violent collision. Note that hard bodies always maintain their shape, do not deform and therefore cannot rebound. This makes it impossible for them to conserve the kinetic energy they had before the impact. Leibniz rejects this type of ideal body; emphasizing rather the role of *elastic bodies*, which, by changing their shape on impact, conserve energy and therefore can rebound on impact.

Leibniz also stresses that "our mind seeks conservations" (Descartes also followed this idea, justifying it with the conservation of the movement impressed by God at the time of the creation of the world). In other words, the many theoretical possibilities of understanding reality induce our mind to look for something stable, invariant. The controversy with Descartes leads him to consider as the main invariant not the *vis* (understood by Descartes as *mv*) but rather the *vis viva*, *mv2*, that is, without the constant factor, today's kinetic energy, ½ *mv2*. This technical contribution is universally recognized: it is he who ascribed far greater importance to mechanical energy than Huygens, his physics teacher, had done.

Leibniz was therefore able to write the fundamental laws of the theory of collisions of elastic bodies as conservations of certain quantities, the invariants of motion. These laws had been anticipated by Huygens, but Leibniz was the first to give them prominence and generalized them as follows (leaving out the vector signs):

 

 

 

Although I put together all three equations for the sake of beauty and harmony, two of them would suffice, since, taking any two of these equations, we can derive the other.

Thus the first and second give the third as follows. From the first we have *v+v'=V'+V*; from the second we have *m(v-v')=M(V'-V)*; multiplying LHS’s and RHS’s of these and equating, *m(v+v')(v-v')=M(V+V')(V-V')*; which gives *m(vv-v'v')=M(VV-V'V')*, i.e. the third equation.

 Similarly the first and third give the second, since dividing LHS (RHS) of *m(vv-v'v') = M(VV-V'V')*, which is the third, by LHS (RHS) of the first, *v+v' = V'+V,* and equating, gives *m(vv-v'v')/(v+v') = M(VV-V'V')/(V'+V)* from which it follows that  *m(v-v') = M(V'-V)* i.e. the second equation.

 Finally, the second and third equation give the first, since the third, *m(vv-v'v') = M(VV-V'V'),* divided by the second, i.e. by *m(v-v') = M(V'-V)* gives *m(vv-v'v')/m(v+v') = M(VV-V'V')/M(V'+V)* which gives *v+v'=V'+V,* which is precisely the first equation.

I will add only one observation, namely that many distinguish between [perfectly] hard and soft bodies, and again distinguish the hard bodies between elastic and non-elastic hard bodies; and for each type of body they construct different laws. But it can be conceived that in nature bodies are hard-elastic [or rather more or less elastic] .... Well, Nature needs this Elasticity of bodies to obtain the execution of the great and beautiful laws that the its infinitely wise Author designed; among which those two laws of Nature, which I was the first to make known, are not of lesser importance; of which *the first is the law of conservation of energy...*; and *the second is the law of continuiy,* by virtue of which, among other effects, every change must take place in imperceptible steps and never by leaps. This means that nature does not admit non-elastic hard bodies [that is, ideal perfectly hard bodies]....

However, it must be admitted that although bodies must be elastic in nature, in the sense I have just explained, nevertheless elasticity often does not appear much in the masses or bodies we use, even when these masses are formed by elastic parts: they resemble a sack full of hard balls, which also give way to a modest impact, without restoring the shape of the sack, as do soft bodies or those bodies that yield without sufficiently recovering [their initial shape]. This is because their parts are not sufficiently united to transfer their changes to the whole. From this it follows that during the collision of these bodies a part of the energy is absorbed by the particles that make up the mass, without this energy being restored to the whole; and this must always happen when the compressed mass does not recover [its initial shape] perfectly. [This is true] even if it often happens that a mass proves to be more or less elastic depending on the different types of collision; it is also proved by water, which yields to modest pressure yet causes a cannonball to rebound.

When the parts of the bodies totally absorb the energy of impact (as when two pieces of loam or clay collide), or in part (as in the collision of two wooden balls, which are much less elastic than two balls of marble or tempered steel); when a certain amount of energy is absorbed by the parts, there is an equal loss of absolute energy and relative speed; that is, [this is the case] in the first and third equations; which are no longer true, since what remains after impact has become less than what existed before the collision due to that part of the energy that was diverted elsewhere. But this does not at all concern the amount of progress [read: vector momentum], or the second Equation. Moreover the movement of this total progress is maintained, it alone, even when the two bodies [being perfectly soft bodies] after impact accompany each other with the speed of their common centre of gravity, as do two balls of loam or clay. But for the semi-elastic bodies (such as two wooden balls) it also occurs that after impact the bodies move away from each other, even if with a decrease of the first Equation, owing to that energy that was absorbed by them on impact causing them to be deformed. And, based on experiments on the degree of elasticity of this wood, what should happen to two balls of this material in any type of collision or impact could be predicted. However, this total energy loss, or this defect of the third Equation, does not derogate at all from the inviolable truth of the law of conservation of the same energy in the world, since what is absorbed by the particles is by no means lost in the universe, although it is true for the total energy of the competing bodies.[[4]](#footnote-4)

Note the description of impact with practically modern concepts, and above all how the last sentence prefigures conservation of energy other than merely mechanical energy.

One novelty of this approach to mechanical theory is that the phenomenon of impact does not concern the trajectories of points in space, but only the invariants expressed as a function of the velocities of bodies; consequently the impact equations must be solved only for velocities. The previous conservations lead therefore simply to algebraic equations in unknown velocities. This agrees with Leibniz's rejection of infinitesimal analysis in the Foundations of Physics and substantiates that relationship with elementary mathematics that Leibniz had envisioned for theoretical physics.

 It should also be noted that for Leibniz the case of continuous forces can be obtained by passing to the limit of an infinite series of impulsive forces, that is to say, collisions. Therefore, all the physics treated by Newton can be subsumed under the formulation of mechanical theory based oncollisions.

 However, it should be noted that the mathematical laws of Leibniz’s dynamics lack the mathematical treatment of the collisions of plastic bodies (which Wallis had given separately). Leibniz did not therefore obtain all the laws necessary for a complete theory of collisions of bodies.

 Leibniz also studied terrestrial mechanical phenomena, of which those of heavy bodies are of particular importance. For ease of reasoning, Leibniz limited himself to the exemplary case (used by him many times) of a single falling body, which falls or ascends a certain distance. It is in this context that Leibniz perfected what he had already specified in his controversy with Descartes, a first definition of work: force times distance travelled, *f∙ds* (in his language: power times velocity), but he is not able to take into account the angle formed between the two quantities.

Leibniz also studied the interaction between bodies according to the principle of action and reaction and Torricelli's principle: "The centre of gravity of a system of bodies subject to the action of gravity cannot ascend." Today we are well aware that this principle is a restricted (although more precise than several others used since ancient times) form of the principle of virtual work, the mathematical statement of which was formulated for the first time by a follower of Leibniz (J. Bernoulli), just one year after his death (1617) With it the theory of Leibniz, which was always in continuous evolution, would have had a more specific and formal physical principle than that of the impossibility of perpetual motion. With his formula he would have been able to achieve a complete reformulation (as we will see below).

*2.6 A second method of founding mechanical theory: Lazare Carnot’s formulation*

*The sciences are like a beautiful river, whose course is easy to follow, when it has assumed a certain regularity; but if we want to go back to the source, we never find it, because it is everywhere; in a way it is spread over the entire surface of the earth: similarly, when we want to go back to the origins of the sciences we find nothing but obscurity, vague ideas, vicious circles; and we lose ourselves in primitive ideas.* (L. Carnot 1783)

In the following we will describe a formulation of Mechanics that was wrongly neglected until a few decades ago and would prove to be be very important: that of Lazare Nicolas Marguerite Carnot (father of Sadi Carnot, the founder of Thermodynamics), famous leader of the French revolution, head of the armed forces during the Victory (against the European monarchist armies, allied in 1793 to crush the newborn revolution), at the time well known in Europe as a geometer, analyst and theoretical physicist of Mechanics. Strangely it was then completely neglected, until in 1971, one of the major historians of physics, C.C. Gillispie, published an authoritative study which re-evaluated his formulation of mechanics and in general the whole of his scientific work. This revaluation, however, did not emphasize what follows below, namely that Carnot’s Mechanics constitutes a real alternative to that of Newton, being a concretization of Leibniz's project (L. Carnot declared himself a follower of Leibniz in his works of Geometry and Analysis).

Carnot published two books on mechanics.[[5]](#footnote-5) The first book considers machines in the most general terms possible, that is, according to the definition: "machine" is *everything that transmits movement*; and since according to the Carnotian concept space is full, then no mechanical phenomenon is outside such a theory of machines, which is therefore defined as "the science of the communication of movements".

Furthermore, Carnot clearly distinguished two different organizations of mechanical theory. At the end of the first edition and in the preface of the second he wrote some pages on the two ways of organizing the theory and on the "two methods of considering the principles of mechanics". The first method consists in considering the principles as metaphysical; in particular that of force, from which the whole theory is derived and is only verified at the end; the second method aims to organize a theory on the basis of simple and real ideas (remember Leibniz's letter to H. Fabri), and then considers all the principles of the theory in the same way as other physical and mathematical entities, as ideas derived from experience. We can add that the objective is, according to Leibniz’s teachings, to obtain the invariants of motion that our mind is seeking. Carnot specified that the theory can be developed as an alternative to that of Newton's, which starts from the static and then proceeds to the dynamic. Instead one can, as he himself does in the *Essai*, proceed from the dynamic and then consider the static of system equilibria as a special case.

