# Deleuze and *The Fold*: A Critical Reader

Edited by

Sjoerd van Tuinen

and

Niamh McDonnell

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First published 2010 by
PALGRAVE MACMILLAN

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Palgrave Macmillan in the US is a division of St Martin's Press LLC, 175 Fifth Avenue, New York, NY 10010.

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ISBN-13: 978-0-230-55287-6 hardback

This book is printed on paper suitable for recycling and made from fully managed and sustained forest sources. Logging, pulping and manufacturing processes are expected to conform to the environmental regulations of the country of origin.

A catalogue record for this book is available from the British Library.

A catalog record for this book is available from the Library of Congress.

10 9 8 7 6 5 4 3 2 1
19 18 17 16 15 14 13 12 11 10

Printed and bound in Great Britain by
CPI Antony Rowe, Chippenham and Eastbourne

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23. Under the terms of 'the play of the world [that] has changed in a unique way', the choice of the properties of inflexion no longer depends on a God but on pure process. Deleuze shows how a theory of synthesis particular to the monads bears upon modern mathematics: "monads' test the paths in the universe and enter in syntheses associated with each path' (*TF* 81). The notion of paths that are *tested* can be seen in the context of problems that 'escape demonstration', an experimentation in which a performance of thresholds maintaining a 'baroque equilibrium or disequilibrium' informs the affective state of the whole in variation. For the discussion of 'capture of code' in relation to 'becoming', (TP 10). See also 'statements' and 'visibilities' of capture in the diagram read through Foucault, (F 67).

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# 4

# Leibniz, Mathematics and the Monad

*Simon Duffy*

The reconstruction of Leibniz's metaphysics that Deleuze undertakes in *The Fold* provides a systematic account of the structure of Leibniz's metaphysics in terms of its mathematical foundations. However, in doing so, Deleuze draws not only upon the mathematics developed by Leibniz – including the law of continuity as reflected in the calculus of infinite series and the infinitesimal calculus – but also upon developments in mathematics made by a number of Leibniz's contemporaries – including Newton's method of fluxions. He also draws upon a number of subsequent developments in mathematics, the rudiments of which can be more or less located in Leibniz's own work – including the theory of functions and singularities, the Weierstrassian theory of analytic continuity, and Poincaré's theory of automorphic functions. Deleuze then retrospectively maps these developments back onto the structure of Leibniz's metaphysics. While the Weierstrassian theory of analytic continuity serves to clarify Leibniz's work, Poincaré's theory of automorphic functions offers a solution to overcome and extend the limits that Deleuze identifies in Leibniz's metaphysics. Deleuze brings this elaborate conjunction of material together in order to set up a mathematical idealisation of the system that he considers to be implicit in Leibniz's work. The result is a thoroughly mathematical explication of the structure of Leibniz's metaphysics. This essay is an exposition of the very mathematical underpinnings of this Deleuzian account of the structure of Leibniz's metaphysics, which, I maintain, subtends the entire text of *The Fold*.

Deleuze's project in *The Fold* is predominantly oriented by Leibniz's insistence on the metaphysical importance of mathematical speculation. What this suggests is that mathematics functions as an important heuristic in the development of Leibniz's metaphysical theories. Deleuze puts this insistence to good use by bringing together the different aspects of Leibniz's metaphysics with the variety of mathematical themes that run throughout his work, principally the infinitesimal calculus. Those aspects of Leibniz's metaphysics that Deleuze undertakes to clarify in this way, and upon which this essay will focus, include the definition of the monad and the theory of compossibility. However, before providing the details of Deleuze's reconstruction of the structure of Leibniz's metaphysics, it will be necessary to give an introduction to Leibniz's infinitesimal calculus and to some of the other developments in mathematics associated with it.

## Leibniz's law of continuity and the infinitesimal calculus

Leibniz was both a philosopher and a mathematician. His infinitesimal analysis encompassed the investigation of infinite sequences and series, the study of algebraic and transcendental curves and the operations of differentiation and integration upon them, and the solution of differential equations; integration and differentiation are the two fundamental operations of the infinitesimal calculus developed by him.

Leibniz applied the calculus primarily to problems about curves and the calculus of finite sequences, which had been used since antiquity to approximate the curve by a polygon in the Archimedean approach to geometrical problems by means of the method of exhaustion. In his early exploration of mathematics, Leibniz applied the theory of number sequences to the study of curves and showed that the differences and sums in number sequences correspond to tangents and quadratures respectively, and he developed the conception of the infinitesimal calculus by supposing the differences between the terms of these sequences infinitely small (See Bos 1974, p. 13). One of the keys to the calculus that Leibniz emphasised was to conceive the curve as an infinitangular polygon:

> I feel that this method and others in use up till now can all be deduced from a general principle which I use in measuring curvilinear figures, that a curvilinear figure must be considered to be the same as a polygon with infinitely many sides.
>
> (*GM* V 126)

Leibniz based his proofs for the infinitangular polygon on a law of continuity, which he formulated as follows: 'In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning,



in which the final terminus may also be included' (Leibniz 1920, p. 147). Leibniz also thought the following to be a requirement for continuity: 'Two instances [...] approach each other continuously [if] the difference between [the] two instances [...] can be diminished until it becomes smaller than any given quantity whatever' (*L* 351). Leibniz used the adjective *continuous* for a variable ranging over an infinite sequence of values. In the infinite continuation of the polygon, its sides become infinitely small and its angles infinitely many. The infinitangular polygon is considered to coincide with the curve, the infinitely small sides of which, if prolonged, would form tangents to the curve; where a tangent is a straight line that touches a circle or curve at only one point. Leibniz applied the law of continuity to the tangents of curves as follows: he took the tangent to be continuous with, or as the limiting case (*terminus*) of the secant. To find a tangent is to draw a straight line joining two points of the curve – the secant – which are separated by an infinitely small distance or vanishing difference, which he called a differential. (*GM* V 223) The Leibnizian infinitesimal calculus was built upon the concept of the differential. The differential, *dx*, is the difference in *x* values between two consecutive values of the variable at P (See Figure 4.1), and the tangent is the line joining such points.

