A model-theoretic analysis of Fidel-structures for **mbC**

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Abstract

In this paper the class of Fidel-structures for the paraconsistent logic **mbC** is studied from the point of view of Model Theory and Category Theory. The basic point is that Fidel-structures for **mbC** (or **mbC**-structures) can be seen as first-order structures over the signature of Boolean algebras expanded by two binary predicate symbols N (for negation) and O (for the consistency connective) satisfying certain Horn sentences. This perspective allows us to consider notions and results from Model Theory in order to analyze the class of **mbC**-structures. Thus, substructures, union of chains, direct products, direct limits, congruences and quotient structures can be analyzed under this perspective. In particular, a Birkhoff-like representation theorem for **mbC**-structures as subdirect poducts in terms of subdirectly irreducible **mbC**-structures is obtained by adapting a general result for first-order structures due to Caicedo. Moreover, a characterization of all the subdirectly irreducible **mbC**-structures is also given. An alternative decomposition theorem is obtained by using the notions of weak substructure and weak isomorphism considered by Fidel for C_n-structures.

1 Paraconsistency and non-deterministic semantics

Paraconsistency is the study of logic systems having a negation $\neg$ which is not *explosive*, that is, there exist formulas α and β in the language of the logic such that β is not derivable from the contradictory set $\{\alpha, \neg\alpha\}$. In other words, the logic has contradictory but non-trivial theories.

There are several approaches to paraconsistency in the literature since the introduction in 1948 of Jaskowski's system of *Discussive logic* (see [20]), such as Relevant logics, Adaptive logics, Many-valued logics, and many others.[1] The well-known 3-valued logic **LP** (*Logic of Paradox*) was introduced by G. Priest in [24] with the aim of formalizing the philosophical perspective underlying G. Priest and R. Sylvan's *Dialetheism* (see, for instance, [28] and [25]). As it is well-known, the main thesis behind Dialetheism is that there are *true contradictions*, that is, that some sentences can be both true and false at the same time and in the same way. Since then, the logic **LP** was intensively studied and developed by several authors poposing, for instance, extensions to first-order languages and applications to Set Theory (see, among others, [30, 31, 23]).

The publication in 1963 of N. da Costa's Habilitation thesis *Sistemas Formais Inconsistentes* (*Inconsistent Formal Systems*, in Portuguese, see [15]) constitutes a landmark in the history of paraconsistency. In that thesis, da Costa introduces the hierarchy C_n (for $n \geq 1$) of C-systems. This approach to paraconsistency differs from others, as it is based on the idea of locally recovering the classical reasoning (in particular, the explosion law for negation) by means of a derived unary connective of *well-behavior*, $(\cdot)^\circ$. Being so, a contradiction is not explosive in general in such systems (namely, $\alpha, \neg\alpha \nvdash \beta$ for some α and β). But assuming additionally that α is well-behaved, then such a contradiction must be trivializing; namely, $\alpha, \neg\alpha, \alpha^\circ \vdash \beta$ for every β. The idea of C-systems was afterwards generalized by W. Carnielli and J. Marcos in [12] through the class of *Logics of Formal Inconsistency*, in short **LFI**s. In such logics, da Costa's well-behavior derived connective $(\cdot)^\circ$ is replaced by a (possibly primitive) *consistency* unary conective $\circ$. The basic idea of **LFI**s, as in da Costa's C-systems, is that $\alpha, \neg\alpha \nvdash \beta$ in general, but $\alpha, \neg\alpha, \circ\alpha \vdash \beta$ always. Of course the C-systems are particular cases of **LFI**s.

Giving a semantical interpretation for C-systems, and for **LFI**s in general, is not a simple task: most of the **LFI**s introduced in the literature (see [12, 10, 9]) are not algebraizable by means of the standard techniques of algebraic logic (including Blok and Pigozzi's method, see [5]). Being so, the use of semantics of a non-deterministic character have shown to be a useful alternative way for dealing with **LFI**s. Several non-deterministic semantical tools were introduced in the literature in order to analyze such systems, allowing so decision procedures for them: non-truth-functional bivaluations (proposed by N. da Costa and E. Alves in [16, 17]), possible-translations semantics (proposed by W. Carnielli in [7]), and non-deterministic matrices (or Nmatrices), proposed by A. Avron and I. Lev in [1]. In particular, the Nmatrix semantics can be analyzed from a general (non-deterministic) algebraic perspective through the notion of *swap structures* (see [9, Chapter 6] and [14]). However, as shown in [2], not every **LFI** can be characterized by a single finite Nmatrix. In particular, da Costa's logic C_1 cannot be characterized by a single finite Nmatrix.

Another interesting approach to non-deterministic semantics for non-classical logics was proposed for G. Priest in [26], through the notion of *plurivalent semantics*. Let $\mathcal{M} = \langle \mathcal{A}, D \rangle$ be a matrix semantics for a propositional signature Ξ (that is, $\mathcal{A}$ is an algebra over Ξ and D is a non-empty subset of the domain A of $\mathcal{A}$, called the set of *designated values*). Then, a plurivalent semantics over $\mathcal{M}$ is a pair $\mathcal{M}_\triangleright = \langle \mathcal{M}, \triangleright \rangle$ such that $\triangleright \subseteq \mathcal{V} \times A$ is a relation from the set $\mathcal{V}$ of propositional variables to A. It is assumed that, for every $p \in \mathcal{V}$, there is $a \in A$ such that

[1] For a good introductory article on Paraconsistency see [29] and the references therein.

$p \triangleright a$.[2] Given $\mathcal{M}_\triangleright$, a non-deterministic (or plurivalent) interpretation $[\cdot]^\triangleright_\mathcal{M}$ for the algebra of formulas over Ξ generated by $\mathcal{V}$ is defined recursively: $[p]^\triangleright_\mathcal{M} = \{a \in A \ : \ p \triangleright a\}$, if $p \in \mathcal{V}$; and $[c(\varphi_1, \ldots, \varphi_n)]^\triangleright_\mathcal{M} = \bigcup\{c^\mathcal{A}(a_1, \ldots, a_n) \ : \ a_i \in [\varphi_i]^\triangleright_\mathcal{M}$ for $1 \leq i \leq n\}$, for every n-ary connective c in Ξ. Given a set of formulas $\Gamma \cup \{\varphi\}$, Γ *infers* φ in $\mathcal{M}_\triangleright$, denoted by $\Gamma \models^\triangleright_\mathcal{M} \varphi$, if $[\varphi]^\triangleright_\mathcal{M} \cap D \neq \emptyset$ whenever $[\beta]^\triangleright_\mathcal{M} \cap D \neq \emptyset$ for every $\beta \in \Gamma$. The plurivalent consequence relation $\models^\mathcal{M}_p$ generated from $\mathcal{M}$ is defined as expected: $\Gamma \models^\mathcal{M}_p \varphi$ iff $\Gamma \models^\triangleright_\mathcal{M} \varphi$ for every $\mathcal{M}_\triangleright$. Priest have shown in [26] how to produce a family of plurivalent logics related to the first-degree entailment *FDE*. Additionally, in [27] he applies the notion of plurivalent semantics in order to analyze Indian Buddhist logic from the pespective of formal logic.

Before all these efforts, M. Fidel have proved in 1969 (despite it was only published in 1977, see [19]), for the first time in the literature, the decidability of da Costa's calculi C_n by means of a novel algebraic-relational class of structures called C_n-structures. Afterwards, this kind of structure was called *Fidel-structures* or **F**-*structures* (see [22]). A C_n-structure is a triple $\langle \mathcal{A}, \{N_a\}_{a \in A}, \{N_a^{(n)}\}_{a \in A}\rangle$ such that $\mathcal{A}$ is a Boolean algebra with domain A and each N_a and $N_a^{(n)}$ is a non-empty subset of A. The intuitive meaning of $b \in N_a$ and $c \in N_a^{(n)}$ is that b and c are possible values for the paraconsistent negation $\neg a$ of a and for the well-behavior a° of a in C_n, respectively. The use of relations instead of functions for interpreting the paraconsistent negation $\neg$ and the well-behavior connective $(\cdot)^\circ$ of the C_n calculi is justified by the fact that these connectives are not truth-functional. That is, they cannot be characterized by means of truth-functions, and so the use of relations seems to be a good choice, constituting the first non-deterministic semantics for da Costa's calculi C_n proposed in the literature. Given that every C_n can be characterized by **F**-structures over the 2-element Boolean algebra, as Fidel has shown in [19], this result evidences the greater expressive power of **F**-structures with respect to Nmatrices. A discussion about this topic can be found in [9, Sections 6.6 and 6.7].

Fidel-structures can be defined for a wide class of logics, not only paraconsistent ones. Concerning **LFI**s, Fidel structures were defined in [8] and [9, Chapter 6] for several **LFI**s which are weaker than da Costa's C_1, starting from a basic but very interesting **LFI** called **mbC**.

This paper proposes the study of Fidel-structures for **mbC** from the point of view of Model Theory. Under this perspective, substructures, union of chains, direct products, direct limits, congruences and quotient structures can be defined. In particular, a generalization to first-order structures of Birkhoff's representation theorem for algebras, due to Caicedo, can be obtained for **mbC**-structures (see Theorem 56). Moreover, a characterization of the subdirectly irreducible **mbC**-structures is given in Theorem 55. This representation theorem is compared to a similar one, obtained by Fidel for the calculi C_n.

2 The logic mbC

The logic **mbC** is the most basic **LFI** analyzed by Carnielli, Coniglio and Marcos in [10], which later on it was studied by several authors. This logic is based on propositional positive classical

[2] The general case in which this conditions is dropped is briefly analyzed by Priest in [26].

logic and, despite its apparent simplicity, it enjoys extremely interesting features. This section is devoted to briefly describe the logic **mbC** for the reader's convenience.

Consider the propositional signature $\Sigma = \{\wedge, \vee, \to, \neg, \circ\}$ for **LFI**s, and let $\mathcal{V} = \{p_i \ : \ i \in \mathbb{N}\}$ be a denumerable set of propositional variables. The algebra of propositional formulas over Σ generated by $\mathcal{V}$ will be denoted by $For(\Sigma)$.