Probably he had to submit the second edition of the book to the judgment of those who, like Laplace, considered deductive organization and Newton’s principles to be the pinnacle of theoretical physics. Carnot decided therefore, in his last work, to accept the challenge of organizing Mechanics also deductively from principles and starting from the static, but he did so in his own way.

* 1. The seven hypotheses

The first part of the *Essai sur les machines en général* is dedicated to *principles*, called "hypotheses", thus emphasizing the need for verification through experience; they are declared by the author to be "inductively derived from the best observed phenomena ".[[6]](#footnote-6) Here, Lazare Carnot has the merit of being the first to state the principles of mechanics that are exclusively experimental (that is, in a way that is today considered necessary, especially following the profound crisis in the foundations caused by the failure of the notion of an abstract ether and Einstein's critical analysis of the concept of simultaneity with respect to space and time measurements). Furthermore, it should be noted that he declared that he did not want to go back to explaining many fundamental concepts since it would mean engaging in metaphysical discussions. Matter, time, space, motion, movement therefore must be taken as primitive concepts.

Let us then consider Lazare Carnot’s seven hypotheses (p. 49).

First hypothesis: A body, once put in a state of rest, would not be able by itself to leave that state, and, once set in motion, would not be able by itself to change either its speed or its direction.

Second hypothesis: If different parts of any system of bodies in equilibrium are acted upon by new forces, which alone would be in mutual equilibrium, then the equilibrium of the system will not be disturbed.

Third hypothesis: When several forces, whether active and passive, are in equilibrium with each other, each of them is always equal and directly opposite to the resultant of all the others.

Fourth hypothesis: In a system of bodies the motions or motive forces, which destroy each other in each instant, can always be decomposed into equal and directly opposite pairs, relative to the straight line that joins the moving bodies to which they belong. In each of these bodies these forces can be seen as destroyed by the action of the other.

Fifth hypothesis: The action that two contiguous bodies exert on each other by impact, pressure or traction, does not depend at all on their absolute velocity, but only on their relative velocity. That [action] which two bodies communicate to each other by means of bodies interposed between them, is gradually transmitted from one to the other by means of these intermediate bodies: in this way the action always resolves itself into a series of immediate actions between two contiguous bodies.

Sixth hypothesis: The quantities of motion, or motive forces, which bodies impart to each other by means of wires or rods, are directed along such wires or rods; and those that bodies impart to each other by impact or pressure, are directed along the perpendicular to their common surface, coming out of the point of contact.

Seventh hypothesis: When two equal bodies that collide centrally are perfectly hard [read: plastic], they always move in company after the collision; that is, along the line of their reciprocal action which, according to the previous hypothesis, is always perpendicular to their common surface and through the point of contact. When bodies are perfectly elastic, they separate after the impact with a relative velocity equal, but in the opposite direction, to that at which they were travelling immediately before the impact. If the bodies are neither perfectly plastic nor perfectly elastic, they separate with a greater or lesser relative velocity, according to the degree of elasticity.

(In the statement of the last hypothesis we have substituted "plastic" for "hard", because, according to Gillispie, that is the meaning of this last word in L. Carnot’s theory).

2.8 The principle of inertia

We now turn to a detailed analysis of the hypotheses of Carnot’s mechanics. He himself, at the end of his commentary on the first hypothesis, wrote about the "principle known as the law of inertia". However, Carnot's version differs markedly from Newton’s.

One first difference between the two versions consists in the fact that Newton refers to all bodies at all times and in all places, while Carnot circumscribes the proposition to a restricted set of situations, in which it can be affirmed that a body is at rest or is in motion. These situations are indicated with a deliberately imprecise premise: "A body, once put at rest ...". It is thanks to this premise that Carnot's version avoids the problem implicit in Newton's statement, that is, being able to select a perfectly uniform and rectilinear motion, which would take place "as long as ..."; that is, it avoids the problem of having to decide when, on the path (possibly infinite) of the body, there exists a force which has an exact non-zero value. Therefore, in the principle of inertia, Carnot, correctly, does not require the verification of the absence of forces (*F = 0*) on the whole path, nor does he assert their presence.

Carnot is well aware of this. He argued that it is not possible to judge with certainty "if a motion is absolute [as does Newton since he is referring to absolute space and absolute time], or if there is a movement or a drag force" and that it took "a lot of effort to correct this error". Therefore the statement of the first hypothesis does not consider the general verification of the absence of forces nor does it claim to provide rules for the verification of the state of rest or motion. In general such verifications are impossible, and would be circular anyway with the definition of inertial reference system.

Therefore Carnot deliberately used the expression " A body, once put at rest ...", referring to the particular circumstances (experimental and / or theoretical) in which we know how to decide whether a body is at rest or in uniform rectilinear motion. This evaluation remains our judgment, of an empirical and occasional, rather than of a general nature. In fact, in the history of Mechanics, the principle of inertia has been applied only when, taking the earth, considered as stationary, as the initial reference system it was believed that it was possible to indicate approximately a state of uniform rectilinear motion or rest, with gradually more and more sophisticated corrections for more sophisticated situations. At this point we understand that the situation described by Newton's statement, the absence of forces, is an idealization of the real one, an idealization which he obtained both by passing to the limit F=0 exactly and by positing a reference frame as absolute.

A further problem, equivalent to the previous one (to establish if *F = 0* exactly), consists in Newton's claim to establish exactly when a body is in a state of rest, and to distinguish it from a state of motion; that is, decide whether *v = 0* exactly (not whether *v<*!). Carnot's wording "Once ..." also avoids this problem of deciding whether *v* = 0 exactly, which is impossible in reality.

Moreover, since the Carnot statement requires, in the premise, that we have already somehow succeeded in defining operationally the state of rest or uniform rectilinear motion, this motion, once identified, *can* be used, in that particular situation, in such a way as to operationally define the inertial reference system and the instrument for measuring time. Then indeed, starting from Carnot’s first hypothesis we can progressively construct the entire conceptual scheme of dynamics without any vicious circle. In Newton's Mechanics, on the other hand, the three concepts, posited at the idealized limit of experience, 'statement of the principle', 'reference system', 'clock' must be defined by referring one to the other, that is, forming a circular definition of such entities.

Carnot therefore suggested a version of the principle of inertia that fits very well with what has actually been and what is being done, while that of Newton (as also that taught today), which claims to establish universally valid properties, turns out to be a tautological proposition.

* 1. Inertia and animate beings

 Carnot's statement goes on to state that in the above conditions the body cannot “by itself” change its state of motion. The expression goes back to Aristotle and it is legitimate to wonder why Carnot repeats it. In reality Carnot wants to exclude definitively the theory of *impetus*, according to which a body, if it is in motion, has previously had impressed on it a propulsive force. If one pays close attention, this theory is still implied in Newton's proposition, when he uses the (moralistic) words "persevere", or "continue" (it is precisely their meaning of an effort performed by the body that makes the imprecise words "as long as", discussed above, seem natural).

Let us ask ourselves if Aristotle's words could have generated confusion. He attributed capacity for autonomous movement only to beings endowed with a soul and denied it to inanimate beings. Carnot repeats the aforementioned words, but substantially disagreeing with Aristotle, because for him the principle of inertia can be applied equally well to animate beings, as he writes expressly.

An animal is subject, like inanimate bodies, to the law of inertia; that is, the general system of the parts that compose it cannot on its own give itself any progressive movement in any sense. For example, if you place a horse on a horizontal plane [without friction, slippery], it will be impossible for the horse to impress the slightest movement on its center of gravity in any horizontal direction; nevertheless the horse has the capacity to advance each of its limbs in the direction it wants and this distinguishes it from inanimate bodies; but at the same time as it moves a paw to one side, another part of his body will recoil to the same degree, since in the internal system of this animal the principle of equality between action and reaction is as valid as in the inert matter; in such a way that it is only thanks to the friction of its legs against the ground on which it finds itself, that it manages to move forward, impressing on the earth itself a quantity of motion, imperceptible to us, equal and opposite to that which it acquires.[[7]](#footnote-7)

It is the first time in the history of physics that such considerations are lucidly developed. In Carnot they derive from his refusal to attribute the variations of motion to forces of a metaphysical kind, in this case animistic. In fact when he says "by itself" he is stating that a body has no autonomous force, so that the explanation for changes of motion, whatever it may be, must be sought outside that body. Carnot himself maintains it:

[...] Any body that changes its state of rest or uniform and rectilinear movement always does so as a result of the influence or action of some other body, to which it at the same time impresses a momentum equal to and directly opposite to that which it receives [...]. Every body therefore resists its change of state; and this resistance, which is called the force of inertia, is always the same and directly opposite to the quantity of motion it receives.

This formulation therefore avoids the need in theoretical physics to discuss causes which are not acting on material bodies (metaphysical force-causes) and opens the way to the statement of the principle of action and reaction, which concerns precisely the experimentally verifiable interaction between material bodies.

* 1. *Comparison of the versions of the principle of inertia in L. Carmot and Newton*

Carnot therefore restricts his reasoning/formulation to what is experimentally observable, considering motion only in a finite space, that which can be controlled with measurements which take into account experimental error. Understood in this way, the principle of inertia indicates a method of investigation: if in the observed body there is a change of motion, other bodies should be looked at, in order to find those acting on it.