The differential relation, that is, the quotient between two differentials of the type *dy/dx*, serves in the determination of the gradient of the tangent to the circle or curve. The gradient of a tangent indicates the slope or rate of change of the curve at that point, that is, the rate

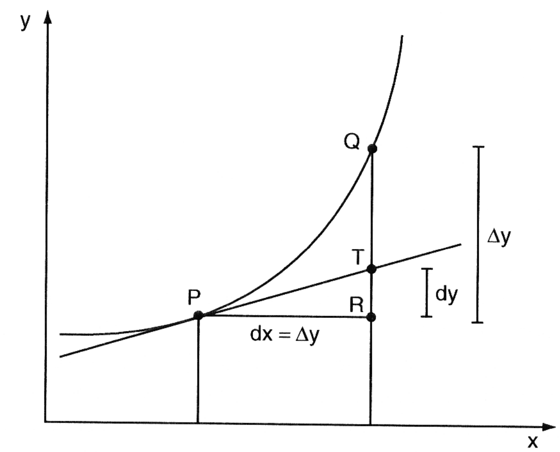


*Figure 4.1* The tangent to the curve at P.

at which the curve changes on the $y$-axis relative to the $x$-axis. Leibniz thought of the $dy$ and $dx$ in $dy/dx$ as 'infinitesimal' quantities. Thus $dx$ was an infinitely small nonzero increment in $x$ and $dy$ was an infinitely small nonzero increment in $y$.

Leibniz brings together the definition of the differential as it operates in the calculus of infinite series, in regard to the infinitangular triangle, and the infinitesimal calculus, in regard to the determination of tangents to curves, as follows:

> Here $dx$ means the element, that is, the (instantaneous) increment or decrement, of the (continually) increasing quantity $x$. It is also called difference, namely the difference between two proximate $x$'s which differ by an element (or by an unassaignable), the one originating from the other, as the other increases or decreases (momentaneously).
>
> (*GM* VII 223)

The differential can therefore be understood on the one hand, in relation to the calculus of infinite series as the infinitesimal difference between consecutive values of a continuously diminishing quantity, and on the other, in relation to the infinitesimal calculus as an infinitesimal quantity. The operation of the differential in the latter actually demonstrates the operation of the differential in the former, because the operation of the differential in the infinitesimal calculus in the determination of tangents to curves demonstrates that the infinitely small sides of the infinitangular polygon are continuous with the curve.

In one of his early mathematical manuscripts entitled 'Justification of the Infinitesimal Calculus by that of Ordinary Algebra', Leibniz offers an account of the infinitesimal calculus in relation to a particular geometrical problem that is solved using ordinary algebra (*L* 545). An outline of the demonstration that Leibniz gives is as follows (Figure 4.2):[1] since the two right triangles, ZFE and ZHJ, that meet at their apex, point Z, are similar, it follows that the ratio $y/x$ is equal to $(Y - y)/X$. As the straight line EJ approaches point F, maintaining the same angle at the variable point Z, the lengths of the straight lines FZ and FE, or $y$ and $x$, steadily diminish, yet the ratio of $y$ to $x$ remains constant. When the straight line EJ passes through F, the points E and Z coincide with F, and the straight lines, $y$ and $x$, vanish. Yet $y$ and $x$ will not be absolutely nothing since they preserve the ratio of ZH to HJ, represented by the proportion $(Y - y)/X$, which in this case reduces to $Y/X$, and obviously does not equal zero. The relation $y/x$ continues to exist even though the terms have vanished since the relation is determinable as equal to $Y/X$. In this



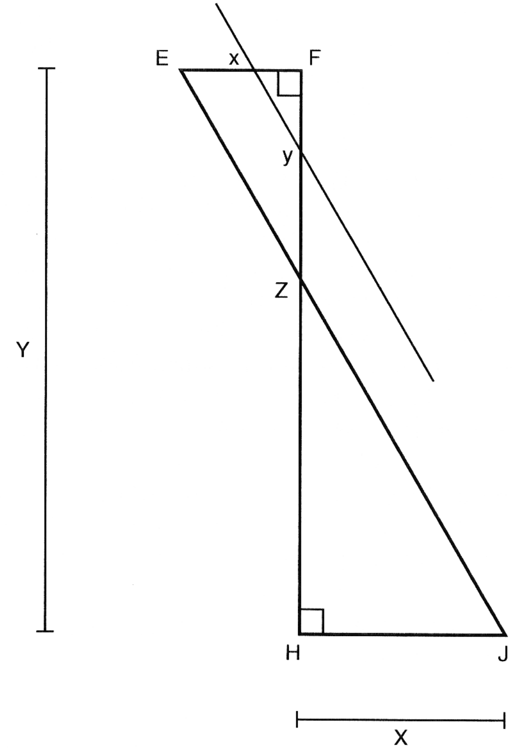


*Figure 4.2* Leibniz's example of the infinitesimal calculus using ordinary algebra.

algebraic calculus, the vanished lines $x$ and $y$ are not taken for zeros since they still have an algebraic relation to each other. 'And so [Leibniz argues], they are treated as infinitesimals, exactly as one of the elements which [...] differential calculus recognises in the ordinates of curves for momentary increments and decrements' (*L* 545). That is, the vanished lines $x$ and $y$ are determinable in relation to each other only insofar as they can be replaced by the infinitesimals $dy$ and $dx$, by making the supposition that the ratio $y/x$ is equal to the ratio of the infinitesimals, $dy/dx$. When the relation continues even though the terms of the relation have disappeared, a continuity has been constructed by algebraic means that is instructive of the operations of the infinitesimal calculus.