Definition 1 (Logic mbC, [10]) *The calculus* **mbC** *over the language* $For(\Sigma)$ *is defined by means of the following Hilbert calculus :*

(A1) $\alpha \to (\beta \to \alpha)$

(A2) $(\alpha \to \beta) \to ((\alpha \to (\beta \to \gamma)) \to (\alpha \to \gamma))$

(A3) $\alpha \to (\beta \to (\alpha \wedge \beta))$

(A4) $(\alpha \wedge \beta) \to \alpha$

(A5) $(\alpha \wedge \beta) \to \beta$

(A6) $\alpha \to (\alpha \vee \beta)$

(A7) $\beta \to (\alpha \vee \beta)$

(A8) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

(A9) $\alpha \vee (\alpha \to \beta)$

(A10) $\alpha \vee \neg\alpha$

(A11) $\circ\alpha \to (\alpha \to (\neg\alpha \to \beta))$

(MP) From α and $\alpha \to \beta$ infer β

It is worth noting that **mbC** is obtained from a calculus for the positive cassical logic $\mathbf{CPL}^+$ by adding the axiom schemas (A10) and (A11), concerning the paraconsistent negation $\neg$ and the consistency operator $\circ$. As observed in [12], each calculus C_n is a particular case of **LFI** in which the consistency connective $\circ$ is defined in term of the others: for instance, $\circ\alpha \stackrel{\text{def}}{=} \alpha^\circ = \neg(\alpha\wedge\neg\alpha)$ in C_1. The logic C_1 can be seen as an axiomatic extension of **mbC** up to language (see [10, 9]).

Being weaker than C_1, the logic **mbC** cannot be characterized by any standard algebraic semantics, even in the wide sense of Blok-Pigozzi [5]. Moreover, it cannot be characterized by a single finite logical matrix. From this, it is clear that alternative semantics for **mbC** are necessary. In [8] and [9], Fidel-structures for **mbC** and for several axiomatic extensions of it were presented. However, the formal study of the properties of the class of **F**-structures for such **LFI**s was never developed, in contrast with the study of the **F**-structures for C_n carry out by Fidel in [19].

In the following sections the class of **F**-structures for **mbC** will be studied as a basic but relevant example through an original approach based on elementary concepts of Model Theory and Category Theory.[3]

3 Fidel-structures for mbC

In this section, the class of **F**-structures for **mbC** introduced in [8] and [9] will be recast in the language of first-order structures. This constitutes a novel approach to Fidel-structures in general.

Consider classical first-order theories defined over first-order signatures based on the following logical symbols: the connectives $\wedge$, $\vee$, $\to$ and $\sim$ (for conjunction, disjunction, implication and negation, respectively), the quantifiers $\forall$ and $\exists$, and the symbol $\approx$ for the equality predicate which is always interpreted as the identity relation. Assume also a denumerable set $\mathcal{V}_{ind} = \{v_i : i \in \mathbb{N}\}$ of variables. The letters u, w, z will be used to refer to arbitrary variables. If Ξ is a first-order signature then Ξ-**str** denotes the category of Ξ-structures, that is, the category of first-order structures over the signature Ξ.

Definition 2 *The* signature for **F**-structures for **LFI**s *is the first-order signature Θ composed by the following symbols:*

(i) Two binary predicate symbols N and O, for the paraconsistent negation and the consistency operator, respectively;

(ii) Two binary function symbols $\sqcap$ and $\sqcup$, and an unary function symbol $-$, for Boolean meet, Boolean join and Boolean complement, respectively;

(iii) Two constant symbols $\mathbf{0}$ and $\mathbf{1}$ for the bottom and the top element, respectively.

Observe that the subsignature Θ_{BA} of Θ obtained by dropping the predicate symbols N, O is the usual signature for Boolean algebras.

Definition 3 *An* **F***-structure for* **mbC** *(in short, an* **mbC***-structure) is a Θ-first order structure*

$$\mathcal{E} = \langle A, \sqcap^{\mathcal{E}}, \sqcup^{\mathcal{E}}, -^{\mathcal{E}}, \mathbf{0}^{\mathcal{E}}, \mathbf{1}^{\mathcal{E}}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$$

such that:

(a) the Θ_{BA}-reduct $\mathcal{A} = \langle A, \sqcap^{\mathcal{E}}, \sqcup^{\mathcal{E}}, -^{\mathcal{E}}, \mathbf{0}^{\mathcal{E}}, \mathbf{1}^{\mathcal{E}} \rangle$ of $\mathcal{E}$ is a Boolean algebra; that is, $\mathcal{A}$ satisfies the usual equations axiomatizing Boolean algebras in the signature Θ_{BA}, see for instance [13, Example 1.4.3];[4]

[3] For general notions on Model Theory and Category Theory the reader can consult [13] and [21], respectively.

[4] In difference with several authors, we admit the trivial one-element Boolean algebra, see Remark 39.

(b) $\mathcal{E}$ *satisfies the following* Θ*-sentences:*

(i) $\forall u \exists w N(u,w)$,

(ii) $\forall u \exists w O(u,w)$,

(iii) $\forall u \forall w (N(u,w) \to (u \sqcup w \approx \mathbf{1}))$,

(iv) $\forall u \forall w (N(u,w) \to \exists z (O(u,z) \wedge ((u \sqcap w \sqcap z) \approx \mathbf{0}))$.

The class of **mbC***-structures will be denoted by* **FmbC***.*

Notation 4 *From now on, an* **mbC***-structure* $\mathcal{E}$ *will be denoted by* $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ *such that* $\mathcal{A} = \langle A, \sqcap^{\mathcal{E}}, \sqcup^{\mathcal{E}}, -^{\mathcal{E}}, \mathbf{0}^{\mathcal{E}}, \mathbf{1}^{\mathcal{E}} \rangle$ *is a Boolean algebra. We will frequently write* $\#^{\mathcal{A}}$ *instead of* $\#^{\mathcal{E}}$ *for* $\# \in \{\sqcap, \sqcup, -, \mathbf{0}, \mathbf{1}\}$*. Given an* **mbC***-structure* $\mathcal{E}$ *and* $a, b \in A$*, alternatively we can write* $b \in N^{\mathcal{E}}_a$ *instead of* $N^{\mathcal{E}}(a,b)$*. Similar notation will be adopted for the predicate symbol* O*, alternatively writing* $b \in O^{\mathcal{E}}_a$ *instead of* $O^{\mathcal{E}}(a,b)$*. This is in line with the traditional presentation of* **F***-structures where the non-truth-functional unary connectives are interpreted by families of non-empty subsets of the domain of the structure (recall the definition of* C_n*-structure outlined in Section 1).*

The intuitive reading for $b \in N^{\mathcal{E}}_a$ and $c \in O^{\mathcal{E}}_a$ is that b is a *possible negation* $\neg a$ of a, and that c is a *possible consistency* $\circ a$ of a coherent with b. This is justified by Definition 6 below.

Remark 5 *Observe that* $N^{\mathcal{E}}(\mathbf{0}^{\mathcal{E}}, b)$ *iff* $b = \mathbf{1}^{\mathcal{E}}$*, for every* **F***-structure* $\mathcal{E}$ *for* **mbC***. On the other hand, if* $(\mathbf{1}^{\mathcal{E}}, \mathbf{1}^{\mathcal{E}}) \in N^{\mathcal{E}}$ *then* $(\mathbf{1}^{\mathcal{E}}, \mathbf{0}^{\mathcal{E}})$ *must belong to* $O^{\mathcal{E}}$*. Indeed, it must exists* $z \in A$ *such that* $(\mathbf{1}^{\mathcal{E}}, z) \in O^{\mathcal{E}}$ *and* $\mathbf{1}^{\mathcal{E}} \sqcap \mathbf{1}^{\mathcal{E}} \sqcap z = \mathbf{0}^{\mathcal{E}}$*. Therefore* $z = \mathbf{0}^{\mathcal{E}}$ *and so* $(\mathbf{1}^{\mathcal{E}}, \mathbf{0}^{\mathcal{E}}) \in O^{\mathcal{E}}$*.*

Recall now the semantics for **mbC** defined from **F**-structures (see [8, 9]). Given an **mbC**-structure $\mathcal{E}$, consider the Boolean implication defined as usual by $a \Rightarrow^{\mathcal{E}} b \stackrel{\text{def}}{=} -^{\mathcal{E}} a \sqcup^{\mathcal{E}} b$ for every $a, b \in A$. For $\# \in \{\wedge, \vee, \to\}$ let $\widehat{\#}^{\mathcal{E}}$ be the corresponding operation in $\mathcal{E}$, that is: $\widehat{\wedge}^{\mathcal{E}} = \sqcap^{\mathcal{E}}$; $\widehat{\vee}^{\mathcal{E}} = \sqcup^{\mathcal{E}}$; and $\widehat{\to}^{\mathcal{E}} = \Rightarrow^{\mathcal{E}}$. As stated above, $For(\Sigma)$ denotes the algebra of formulas for the logic **mbC**.

Definition 6 *A* valuation *over an* **mbC***-structure* $\mathcal{E}$ *is a map* $v : For(\Sigma) \to A$ *satisfying the following properties, for every formulas* α *and* β*:*

(1) $v(\alpha \# \beta) = v(\alpha) \widehat{\#}^{\mathcal{E}} v(\beta)$*, for* $\# \in \{\wedge, \vee, \to\}$*;*

(2) $v(\neg\alpha) \in N^{\mathcal{E}}_{v(\alpha)}$ *(that is,* $N^{\mathcal{E}}(v(\alpha), v(\neg\alpha))$*);*

(3) $v(\circ\alpha) \in O^{\mathcal{E}}_{v(\alpha)}$ *(that is,* $v(\circ\alpha)$ *is such that* $O^{\mathcal{E}}(v(\alpha), v(\circ\alpha))$*) and* $v(\alpha) \sqcap^{\mathcal{E}} v(\neg\alpha) \sqcap^{\mathcal{E}} v(\circ\alpha) = \mathbf{0}^{\mathcal{E}}$*.*

Observe that item (3) of the previous definition is well-defined by item (b)(iv) of Definition 3 and item (2) of the last definition.