We also note that, like all methodological principles, it is precisely a doubly negated sentence: "... it cannot change ...." where the word "change" is to be considered negative because it requires an explanation on the part of the physicist regarding the state of rest.

This double negation is not merely a figure of speech, without importance for physics, because, unlike figures of speech, it does not possess an affirmative equivalent. Indeed if the two negations are removed we would have an abstract word, with no physical meaning. In fact we obtain precisely the affirmative sentence of Newton's version that corresponds to L. Carnot’s doubly negated sentence: when from this the two negations are removed one obtains precisely the "perseveres" or the "continues" of Newton's version! Here we see the logical distance between the two types of organization that the two physicists used for their theories of mechanics.

Now let us consider for a moment the theoretical importance of a doubly negated proposition, as it is used here by L. Carnot. It is the consequence of a way of conceiving the whole theory to which it belongs. In fact, if a proposition of non-classical logic is translated (removing the two negatives) into the corresponding affirmative proposition of classical logic, we obtain unverifiable content; therefore in a physical theory it can be valid as long as the theory allows idealization. In an Aristotelian organization (of a theory) some principles may be very abstract compared to experimental facts so that some consequent deductions may also be non-operative with respect to reality, provided they are compatible with experimental data (for example, in Newtonian mechanics, absolute motion as derived from the fundamental concepts of space and time. This gives a large degree of freedom to the theoretical physicist.

Note, however, that there are chemical, physical and mathematical theories that have a different organization; they been formulated without abstract principles, but are each based on a problem (PO). First, classical chemistry (which wants to solve the problem of how many elements of matter there are) and, originally, thermodynamics (which wants to solve the problem of the efficiency of conversions of heat into work); in addition some mathematical theories, such as Lobachevsky's non-Euclidean geometry (which wants to solve the problem of the number of parallels).

The fundamental problem of such theoretical organization is expressed with a proposition for which the law *¬¬AA* does not apply. In this type of organization *¬¬A* cannot be idealized in *A*, because it would mean eliminating the problem. E.g. if it is stated that heat equals work, then there is no longer a problem; whereas the problem is expressed by the corresponding doubly negated proposition: "It is not true that heat is not work"; and therefore it is necessary to find how it is converted.

 Thus the formal characteristic of the PO appears to be the presence of doubly negated propositions, beginning with the one that expresses the problem of the theory; they cannot be replaced by equivalent positive statements because there are no operational means to prove them. Then it is clear that, while in a deductive theory we deduce from the vertex of the first principles a pyramid of theorems, which succeed one another to infinity without ever closing their series, in a PO theory the reasoning is essentially cyclic, in the sense that, given the problem posed ¬¬*A* and the direction indicated to solve it, the theory reveals as many contents as possible of that proposition *A* which at the beginning could not be made true with the simple intellectualistic operation of suppressing the two negations.

Returning to the principle of inertia, note that in general in Lazare Carnot’s version there is no:

* + 1. force-cause;
		2. infinite motion in time (*t* varies between + ∞and - ∞);
		3. perfectly rectilinear and uniform motion;
		4. motion that takes place in a perfectly empty space;
		5. space and time as absolutes.

These concepts, on the other hand, are present in Newton's version, which seems to postulate an infinite inertial rectilinear motion, which in fact would be a perpetual motion. Lazare Carnot’s thinking is far from stating this type of infinite motion because in the commentary to the seven hypotheses he notes that there are always limitations due to other bodies, as well as inevitable energy losses. And therefore for him the fundamental principle holds (referred to in the “Preface” of the book *Principles*): "Perpetual motion is impossible".

We then conclude that Carnot omitted undue idealizations and, in particular, the concepts obtained by passing to the ideal limit concept of the experimental properties (for example the perfectly hard body, there exist bodies that approximate to it, but is not realized by any real body). Already in paragraph 1.6 it was said that this principle of the impossibility of perpetual motion can be translated operationally by stating that there is no finite set of physical operations (i.e. a machine) that allows us to achieve an endless movement. This principle corresponds therefore to the physical translation of the basic principle of constructivism, that is, the restriction of our ability to conceive something to the case where we can calculate it with precise operations.

Having established this connection, it is natural that Carnot's version of the principle of inertia can be expressed through constructive mathematics. The language of Carnot ("A body, once put at rest ...") seems to be taken precisely from constructive mathematics which would express itself in this way in the case of an undecidable problem in general, but solved in particular cases. For example, in constructive mathematics it is undecidable for which point a function *f(x)* has an endpoint, but when the function is the quadratic expression of two variables, then we can establish the point for which *f(x)* has a null derivative by examining whether the discriminant is 0 (this problem is decidable in constructive mathematics because there is an algorithm to calculate the square root of the discriminant). In the same way, Lazare Carnot's statement means that we should restrict ourselves to those particular cases in which we are able, in some opportune manner, to actually determine whether *v=0* o *v=cost*.

* 1. From a static to a dynamic force; interactions

The second hypothesis concerns the forces applied to the systems of bodies in equilibrium with each other. Here we are dealing with statics, where the forces are well defined and therefore are also accepted by Carnot. The idea of the principle in question translates the fundamental idea of Leibniz and D'Alembert (superimposing on the given system a system of forces with null resultant). In modern terms, we should add that the bodies are rigid, since otherwise, in addition to the indicated effects of translation and rotation, the effects of deformation or even breaking would also have to be taken into account . But this omission by Carnot has no influence on his theory, since in fact he does not then apply his hypothesis to deformable bodies.

The third hypothesis concerns the gradual transition from a static (equilibrium) to a dynamic situation (subject to forces). Carnot was able to establish this transition to the extent in which he also considers passive forces, that is forces opposed to the velocity of the body. From the Newtonian point of view of the metaphysical force-cause they are a contradiction in terms. Lazare Carmot was the first to define them correctly.

In such a system of forces in equilibrium each of them is equal, but in the opposite direction, to the resultant of all the others; the latter is defined by Carnot as that force which, evaluated in any given direction, is equal to the sum of all the others evaluated in the same direction. Note that the (still undefined) vectorial character of the forces is well highlighted here, although in the simplified form of specifying the mathematical representation of the vectors by means of their projections along a specified direction. Newton, on the other hand, is unclear on this point; e.g. he wants to deduce the well-known rule of the parallelogram of forces as a Corollary of his three principles, that is, improperly, from physical theory.

With the fourth hypothesis we pass to dynamics in the strict sense, since momenta are considered as well as the "motive powers", which however in the usual definition are each the product of mass by acceleration and therefore differs from the momentum by virtue of an operation of differentiation of the velocity. Here Carnot wants to consider on the same par the impact of bodies and continuous motion in a synthetic way.

The contribution of this hypothesis constitutes a broadening of the principle of inertia in that it expresses the conservation of the overall momentum for an (isolated) system of interacting bodies, rather than of one body.

From this principle derives the conception that any change in the quantity of motion derives from an impact (or a continuous series of infinitesimal impacts) with other bodies and that, provided that one extends the system to include the acting bodies, it is possible to abandon the Newtonian schematizaion of a system as a single body subject to unspecified forces coming from its environment. In reality this hypothesis is the traditional principle of action and reaction, as also stated in Carnot’s last sentence:

It therefore seems certain that in general, whenever one body impresses movement on another, it in turn receives the same amount in the opposite direction; [this occurs] at least as long as the collision is direct and is exercised between two bodies only. But the analogy suggest that the same thing must happen regardless of the number of bodies and whatever the directions of their movements. All the phenomena of nature confirm this important law, which is usually expressed by saying that the reaction is always equal and opposite to the action. (L. Carnot 1803, p. 60)

Note that Carnot speaks of "analogy" because in the statement of the fourth hypothesis he refers, for the first time in the history of theoretical physics, to a partition into pairs of quantities of motion (called, generically, "forces") which the bodies of a system exchange with each other. This partition is not an experimental operation; it is superimposed on reality, but it is compatible with experiments and everything appears as if it were actually being done. It is very important to express it as a specific principle (Some call it the fourth principle of Newton's dynamics).

Note that here also Carnot idealizes reality. This is not, however, suggested by an operation whereby an unattainable situation is approached as a limit, as is the case with Newton with the absence of friction or with the absolute void, but rather by mathematical operations which at present pertain to vector algebra.

2.12 Analysis of the principles of mechanics through mathematical logic

The previous critical comments on the principles of mechanics can be summarized and illuminated in modern terms by means of an simple formalism of mathematical logic. It will show that the principles of dynamics cannot have many versions and that the physicists mentioned have already explored them all, even if they were reasoning only intuitively.

 It has been pointed out (Nagel) that all the laws of physics can be expressed by a statement preceded by two quantifiers: "Every" (**), and "There exist one" (**). A possible form of the current statement of the principle of inertia might be the following: "For every body *x* there exists a *y* (i.e. a system, composed of an inertial system, a closed system and a clock) such that if the body *x* is in *y*, then it is either at rest or is in rectilinear and uniform motion" (that *y* must include the three elements indicated is confirmed by every analysis of the principle of inertia).

 The statement can be formalized as follows

*A= x y : P(x,y)* (2.1)

*A* the statement of the principle of inertia;

*x* a body;

*y* a compound system, formed by a closed system, which is inertial and in which a clock has been defined

*P(x,y)* a predicate regarding *x* and *y*: "if *x* is in *y*, then it is either at rest or in rectilinear and uniform motion ".