What Leibniz demonstrates in this example are the conditions according to which any unique triangle can be considered as the extreme case of two similar triangles opposed at the vertex. Deleuze argues that, in the case of a figure in which there is only one triangle, the other triangle is there, but it is there only virtually (*CGD* 22 April 1980). The *virtual* triangle has not simply disappeared, but rather it has become unassignable, all the while remaining completely determined. The hypotenuse of the virtual triangle can be mapped as a side of the infinitangular polygon, which, if prolonged, forms a tangent line to the curve. There is therefore continuity from the polygon to the circle, just as there is continuity from two similar triangles opposed at the vertex to a single triangle. Hence this relation is fundamental for the application of differentials to problems about tangents.

In the first published account of the calculus, Leibniz defines the ratio of infinitesimals as the quotient of first-order differentials, or the associated differential relation. He says that 'the differential $dx$ of the abscissa $x$ is an arbitrary quantity, and that the differential $dy$ of the ordinate $y$ is defined as the quantity which is to $dx$ as the ratio of the ordinate to the subtangent' (Boyer 1959, p. 210) (see Figure 4.1). Leibniz considers differentials to be the fundamental concepts of the infinitesimal calculus, the differential relation being defined in terms of these differentials.

## Newton's method of fluxions and infinite series

Newton began thinking of the rate of change, or fluxion, of continuously varying quantities, which he called fluents such as lengths, areas, volumes, distances, temperatures, in 1665, which predates Leibniz by about ten years. Newton regarded his variables as generated by the continuous motion of points, lines, and planes, and offered an account of the fundamental problem of the calculus as follows: 'Given a relation between two fluents, find the relation between their fluxions, and conversely' (Newton 1736). Newton thinks of the two variables whose relation is given as changing with time, and, although he does point out that this is useful rather than necessary, it remains a defining feature of his approach and is exemplified in the geometrical reasoning about limits, which Newton was the first to come up with. Put simply, to determine the tangent to a curve at a specified point, a second point on the curve is selected, and the gradient of the line that runs through both of these points is calculated. As the second point approaches the point of tangency, the gradient of the line between the two points approaches the gradient of the tangent. The gradient of the tangent is, therefore, the limit of the gradient of the line between the two points as the points become increasingly close to one another (Figure 4.3).



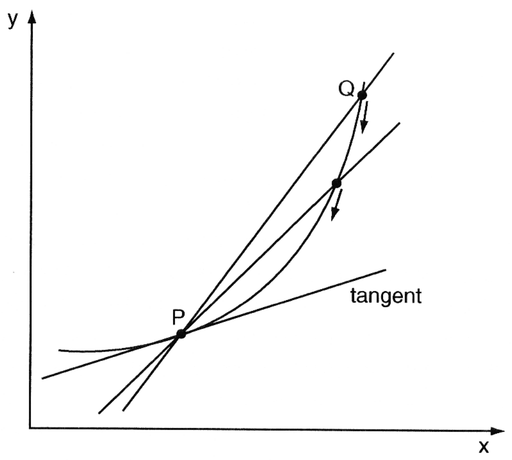


*Figure 4.3* Newton's geometrical reasoning about the gradient of a tangent as a limit.

He conceptualised the tangent geometrically, as the limit of a sequence of lines between two points, P and Q, on a curve, which is a secant. As the distance between the points approached zero, the secants became progressively smaller, however they always retained 'a real length'. The secant therefore approached the tangent without reaching it. When this distance 'got arbitrarily small (but remained a real number)' (Lakoff, Núñez 2000, p. 224), it was considered insignificant for practical purposes, and was ignored. What is different in Leibniz's method is that he 'hypothesized infinitely small numbers – infinitesimals – to designate the size of infinitely small intervals.' (ibid.) (See Figure 4.1) For Newton, on the contrary, these intervals remained only small, and therefore real. When performing calculations, however, both approaches yielded the same results. But they differed ontologically, because Leibniz had hypothesised a new kind of number, a number Newton did not need, since 'his secants always had a real length, while Leibniz's had an infinitesimal length' (ibid.). Leibniz's symbolism also treats quantities independently of their genesis, rather than as the product of an explicit functional relation.

Deleuze uses this distinction between the methods of Leibniz and Newton to characterise the mind–body distinction in Leibniz's account of the monad. Deleuze distinguishes according to the distinction canvassed earlier between the functional definition of the Newtonian fluxion and the Leibnizian infinitesimal as a concept. 'The physical mechanism of bodies (fluxion) is not identical to the psychic mechanism of perception (differentials), but the latter resembles the former' (*TF* 98). So Deleuze maintains that 'Leibniz's calculus is adequate to psychic mechanics where Newton's is operative for physical mechanics' (ibid.), and here again draws from the mathematics of Leibniz's contemporaries to determine a distinction between the mind and body of a monad in Leibniz's metaphysics.

Both Newton and Leibniz are credited with developing the calculus as a new and general method, and with having appreciated that the operations in the new analysis are applicable to infinite series as well as to finite algebraic expressions. However, neither of them clearly understood nor rigorously defined their fundamental concepts. Newton thought his underlying methods were natural extensions of pure geometry, while Leibniz felt that the ultimate justification of his procedures lay in their effectiveness. For the next 200 years, various attempts were made to find a rigorous arithmetic foundation for the calculus; one that relied on neither the mathematical intuition of geometry, with its tangents and secants, (perceived as imprecise because its conception of limits was not properly understood); nor the vagaries of the infinitesimal, which cannot be justified either from the point of view of classical algebra or from the point of view of arithmetic; the latter made many mathematicians wary, so much so that they refused the hypothesis outright despite the fact that Leibniz 'could do calculus using arithmetic without geometry – by using infinitesimal numbers' (Lakoff, Núñez 2000, pp. 224–5).