The semantical consequence relation associated to **F**-structures for **mbC** is naturally defined:

Definition 7 *Let $\Gamma \cup \{\alpha\} \subseteq For(\Sigma)$ be a finite set of formulas.*
(i) Given a Fidel-structure $\mathcal{E}$ for **mbC**, *we say that α is a semantical consequence of Γ (w.r.t. $\mathcal{E}$), denoted by $\Gamma \Vdash_{\mathcal{E}}^{\mathbf{mbC}} \alpha$, if, for every valuation v over $\mathcal{E}$: $v(\alpha) = 1$ whenever $v(\gamma) = 1$ for every $\gamma \in \Gamma$.*
(ii) We say that α is a semantical consequence of Γ (w.r.t. Fidel-structures for **mbC**), *denoted by $\Gamma \Vdash_{\mathbf{F}}^{\mathbf{mbC}} \alpha$, if $\Gamma \Vdash_{\mathcal{E}}^{\mathbf{mbC}} \alpha$ for every* **F**-*structure $\mathcal{E}$ for* **mbC**.

Theorem 8 (Soundness and completeness of mbC w.r.t. F-structures, [8, 9]) *Let $\Gamma \cup \{\alpha\}$ be a finite set of formulas in $For(\Sigma)$. Then: $\Gamma \vdash_{\mathbf{mbC}} \alpha$ iff $\Gamma \Vdash_{\mathbf{F}}^{\mathbf{mbC}} \alpha$.*

Moreover, let $\mathbb{A}_2$ be the two-element Boolean algebra with domain $\{0, 1\}$, and let $\Vdash_{\mathbf{F}_2}^{\mathbf{mbC}}$ be the semantical consequence relation with respect to the class of **mbC**-structures defined over $\mathbb{A}_2$. By adapting the proof of [9, Theorem 6.2.16] it is easy to obtain the following result:

Theorem 9 (Soundness and completeness of mbC w.r.t. F-structures over $\mathbb{A}_2$) *Let $\Gamma \cup \{\alpha\}$ be a finite set of formulas in $For(\Sigma)$. Then: $\Gamma \vdash_{\mathbf{mbC}} \alpha$ iff $\Gamma \Vdash_{\mathbf{F}_2}^{\mathbf{mbC}} \alpha$.*

The latter result give us a decision procedure for checking validity in **mbC**.

4 On the axiomatization of mbC-structures

In this section we briefly discuss the class **FmbC** of **F**-structures from the point of view of the syntactic form of its axioms (recall Definition 3). From this, some well-known results from Model Theory can be applied to **FmbC** in a direct way, as we shall see along this paper.

Recall from [13, Page 407] that a *basic Horn formula* over a first-order signature Ξ is a formula of the form $\sigma_1 \vee \ldots \vee \sigma_n$ (for $n \geq 1$) such that at most one formula is atomic and the rest is the negation of an atomic formula over Ξ. In particular, formulas of the form $\sigma_1 \wedge \ldots \wedge \sigma_n \to \sigma_{n+1}$, where each σ_i is atomic, are (logically equivalent to) basic Horn formulas. A *Horn formula* over Ξ is any formula over Ξ built up from basic Horn formulas by using exclusively the connective $\wedge$ and the quantifiers $\forall$ and $\exists$. A *Horn sentence* is a Horn formula with no free variables.

Remark 10 (FmbC as a Horn theory) *From Definition 3, it is easy to see that* **FmbC** *can be axiomatized by means of Horn sentences. Indeed, the axioms of Boolean algebras are sentences of the form $\forall x_1 \cdots \forall x_n \sigma$, where σ is an atomic formula. On the other hand, axioms (b)(i) and (b)(ii) are of the form $\forall u \exists w \sigma$, where σ is an atomic formula. Axiom (b)(iii) is of the form $\forall u \forall w(\sigma_1 \to \sigma_2)$, where σ_1 and σ_2 are atomic formula. Finally, axiom (b)(iv) is logically equivalent to a sentence of the form $\forall u \forall w \exists z((\sigma_1 \to \sigma_2) \wedge (\sigma_1 \to \sigma_3))$ where each σ_i is atomic. This means that* **FmbC** *can be axiomatized by Horn sentences.*

Now, recall from [13, Pages 142-143] that an *universal-existencial sentence*, or a $\forall\exists$-*sentence*, or a Π^0_2-*sentence* over a first-order signature Ξ, is a sentence of the form $\forall x_1 \cdots \forall x_n \exists y_1 \cdots \exists y_k \sigma$, where σ is a formula over Ξ without quantifiers.

Remark 11 (FmbC as a Π^0_2-theory) *It is immediate to see that the class of* **mbC**-*structures can be axiomatized by means of* $\forall\exists$-*sentences (that is,* Π^0_2-*sentences). Indeed, the axioms of Boolean algebras are of the form* $\forall x_1 \cdots \forall x_n \sigma$, *where* σ *is an atomic formula. On the other hand, axioms (b)(i) and (b)(ii) are of the form* $\forall u \exists w \sigma$, *where* σ *is an atomic formula. Axiom (b)(iii) is of the form* $\forall u \forall w \sigma$, *where* σ *is without quantifiers. Finally, axiom (b)(iv) is logically equivalent to a sentence of the form* $\forall u \forall w \exists z \sigma$ *where* σ *has not quantifiers. This means that* **FmbC** *can be axiomatized by* Π^0_2-*sentences over signature* Θ.

Finally, it is worth noting that **FmbC** can also be axiomatized by sentences over signature Θ of the form $\forall x_1 \cdots \forall x_n (\sigma_1 \to \exists y_1 \cdots \exists y_k \sigma_2))$, where σ_1 and σ_2 are positive formulas (that is, built up from conjunctions and disjunctions only) without quantifiers.

The fact that **FmbC** can be axiomatized by sentences of a special form will be used along this paper, as it will be pointed out.

5 Homomorphisms and substructures

From the definitions of the previous section, the category of **mbC**-structures can be defined in a natural way.

Definition 12 *Let* $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ *and* $\mathcal{E}' = \langle \mathcal{A}', N^{\mathcal{E}'}, O^{\mathcal{E}'} \rangle$ *be two* **mbC**-*structures. An* **mbC**-homomorphism *h from* $\mathcal{E}$ *to* $\mathcal{E}'$ *is a homomorphism* $h : \mathcal{E} \to \mathcal{E}'$ *in the category of* Θ-*structures.*

Remark 13 *By definition, an* **mbC**-*homomorphism* $h : \mathcal{E} \to \mathcal{E}'$ *is a function* $h : A \to A'$ *satisfying the following conditions, for every* $a, b \in A$:

(i) $h : \mathcal{A} \to \mathcal{A}'$ *is a homomorphism between Boolean algebras,*

(ii) if $N^{\mathcal{E}}(a, b)$ *then* $N^{\mathcal{E}'}(h(a), h(b))$;

(iii) if $O^{\mathcal{E}}(a, b)$ *then* $O^{\mathcal{E}'}(h(a), h(b))$.

From the notions above, it is defined a category $\mathbb{F}$**mbC** of **mbC**-structures having the class **FmbC** of **F**-structures as objects and with **mbC**-homomorphisms as its morphisms. Clearly, it is a full subcategory of the category Θ-**str** of Θ-structures. For every **mbC**-structure $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$, the identity homomorphism given by the identity mapping over A will be denoted by $id_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$. If $h : \mathcal{E} \to \mathcal{E}'$ and $h' : \mathcal{E}' \to \mathcal{E}''$ are two homomorphisms then the composite homomorphism from $\mathcal{E}$ to $\mathcal{E}''$ will be denoted by $h' \circ h$. As a consequence of the definitions, well-known basic notions and results from Model Theory can be applied to $\mathbb{F}$**mbC**. The only detail to be taken into account in the constructions is that **mbC**-structures are first-order structures over Θ satisfying certain Θ-sentences, as it was discussed in Section 4.

Definition 14 *Let $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ and $\mathcal{E}' = \langle \mathcal{A}', N^{\mathcal{E}'}, O^{\mathcal{E}'} \rangle$ be two* **mbC***-structures. The structure $\mathcal{E}$ is said to be a* substructure *of $\mathcal{E}'$, denoted by $\mathcal{E} \subseteq \mathcal{E}'$, if the following conditions hold:*

(i) *$\mathcal{A}$ is a Boolean subalgebra of $\mathcal{A}'$ (which will be denoted as $\mathcal{A} \subseteq \mathcal{A}'$),*[5]

(ii) *$N^{\mathcal{E}} = N^{\mathcal{E}'} \cap (A')^2$ and $O^{\mathcal{E}} = O^{\mathcal{E}'} \cap (A')^2$.*

Remark 15 *The notion of substructure considered in Model Theory (and, in particular, in Definition 14) differ slightly from that used in some areas of Mathematics. For instance, in Graph Theory a graph $H = \langle V, E \rangle$ (where V and E denote, respectively, the set of vertices and edges) is said to be a* subgraph *of another grap $G = \langle V', E' \rangle$ provided that $V \subseteq V'$ and $E \subseteq E'$. Then, it can be possible to have a subgraph H of a graph G such that H is not a substructure of G seen as first-order structures (over the signature of graphs having, besides the equality predicate symbol $\approx$, a binary predicate symbol for the* edge *relation). Indeed, H is a substructure of G if and only if H is what is called in Graph Theory a* induced *subgraph of G.*

In Category Theory and in Universal Algebra the subobjects (the subalgebras, respectively) are characterized by means of the the notion of monomorphism. Recall from Category Theory (see, for instance, [21]) that a *monomorphism* in a category $\mathbb{C}$ is a homomorphism $h : A \to B$ in $\mathbb{C}$ such that, for every pair of parallel homomorphisms $h', h'' : C \to A$ for which $h \circ h' = h \circ h''$ in $\mathbb{C}$, it is the case that $h' = h''$. On the other hand, recall the following notion from Model Theory (see, for instance, [13]):

Definition 16 *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two first-order structures over a signature Ξ. A homomorphism $h : \mathfrak{A} \to \mathfrak{A}'$ in Ξ-***str** *is an* embedding *if it is injective and, for every n-ary predicate symbol P and every $(a_1, \ldots, a_n) \in |\mathfrak{A}|^n$, $(a_1, \ldots, a_n) \in P^{\mathfrak{A}}$ if and only if $(h(a_1), \ldots, h(a_n)) \in P^{\mathfrak{A}'}$.*[6]

It is well-known that a homomorphism $h : \mathfrak{A} \to \mathfrak{A}'$ in Ξ-**str** is an embedding if and only if it is a monomorphism in the subcategory Ξ-**emb** of Ξ-**str** formed by Ξ-structures as objects and embeddings as morphisms. Clearly, if Ξ has only function symbols besides the identity predicate $\approx$ (this is the case of Universal Algebra) then Ξ-**emb** is the subcategory of Ξ-**str** in which every morphism is a monomorphism. Moreover, the induced substructures correspond to the subobjects in Ξ-**str** .