All this, however, belongs to classical logic, which does not concern itself with establishing whether the functions and operations indicated by the statement can actually be calculated (let alone declared operational). In this context the principle of inertia announces that *y exists* but does not explain how to find it.

At the beginning of the century a technique was suggested (by the logician T. Skolem) to formally translate the existential quantifier on the variable *y* of the predicate into a constructive function. It is a question of finding in the set *C* of functions of constructive mathematics a function *(x)* such that it actually calculates for each *x* the corresponding *y*, and then replacing the existential quantifier ** with it.

  (2.2)

Now (2.2) says that for every *x* the predicate *P* holds for *x* and for *(x)*: "Every body *x* is at rest or in uniform rectilinear motion in a system *y* given by *(x)*", where *(x)* is a function that belongs to constructive mathematics and actually associates a compound system *y* to every body x

If we consider constructivism as physical operationalism, then to translate (2.1) into (2.2) we must find, for each body *x*, an effective procedure *(x)* that, from the knowledge of body *x* alone, provides an inertial system, an isolated system and a clock. Two centuries of fruitless research convincingly demonstrate that it is not possible to derive so many entities from *x* operationally.

However the aim of relativizing (2.1) operationally can be achieved by forcing the predicate, that is through one of these choices:

1) Replace the quantifier ** in (2.1) by a constant value *y0*. This proposition follows:

 (2.3)

that is: "Every body *x* is at rest or in uniform rectilinear motion if it is placed in *y0*"; that is, if its motion is measured with respect to a precise inertial system, in a precise closed system and equipped with a precise clock.

(2.3) corresponds to setting the clock and the reference system in two very different ways:

*a*) in the manner followed by physicists since the time of Galilei, that is, with an empirical clock, with a closed system experimentally verified and with the terrestrial reference frame, but changing them from time to time according to subsequent improvements for specific cases;

*b*) in the idealistic way suggested by Newton, that is idealizing the previous empirical method with the transition to the limit, a limit that transcends experience itself: introducing the ideal concepts of space and absolute time, which stipulate the clock once and for all (as an actualization of absolute time) and the inertial system (as an actualization of absolute space), also implicitly suggesting that we are always capable, in principle, of verifying whether our system is isolated or not, or whether *F = 0* or not.

2) Accept the fact that we generally do not know the function *(x)*, but sidestep the problem of the existential quantifier by saying: "In particular circumstances that experimental physics can specify: *x A(x,y)*", without giving further explanation as to what experimental physicists should do. This is what Mach does (in his famous criticism of Newton's principles that paved the way for Einstein).

3) Deny the physical importance of these problems, classifying them as metaphysical and state only what we are able to derive from the experimental observations of body *x*: the impossibility that the single body under observation changes its state of motion on its own when it is at rest or in rectilinear and uniform motion; ultimately, no quantifiers ** and**. This is what L. Carnot does, expressing everything through finite quantities only and experimentally.

We can also apply this technique of mathematical logic to the third principle of dynamics: "For every action there is an equal and opposite reaction". To express this principle we will not take into account the problem of determining in which inertial system it holds, since this problem will have been solved already with the first principle.

To translate it into a formula, we give the imprecise words "action" and "reaction" the only possible meaning, that of "forces". Then we can specify it this way: "For every force  measured on a body *x*, there exists a force  resultant of the forces acting on a system of

bodies  such that:

, with  equal and opposite to . Formulated in mathematical logic:

 (2.4)

 The principle says that if a physicist measures a force on a certain body x, he is able to derive the "reactions" on the other bodies *xi*, which acted on the first. We know from the previous case of the principle of inertia that the crucial point of the formula (2.4) is the existential quantifier. In fact, the quantifier index is already a problem: how many bodies? Their number could be higher than any preset limit. For example, before the discovery of Neptune (1846) suggested by Leverrier's calculations, the actions of the bodies of the solar system seemed complete, that is, they seemed to come from a precise number of planets; later, however, due to unforeseen deviations of the known planets from the calculated trajectories, we had to add other planets. Then, since we cannot generally know the exact number of bodies involved in (2.4), we must conclude that the statement is metaphysical.

If then somehow we come to know that the bodies are *n*, with *n>* 2, what is the procedure required to find the reaction? To answer, it is necessary to know the vector calculation of forces. But, according to Newton, the parallelogram rule, instead of preceding principle III, is a consequence of it (Corollaries I and II). The Newtonian statement of principle III is therefore once again metaphysical, unable to tell us how to compose forces in order to discover the reaction for the case *n*> 2.

And if in any case the parallelogram rule is valid between the given *n* vectors, on the Cartesian axes it gives three mathematical relations, which are added to the three mathematical relations ; in total we have on each axis two relations to solve a problem which, however, depends on *n* degrees of freedom (the bodies). There remain therefore ∞*n-2* possible solutions. Clearly we will never have an operative procedure to determine in general a single solution.

 Let us therefore try to replace the existential quantifier with a constructive function:

  (2.5)

where the second and last term,  , represent actual procedures for finding the various reactions (in the second case  and thus be able to calculate the equality of the second proposition of (2.5)). Given everything that has been said above this function  clearly cannot be found.

Let us therefore try to solve the problem, as in the case of inertia, by establishing the values on which it is existentially quantified. This operation is equivalent to establishing the number of bodies and the procedure for finding them (in particular, assuming a priori the dependence of the force on distance *f(r)*). This is what is done experimentally.

If, on the other hand, we want to transcend experience and move to an ideally universal situation, then we can fix the number of bodies, certainly not an infinite number, but the simplest situation, two alone, hoping that then the procedure for the interaction of *n* bodies can be repeated in pairs. So in the case of two bodies, this addition gives us, assuming we know the vector calculus, an indication: the reaction sought is along the line of action, in the opposite direction to the action. All this constitutes the only, manifestly very small, indication that Newton's statement offers regarding the complexity of the problem.

But even in this way we do not know at what distance the body we are looking for is, unless we already know the force function *F(r)=1/r2* rather than, for example, *1/r5*. Once again Newton's version of the principle solves the problem by assuming a particular force, gravity, as universal, that is, imagining that this is the only possible one in nature.

Now it is clear that Newton wanted to infer from a very particular case of interaction (*n* = 2 with a known force), a statement about a situation generalized to any force and to an unlimited number of bodies. That is, Newton has removed almost all the propositions that should constitute and limit (2.4), leaving only the last one. In it, , left alone and detached from any body, can only be understood metaphysically: it is considered a cause. Therefore also the first force  is detached from the body it is acting on and becomes a force in a metaphysical sense; as a result the proposition is completely abstracted from bodies. Moreover, it is further generalized: the concept of force is replaced by the concept of "action ".

For his part, Mach says nothing of interest in this regard, although he uses this principle as the defining basis for the initial concept of mass.

Lazare Carnot, on the other hand, not knowing how to provide generally a constructive function ** that finds the reactions, correctly begins his third hypothesis by saying: "Once ..." we are able to establish experimentally that the equality between a set of forces ...; that is, Carnot restricts himself to those cases in which we can establish experimentally and conventionally that equality that Newton assumes established a priori. Moreover Carnot does not abstract from bodies, since he specifies that "force" is essentially the variation of the momentum, which is linked to the body by mass. Furthermore he limits himself to a finite region of space, and implicitly limits the number of bodies to what can be found there given the bodies have finite dimensions; that is, he limits it to a finite number (even if unspecified). Finally the vector calculation necessary to compose forces is correctly expressed by Carnot. (For this calculation he adds a further hypothesis to clarify his procedure: his fourth hypothesis suggests that between the forces in equilibrium everything occurs as if each pair of bodies is independent of all the others). Given all this he can state his third hypothesis as a correspondence between Physics and Mathematics, that is, between physical equilibrium of forces and vectorial calculation of opposing forces. Having done this he has exhausted both its mathematical and physical content, since the function ** summarizing a general method to find operationally the reactions of the single bodies, cannot be specified further.

It must be concluded that here once again Carnot has clarified the actual content of the third Newtonian principle and that, from the point of view of mathematical logic, no other type of solution to the problem of how to formalize the principles of dynamics can be expected.

2.13 Hypotheses on Impact

The content of the fifth hypothesis is that in a collision, or pressure or traction between two bodies, everything depends on the relative velocities of the bodies, not on the absolute velocities. This type of hypothesis concerns only interactions. So we have before us the first formulation of an entire physical theory based on interactions and not on the schema of causes and effects.

The second part of this hypothesis takes into consideration the interaction between two bodies (mediated by intermediate bodies) and states that it is always transmitted through bodies that are in direct contact with each another. Note that, according to the Cartesian conception of space, Lazare Carnot identifies it with matter. For him, therefore, there is no void, rather there are only more or less light bodies, all in contact with each other, direct or indirect. From this follows the complete generality of Carnot's position with respect to interactions between bodies, valid for any interaction (known at the time) in space. The Newtonian gravitational force might also be explained in terms of exchange of momentum between contiguous bodies.

In modern language, the content of the sixth hypothesis specifies the geometry of interactions. According to Carnot this can be derived either inductively from experience, or through reasoning alone. For example, in the case of impact, he argues that, with respect to the aforementioned perpendicular, there is no reason why the direction of the exchanged momentum is inclined in one direction rather than the other (this is an example of the use in Science of the principle of sufficient reason).