## The emergence of the concept of the function

Seventeenth-century analysis was a corpus of analytical tools for the study of geometric objects, the most fundamental object of which, (thanks to the development of a curvilinear mathematical physics by Huygens) was the curve, or curvilinear figures generally. The latter were understood to embody relations between several variable geometrical quantities defined with respect to a variable point on the curve. The variables of geometric analysis referred to geometric quantities, which were conceived not as real numbers, but rather as having a dimension: for example, 'the dimension of a line (e.g. ordinate, arc length, subtangent), of an area (e.g. the area between curve and axis) or of a solid (e.g. the solid of revolution)' (Bos 1974, p. 6). The relations between these variables were expressed by means of equations. Leibniz actually referred to these variable geometric quantities as the *functiones* of a curve,[2] and thereby introduced the term 'function' into mathematics. However, it is important to note the absence of the fully developed concept of function in the Leibnizian context of algebraic relations between variables. Today, a function is understood to be a relation that uniquely associates members of one set with members of another set. For Leibniz, neither the equations nor the variables are functions in this modern sense, rather the relation between x and y was considered to be one entity. The curves were thought of as having a primary existence apart from any analysis of their numeric or algebraic properties. In seventeenth-century analysis, equations did not create curves, curves rather gave rise to equations (Dennis, Confrey 1995, p. 125). Thus the curve was not seen as a graph of a function but rather as 'a figure embodying the relation between x and y' (see Bos 1974, p. 6). In the first half of the eighteenth century, a shift of focus from the curve and the geometric quantities themselves to the formulas which expressed the relations among these quantities occurred, thanks in large part to the symbols introduced by Leibniz. The analytical expressions involving numbers and letters, rather than the geometric objects for which they stood, became the focus of interest. It was this change of focus towards the formula that made the emergence of the concept of function possible. In this process, the differential underwent a corresponding change; it lost its initial geometric connotations and came to be treated as a concept connected with formulas rather than with figures.



With the emergence of the concept of the function, the differential was replaced by the derivative, which is the expression of the differential relation as a function, first developed in the work of Euler. One significant difference, reflecting the transition from a geometric analysis to an analysis of functions and formulas, is that the infinitesimal sequences are no longer induced by an infinitangular polygon standing for a curve, according to the law of continuity as reflected in the infinitesimal calculus, but by a function, defined as a set of ordered pairs of real numbers.

## Subsequent developments in mathematics: The problem of rigour

The concept of the function however did not immediately resolve the problem of rigour in the calculus. It was not until the late nineteenth century that an adequate solution to this problem was found. It was Karl Weierstrass who 'developed a pure nongeometric arithmetization for Newtonian calculus' (Lakoff, Núñez 2000, p. 230), which provided the rigour that had been lacking. The Weierstrassian programme determined that the fate of calculus need not be tied to infinitesimals, and could rather be given a rigorous status from the point of view of finite representations. Weierstrass's theory was an updated version of an earlier account by Augustin Cauchy, which had also experienced problems conceptualising limits.

It was Cauchy who insisted on specific tests for the convergence of series, so that divergent series could henceforth be excluded from being used to try to solve problems of integration because of their propensity to lead to false results (see Boyer 1959, p. 287). Extending sums to an infinite number of terms caused problems to emerge if the series did not converge, since the sum or limit of an infinite series is only determinable if the series converges. It was considered that reckoning with divergent series, which have no sum, would therefore lead to false results.

Weierstrass considered Cauchy to have actually begged the question of the concept of limit in his proof.[3] In order to overcome this problem of conceptualising limits, Weierstrass 'sought to eliminate all geometry from the study of [...] derivatives and integrals in calculus' (Lakoff, Núñez 2000, p. 309). In order to characterise calculus purely in terms of arithmetic, it was necessary for the idea of a curve in the Cartesian plane defined in terms of the motion of a point, to be completely replaced with the idea of a function. The geometric idea of 'approaching a limit' had to be replaced by an arithmetised concept of limit that relied on static logical constraints on numbers alone. This approach is commonly referred to as the epsilon-delta method (see Potter 2004, p. 85). The calculus was thereby reformulated without either geometric secants and tangents or infinitesimals; only the real numbers were used.

Because there is no reference to infinitesimals in this Weierstrassian definition of the calculus, the designation 'the infinitesimal calculus' was considered to be 'inappropriate' (Boyer 1959, p. 287). Weierstrass's work not only effectively removed any remnants of geometry from what was now referred to as the differential calculus, but it eliminated the use of the Leibnizian-inspired infinitesimals in doing the calculus for over half a century. It was not until the late 1960s, with the development of the controversial axioms of non-standard analysis by Abraham Robinson, that the infinitesimal was given a rigorous foundation (see Bell 1998), thus allowing the inconsistencies to be removed from the Leibnizian infinitesimal calculus without removing the infinitesimals themselves.[4] Leibniz's ideas about the role of the infinitesimal in the calculus have therefore been be 'fully vindicated' (Robinson 1996, p. 2), as Newton's had been thanks to Weierstrass.[5]



In response to these developments, Deleuze brings renewed scrutiny to the relationship between the developments in the history of mathematics and the metaphysics associated with these developments, which were marginalised as a result of efforts to determine the rigorous foundations of the calculus. This is a part of Deleuze's broader project of constructing an alternative lineage in the history of philosophy that tracks the development of a series of metaphysical schemes that respond to and attempt to deploy the concept of the infinitesimal. The aim of the project is to construct a philosophy of difference as an alternative speculative logic that subverts a number of the commitments of the Hegelian dialectical logic which supported the elimination of the infinitesimal in favour of the operation of negation, the procedure of which postulates the synthesis of a series of contradictions in the determination of concepts.[6]

## The theory of singularities

Another development in mathematics, the rudiments of which can be found in the work of Leibniz is the theory of singularities. A singularity or singular point is a mathematical concept that appears with the development of the theory of functions, which historians of mathematics consider to be one of the first major mathematical concepts upon which the development of modern mathematics depends. Even though the theory of functions does not actually take shape until later in the eighteenth century, it is in fact Leibniz who contributes greatly to this development. Indeed, it was Leibniz who developed the first theory of singularities in mathematics, and, Deleuze argues, it is with Leibniz that the concept of singularity becomes a mathematico-philosophical concept. (*CGD* 29 April 1980) However, before explaining what is philosophical in the concept of singularity for Leibniz, it is necessary to offer an account of what he considers singularities to be in mathematics, and of how this concept was subsequently developed in the theory of analytic functions, which is important for Deleuze's account of (in)compossibility in Leibniz, despite it not being developed until long after Leibniz's death.