Remark 17 *As a direct consequence of the definitions, a homomorphism $h : \mathcal{E} \to \mathcal{E}'$ in $\mathbb{F}$***mbC** *is an embedding if and only if it is an injective homomorphism where conditions (ii) and (iii) of Remark 13 are replaced by*

(ii)' $N^{\mathcal{E}}(a, b)$ *if and only if* $N^{\mathcal{E}'}(h(a), h(b))$*;*

[5] This means that $A \subseteq A'$, $\mathbf{0}^{\mathcal{A}} = \mathbf{0}^{\mathcal{A}'}$, $\mathbf{1}^{\mathcal{A}} = \mathbf{1}^{\mathcal{A}'}$ and, for every $a, b \in A$: $a \#^{\mathcal{A}} b = a \#^{\mathcal{A}'} b$ and $-^{\mathcal{A}} a = -^{\mathcal{A}'} a$ for $\# \in \{\sqcap, \sqcup\}$. Note that we write $s^{\mathcal{A}}$ instead of $s^{\mathcal{E}}$ when s correspond to a symbol of the subsignature Θ_{BA} of Θ.

[6] Since in Model Theory the equality predicate $\approx$ is considered as a predicate symbol which is always interpreted as the standard equality, the injectivity of an embedding is a consequence of the definition.

(iii)' $O^{\mathcal{E}}(a,b)$ if and only if $O^{\mathcal{E}'}(h(a),h(b))$.

Clearly, $\mathcal{E} \subseteq \mathcal{E}'$ if and only if $A \subseteq A'$ and the inclusion map $i : A \to A'$ induces an embedding $i : \mathcal{E} \to \mathcal{E}'$ of Θ-structures.

Remark 18 *From the definitions above, it is clear that in the category $\mathbb{F}$**mbC** the monomorphism (that is, the substructures in Θ–**str**) are not strong enough: in order to obtain a substructure in $\mathbb{F}$**mbC** the monomorphism must be, in addition, an embedding. Consider, for instance, the **mbC**-structures $\mathcal{E} = \langle \mathbb{A}_2, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ and $\mathcal{E}' = \langle \mathbb{A}_2, N^{\mathcal{E}'}, O^{\mathcal{E}'} \rangle$ defined over the two-element Boolean algebra $\mathbb{A}_2$ such that $N^{\mathcal{E}} = \{(0,1),(1,0)\}$, $O^{\mathcal{E}} = \{(0,1),(1,1)\}$ and $N^{\mathcal{E}'} = O^{\mathcal{E}'} = \{(0,1),(1,0),(1,1)\}$. Clearly, the identity $h : \{0,1\} \to \{0,1\}$ induces a monomorphism $h : \mathcal{E} \to \mathcal{E}'$ in $\mathbb{F}$**mbC** (since it is an injective homomorphism in $\mathbb{F}$**mbC**). But $\mathcal{E}$* is not *a substructure of $\mathcal{E}$' since, for instance, $(1,0) \in O^{\mathcal{E}'} \cap |\mathcal{E}|^2$ but $(1,0) \notin O^{\mathcal{E}}$.*

A weaker notion of substructure was considered in the literature of Model Theory under the name of *weak substructures*. Thus, $\mathfrak{A}$ is said to be a *weak substructure* of $\mathfrak{A}'$ in Ξ-**str** provided that $|\mathfrak{A}| \subseteq |\mathfrak{A}'|$ and $P^{\mathfrak{A}} \subseteq P^{\mathfrak{A}'}$, for every predicate symbol P. This is equivalent to say that the inclusion mapping $i : |\mathfrak{A}| \to |\mathfrak{A}'|$ induces a homomorphism $i : \mathfrak{A} \to \mathfrak{A}'$ in Ξ-**str**. For instance, in Graph Theory a graph H is a subgraph of a graph G provided that H is a weak substructure of G, as observed in Remark 15. In $\mathbb{F}$**mbC** it can be considered a intermediate notion between weak substructures and substructures (as defined in Model Theory):

Definition 19 *Let $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ and $\mathcal{E}' = \langle \mathcal{A}', N^{\mathcal{E}'}, O^{\mathcal{E}'} \rangle$ be two **mbC**-structures. We say that $\mathcal{E}$ is a* weak substructure *of $\mathcal{E}'$ in $\mathbb{F}$**mbC**, denoted by $\mathcal{E} \subseteq_W \mathcal{E}'$, if $\mathcal{A}$ is a Boolean subalgebra of $\mathcal{A}'$, $N^{\mathcal{E}} \subseteq N^{\mathcal{E}'}$, and $O^{\mathcal{E}} \subseteq O^{\mathcal{E}'}$. This is equivalent to say that the inclusion map $i : A \to A'$ induces an injective homomorphism $i : \mathcal{E} \to \mathcal{E}'$ in $\mathbb{F}$**mbC**.*

Remark 20 *Observe that if a Ξ-structure $\mathfrak{A}$ is a substructure of $\mathfrak{A}'$ then it is a weak substructure of $\mathfrak{A}'$. On the other hand, if $\mathcal{E}$ is a substructure of an **mbC**-structure $\mathcal{E}$' (hence, a weak-substructure) in the category Θ-**str** then $\mathcal{E}$ is not necessarily an **mbC**-structure and so it is not necessarily a weak substructure of $\mathcal{E}'$ in $\mathbb{F}$**mbC**. For instance, consider the four-element Boolean algebra $\mathbb{A}_4$ with domain $FOUR = \{0, \mathsf{a}, \mathsf{b}, 1\}$ (observe that $\mathbb{A}_4$ is isomorphic, as a Boolean algebra, to the powerset of $\{0,1\}$) and let $\mathcal{E}' = \langle \mathbb{A}_4, N^{\mathcal{E}'}, O^{\mathcal{E}'} \rangle$ be the **mbC**-structure over $\mathbb{A}_4$ such that $N^{\mathcal{E}'} = \{(0,1),(\mathsf{a},\mathsf{b}),(\mathsf{b},\mathsf{a}),(1,\mathsf{a})\}$ and $O^{\mathcal{E}'} = \{(0,\mathsf{a}),(\mathsf{a},0),(\mathsf{b},0),(1,0)\}$. Let $\mathcal{E} = \langle \mathbb{A}_2, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ be the Θ-structure with domain $\{0,1\}$ such that $N^{\mathcal{E}} = \{(0,1)\} = N^{\mathcal{E}'} \cap \big(\{0,1\}\big)^2$ and $O^{\mathcal{E}} = \{(1,0)\} = O^{\mathcal{E}'} \cap \big(\{0,1\}\big)^2$. Then $\mathcal{E}$ is a substructure of $\mathcal{E}$' which is not an **mbC**-structure, hence it is not a weak substructure of $\mathcal{E}$' in $\mathbb{F}$**mbC**.*

*Finally, it is worth noting that if $\mathcal{E}$ and $\mathcal{E}$' are two **mbC**-structures such that $\mathcal{E}$ is a weak substructure of $\mathcal{E}$' then every valuation over $\mathcal{E}$ (recall Definition 6) is a valuation over $\mathcal{E}$'.*

It is interesting to notice that any **mbC**-structure $\mathcal{E}$ over $\mathbb{A}_2$ can be seen as a substructure (in $\mathbb{F}$**mbC**) of an **mbC**-structure $\mathcal{E}(\mathcal{A})$ over $\mathcal{A}$, for any Boolean algebra $\mathcal{A}$ with more than two elements:

Proposition 21 *Let $\mathcal{A}$ be a Boolean algebra with more that two elements. Given an* **mbC**-*structure $\mathcal{E} = \langle \mathbb{A}_2, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ defined over $\mathbb{A}_2$ let $\mathcal{E}(\mathcal{A}) \stackrel{def}{=} \langle \mathcal{A}, N^{\mathcal{E}}_{\mathcal{A}}, O^{\mathcal{E}}_{\mathcal{A}} \rangle$ be the Θ-structure defined as follows:*

- $N^{\mathcal{E}}_{\mathcal{A}} = N^{\mathcal{E}} \cup \{(a, {\sim}a) \ : \ a \in |\mathcal{A}| \setminus \{0,1\}\}$;
- $O^{\mathcal{E}}_{\mathcal{A}} = O^{\mathcal{E}} \cup \{(a, 1) \ : \ a \in |\mathcal{A}| \setminus \{0,1\}\}$.

Then, $\mathcal{E}(\mathcal{A})$ is an **mbC**-*structure over $\mathcal{A}$ such that $\mathcal{E} \subseteq \mathcal{E}(\mathcal{A})$ in* $\mathbb{F}$**mbC**.

Proof. It is straightforward from the definitions. □

As a consequence of the latter result, combined with Theorem 9, it can be seen that the logic of the **mbC**-structures over $\mathcal{A}$ is exactly **mbC**.