The seventh hypothesis expresses the behaviour of bodies in collisions. In this principle we note, in effect, a repetition concerning the direction in which the bodies interact through contact. Indeed the part of the seventh principle that deals with this problem could easily be omitted, since it was already contained in the sixth principle. But for Carnot repetitions merely detract from elegance.

This hypothesis is crucial, since the schematization of perfectly hard bodies hindered, for a century and a half after Newton, an understanding of conservation laws, in particular the conservation of energy. The following table 2.1 illustrates the ancient schematizations (of Wallis and Newton: on the left) and modern ones (on the right) of the bodies with respect to impact (see summary table below).

Carnot’s hypothesis also includes elastic bodies, for which (if the masses are equal) the relative velocity after the impact is the same, but with the opposite sign of the one before impact, and partially elastic bodies, for which the relative velocity after impact has a lower absolute velocity value than it had before impact. This consideration is only qualitative, but it is specified by Carnot subsequently through an index of elasticity of bodies (as is done in our times). Consequently Carnot’s schematization for colliding bodies turns out to coincide with the modern one: it considers perfectly plastic bodies and ~~the~~ perfectly elastic bodies, where bodies between those extremes can be specified by an index (the modern one varies between 0 and 1). This shows that he had gone completely beyond the Newtonian conception of perfectly hard bodies.

**Table 2.1**

*THE TWO CLASSIFICATIONS OF BODIES WITH RESPECT TO IMPACT: NEWTONIAN AND MODERN*

 HARD

Plastic - - - - - - - - - - - - - - - - - Elastic

 Wallis and Newton Huygens, Leibniz and L. Carnot

|  |  |  |
| --- | --- | --- |
|  | *fixed form*  | *variable form* |
| *Without rebound* | **[perfectly hard]** | **plastic**  |
| *With rebound* | ------------ | **elastic** |

It should also be noted that by imagining a series of impacts or impulses any continuous force can (in principle) be obtained, so that the Newtonian case of continuous force is brought back to the particular case of impact. The fact that a process requiring a limit is needed for this step shows clearly that by doing so we make the transition from an operational and experimental theory like that of impact to a theory involving a limit, which can therefore lead to non-operational idealizations, precisely as occurs in Newton.

2.14 Critical considerations on the seven hypotheses

We observe that Carnot’s seven hypotheses contain both the first principle (first hypothesis) and the third principle (fourth hypothesis) of Newtonian mechanics. But the second principle, *F = ma*, is missing. In fact, this formula is never used by Carnot (in any his works on Mechanics), since for him, as for D'Alembert, it represents a metaphysical cause; for both of them the force of the dynamics (that according to them precedes the static), *F*, is not equal to the static force and can therefore be defined by only identifying it with *ma*. In fact, there is no way to define the dynamic force of a body that had been suspended from a gauge for measuring its static force and then separated from it, since it is not known whether the dynamic force exercised on the body after the separation is equal to the previous static force.

Let us now see how Carnot’s mechanics develops mathematically. The concept of mass has no operational definition in Carnot's works. In the *Essai* mass is not defined, while in the final work we read that the mass of a body is (repeating the definition of Descartes) "the actual space" that it occupies, or (repeating Newton's definition ) its "real quantity of matter". Jammer criticizes Carnot for failing to define it,[[8]](#footnote-8) but at that time the defect was common to all, particularly to the Cartesian school. Note that it was only a hundred years later that an operational definition of mass was formulated, when Mach put forward a definition, which nevertheless received heavy criticism, as Jammer himself notes; it starts from the equality *m1a1 = m2a*2 of the interaction of two bodies, yet in order actually to give the definition of a single, preliminary concept it assumes the third principle. This is not surprising, as so far it actually has not been possible to define the mass without the second or third principle. We therefore conclude that Carnot certainly did not solve the problem: he, like everyone else, fell back on an intuitive definition. He does, however, have the merit of having abstained from false solutions, such as those of current authors.

We now note the order of the seven hypotheses. Carnot, like Newton, begins with the one that signals the divergence of modern mechanics from the medieval theory of impetus. In Carnot's statement, the expression "itself" may recall Aristotle, but in fact that expression ("if not by the work of other bodies") emphasizes that Carnot’s physics is the physics of interaction, as was Leibniz’s and D'Alembert’s theories.

The principle of inertia states that being at rest and uniform rectilinear motion are equivalent. But what does equivalent mean? Newton's statement treats the two cases as if they were the same ("... at rest or in motion ...”). Carnot's, on the other hand, is more cautious and separates out two parallel but distinct statements. It does not therefore consider the transition from static to dynamic to be obvious or easy. After the first hypothesis, the other hypotheses articulate this equivalence in gradual steps. While the second hypothesis still concerns static situations, the third and fourth finally include the dynamic We conclude that Carnot's hypotheses, after the first one, represent a precise strategy for the transition from statics to dynamics.

We also note that all the aforementioned hypotheses are constructive because they are essentially experimental, except the fourth one which is considered by Carnot to be a mathematical convention.

*2.15 The development of mechanical theory: the index of elasticity and the conservation of kinetic energy*

 From the seven hypotheses of Carnot it is possible to develop mechanical theory in two ways (as specified by the author at the end of the *Essai* and in the preface of the *Principes*): as a theory based on the artificial Newtonian concept of force (that is, reformulating the Newtonian theory withour abstractions and undue idealizations) or as "the science of the communication of movements" (which is the path chosen by Carnot himself), which studies the ways in which movement is transmitted from one body to another through mutual interactions.

To follow the first possibility we know that it would be necessary to extend the static definition of force to dynamics (as does Newton), to postulate as an axiom *F = ma* and then proceed using differential calculus. However, Carnot believes that mechanics, understood in this way, has two fundamental defects: the "metaphysical and obscure" concept of force-cause and the use of infinitesimal analysis, at the time openly metaphysical. (A book by Lazare became famous in Europe for its radical critique of infinitesimal metaphysics; he reduced this type of calculation to a mere operational technique). Like Leibniz, Carnot also did not want to introduce infinitesimal calculus into the foundations of Physics.

 Let us now see how Carnot develops mechanics in the second manner indicated by him (the "empirical" manner of his first work, *Essai*), that is, as "the science of the communication of movements". He starts from an equation (the “first fundamental equation”) valid for plastic bodies (although called "hard" by him), whose derivation is clear~~:~~ the principle of action and reaction between bodies (of an isolated system) results in:

 
where  represents the mass of the i-th body,  the velocity it loses in the interaction i.e. the difference between the velocity  before the interaction and the velocity  after. Therefore the equation is equivalent to . Now for completely plastic bodies the final velocity  is the same for every body, hence it can be written formally as:

 

 Carnot is able to apply the above equality to all types of bodies, introducing the elasticity index *n*. Reasoning by analogy he says that on impact the products change by a factor that ranges from 1 (for completely plastic bodies) to 2 (for elastic bodies). Indeed, given that , the preceding equation can be written as:

 

 Now dividing  by *n* in accordance with the above reasoning, we have a new equation:

 

or

 

 But since :

 

 Then if *n*=1 we have the previous equation, valid for plastic bodies. Instead for *n*=2 we have the case of elastic bodies. In fact,

  ,

which, by virtue of the trigonometric relation , reduces to the well-known conservation of kinetic energy, characteristic of elastic bodies:

 

*2.16 Geometric motions. The other conservations*

Then to replace differential equations a mathematical calculation tool is introduced; it is based on the concept of "geometric motion", whose clearest definition is given in the *Essai*: "a motion assigned to a system of bodies is geometric if it is such that the opposite movement is also possible". The introduction of this concept allows Carnot to establish the second fundamental equation, to be applied to a system of interacting bodies (by impact and by contact mediated by wires or rods). This is the crux of his theory:

  (2.6)

where  is the velocity of any geometric motion, attributable to the system. In other words, the velocity  is an indeterminate magnitude and any specification of it gives rise, in the relation (2.6), to an equation that is applicable to the system.

 For example, we assign to  the same value *u* = const (we can do this, since a uniform translation of an entire system of bodies is certainly a geometric motion); we have:

 

From this, because is arbitrary, it follows that:

 .

This is precisely the conservation of the total momentum of the system.

 Then we can assign another geometric motion, consisting in rotating the whole system around a fixed axis with angular velocity . In this case we have:

 

By the property of the mixed product, we have:

 

By the arbitrariness of  we have:

  

Finally, since , we have:

 .

This last equation expresses the conservation of the moment of momentum.

We could still attribute to *ui* other values, but this would be pointless and lead to some equations already contained in the previous ones .....

These calculations by Lazare Carnot show the extent to which the line of development and the methods of his Mechanics are different from those of the Mechanics of Newton.

Lazare Carnot was proud to have introduced the new concept of geometric motion: its general purpose was to build "a new Science, intermediate between Geometry and Mechanics". Let us ask ourselves if the project declared by Carnot was realistic.