The great mathematical discovery that Deleuze refers to is that singularity is no longer thought of in relation to the universal, but rather in relation to the ordinary or the regular (*CGD* 29 April 1980). In classical logic, the singular was thought of with reference to the universal, however that does not necessarily exhaust the concept since in mathematics, the singular is distinct from or exceeds the ordinary or regular. Mathematics refers to the singular and the ordinary in terms of the points of a curve, or more generally concerning complex curves or figures. A curve, a curvilinear surface, or a figure includes singular points and others that are regular or ordinary. Therefore, the relation between singular and ordinary or regular points is a function of curvilinear problems which can be determined by means of the Leibnizian infinitesimal calculus.

The differential relation is used to determine the overall shape of a curve primarily by determining the number and distribution of its singular points or singularities, which are defined as points of articulation where the shape of the curve changes or alters its behaviour. For example, when the differential relation is equal to zero, the gradient of the tangent at that point is horizontal, indicating, for example, that the curve peaks or dips, determining therefore a maximum or minimum at that point. These singular points are known as stationary or turning points.

The differential relation characterises not only the singular points which it determines, but also the nature of the regular points in the immediate neighbourhood of these points, that is, the shape of the branches of the curve on either side of each singular point. Where the differential relation gives the value of the gradient at the singular point, the value of the second-order differential relation, that is if the differential relation is itself differentiated and which is now referred to as the second derivative, indicates the rate at which the gradient is changing at that point. This allows a more accurate approximation of the shape of the curve in the neighbourhood of that point.

Leibniz referred to the stationary points as *maxima* and *minima* depending on whether the curve was concave up or down respectively. A curve is concave up where the second-order differential relation is positive and concave down where the second-order differential relation is negative. The points on a curve that mark a transition between a region where the curve is concave up and one where it is concave down are points of inflexion. The second-order differential relation will be zero at an inflexion point. Deleuze distinguishes a point of inflexion, as an intrinsic singularity, from the *maxima* and *minima*, as extrinsic singularities, on the grounds that the former 'does not refer to coordinates [but rather] corresponds' to what Leibniz calls an 'ambiguous sign' (*TF* 15), that is, where concavity changes, the sign of the second-order differential relation changes from + to –, or vice versa.

The value of the third-order differential relation indicates the rate at which the second-order differential relation is changing at that point.



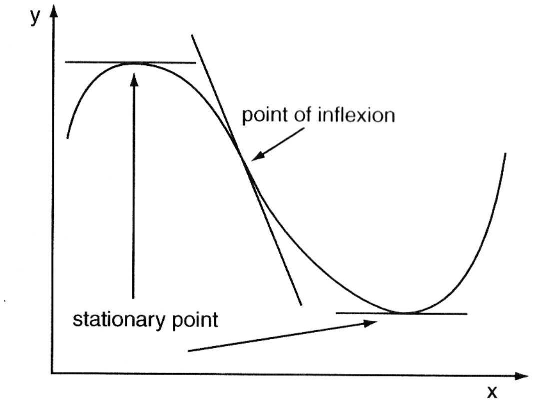


*Figure 4.4* The singular points of a curve.

In fact, the more successive orders of the differential relation that can be evaluated at the singular point, the more accurate the approximation of the shape of the curve in the 'immediate' neighbourhood of that point. Leibniz even provided a formula for the nth-order differential relation, as n approaches infinity. The nth-order differential relation at the point of inflexion would determine the continuity of the variable curvature in the immediate neighbourhood of the inflexion with the curve. Because the point of inflexion is where the tangent crosses the curve (see Figure 4.4) and the point where the nth-order differential relation is continuous with the curve, Deleuze characterises the point of inflexion as a *point-fold*; which is the trope that unifies a number of the themes and elements of *The Fold*.[7]

## Subsequent developments in mathematics: Weierstrass and Poincaré

The important development in mathematics, the rudiments of which Deleuze considers to be in Leibniz's work and that he retrospectively maps back onto Leibniz's account of (in)compossibility is the Weierstrassian theory of analytic continuity. The Leibnizian method of approximation using successive orders of the differential relation is formalised in the calculus according to Weierstrass's theory by a Taylor series or power series expansion. A power series expansion can be written as a polynomial, the coefficients of each of its terms being the successive derivatives evaluated at the singular point. The sum of such a series represents the expanded function provided that any remainder approaches zero as the number of terms becomes infinite; the polynomial then becomes an infinite series which converges with the function in the neighbourhood of the singular point.[8] This criterion of convergence repeats Cauchy's earlier exclusion of divergent series from the calculus. A power series operates at each singular point by successively determining the specific qualitative nature of the function at that point, that is, the shape and behaviour of the graph of the function or curve. The power series determines not only the nature of the function at the point in question, but also the nature of all of the regular points in the neighbourhood of that singular point, such that the specific qualitative nature of a function in the neighbourhood of a singular point insists in that one point. By examining the relation between the differently distributed singular points determined by the differential relation, the regular points which are continuous between the singular points can be determined, which in geometrical terms are the branches of the curve. In general, the power series converges with a function by generating a continuous branch of a curve in the neighbourhood of a singular point. To the extent that all of the regular points are continuous across all of the different branches generated by the power series of the singular points, the entire complex curve or the whole analytic function is generated.