Theorem 22 (Soundness and completeness of mbC w.r.t. F-structures over a non-trivial Bolean algebra $\mathcal{A}$) *Let $\mathcal{A}$ be a non-trivial Boolean algebra (that is, $\mathbf{0}^{\mathcal{A}} \neq \mathbf{1}^{\mathcal{A}}$). Let $\Vdash^{\mathbf{mbC}}_{\mathbf{F}_{\mathcal{A}}}$ be the semantical consequence relation with respect to the class $\mathbf{F}_{\mathcal{A}}$ of* **mbC**-*structures defined over $\mathcal{A}$, and let $\Gamma \cup \{\alpha\}$ be a finite set of formulas in $For(\Sigma)$. Then: $\Gamma \vdash_{\mathbf{mbC}} \alpha$ iff $\Gamma \Vdash^{\mathbf{mbC}}_{\mathbf{F}_{\mathcal{A}}} \alpha$.*

Proof. If $\mathcal{A} = \mathbb{A}_2$ then the result follows by Theorem 9. Assume now that $\mathcal{A}$ has more than two elements. The 'only if' part is a consequence of Theorem 8. Now, suppose that $\Gamma \nvdash_{\mathbf{mbC}} \alpha$. By Theorem 9, there exists an **mbC**-structure $\mathcal{E}$ over $\mathbb{A}_2$, and a valuation v over $\mathcal{E}$ such that $v[\Gamma] \subseteq \{1\}$ but $v(\alpha) = 0$. By Proposition 21, there exists an **mbC**-structure $\mathcal{E}(\mathcal{A})$ over $\mathcal{A}$ such that $\mathcal{E} \subseteq \mathcal{E}(\mathcal{A})$. In particular, $\mathcal{E}$ is a weak substructure of $\mathcal{E}(\mathcal{A})$. By the last observation in Remark 20 it follows that v is also a valuation over $\mathcal{E}(\mathcal{A})$. This shows that $\Gamma \nVdash^{\mathbf{mbC}}_{\mathbf{F}_{\mathcal{A}}} \alpha$. □

It is worth noting that Fidel considered in [19] the notion of **F**-substructures stated in Definition 19 in order to obtain a decomposition result for **F**-structures for da Costa's calculi C_n in terms of irreducible structures. The adaptation of this result to **mbC**-structures will be briefly analyzed in Subsection 8.4.

Recall that an *epimorphism* in a category $\mathbb{C}$ is a homomorphism $h : A \to B$ such that, for every pair of parallel homomorphisms $h', h'' : B \to C$ such that $h' \circ h = h'' \circ h$, it is the case that $h' = h''$, see for instance [21]. Then:

Proposition 23 *A homomorphism $h : \mathcal{E} \to \mathcal{E}'$ is an epimorphism in* $\mathbb{F}$**mbC** *if and only if h is onto as a mapping.*

Recall now that an *isomorphism* in a category $\mathbb{C}$ is a homomorphism $h : A \to B$ such that there exists a homomorphism $h' : B \to A$ where $h' \circ h = id_A$ and $h \circ h' = id_B$, see for instance [21]. Then:

Proposition 24 *A homomorphism $h : \mathcal{E} \to \mathcal{E}'$ is an isomorphism in* $\mathbb{F}$**mbC** *if and only if h is an embedding which is onto, that is, h is a bijective embedding.*

This means that, in the category $\mathbb{F}$**mbC** of **mbC**-structures, h is an isomorphism if and only if it is both a monomorphism and an epimorphism.

6 Union of chains of mbC-structures

As a consequence of the proposed approach to **F**-structures as being a class of Θ-structures axiomatized by a set of sentences of a certain form (recall Section 4), some basic results from Model Theory can be applied to its study. In this section, the union of chains of **mbC**-structures will be analyzed.

Definition 25 *A* chain *of* **F***-structures for* **mbC** *is a family* $(\mathcal{E}_\lambda)_{\lambda<\mu}$ *for an ordinal* μ *such that* $\mathcal{E}_\xi \subseteq \mathcal{E}_\lambda$ *whenever* $\xi < \lambda < \mu$.

A chain can be displayed as $\mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \ldots \subseteq \mathcal{E}_\lambda \subseteq \ldots$ for $\lambda < \mu$. From Model Theory, it is known that, for every chain $(\mathcal{E}_\lambda)_{\lambda<\mu}$ of **F**-structures for **mbC** (seen as first-order structures), there exists a Θ-structure $\mathcal{E}$ which is its union, such that $\mathcal{E}_\lambda \subseteq \mathcal{E}$ for every $\lambda < \mu$. The structure $\mathcal{E}$ is defined as follows:

(i) $A = \bigcup_{\lambda<\mu} A_\lambda$;

(ii) $N^{\mathcal{E}} = \bigcup_{\lambda<\mu} N^{\mathcal{E}_\lambda}$ and $O^{\mathcal{E}} = \bigcup_{\lambda<\mu} O^{\mathcal{E}_\lambda}$;

(iii) for $a, b \in A$: $a \sqcap^{\mathcal{E}} b = a \sqcap^{\mathcal{E}_\lambda} b$; $a \sqcup^{\mathcal{E}} b = a \sqcup^{\mathcal{E}_\lambda} b$; and $-^{\mathcal{E}} a = -^{\mathcal{E}_\lambda} a$, if $a, b \in A_\lambda$;

(iv) $\mathbf{0}^{\mathcal{E}} = \mathbf{0}^{\mathcal{E}_0}$ and $\mathbf{1}^{\mathcal{E}} = \mathbf{1}^{\mathcal{E}_0}$.

Observe that items (ii) and (iii) are well-defined given that, for $\xi < \lambda < \mu$, $\mathcal{E}_\xi \subseteq \mathcal{E}_\lambda$ and so $A_\xi \subseteq A_\lambda$. Clearly $\mathbf{0}^{\mathcal{E}} = \mathbf{0}^{\mathcal{E}_\lambda}$ and $\mathbf{1}^{\mathcal{E}} = \mathbf{1}^{\mathcal{E}_\lambda}$ for every $\lambda < \mu$.

Given that, as observed in Remark 11, the class **FmbC** of **mbC**-structures can be axiomatized by means of $\forall\exists$-sentences (that is, Π^0_2-sentences), that class is closed under union of chains. This is a consequence of [13, Theorem 3.2.3]. In other words, the following result holds:

Proposition 26 *Let* $(\mathcal{E}_\lambda)_{\lambda<\mu}$ *be a chain of* **mbC***-structures, and let* $\mathcal{E}$ *be its union. Then* $\mathcal{E}$ *is the least* **mbC***-structure having every* $\mathcal{E}_\lambda$ *as a substructure.*

The last result can be interpreted in terms of the propositional logics generated by single **mbC**-structures (see Proposition 28 below). Indeed: recall from Definition 7 the propositional consequence relation $\Vdash^{\mathbf{mbC}}_{\mathcal{E}}$ generated by an **mbC**-structure $\mathcal{E}$.[7] Then, the following useful result can be stated:

[7]The propositional consequence relation $\Vdash^{\mathbf{mbC}}_{\mathbf{K}}$ generated by a class $\mathbf{K}$ of **mbC**-structures should not be confused with the first-order consequence relation $\models_{\mathbf{K}}$ defined over first-order Θ-sentences as follows: $\Upsilon \models_{\mathbf{K}} \sigma$ if, for every $\mathcal{E} \in \mathbf{K}$, it holds: $\mathcal{E} \models \sigma$ whenever $\mathcal{E} \models \varrho$ for every $\varrho \in \Upsilon$.

Lemma 27 *Let $\mathcal{E}$ and $\mathcal{E}'$ be two* **mbC***-structures such that $\mathcal{E} \subseteq \mathcal{E}'$. Then $\Vdash^{\mathbf{mbC}}_{\mathcal{E}'} \subseteq \Vdash^{\mathbf{mbC}}_{\mathcal{E}}$.*

Proof. Suppose that $\mathcal{E} \subseteq \mathcal{E}'$ and let $\Gamma \cup \{\alpha\}$ be a finite set of formulas in $For(\Sigma)$ such that $\Gamma \Vdash^{\mathbf{mbC}}_{\mathcal{E}'} \alpha$. Consider a valuation v over $\mathcal{E}$ such that $v[\Gamma] \subseteq \{\mathbf{1}^{\mathcal{E}}\}$. Then, the mapping $\bar{v} : For(\Sigma) \to A'$ such that $\bar{v}(\gamma) = v(\gamma)$ for every $\gamma \in For(\Sigma)$ is a valuation over $\mathcal{E}$' such that $\bar{v}[\Gamma] \subseteq \{\mathbf{1}^{\mathcal{E}'}\}$. By hypothesis, $\bar{v}(\alpha) = \mathbf{1}^{\mathcal{E}'}$ whence $v(\alpha) = \mathbf{1}^{\mathcal{E}}$. This means that $\Gamma \Vdash^{\mathbf{mbC}}_{\mathcal{E}} \alpha$. □

From this, Proposition 26 can be interpreted in terms of the propositional logics associated to **mbC**-structures:

Proposition 28 *Let $(\mathcal{E}_\lambda)_{\lambda<\mu}$ be a chain of* **mbC***-structures, and let $\mathcal{E}$ be its union. Then $\Vdash^{\mathbf{mbC}}_{\mathcal{E}} = \bigcap_{\lambda<\mu} \Vdash^{\mathbf{mbC}}_{\mathcal{E}_\lambda}$.*

Proof. It is clear from Lemma 27 that $\Vdash^{\mathbf{mbC}}_{\mathcal{E}} \subseteq \Vdash^{\mathbf{mbC}}_{\mathcal{E}_\lambda}$ for every $\lambda < \mu$, whence $\Vdash^{\mathbf{mbC}}_{\mathcal{E}} \subseteq \bigcap_{\lambda<\mu} \Vdash^{\mathbf{mbC}}_{\mathcal{E}_\mu}$. Now, let $\Gamma \cup \{\alpha\}$ be a finite set of formulas in $For(\Sigma)$ such that $\Gamma \nVdash^{\mathbf{mbC}}_{\mathcal{E}} \alpha$. Then, there is a valuation v over $\mathcal{E}$ such that $v[\Gamma] \subseteq \{\mathbf{1}^{\mathcal{E}}\}$ and $v(\alpha) \neq \mathbf{1}^{\mathcal{E}}$. Since $\Gamma \cup \{\alpha\}$ is a finite set, then $v[\Gamma \cup \{\alpha\}] \subseteq A_n$ for some natural number $n < \mu$. Define a mapping $\bar{v} : For(\Sigma) \to A_n$ such that $\bar{v}(\gamma) = v(\gamma)$ whenever $\gamma \in \Gamma \cup \{\alpha\}$ and, for every formula $\gamma \notin \Gamma \cup \{\alpha\}$:

(0) if $\gamma \in \mathcal{V}$ then $\bar{v}(\gamma)$ is arbitrary;

(1) if $\gamma = \gamma_1 \# \gamma_2$ then $\bar{v}(\gamma) = \bar{v}(\alpha) \bar{\#}^{\mathcal{E}_n} \bar{v}(\beta)$, for $\# \in \{\wedge, \vee, \to\}$;

(2) if $\gamma = \neg\gamma_1$ then $\bar{v}(\gamma)$ is such that $N^{\mathcal{E}_n}(\bar{v}(\gamma_1), \bar{v}(\gamma))$;

(3) if $\gamma = \circ\gamma_1$ then $\bar{v}(\gamma)$ is such that $O^{\mathcal{E}_n}(\bar{v}(\gamma_1), \bar{v}(\gamma))$ and $\bar{v}(\gamma_1) \sqcap^{\mathcal{E}_n} \bar{v}(\neg\gamma_1) \sqcap^{\mathcal{E}_n} \bar{v}(\gamma) = \mathbf{0}^{\mathcal{E}_n}$.[8]

It is clear by the very definition that $\bar{v}$ is a valuation over $\mathcal{E}_n$ such that $\bar{v}[\Gamma] \subseteq \{\mathbf{1}^{\mathcal{E}_n}\}$ and $\bar{v}(\alpha) \neq \mathbf{1}^{\mathcal{E}_n}$. This means that $\Gamma \nVdash^{\mathbf{mbC}}_{\mathcal{E}_n} \alpha$. Therefore $\bigcap_{\lambda<\mu} \Vdash^{\mathbf{mbC}}_{\mathcal{E}_\lambda} \subseteq \Vdash^{\mathbf{mbC}}_{\mathcal{E}}$. □

7 Congruences and quotient structures

In this section, the well-known lattice isomorphism between the lattice of filters over a Boolean algebra $\mathcal{A}$ and the lattice of Boolean congruences over $\mathcal{A}$ will be extended to **mbC**-structures. Additionally, the quotient **mbC**-structures by **mbC**-congruences will be defined.

Definition 29 *Let θ be a relation on an* **mbC***-structure $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$. Then θ is said to be an* **mbC***-congruence over $\mathcal{E}$ if the following conditions hold:*

(i) *θ is a Boolean congruence over $\mathcal{A}$;*[9]

[8] In order to define $\bar{v}$ by induction on the complexity of γ, the complexity measure on $For(\Sigma)$ must be defined in a way such that $\circ\beta$ has a complexity degree strictly greater than that of $\neg\beta$, for every $\beta \in For(\Sigma)$.

[9] That is, θ is an equivalence relation which is preserved by the operations of the Boolean algebra $\mathcal{A}$.

(ii) *if* $(x,x'),(y,y') \in \theta$ *and* $N^{\mathcal{E}}(x,y)$ *then* $N^{\mathcal{E}}(x',y')$;

(iii) *if* $(x,x'),(y,y') \in \theta$ *and* $O^{\mathcal{E}}(x,y)$ *then* $O^{\mathcal{E}}(x',y')$.

Given a Boolean algebra $\mathcal{A}$, let $Con_B(\mathcal{A})$ be the set of Boolean congruences defined on $\mathcal{A}$. It is well-known that the poset $(Con_B(\mathcal{A}),\subseteq)$ partially ordered by the inclusion relation is a distributive lattice.

Definition 30 *Let $\mathcal{A}$ be a Boolean algebra, and let $F \subseteq A$. Then F is a* filter *over $\mathcal{A}$ if the following holds: (i) $\mathbf{1}^{\mathcal{A}} \in F$; (ii) if $x,y \in F$ then $x \sqcap y \in F$; and (iii) if $x \in F$ and $x \leq y$ then $y \in F$. We denote by $F(\mathcal{A})$ the set of filters over $\mathcal{A}$.*

The following is a well-known result:

Theorem 31 *Given a Boolean algebra $\mathcal{A}$, there exists a lattice isomorphism between $F(\mathcal{A})$ and $Con_B(\mathcal{A})$ given by $F \mapsto R(F)$, where $R(F) = \{(x,y) \in A^2 \ : \ x \sqcap z = y \sqcap z \text{ for some } z \in F\}$. The inverse mapping is given by $\theta \mapsto [\mathbf{1}^{\mathcal{A}}]_\theta$, where $[a]_\theta$ denotes the θ-equivalence class of $a \in A$.*

Definition 32 *Given an* **mbC**-*structure $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$, a set $F \subseteq A$, is said to be an* **mbC**-filter *if the following conditions hold:*

(i) *F is a filter over the Boolean algebra $\mathcal{A}$;*

(ii) *$R(F)$ verifies conditions (ii) and (iii) of Definition 29, where $R(F)$ is defined as in Theorem 31.*

Let us denote by $F_{\mathbf{mbC}}(\mathcal{E})$ and by $Con_{\mathbf{mbC}}(\mathcal{E})$ the set of **mbC**-filters and the set of **mbC**-congruences over a given **mbC**-structure $\mathcal{E}$, respectively. Thus, the following result holds:

Theorem 33 *Let $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ be an* **mbC**-*structure. Then, there exists a lattice isomorphism between $F_{\mathbf{mbC}}(\mathcal{E})$ and $Con_{\mathbf{mbC}}(\mathcal{E})$.*

Proof. It is proved by an easy adaptation of proof of the corresponding result for Boolean algebras. $\square$

Now, we are going to define quotient **mbC**-structures. Let $\mathcal{E}$ be an **mbC**-structure, and let θ be an **mbC**-congruence on it. Then A/θ is a Boolean algebra with the operations induced from $\mathcal{A}$; this Boolean algebra will be denoted by $\mathcal{A}/\theta$. Consider the following relations over A/θ induced from $\mathcal{E}$:

$$N^{\mathcal{E}/\theta} \stackrel{\text{def}}{=} \{([x]_\theta,[y]_\theta) \in A/\theta \times A/\theta \ : \ (x,y) \in N^{\mathcal{E}}\}$$

and

$$O^{\mathcal{E}/\theta} \stackrel{\text{def}}{=} \{([x]_\theta,[y]_\theta) \in A/\theta \times A/\theta \ : \ (x,y) \in O^{\mathcal{E}}\}.$$

From Definition 29, it follows that $(x, y) \in N^{\mathcal{E}}$ if and only if $([x]_\theta, [y]_\theta) \in N^{\mathcal{E}/\theta}$; the same holds for the predicate O. From this, it is easy to check that $\mathcal{E}/\theta = \langle \mathcal{A}/\theta, N^{\mathcal{E}/\theta}, O^{\mathcal{E}/\theta} \rangle$ is an **mbC**-structure. Now, consider the canonical projection $q : A \to A/\theta$ given by $q(x) = [x]_\theta$. It is clear that q is a homomorphism of Boolean algebras. Moreover, it is an **mbC**-homomorphism $q : \mathcal{E} \to \mathcal{E}/\theta$ which is onto, that is, an epimorphism in $\mathbb{F}$**mbC** (by Proposition 23).

It is well-known that given a Boolean homomorphism $h : \mathcal{A} \to \mathcal{A}'$, the relation $Ker(h) = \{(x, y) \in A \times A \ : \ h(x) = h(y)\}$ is a Boolean congruence. This allows us to prove the so-called *First Isomorphism theorem* for Boolean algebras. In order to generalize this result to **mbC**-structures, the homomorphisms must satisfy additional coherence properties:

Definition 34 *Let $h : \mathcal{E} \to \mathcal{E}'$ be an* **mbC***-homomorphism.*[10] *Then h is said to be* congruential *if it satisfies the following:*

$$\textit{If } \ N^{\mathcal{E}}(x, y) \ \textit{ and } (x, x'), (y, y') \in Ker(h) \ \textit{ then } N^{\mathcal{E}}(x', y'),$$

and

$$\textit{If } \ O^{\mathcal{E}}(x, y) \ \textit{ and } (x, x'), (y, y') \in Ker(h) \ \textit{ then } O^{\mathcal{E}}(x', y').$$

Proposition 35 *Let $h : \mathcal{E} \to \mathcal{E}'$ be an* **mbC***-homomorphism. Then, h is congruential if and only if the relation $Ker(h) = \{(x, y) \in A \times A \ : \ h(x) = h(y)\}$ is an* **mbC**- *congruence.*

Proof. It follows from the very definitions. □

Theorem 36 (First Isomorphism theorem) *Let $h : \mathcal{E} \to \mathcal{E}'$ be an* **mbC***-homomorphism which is congruential. Then, there is a unique* **mbC***-monomorphism $\overline{h} : \mathcal{E}/Ker(h) \to \mathcal{E}'$ such that $\overline{h} \circ q = h$. In particular, if h is surjective then $\overline{h}$ is an isomorphism between $\mathcal{E}/Ker(h)$ and $\mathcal{E}'$.*

Proof. As mentioned above, it is well-known that the Boolean congruence $\theta = Ker(h)$ is such that there is a unique Boolean monomomorphism $\overline{h} : A/\theta \to A'$ with $\overline{h} \circ q = h$. Thus, in view of Proposition 35, it suffices to prove that $\overline{h}$ is an **mbC**-monomomorphism. The details are left to the reader. □

8 Birkhoff's decomposition theorems for mbC-structures

In this section, Caicedo's generalization to first-order structures (see [6]) of Birkhoff's decomposition theorem for algebras (see [4]) will be applied to the specific case of **mbC**-structures.

[10] Observe that, in particular, $h : \mathcal{A} \to \mathcal{A}'$ is a Boolean homomorphism.