We have seen that by introducing particular geometric motions, he derives the fundamental equations of motion from equation (2.6). In particular, by assigning a uniform translational motion to the system, he derives the conservation of momentum. In doing so, he explicitly observes that the equation expressing this conservation is independent of the particular uniform translational motion assigned, i.e. the result is the same for all motions of that type. Similarly, the result of the introduction of a rotational geometric motion in equation (2.1) does not depend on the particular geometric motion used:

We note that Hermann Weyl gave this definition of symmetry: "A thing is symmetrical if it can be subjected to a certain operation and it remains exactly the same as before". In the light of this definition it can be said that under the transformation given by a geometric motion (as long as it is of the same type; for example, translational), the result of the second fundamental equation being the same manifests a symmetry. In other words, geometric motions lead to symmetries because we can easily verify that they form a group of transfomations: they are associative, commutative, and always possess an inverse (in that they are by definition reversible). We can therefore say that the subgroup of uniform translational motions leads, with the Carnot technique, to the conservation of momentum; while the subgroup of uniform rotatory motions leads to the conservation of moment of momentum. In addition, we know then that conservation of energy is equivalent to invariance with respect to temporal translations; we will see that he derives it in another way, but still from his fundamental equations. We conclude that Carnot essentially founded his Mechanics on the invariants of motion provided by the application of the geometric group of symmetries. Ultimately, Lazare Carnot was the first to introduce the theory of symmetries in theoretical physics, using simple mathematics without differential operations. No surprise if this discovery was for a long time ignored; in Mathematics the symmetries were introduced with a stroke of genius by Galois 50 years later, and that its result remained unknown for the following twenty years.

* 1. Carnot’s fundamental equation and the principle of virtual work

Let us now compare the geometric motions, defined as invertible motions, with the virtual movements of today's mechanics. We will verify the validity of the opinion of Dugas and Gillispie, who consider the former an anticipation of the latter. First of all let us remember that a displacement is possible if it is compatible with the constraints considered fixed, virtual if it is compatible with the constraints even if in motion. Now, in order to compare the two different concepts, we need to translate geometric motions into displacements with *udt**s*, or the virtual displacements into motions with *s/dt**u*, (once again we find a limit operation that separates the two formulations of Mechanics). We will only consider the case of time-independent constraints.

Let us then begin by translating a possible displacements into a motion: it is sufficient to derive it, which is essentially always possible. The result is a motion that is possible but not always reversible (for example the motion of a sliding ring on a rotating bar). Therefore, to obtain a geometric motion, we must add the hypothesis of invertibility.

**Table 2.2**

*THE TRANSITION FROM GEOMETRIC MOTIONS TO POSSIBLE DISPLACEMENTS AND VICE VERSA*

**Geometric**

**displacement**

*Integration with fixed constraints*

*Invertible, virtual displacement*

**Possible displacement**

**Geometric motion**

 **Motion**

*Derivability*

*Invertibility*

Conversely, we pass from a geometric motion, which is invertible, to a geometric displacement. We need to integrate the geometric motion and thus obtain a geometric displacement, which by definition of the first will result in a virtual and invertible displacement. This last operation is essentially always feasible, if the hypothesis of fixed constraints is valid; otherwise the equations of the motion of the constraints will be considered, which can in general be complicated and difficult to integrate. At this point we can conclude *that for time-independent constraints a geometric displacement is equivalent to an invertible virtual displacement, but inversely a possible displacement only if it is invertible gives, by derivation, a geometric motion*.

With this partial equivalence, we wish to demonstrate, in the manner of the scientists of the 1700s (for whom , since all the functions were considered continuous curves), the derivation of the Carnot equation (2.1) from the principle of virtual work for a system of particles. Let us recall the textbook definition of this principle: *A system of particles is in equilibrium if, and only if, the total virtual work of active forces is zero; that is, if, and only if*

 ** (2.7)

But let us recall that for invertible motions the work of the constraints is null

(otherwise there would be free work):

 ** (2.8)

Which expresses mathematically the impossibility of perpetual motion, hence:

 ** **

 We note that, from the point of view of constructive mathematics, the principle of virtual work has a formula that poses no problems, because it is a simple mathematical formula and its equality is to be understood as approximate equality, as it is for all experimental laws; there might be problems with the constraint equations, to the extent that a solution of them is sought without approximations and they may have double solutions, which however seems extraneous to the usual applications.

Let us go back to (2.8). Confusing, as was done in the 1700s, ** with ** (which is correct if the constraints are fixed and the forces are continuous), we have :

  (2.9)

where the last step applies if the virtual displacement, having become the motion , is invertible.

 Now let us proceed with the second fundamental equation of Lazare Carnot. If we consider the following relations:

 

(where  is the geometric displacement if the *i*-th point and  has to be specified),

we obtain from (2.9)

  i.e. 

Switching to infinitesimals we have:

  (2.10)

This equation and (2.7) differ from each other because:

1) in (2.10) there is a geometric motion which gives a and not a virtual displacement 

2)  is the resultant force acting on the *i*-th particle, not its active component .

However, note that if a geometric displacement equals an invertible virtual displacement, the virtual work of the constraining force components is null. Furthermore, when constraints are present, the resulting force acting on the *i*-th particle is given by , where the second represents the constraining force acting on the *i*-th particle. Moreover, in applying the principle of virtual work, it is assumed that if every displacement is invertible (precisely the condition that derives from being geometric motion) then the virtual work of the constraining forces is null

 

 Thus the equation (2.10), removing the latter from it, assumes the form of the principle of virtual works (2.7). The agreement is surprising, for we started from concepts that are quite different from the usual ones.

Carnot himself, in the preface to the *Principes*, explains this novelty:

My theory could not be founded precisely on the principle of virtual velocities ..., which is not applicable, without modification, to the impact of bodies. I therefore start from a principle which is different but very analogous, or rather, which is this same principle of virtual velocities, but suitably extended. (L. Carnot 1803, p. x)

* 1. The historical importance of the Lazare Carnot formulation

Historically it is important to observe that Carnot’s generalization of the principle of virtual work precedes the different one of Lagrange's *Mécanique Analytique*[[9]](#footnote-9) by 6 years; however, it has not been remembered to the same extent as the latter, which is based on the infinitesimal analysis. Yet L. Carnot’s Mechanics did not go unnoticed among the physicists of his time: this is demonstrated by citations in texts, some prestigious (those by Lagrange and Fourier, for example) and by the fact that it gave rise to the tradition of technical Physics in engineering.

The usual history of physics eliminates Lazare Carnot’s contribution, which is charged with being too engineering-oriented, although his mechanics is aimed not so much at studying machines, but at maintaining the experimental character of the theory and the theoretical principles as well. Although it has a fundamentally experimental character, the theoretical level reached is universal, without erring into metaphysics. In other words, the mechanical work of Lazare Carnot constituted a fruitful union of technology and theory, without subordinating the first to the second, but rather with the first demonstrating an innovative theoretical capacity in the second, at that time too tied to metaphysics. (His son, Sadi, did something similar shortly after with thermal machines and Thermodynamic theory).

In fact, in the history of mechanics, Lazare Carnot ranks as the one who brought together two mechanical traditions, artisanal-engineering (Bernoulli, Borda) and classical. Because of this convergence his mechanics must be seen both as a superior, and ultimately adequate, conception of artisanal mechanical techniques, and as a deepening of the theoretical heritage of Physics.

Because of this the mechanics of Carnot realizes an alternative to the conception of Physics of Descartes and above all of Newton, who idealized the principles of Physics according to a conception that has long dominated the history of physical theory. It was only in the 1900s, through two profound crises (the ether, quanta), that the foundations of Physics returned to Carnot’s conception of both the theoretical principles and the mathematical (algebraic) techniques.

Furthermore, his experimental and anti-metaphysical outlook leads Lazare Carnot to a mechanical-mathematical relationship based on a controllable and secure mathematics, different from that (infinitesimal Analysis) which Newton established in *Principia* and which informed all the subsequent physical theory up to the XX century. For this reason we associate with Carnot an unexpected result in the relationship between physics and mathematics: his theory is compatible with the current constructive mathematics. In other words, Lazare Carnot’s restriction of the potential of mathematics used in theoretical physics, allowed him to anticipate the constructive relationship between physics and mathematics. The Mechanics of Lazare Carnot presents a clear alternative choice to that of Newton: potential infinity (PI) rather than actual infinity (AI).

The table below clarifies the contrast between the two mechanical theories and highlights the divergencies existing between the various concepts of Newtonian Mechanics and those of Carnotian Mechanics.

Table 2.3

*DIFFERENCES BETWEEN NEWTON AND L. CARNOT’S FORMULATIONS*

|  |  |  |
| --- | --- | --- |
| **SUBJECT** | **NEWTON** | **LAZARE CARNOT** |
| *Cultural value of the theory* | Theory also philosophical | Completely experimental theory |
| *Organization of the theory* | By princìples | Based on a fundamental problem |
| *Space* | I Infinite and absolute  | Delimited and relational |
| *Time*  | Absolute  | Finite variation in time |
| *Bodies*  | As a set of material points without extension | Extended bodies, machines |
| *Movement* | Property of the body  | Comunication of motion |
| *Inertia*  | As perpetual motion | Impossibility of creating perpetual motion |
| *Basic concept* | Force-cause | Quantity of motion exchanged |
| *Espression of interaction* | Force, as a synthesis of all the influence of the external environment | Work |
| *Mathematical conception* | The same for the 'infinitely large for the infinitely small | Geometric motions relative to the geometric configuration of bodies |
| *Central problem*  |  | Laws of conservation on impact |
| *Mathematical technique* | InfinitesimalsDifferential equations | Geom. Motions. Vectorial calculus  |
| *Mathematical problem* |  | Invariances of  |
| *Solutions* | Trajectories from minus to plus infinity  | Quantities conserved |
| *Machine* | Particular application of the theory | Universal subject of the theory |
| *Capacity of machines* | Possibility of an infinite power | Against the chimera infinite power |

It should be noted that almost all the constituent elements of the theory have radical variations of meaning in the transition from one formulation to another. In particular, the organization of the theory in Newton is decidedly based on principles-axioms, from which to deduce all the laws of the theory. Instead, that of Carnot (excluding the edition of the *Princìpes* of 1803) is based on a *problem*, that of the collision of the bodies (as it is in Leibniz and D'Alembert), with the solution of which the whole of mechanics is constructed. Its purpose is to find a new scientific method that is able to solve this problem; that is, capable of determining quantities that are invariant despite the impact. He himself says: "there is therefore ... in every percussion or communication of movement a quantity which *is not* *altered* ( remains the same) b*y* the impact." (*Essai* p. 47) Note that the sentence is doubly negated, as is any methodological principle. His theory is therefore not organization by principle-axioms as in Newtonian mechanics (OA), but is based on a universal problem.