The mathematical elements of this interpretation are most clearly developed by Weierstrassian analysis, according to the theorem on the approximation of analytic functions. According to Weierstrass, for any continuous analytic function on a given interval, or domain, there exists a power series expansion which uniformly converges to this function on the given domain. Given that a power series approximates a function in such a restricted domain, the task is then to determine other power series expansions that approximate the same function in other domains. An analytic function is differentiable at each point of its domain, and is essentially defined, for Weierstrass, from the neighbourhood of a singular point by a power series expansion which is convergent with a 'circle of convergence' around that point. A power series expansion that is convergent in such a circle represents a function that is analytic at each point in the circle. By taking a point interior to the first circle as a new centre, and by determining the values of the coefficients of this new series using the function generated by the first series, a new series and a new centre of convergence are obtained, whose circle of convergence overlaps the first. The new series is continuous with the first if the values of the function coincide in the common part of



the two circles. This method of 'analytic continuity' allows the gradual construction of a whole domain over which the generated function is continuous. At the points of the new circle of convergence which are exterior to, or extend outside the first, the function represented by the second series is then the analytic continuation of the function defined by the first series; this is defined by Weierstrass as the analytic continuation of a power series expansion outside its circle of convergence. The domain of the function is extended by the successive adjunction of more and more circles of convergence. Each series expansion which determines a circle of convergence is called an element of the function. In this way, given an element of an analytic function, by analytic continuation one can obtain the entire analytic function over an extended domain. The domain of the successive adjunction of circles of convergence, as determined by analytic continuity, actually has the structure of a surface. The analytic continuation of power series expansions can be continued in this way in all directions up to the points in the immediate neighbourhood exterior to the circles of convergence where the series obtained diverge.

Power series expansions diverge at specific 'singular points' or 'singularities' that may arise in the process of analytic continuity. A singular point or singularity of an analytic function, as with a curve, is any point which is not a regular or ordinary point of the function or curve. They are points which exhibit remarkable properties and thereby have a dominating and exceptional role in the determination of the characteristics of the function, or shape and behaviour of the curve. The singular points of a function, which include the stationary points, where $dy/dx = 0$, and points of inflexion, where $d^2y/dx^2 = 0$, are 'removable singular points', since the power series at these points converge with the function. A removable singular point is uniformly determined by the function and therefore redefinable as a singular point of the function, such that the function is analytic or continuous at that point. The specific singularities of an analytic function where the series obtained diverge are called 'poles'. Singularities of this kind are those points where the function no longer satisfies the conditions of regularity which assure its local continuity, such that the rule of analytic continuity breaks down. They are therefore points of discontinuity. A singularity is called a pole of a function when the values of the differential relation, that is, the gradients of the tangents to the points of the function, approach infinity as the function approaches the pole. The function is said to be asymptotic to the pole, it is therefore no longer differentiable at that point, but rather remains undefined, or vanishes. A pole is therefore the limit point of a function. The poles that arise in the process of analytic continuity necessarily lie on the boundaries of the circles of convergence of power series. The effective domain of an analytic function determined by the process of the analytic continuation of power series expansions is therefore limited to that between its poles. The poles of the two discontinuous analytic functions are non-removable, thus analytic continuity between the two functions is not able to be established.

This is the extent of the Weierstrassian theory of analytic continuity that Deleuze retrospectively maps onto Leibniz's theory of singularities and that he deploys in his account of Leibnizian incompossibility, which is explicated in the following section. A singularity is a distinctive point on a curve in the neighbourhood of which the second-order differential relation changes its sign. This characteristic of the singular point is extended into or is continuous with the series of ordinary points that depend on it, all the way to the neighbourhood of subsequent singularities. It is for this reason that Deleuze maintains that the theory of singularities is inseparable from a theory or an activity of continuity, where continuity, or the continuous, is the extension of a singular point into the ordinary points up to the neighbourhood of the subsequent singularity. And it is for this reason that Deleuze considers the rudiments of the Weierstrassian theory to be in the work of Leibniz, and that it is therefore able to be retrospectively mapped back onto the work of Leibniz.

Weierstrass did recognise a means of solving the problem of the discontinuity between the poles of analytic functions by postulating a potential function, the parameters of the domain of which is determined by the poles of the two discontinuous analytic functions, and by extending his analysis to meromorphic functions.[9] A function is said to be meromorphic in a domain if it is analytic in the domain determined by the poles of analytic functions. A meromorphic function is determined by the quotient of two arbitrary analytic functions, which have been determined independently on the same surface by the point-wise operations of Weierstrassian analysis. Such a function is defined by the differential relation:

$$\frac{dy}{dx} = \frac{Y}{X}$$

*Figure 4.5* The meromorphic function.

where *X* and *Y* are the polynomials, or power series of the two local functions. The meromorphic function is the differential relation of



the function between the two discontinuous analytic functions. The expansion of the power series determined by the repeated differentiation of the meromorphic function generates the graph of a composite function that consists of curves with infinite branches, because the series generated by the expansion of the meromorphic function is divergent. The representation of such curves however posed a problem for Weierstrass, which he was unable to resolve, because divergent series fall outside the parameters of the differential calculus, as determined by the epsilon-delta approach, since they defy the criterion of convergence.

Henri Poincaré took up this problem of the representation of composite functions, by extending the Weierstrassian theory of meromorphic functions into what was called 'the qualitative theory of differential equations', or theory of automorphic functions (Kline 1972, p. 732). While such divergent series do not converge to a function, in the Weierstrassian sense, they may indeed furnish a useful approximation to a function if they can be said to represent the function asymptotically. When such a series is asymptotic to the function, it can represent an analytic or composite function even though the series is divergent. The determination of a composite function requires the determination of a new singularity in relation to the poles of the local functions of which it is composed. Poincaré called this new kind of singularity an essential singularity. Poincaré distinguished four types of essential singularity, which he classified according to the behaviour of the function and the geometrical appearance of the solution curves in the neighbourhood of these points: the saddle point or dip (col); the node (nœud); the point of focus (foyer); and, the centre (Barrow-Green 1997, p. 32; *DR* 177). Singularities develop increasingly complex relations with the increasing complexity of curves. The subsequent developments that the Weierstrassian theory of analytic continuity undergoes, up to and including Poincaré's theory of automorphic functions, is the material that Deleuze draws upon to offer a solution to overcome and extend the limits of Leibniz's metaphysics. The details of this critical move on Deleuze's part are examined in the final section of the essay.