In order to adapt Caicedo's result to the present framework, some important constructions over **mbC**-structures will be analyzed: direct and subdirect poducts, as well as the associated notion of subdirectly irreducible **mbC**-structures. Finally, in Subsection 8.4, a related decomposition result will be obtained by adapting the notions introduced by Fidel in [19] in terms of weak substructures, recall Definition 19. Once again, the fact that **FmbC** can be axiomatized by means of sentences of a special form (recall Section 4) will we used, by adapting well-known results from Model Theory.

8.1 Direct products of mbC-structures

As observed in Remark 10 the class **FmbC** of **mbC**-structures can be axiomatized by means of Horn sentences. Then, as a consequence of [13, Proposition 6.2.2], the class **FmbC** is closed under (arbitrary) direct products. That is:

Theorem 37 *The category* $\mathbb{F}\mathbf{mbC}$ *has arbitrary products.*[11]

For the reader's convenience the standard construction of products in $\mathbb{F}\mathbf{mbC}$ will be given in Definition 38 below.

Given a family $\{\mathcal{A}_i\}_{i\in I}$ of Boolean algebras, consider the standard construction $\mathcal{A} = \prod_{i\in I} \mathcal{A}_i$ of its product such that its support is $\prod_{i\in I} A_i = \{x \in (\bigcup_{i\in I} A_i)^I \ : \ x(i) \in A_i \text{ for every } i \in I\}$,[12] and the operations are defined pointwise; in particular, $\mathbf{0}^{\mathcal{A}}(i) = \mathbf{0}^{\mathcal{A}_i}$ and $\mathbf{1}^{\mathcal{A}}(i) = \mathbf{1}^{\mathcal{A}_i}$ for every $i \in I$. If $I = \emptyset$ then $\prod_{i\in I} \mathcal{A}_i$ is the trivial one-point Boolean algebra $\mathbb{A}_\perp$ (see Remark 39). The canonical i-projection $\pi_i : \prod_{i\in I} \mathcal{A}_i \to \mathcal{A}_i$ is given by $\pi_i(x) = x(i)$ for every $x \in \prod_{i\in I} A_i$ and every $i \in I$.

Definition 38 *Let* $\mathcal{E}_i = \langle \mathcal{A}_i, N^{\mathcal{E}_i}, O^{\mathcal{E}_i}\rangle$ *(for* $i \in I$*) be an* **mbC***-structure. The* direct product *of the family* $\{\mathcal{E}_i\}_{i\in I}$ *is the structure* $\prod_{i\in I} \mathcal{E}_i = \langle \prod_{i\in I} \mathcal{A}_i, N^{\prod_{i\in I}\mathcal{E}_i}, O^{\prod_{i\in I}\mathcal{E}_i}\rangle$ *defined as follows:*

(i) $\prod_{i\in I} \mathcal{A}_i$ *is the standard product of the family* $\{\mathcal{A}_i\}_{i\in I}$ *of Boolean algebras;*

(ii) $(x, y) \in N^{\prod_{i\in I}\mathcal{E}_i}$ *if and only if* $(x(i), y(i)) \in N^{\mathcal{E}_i}$ *for every* $i \in I$*;*

(iii) $(x, y) \in O^{\prod_{i\in I}\mathcal{E}_i}$ *if and only if* $(x(i), y(i)) \in O^{\mathcal{E}_i}$ *for every* $i \in I$*.*

It is an easy exercise to check that the construction described in Definition 38 is in fact the product of the given family of **mbC**-structures (that is, $\langle \prod_{i\in I} \mathcal{E}_i, \{\pi_i \ : \ i \in I\}\rangle$ satisfies the universal property of the product in $\mathbb{F}\mathbf{mbC}$).

[11] In addition, also as a consequence of [13, Proposition 6.2.2], it follows that the class **FmbC** is closed under reduced products; in particular, it is closed under ultraproducts.

[12] As usual, if I and Z are two sets, then Z^I denotes the set of mappings from I to Z.

Remark 39 *Observe that the product of the empty family of* **mbC***-structures is the terminal object* $\mathbf{1}_\perp = \langle \mathbb{A}_\perp, N^{\mathbf{1}_\perp}, O^{\mathbf{1}_\perp} \rangle$ *given by the one-element Boolean algebra* $\mathbb{A}_\perp$ *with domain* $A_\perp = \{*\}$*, and where* $N^{\mathbf{1}_\perp} = O^{\mathbf{1}_\perp} = \{(*, *)\}$*. Note that* $\mathbf{0}^{\mathbb{A}_\perp} = \mathbf{1}^{\mathbb{A}_\perp} = *$*.*

8.2 Subdirect products and subdirectly irreducible mbC-structures

Recall from Remark 17 the characterization of embeddings in the category of **mbC**-structures. As discussed in Section 5, the substructures in $\mathbb{F}$**mbC** are defined by means of embeddings. It is worth noting that in [6] the embeddings are called *substructure monomorphisms.*

Definition 40 (Subdirect product, [6]) *For* $i \in I$ *let* $\mathcal{E}_i = \langle \mathcal{A}_i, N^{\mathcal{E}_i}, O^{\mathcal{E}_i} \rangle$ *be an* **mbC***-structure. A* subdirect product *of the family* $\{\mathcal{E}_i\}_{i \in I}$ *is an embedding* $h : \mathcal{E} \to \prod_{i \in I} \mathcal{E}_i$ *(for some* **mbC***-structure* $\mathcal{E}$*) such that* $\pi_i \circ h$ *is onto for every* $i \in I$*. It is also called a* subdirect decomposition *of* $\mathcal{E}$*.*

Definition 41 (Subdirectly irreducible structures, [6]) *An* **mbC***-structure* $\mathcal{E}$ *is said to be* subdirectly irreducible *(s.i.) in* $\mathbb{F}$**mbC** *if for every subdirect descomposition* $h : \mathcal{E} \to \prod_{i \in I} \mathcal{E}_i$ *in* $\mathbb{F}$**mbC** *with* $I \neq \emptyset$*, there is an* $i \in I$ *such that* $\pi_i \circ h$ *is an isomorphism in* $\mathbb{F}$**mbC***.*

Theorem 42 ([6, Lemma 1]) *Let* $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ *be an* **mbC***-structure such that* $\mathcal{E} \neq \mathbf{1}_\perp$*. Then,* $\mathcal{E}$ *is subdirectly irreducible in* $\mathbb{F}$**mbC** *if and only if there exists a predicate* $P \in \{N, O, \approx\}$ *and* $(x, y) \in A^2$ *such that* $(x, y) \notin P^{\mathcal{E}}$*, and for every onto homomorphism* $h : \mathcal{E} \to \mathcal{E}'$ *in* $\mathbb{F}$**mbC** *which is not an isomorphism,* $(h(x), h(y)) \in P^{\mathcal{E}'}$*.*

Remark 43 *It is interesting to characterize, in terms of Theorem 42, the* **mbC***-structures which* are not *subdirectly irreducible. Thus, it follows from the previous result that an* **mbC***-structure* $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle \neq \mathbf{1}_\perp$ *is not subdirectly irreducible in* $\mathbb{F}$**mbC** *if only if the following conditions hold:*

1. *For every* $(x, y) \in A^2$*,* $(x, y) \notin N^{\mathcal{E}}$ *implies that there exists an onto homomorphism* $h : \mathcal{E} \to \mathcal{E}'$ *in* $\mathbb{F}$**mbC** *which is not an isomorphism, such that* $(h(x), h(y)) \notin N^{\mathcal{E}'}$ *(hence* $\mathcal{E}' \neq \mathbf{1}_\perp$*);*
2. *For every* $(x, y) \in A^2$*,* $(x, y) \notin O^{\mathcal{E}}$ *implies that there exists an onto homomorphism* $h : \mathcal{E} \to \mathcal{E}'$ *in* $\mathbb{F}$**mbC** *which is not an isomorphism, such that* $(h(x), h(y)) \notin O^{\mathcal{E}'}$ *(hence* $\mathcal{E}' \neq \mathbf{1}_\perp$*);*
3. *For every* $(x, y) \in A^2$*,* $x \neq y$ *implies that there exists an onto homomorphism* $h : \mathcal{E} \to \mathcal{E}'$ *in* $\mathbb{F}$**mbC** *which is not an isomorphism, such that* $h(x) \neq h(y)$ *(hence* $\mathcal{E}' \neq \mathbf{1}_\perp$*).*

In the rest of this section, the problem of characterizing the subdirectly irreducible **mbC**-structures will be considered.

Proposition 44 *The terminal* **mbC***-structure* $\mathbf{1}_\perp = \langle \mathbb{A}_\perp, N^{\mathbf{1}_\perp}, O^{\mathbf{1}_\perp} \rangle$ *is subdirectly irreducible in* $\mathbb{F}$**mbC***.*

Proof. It follows from the definition of $\mathbf{1}_\perp$ and from Definition 41: the only subdirect decomposition of $\mathbf{1}_\perp$ is $id_{\mathbf{1}_\perp}$, where the codomain of $id_{\mathbf{1}_\perp}$ is the product $\mathbf{1}_\perp$ of the empty family of **mbC**-structures. □

Now, let us consider the general case for **mbC**-structures defined over an arbitrary Boolean algebra $A \neq \mathbb{A}_\perp$.

Given a Boolean algebra $\mathcal{A}$, let $\mathcal{E}^{max}_{\mathcal{A}} \stackrel{\text{def}}{=} \langle \mathcal{A}, N^{max}_{\mathcal{A}}, O^{max}_{\mathcal{A}} \rangle$ be the greatest **mbC**-structure defined over $\mathcal{A}$, where $N^{max}_{\mathcal{A}} = \{(x, y) \in A^2 \ : \ x \sqcup y = 1\}$ and $O^{max}_{\mathcal{A}} = A^2$. By simplicity, $\mathcal{E}^{max}_{\mathbb{A}_2}$ will be denoted by $\mathcal{E}^{max}_2 = \langle \mathbb{A}_2, N^{max}_2, O^{max}_2 \rangle$, recalling that $\mathbb{A}_2$ denotes the two-element Boolean algebra. Observe that $\mathcal{E}^{max}_{\mathbb{A}_\perp}$ is $\mathbf{1}_\perp$.