Note that if an additional column were added for the corresponding concepts of Sadi Carnot’s Thermodynamics, they would almost always be similar to the concepts of Lazare Carnot’s mechanics. This shows that the birth of Thermodynamics was possible only because Lazare Carnot had formulated mechanics differently from Newton, who, on the other hand, had nothing to say to theorists constructing this new theory. It also shows that the difference between formulations of the same theory relating to the same field of phenomena can be more radical than the difference between formulations of different theories concerning different fields of phenomena. Ultimately, the premises of our theories are very strong, sometimes more so than is needed to describe physical phenomena adequately.

One might add that all of this underlines the considerable importance of concepts in Physics. They arise directly from the experiments but are formed by a large degree of intellectualization.

We have seen the great historical importance of Carnot’s formulation of Mechanics. It must be admitted, however, that Carnot’s formulation is not directly applicable in our day, because it is based on the idea of full space, on tensile and pressure forces alone without taking into account forces acting at a distance (both in the definitions, including that of geometric motion, and in mathematical development) and on the concept of continuous force as the limit of a series of impulses; this last idea is intuitively valid, but it would be better to formalize it, taking into account the different ways of founding Mathematics, a task that has not been carried out so far. It undoubtedly has great historical and cultural value, but today it should be adequately reformulated considering interactions at a distance, continuous forces, etc as well. Let us not forget, however, that three centuries of clarifications and improvements of Newton’s mechanics have passed, while the further development of the Lazare Carnot’s mechanics, which was forgotten, is just beginning.

However, all the above-mentioned defects do not detract from the importance of this formulation of mechanics for the history of physics and also for current theoretical physics.

2.1 The history of false demonstrations of the principle of virtual work

The preceding study re-evaluates the principle of virtual work, which is usually given a secondary role in texts of rational mechanics, as if it were a simple appendix of mechanical theory. Let us study the history of this principle.

Some of its simplified formulations date back to antiquity, to epochs before Aristotle. It was applied widely, first in the case of the lever and then for a system of heavy bodies, according to the version of the Torricelli principle (the centre of gravity of that system cannot rise). It was then mathematically formulated by Johann Bernoulli in 1717. At the end of the same century Lazare Carnot and Lagrange were the first to propose this principle as a new way of founding Mechanics. It was the first time in an experimental science, well constituted and systemized (by Newton and Euler), that the problem of finding new foundations in logico -mathematical terms without appealing to metaphysics (as Maupertuis did when he suggested the new principle of least action), was addressed. Now, for the mentality of the physicists of the time, who by then had internalized the AO as the only possible organization, replacing Newton’s principles meant reworking the theory at the metaphysical level of its axioms, far higher than the single laws. Since the principle of virtual work originated, however, in the ongoing practice of artisans, it seemed of too humble origins to be comparable with the previous principles. It was therefore thought that it was necessary "to prove it", that is to derive it from some other elevated principles (similar to that of Maupertuis), or to derive it from experimental facts that were so simple as to appear self-evident.

All the great scientists of the time were engaged in this enterprise: Lagrange, Laplace, Poisson, Poinsot, Navier and Carnot himself. Leaving aside the metaphysical demonstrations, the other proofs tried to reduce the new principle to its simplest case, e.g. the sum of forces (law of the parallelogram) or the very ancient one of the lever. It was essentially a question of reducing the *n* variables of the principle to just two or even one. It was believed that this goal could be easily achieved by exploiting the technical innovation of the time. A few years earlier (1799) Mascheroni had shown that geometric constructions performed by means of ruler and compass, a pair that in the past had had great significance, even religious and philosophical, may be performed by means of the compass alone. Since then, theoretical importance also began to be given to geometric instruments which until then had not been used or had been used only with diffidence, that is, both linkages (rigid hinged rods, e.g. the pantograph) and tracer wheels, with which more results were obtained than those obtainable with ruler and compass.

Lagrange suggested a demonstration in which the forces of the system were connected to each other with a complex system of pulleys in order to reduce it to a single force. Fourier provided four "proofs" of the principle, the broadest of which uses a complicated three-dimensional linkage with the same aim as Lagrange: to connect all the forces together so that they’re all derivable from one.

Each demonstration depended then on the schematization of the machines which was common then: they were treated theoretically as if they had no mass (as is done today for the mathematical pendulum wire; it was Lazare Carnot who established that machines should instead be treated as having mass, but he was not immediately listened to). It was therefore believed that by unifying all the forces into one (the first for example), it would be possible to determine with the same previous displacement  all displacements caused by other forces. If instead we take into account the masses to be moved, then the total force  which is distributed with the same initial forces on the points forming the system cannot give the same *s1*, which are now no longer known and should be calculated precisely using the formula of that principle of virtual work which must be demonstrated. In conclusion these demonstrations all are vicious circles.

There were decades of proofs that were not entirely conclusive even to the minds of contemporaries. Until, as Poinsot testifies, the problem was abandoned due to exhaustion. Only a few had the idea that this principle didn’t need to be proved, but rather generalized, as Lagrange had done with his equations, or even (as we saw earlier) L. Carnot with his second fundamental equation.

The consequence of the inglorious end of the debate on the principle of virtual works was that Mechanics has never studied its foundations in depth. Rather it tried to resolve them by mythologizing them, to such an extent that an extra university exam is imposed on students, called "rational mechanics", which simply repeats in abstract form what is a physical theory that is already studied in General Physics. The myth that underlies the books on this subject is that Mechanics, by completing tying itself to infinitesimal analysis, clarifies its foundations as no other physical theory has ever done; in other words, it is believed that the fundamentals of Physics are all the clearer, the more they are abstracted from reality and mathematized by analysis (philosophically, this attitude is Platonic).

The subsequent historical development of theoretical physics (through the births of electromagnetism, thermodynamics, and then relativity and quantum mechanics) has completely discredited this idea. Of course it is in fact useful to develop Mechanics with Analysis (which can underline the mathematical isomorphism of the same differential equation with completely different physical situations; for example, it can give rise to the Hamilton-Jacobi equations, which are unique in providing a link between Newtonian mechanics and quantum mechanics; etc.); but this perspective did not give any suggestions for solving the conceptual problems that were historically crucial to arriving at the new theories of the nineteenth and twentieth centuries, and to understanding the Foundations of theoretical Physics.

*2.20 The two lines of development of mechanics*

This false foundational perspective was not only detrimental in the subsequent crisis of Physics, but even led to the widespread belief that the foundations of Physics could not be clarified any more than they were in Mechanics. This left uninvestigated many deep misconceptions in the fundamental principles of Mechanics (such as the ancient misconceptions of absolute space and time; or the modern one of the vicious circle between the three definitions of the principle of inertia, the system of inertia and the clock) and it left undecided whether Newton's principles are the most general possible (while admitting that they are technically less effective than the other principles), or whether there are alternative principles, as L. Carnot and Lagrange had begun to maintain. This false perspective led physicists to think that Newtonian Mechanics could include all the other formulations, even those of L. Carnot and Lagrange, which had threatened its monopoly of theory. It was therefore a common opinion of many theorists that the formulations of Mechanics are all equivalent.

This last point can also be clearly seen in current textbooks of rational Mechanics: which, even today, do not agree on the problem of whether or not it is necessary to prove the principle of virtual work (where "proving" can only mean deriving it from the Newtonian principles). In fact, some texts claim to give the "proof", which is what Appell, for example, does. At the crucial stage of the proof, however, he is forced to recognize a difficulty: he has to hypothesize the statement (2.8) that the work of the constraints cannot be positive. He admits that this statement can be proved only in some particular cases, but nevertheless he wants to make a general use of it, even if he does not have a rational argument to offer. (other books attempt incorrect proofs; for example, deducing it from the Lagrange equations, which, as we know, generalized it).

In reality, we know that constraint reactions are our inventions, introduced to justify the statics of bodies in the presence of constraints: it is we who attribute imaginary (or "phantom") forces to rigid bodies, i.e. constraint reactions, exactly equal and opposite to the applied forces. No instrument has directly measured a constrain reaction. But, after admitting these inventions of ours, we must exclude the possibility that they can do positive work (it would be like believing that it was a thought in the head of Jupiter that generated Minerva): if they could, it would be enough to extract it from constraints in an unlimited way, which would allow perpetual motion, which is absurd. But then we see that the statement from which the principle of virtual works derives is nothing but a version of the impossibility of perpetual motion; the latter statement was suggested and accepted as a methodological principle owing to fruitless attempts over millenia to disprove it, and, as such, cannot be demonstrated deductively from other principles, or be posited as an axiomatic principle from which to derive other truths by simple logical analysis.