## Deleuze's 'Leibnizian' interpretation of the theory of compossibility

What then does Deleuze mean by claiming that Leibniz determines the singularity in the domain of mathematics as a philosophical concept? A crucial test for Deleuze's mathematical reconstruction of Leibniz's metaphysics is how to deal with his subject-predicate logic.

Deleuze maintains that Leibniz's mathematical account of continuity is reconcilable with the relation between the concept of a subject and its predicates. The solution that Deleuze proposes involves demonstrating that the continuity characteristic of the infinitesimal calculus is isomorphic to the series of predicates contained in the concept of a subject. An explanation of this isomorphism requires an explication of Deleuze's understanding of Leibniz's account of predication as determined by the principle of sufficient reason.

For Leibniz, every proposition can be expressed in subject-predicate form. The subject of any proposition is a complete individual substance that is a simple, indivisible, dimensionless metaphysical point or monad. Of this subject it can be said that 'every analytic proposition is true', where an analytical proposition is one in which the predicate is identical with the subject. Deleuze suggests that if this principle of identity is reversed, such that it reads: 'every true proposition is necessarily analytic', then this amounts to a formulation of Leibniz's principle of sufficient reason (*CGD* 15 April 1980). According to this principle each time a true proposition is formulated, it must be understood to be analytic, that is, every true proposition is a statement of identity whose predicate is wholly contained in its subject. It follows that if a proposition is true, then the predicate must be either reciprocal with the subject or contained in the concept of the subject. That is, everything that happens to, everything that can be attributed to, everything that is predicated of a subject – past, present and future – must be contained in the concept of the subject. So for Leibniz, all predicates, that is, the predicates that express all of the states of the world, are contained in the concept of each and every particular or singular subject.

There are however grounds to distinguish truths of reason or essence, from truths of fact or existence. An example of a truth of essence would be the proposition 2+2=4, which is *analytic*, therefore, there is an identity of the predicate, 2+2, with the subject, 4. This can be proved by analysis, that is, in a finite or limited number of quite determinate operations, it can be demonstrated that 4, by virtue of its definition, and 2+2, by virtue of their definition, are identical. So, the identity of the predicate with the subject in an analytic proposition can be demonstrated in a finite series of determinate operations. While 2+2=4 occurs in all time and in all places, and is therefore a necessary truth, the proposition that 'Adam sinned', is specifically dated, that is, Adam will sin in a particular place at a particular time. It is therefore a truth of existence, and as we shall see, a contingent truth. According to the principle of sufficient reason, the proposition 'Adam sinned' must be analytic. If we pass from one predicate to another to retrace all the causes and follow up all the effects, this would involve the entire series of predicates contained in the subject Adam, that is, the analysis would extend to infinity. So, in order to demonstrate the inclusion of 'sinner' in the concept of 'Adam,' an infinite series of operations is required. However, we are not capable of completing such an analysis to infinity.



While Leibniz is committed to the idea of potential 'syncategorematic' infinity, that is, to infinite pluralities such as the terms of an infinite series which are indefinite or unlimited, he ultimately accepted that in the realm of quantity infinity could in no way be construed as a unified whole by us. As Bassler clearly explains: 'So if we ask how many terms there are in an infinite series, the answer is not: an infinite number (if we take this either to mean a magnitude which is infinitely larger than a finite magnitude or a largest magnitude) but rather: more than any given finite magnitude' (Bassler 1998, p. 65). The performance of such an analysis is indefinite both for us, as finite human beings, because our understanding is limited, and for God, since there is no end of the analysis, that is, it is unlimited. However, all the elements of the analysis are given to God in an actual infinity. We cannot grasp the actual infinite, nor reach it via an indefinite intuitive process. It is only accessible for us via finite systems of symbols that approximate it. The infinitesimal calculus provides us with an 'artifice' to operate a well-founded approximation of what happens in God's understanding. We can approach God's understanding thanks to the operation of infinitesimal calculus, without ever actually reaching it. While Leibniz always distinguished philosophical truths and mathematical truths, Deleuze maintains that the idea of infinite analysis in metaphysics has 'certain echoes' in the calculus of infinitesimal analysis in mathematics. The infinite analysis that we perform as human beings in which sinner is contained in the concept of Adam is an indefinite analysis, just as if the terms of the series that includes sinner were isometric with 1/2+1/4+1/8, etc., to infinity. In truths of essence, the analysis is finite, whereas in truths of existence, the analysis is infinite under the above-mentioned conditions of a well-determined finitude.

So what distinguishes truths of essence from truths of existence is that a truth of essence is such that its contrary is contradictory and therefore impossible, that is, it is impossible for 2 and 2 not to equal 4. Just as the identity of 4 and 2+2 can be proved in a series of finite procedures, so too can the contrary, 2+2 not equalling 4, be proved to be contradictory and therefore impossible. While it is impossible to think what 2+2 not equalling 4 or a squared circle may be, it is possible to think of an

Adam who might not have sinned. Truths of existence are therefore contingent truths. A world in which Adam might not have sinned is a logically possible world, that is, the contrary is not necessarily contradictory. While the relation between Adam sinner and Adam non-sinner is a relation of contradiction since it is impossible that Adam is both sinner and non-sinner, Adam non-sinner is not contradictory with the world where Adam sinned, it is rather incompossible with such a world. Deleuze argues that to be incompossible is therefore not the same as to be contradictory, it is another kind of relation that exceeds the contradiction.[10] Deleuze characterises the relation of incompossibility as 'a difference and not a negation' (*TF* 150). Incompossibility conserves a very classical principle of disjunction: it is either this world or some other one. So, when analysis extends to infinity, the type or mode of inclusion of the predicate in the subject is compossiblity. What interests Leibniz at the level of truths of existence is not the identity of the predicate and the subject, but rather the process of passing from one predicate to another from the point of view of an infinite analysis, and it is this process that is characterised by Leibniz as having the maximum of continuity. While truths of essence are governed by the principle of identity, truths of existence are governed by the law of continuity.