There are also wise books on rational mechanics (eg Levi Civita and Amaldi; Sommerfeld) which declare that they refrain from giving a proof of it, rightly recognizing that the crucial statement on the work of constraints is equivalent to the impossibility of perpetual motion; but by doing so these authors seem to say that this fact concerns a grey area of theoretical physics, which is the boundary between genuine theory and empirical practice that has not yet been theorized (or mathematized by analysis). Some other texts suggest that the principle of virtual work is actually a method to solve problems, but they do not develop the above generic statement.

 In conclusion, none of these authors make it clear whether this principle is independent of Newtonian principles or not. Yet simple reasoning easily recognizes it easily: since the Newtonian principles concern an isolated particle, or little more, then they cannot deal with systems of extended bodies, which include constraints (which is what Lagrange was the first to say);the formula (2.8) is an essential novelty. It is therefore easy to come to the conclusion that, in general, the principle of virtual work cannot be derived from Newtonian principles and that Euler had already made the maximum effort to extend the theoretical scheme of Newton's equations to discrete or continuous particle systems (in the latter case, valid for fluids, the continuum is unlimited and does not admit impassable boundaries which is what constraints are).

This great foundational problem, which persisted despite two hundred years of fruitless reflections on the foundations of Mechanics, becomes even clearer if it is recognized that there exist two ways of organizing a scientific theory; and that a principle can be either methodological or axiomatic. Indeed, if mechanical theory is conceived as a problem-based theory (PO) (whose fundamental problem is: how do the constraints react?), then the principle of virtual work is a methodological principle, as is the principle of the impossibility of perpetual motion. Consequently the usual treatment of textbooks of rational mechanics has confused what constitutes a problem with the corresponding positive statement (as does the famous phrase of the marquis de la Palisse: If you do not die, then you will live on), obtaining a version of the principle of virtual work which is a non-physical abstraction.

It is important to note that this great scientific debate on the foundations of mechanics occurred almost simultaneously and in a similar way to the other great debate on the foundations of the first scientific theory, Geometry. The debate on the elimination of the only "blemish" of Euclidean geometry had been going on since antiquity, namely that it was not possible to prove the fifth postulate by deriving it from the other four. The motivation seemed obvious since Euclid had introduced that postulate only at the 29th proposition (that is, as late as possible) and since its inverse proposition (obtained by exchanging the hypothesis for the thesis) was provable, which is the case for all the other theorems of Euclidean geometry. We know that this debate was resolved when finally (in the first half of the nineteenth century) some mathematicians (Gauss, Lobacevskij, Bolyai) decided to consider it a postulate that is independent of the others. As a result, they developed a new geometry. In this way began the revolution of the foundations of geometry and then of all mathematics.

Just as finally accepting the independence of the parallel postulate led on to a new History of Mathematics, in which the problem of the Foundations constituted the first problem, and to different ways of doing both Geometry and Mathematics in general, so, analogously, accepting the independence of the principle of virtual works from Newtonian principles, it would have led on to recognizing the radical novelty of L. Carnot’s and Lagrange’s Mechanics; and research in theoretical physics would have been attentive to the problem of the foundations (first of mechanics, and then of all physics) and a new history of physics would have begun, that of the plurality of lines of development. We have started to see that in the previous paragraphs, albeit after a delay of two centuries.

*2.21 The Foundations of Physics*

To recognize the Foundations of Physics, it is necessary to highlight an unconscious choice by the vast majority of theoretical physicists. Following Newton, they adopted infinitesimal analysis, whereas Leibniz wanted to exclude it from the foundations of a physical theory, as did L. Carnot and Sadi Carnot. In fact he latter were ignored. However, at the beginning of the twentieth century, the discovery of quanta forced physicists to take an interest in the discrete even in the most advanced theory. It is therefore clear that in the Foundations of Physics there is a choice regarding the type of Mathematics.

Another unconscious choice made by theoretical physicists following Newton (and Euclid) was to evaluate a theory as appropriate only if it was organized deductively, that is, founded on a few self-evident principles. from which the entire theory was deduced. Leibniz. L. Carnot and Sadi Carnot, on the other hand, had founded their theories on a problem, without principles. But this innovation was also ignored, although D’Alembert and L. Carnot had clearly indicated that there are two types of theoretical organization. Then, at the beginning of the 1900s, first Poincaré and Lorentz, then Einstein, reflecting on the theories built up by physics, distinguished two types, those deduced from a priori principles and those, like Thermodynamics, based on the impossibility of perpetual motion.[[10]](#footnote-10)

Everything that has been said so far leads us to the conclusion that the Foundations of Physics *include two fundamental dichotomies, one regarding the type of infinity (PI or AI) each characterizing a particular type of Mathematics, and one regarding the type of organization of a theory, deductive or based on a problem. In every theory of the second type there is at least one crucial doubly negated proposition (i.e. a principle) which is not equivalent to the corresponding affirmative, implying non-classical logic. Thus the type of theoretical organization is characterized by the type of logic, which therefore distinguishes these theories with formal precision from AO theories. We can therefore treat both dichotomies formally, that is, the first in mathematical and the second in logical terms.*

Table 2.4

 THE FOUR CHOICES EXPRESSED BOTH IN INTUITIVE AND FORMAL TERMS

|  |  |  |
| --- | --- | --- |
|  | **INFINITY** | **ORGANIZATION** |
|  | **AI** | **PI** | **AO** | **PO** |
| *PHILOSOPHICAL CONCEPT* | *Actual infinity* | *Potential infinity*  | *Aristotelian Organization*  | *Problem-based organization* |
| *FORMAL**MATHEMATICAL SYSTEM* | *Classical mathematics* | *Constructive mathematics* | *Classical logic*  | *Non* *classical logic* |

 With these two dichotomies, a cultural division between the various foundational lines of Mechanics has been evident since the time of Galilei (who saw them but accepted neither AI nor AO) and by the end of the 1700s was manifested with theories based on the principle of virtual work. This division was, however, hidden and / or ignored.

 We can therefore represent the history of Mechanics with the following table, which is far more comprehensive than the traditional one of a unilinear type.

**Table 2.5**

*THE LINES OF DEVELOPMENT OF MECHANICAL THEORY*

 Engineers’ tradition Bernoulli Borda

 L. Carnot

Galilei Huygens Leibniz D’Alembert

 Classical tradition Lagrange

 Descartes Newton Euler Laplace

In the table above we have the lower line which from Newton onwards followed the two choices of infinitesimal analysis (IA) and of the entirely deductive organization of a theory (AO), while the upper lines up to the mechanics of L. Carnot represent the two choices of a simple mathematics (IP) and of a problem-based organization (PO). Lagrange, on the other hand, represents an intermediate position: he uses a new mathematical technique with AI, but it is aimed at solving any problem of Mechanics (PO).

1. G.W. Leibniz 1677, “Letter to H. Fabri” (my emphasis) [↑](#footnote-ref-1)
2. G.W. Leibniz 1686, “Letter to Arnaud”, July. [↑](#footnote-ref-2)
3. See the texts indicated for non-classical logic. The distinction was defined precisely in the work of D. Prawitz e P.-E. Melmnaas, “A survey of some connections between classical intuitionistic and minimal logic”, in *Contributions to Mathematical Logic,* Amsterdam: North-Holland, in H. A. Schmidt, K. Schuette e H.-J. Thiele, eds., 1968, pp. 215-229; J.B. Grize,"Logique", in J. Piaget (ed..): Logique et la connaissance scientifique, in *Encyclopédie de la Pleyade*, Paris: Gallimard, 1970, 135-288, 206-210. [↑](#footnote-ref-3)
4. G.W. Leibniz, *Essay de dynamique sur les lois du mouvement*, 1698, in A. Drago, *La riforma della dinamica di G.W. Leibniz*, Benevento: Hevelius, 2004, pp. 122-131. Refer to the “Introduction” to this book for a more detailed presentation and illustration of Leibniz's anticipation of an alternative formulation of mechanics. For a general view on this mechanics see: “The birth of an alternative mechanics: Leibniz’ principle of sufficient reason”. in H. Poser et al. (eds.): *Leibniz-Kongress. Nihil Sine Ratione*, 2001, Berlin. voI. 1, pp. 322-330. [↑](#footnote-ref-4)
5. L. Carnot 1783, *Essai sur les machines en général*, Dijon: Defay (Italian transl. *Saggio sulle Macchine*, Napoli : CUEN, 1994; English transl. Berlin : Springer, 2020); L. Carnot, *Principes fondamentaux de l'équilibre et du mouvement*, Paris : Deterville, 1803. [↑](#footnote-ref-5)
6. R. Dugas, *Histoire de la Mécanique*, Neuchâtel:Griffon, 1950, p. 312. [↑](#footnote-ref-6)
7. L. Carnot 1803, p. 246. [↑](#footnote-ref-7)
8. M. Jammer, *Concepts of mass in classical and modern physics, Cambridge*, Harvard University press, 1961, chp. 11. [↑](#footnote-ref-8)
9. J.-L. Lagrange, *Mécanique Analytique,* Paris: Desaint, 1788*.* [↑](#footnote-ref-9)
10. Flores, F. (1999). Einstein’s Theory of Theories and Types of Theoretical Explanation. *International Studies in Philosophy of Science, 13,* 123-134. <http://dx.doi.org/10.1080/02698599908573613> [↑](#footnote-ref-10)