Rather than discovering the identical at the end or limit of a finite series, infinite analysis substitutes the point of view of continuity for that of identity. There is continuity when the extrinsic case, for example, the circle, the unique triangle or the predicate, can be considered as included in the concept of the intrinsic case, that is, the infinitangular polygon, the virtual triangle, or the concept of the subject. The domain of incompossibility is therefore a domain different from that of the identity/contradiction. There is no logical identity between sinner and Adam, but there is a continuity. Two elements are in continuity when an infinitely small or vanishing difference is able to be assigned between these two elements. Here Deleuze shows in what way truths of existence are reducible to mathematical truths.

Deleuze offers a 'Leibnizian' interpretation of the difference between compossibility and incompossibility 'based only on divergence or convergence of series' (*TF* 150). He proposes the hypothesis that there is compossibility between two singularities when their 'series of ordinaries converge', that is, when the values of the 'series of regular points that derive from two singularities [...] coincide, otherwise there is discontinuity. In one case, you have the definition of compossibility, in the other case, the definition of incompossibility' (*CGD* 29 April 1980). If the series of ordinary or regular points that derive from singularities diverge, then you have a discontinuity. When the series diverge, when you can no longer compose the continuity of this world with the continuity of this other world, then it can no longer belong to the same world. There are therefore as many worlds as divergences. All worlds are possible, but they are incompossibles with each other. God conceives an infinity of possible worlds that are not compossible with each other, from which He chooses the best of possible worlds, which happens to be the world in which Adam sinned. A world is therefore defined by its continuity. What separates two incompossible worlds is the fact that there is discontinuity between the two worlds. It is in this way that Deleuze maintains that compossibility and incompossibility are the direct consequences of the theory of singularities.



## Overcoming the limits of Leibniz's metaphysics

When Deleuze makes the comment that '[t]he differential relation thus acquires a new meaning, since it expresses the analytical extension of one series into another, and no more the unity of converging series that would not diverge in the least from each other' (*TF* 8), this should be understood in relation to what is presented in this essay as the Weierstrassian development of the meromorphic function as a differential relation. Poincaré's subsequent development of the Weierstrassian meromorphic function means that a continuity can be established across divergent series. What this means is that the Leibnizian account of compossibility as the unity of convergent series, which relies on the exclusion of divergence, is no longer required by the mathematics. The mathematical idealisation has therefore exceeded the metaphysics, so, in keeping with Leibniz's insistence on the metaphysical importance of mathematical speculation, the metaphysics requires recalibration. Leibniz's metaphysics is limited by the part–whole or one–multiple structure according to which this unity of convergent series is fundamentally determined, whether in terms of the one monad containing the infinite series of predicates which express all of the states of the world, as determined by the principle of sufficient reason; or in terms of one God establishing the harmony of a multiplicity of monads, as determined by the pre-established harmony.

What Poincaré's theory of automorphic functions does is offer a way for the part–whole structure of Leibniz's metaphysics to be problematised and overcome. Post Poincaré, the infinite series of states of the world is no longer contained in each monad. There is no pre-established harmony. The continuity of the states of the actual world and the discrimination between what is compossible and what is incompossible with this world is no longer pre-determined. The logical possibilities of all incompossible worlds are now real possibilities, all of which have the potential to be actualised by monads as states of the current world. As Deleuze argues 'To the degree that the world is now made up of divergent series (the chaosmos), [...] the monad is now unable to contain the entire world as if in a closed circle that can be modified by projection' (*TF* 137). So while the Weierstrassian theory of analytic continuity is retrospectively mappable onto the Leibnizian account of the unity of convergent series, the subsequent developments by Poincaré provide a solution that can be understood to overcome the explicit limits of Leibniz's metaphysics. It is these aspects of Deleuze's project in *The Fold* that foreshadow the 'new Baroque and Neo-Leibnizianism' (*TF* 136) that Deleuze explores elsewhere in his body of work – the mathematical account of which is offered most explicitly in *Difference and Repetition*.

## Notes

1. The lettering has been changed to reflect more directly the isomorphism between this algebraic example and Leibniz's notation for the infinitesimal calculus.
2. Leibniz, 'Methodus tangentium inversa, seu de fuctionibus' (1673), see Katz (2007, p. 199), seu de fuctionibus' (1673), see Katz (2007, p. 199).
3. For an account of this problem with limits in Cauchy, see Potter (2004, pp. 85–6).
4. The infinitesimal is now considered to be a hyperreal number that exists in a cloud of other infinitesimals or hyperreals floating infinitesimally close to each real number on the hyperreal number line (Bell 2005, 262). The development of non-standard analysis however has not broken the stranglehold of classical analysis to any significant extent, however this seems to be more a matter of taste and practical utility rather than of necessity (Potter 2004, p. 85).
5. Non-standard analysis allows 'interesting reformulations, more elegant proofs and new results in, for instance, differential geometry, topology, calculus of variations, in the theories of functions of a complex variable, of normed linear spaces, and of topological groups' (Bos 1974, p. 81).
6. For a more extensive discussion of this aspect of Deleuze's project, see Duffy (2006a).
7. In addition to several mathematical examples of the inflexion as a *point-fold*, including the transformations of René Thom and the continuously deferred inflexion of the Koch curve (*TF* 16–7), Deleuze offers an example drawn from baroque architecture, according to which an inflexion serves to hide or round out the right angle, which is figured in the Gothic arch that has the geometrical shape of an ogive.



8. For a more extensive account of Deleuze's deployment of the Weierstrassian theory of analytic continuity and the role of power series, see Duffy (2006b).
9. It was Charles A. A. Briot and Jean-Claude Bouquet who introduced the term 'meromorphic' for a function which possessed just poles in that domain (Kline 1972, p. 642).
10. Deleuze characterises this as 'vice-diction' (*TF* 59).

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