Lemma 45 *Let* $h : \mathcal{E}^{max}_2 \to \mathcal{E}$ *be an* **mbC***-homomorphism such that* $\mathcal{E} = \langle \mathbb{A}_2, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ *is defined over* $\mathbb{A}_2$*. Then* h *is the identity map,* $\mathcal{E} = \mathcal{E}^{max}_2$ *and* h *is an isomorphism in* $\mathbb{F}$**mbC***.*

Proof. Since h is, in particular, a Boolean homomorphism, $h(0) = 0$ and $h(1) = 1$, hence h is the identity map. Since h preserves the predicates and h is the identity map, $P^{max}_2 = h[P^{max}_2] \subseteq P^{\mathcal{E}} \subseteq P^{max}_2$, and so $P^{\mathcal{E}} = P^{max}_2$ for $P \in \{N, O\}$. □

Proposition 46 $\mathcal{E}^{max}_2$ *is subdirectly irreducible in* $\mathbb{F}$**mbC***.*

Proof. Consider in Theorem 42 the predicate symbol N and the point $(0, 0)$. By item (b)(iii) of Definition 3, $(0, 0) \notin N^{max}_2$. Let $h : \mathcal{E}^{max}_2 \to \mathcal{E}$ be an onto homomorphism which is not an isomorphism in $\mathbb{F}$**mbC**. In particular, $h : \mathbb{A}_2 \to \mathcal{A}$ is an onto homomorphism of Boolean algebras which is not an isomorphism in $\mathbb{F}$**mbC**, hence $\mathcal{A}$ is the one-point Boolean algebra $\mathbb{A}_\perp$ or $\mathcal{A}$ is $\mathbb{A}_2$. By Lemma 45 $\mathcal{A}$ must be $\mathbb{A}_\perp$ (otherwise h will be an isomorphism in $\mathbb{F}$**mbC**). From this $\mathcal{E}$ is the terminal **mbC**-structure $\mathbf{1}_\perp$ and so $(h(0), h(0)) \in N^{\mathcal{E}} = \{(*, *)\}$. By Theorem 42, it follows that $\mathcal{E}$ is subdirectly irreducible in $\mathbb{F}$**mbC**. □

The following is a well-known result concerning Boolean algebras which will be useful for our purposes:

Proposition 47 *Let* $\mathcal{A}$ *be a Boolean algebra such that* $\mathcal{A} \neq \mathbb{A}_\perp$*.*

(1) Let $x, y \in A$ *such that* $x \neq y$*. Then, there exists a homomorphism* $h : \mathcal{A} \to \mathbb{A}_2$ *of Boolean algebras such that* $h(x) \neq h(y)$*.*

(2) Let $x \in A$ *such that* $x \neq 1$*. Then, there exists a homomorphism* $h : \mathcal{A} \to \mathbb{A}_2$ *of Boolean algebras such that* $h(x) = 0$*.*

(3) If $h : \mathcal{A} \to \mathcal{A}'$ *is a non-injective homomorphism of Boolean algebras, there exists* $z \in A$ *such that* $z \neq 0$ *but* $h(z) = 0$*.*

(4) If $h : \mathcal{A} \to \mathcal{A}'$ *is a non-injective homomorphism of Boolean algebras, there exists* $z \in A$ *such that* $z \neq 1$ *but* $h(z) = 1$*.*

From now on, the cardinal of a set X will be denoted by $card(X)$.

Proposition 48 *Let $\mathcal{E}_{\mathcal{A}}^{max}$ an* **mbC**-*structure over a Boolean algebra $\mathcal{A}$ such that $card(A) > 2$. Then, $\mathcal{E}_{\mathcal{A}}^{max}$ is not subdirectly irreducible in* $\mathbb{F}$**mbC**.

Proof. Let $(x, y) \notin N_{\mathcal{A}}^{max}$. Then, $x \sqcup y \neq 1$ and so, by Proposition 47(2), there exists a homomorphism $h : \mathcal{A} \to \mathbb{A}_2$ of Boolean algebras such that $h(x \sqcup y) = 0$, whence $h(x) = h(y) = 0$. Clearly h induces an onto homomorphism $h : \mathcal{E}_{\mathcal{A}}^{max} \to \mathcal{E}_{2}^{max}$ in $\mathbb{F}$**mbC** which is not an isomorphism (since $card(A) > 2$), such that $(h(x), h(y)) = (0, 0) \notin N_{2}^{max}$.

Now, let $(x, y) \in A^2$ such that $x \neq y$. By Proposition 47(1), there exists a homomorphism $h : \mathcal{A} \to \mathbb{A}_2$ of Boolean algebras such that $h(x) \neq h(y)$. It is immediate to see that h induces an onto homomorphism $h : \mathcal{E}_{\mathcal{A}}^{max} \to \mathcal{E}_{2}^{max}$ in $\mathbb{F}$**mbC** which is not an isomorphism, such that $h(x) \neq h(y)$.

Finally, there is no $(x, y) \in A^2$ such that $(x, y) \notin O_{\mathcal{A}}^{max}$. Therefore, it follows that $\mathcal{E}_{\mathcal{A}}^{max}$ is not s.i. in $\mathbb{F}$**mbC**, by Remark 43. $\square$

Proposition 49
(1) Let $(x, y) \in N_{\mathcal{A}}^{max} \setminus \{(0, 1)\}$. Then $\mathcal{E}_{\mathcal{A},N(x,y)}^{max} \stackrel{def}{=} \langle \mathcal{A}, N_{\mathcal{A}}^{max} \setminus \{(x, y)\}, O_{\mathcal{A}}^{max} \rangle$ is an **mbC**-*structure which is subdirectly irreducible in* $\mathbb{F}$**mbC**.

(2) Let $(x, y) \in A^2 \setminus \{(1, 0)\}$. Then $\mathcal{E}_{\mathcal{A},O(x,y)}^{max} \stackrel{def}{=} \langle \mathcal{A}, N_{\mathcal{A}}^{max}, O_{\mathcal{A}}^{max} \setminus \{(x, y)\} \rangle$ is an **mbC**-*structure which is subdirectly irreducible in* $\mathbb{F}$**mbC**.

Proof. (1) Let $\mathcal{E} \stackrel{\text{def}}{=} \mathcal{E}_{\mathcal{A},N(x,y)}^{max}$. Clearly, $\mathcal{E}$ is an **mbC**-structure. In order to apply Theorem 42, consider the predicate symbol N and the point (x, y). By definition, $(x, y) \notin N^{\mathcal{E}}$. Now, suppose that we have an onto **mbC**-homomorphism $h : \mathcal{E} \to \mathcal{E}'$ which is not an isomorphism in $\mathbb{F}$**mbC**. Then, $h : \mathcal{A} \to \mathcal{A}'$ is an onto homomorphism of Boolean algebras. There are two cases:

Case 1: h is an isomorphism of Boolean algebras. Then, the only reason for h (an isomorphism of Boolean algebras) not being an isomorphism in $\mathbb{F}$**mbC** is that there exist $(c, d) \in A^2$ and a predicate symbol $P \in \{N, O, \approx\}$ such that $(c, d) \notin P^{\mathcal{E}}$ but $(h(c), h(d)) \in P^{\mathcal{E}'}$. Given that $O^{\mathcal{E}} = A^2$ it follows that $P \neq O$. On the other hand, since h is injective then $P \neq \approx$. Therefore, $P = N$ and $(c, d) = (x, y)$, by definition of $\mathcal{E}$. This implies that $(h(x), h(y)) \in N^{\mathcal{E}'}$ (and $\mathcal{E}' = \mathcal{E}_{\mathcal{A}'}^{max}$).

Case 2: h is not an isomorphism of Boolean algebras. Then, h is not injective. By Proposition 47(3), there exists $z \in A$ such that $z \neq 0$ but $h(z) = 0$. There are two subcases to analyze:

Case 2.1: Either $z \not\leq x$ or $z \not\leq y$.

Case 2.1.1: $z \not\leq x$. Let $x' = x \sqcup z$. Then, $x' \neq x$ (otherwise, if $x = x'$ then $z = z \sqcap (x \sqcup z) = z \sqcap x$ and so $z \leq x$, a contradiction). Clearly $h(x') = h(x)$. Moreover, $x' \sqcup y = (x \sqcup z) \sqcup y = 1$ (since $x \sqcup y = 1$). Hence $(x', y) \neq (x, y)$ such that $(x', y) \in N^{\mathcal{E}}$ and so $(h(x), h(y)) = (h(x'), h(y)) \in N^{\mathcal{E}'}$ since h is homomorphism in $\mathbb{F}$**mbC**.

Case 2.1.2: $z \not\leq y$. As in the proof of Case 2.1.1 it can be shown that $(h(x), h(y)) \in N^{\mathcal{E}'}$ (now by taking the pair $(x, y \sqcup z)$).

Case 2.2: Both $z \leq x$ and $z \leq y$. Let $x' = x \sqcap {\sim}z$. Then $x \neq x'$ (otherwise, if $x = x'$ then $z = z \sqcap x = z \sqcap (x \sqcap {\sim}z) = 0$, a contradiction). On the other hand $x' \sqcup y = (x \sqcap {\sim}z) \sqcup y = (x \sqcup y) \sqcap ({\sim}z \sqcup y) = 1 \sqcap ({\sim}z \sqcup y) = {\sim}z \sqcup y = {\sim}z \sqcup (y \sqcup z) = 1$. Hence $(x', y) \neq (x, y)$ such that $(x', y) \in N^{\mathcal{E}}$ and so $(h(x), h(y)) = (h(x'), h(y)) \in N^{\mathcal{E}'}$ since h is homomorphism in $\mathbb{F}$**mbC**.

By Theorem 42, $\mathcal{E}$ is s.i. in $\mathbb{F}$**mbC**.

(2) The proof is analogous to that of (1), but now considering the predicate symbol O and the point (x, y). □

Remark 50 *In Proposition 49 item (1) the point $(x, y) = (0, 1)$ was not considered because of the first observation in Remark 5 (namely, $(0, 1) \in N^{\mathcal{E}}$ for every $\mathcal{E}$). Similarly, in Proposition 49(2) it is required that $(x, y) \neq (1, 0)$ because of the second observation in Remark 5. Indeed, since $(1, 1) \in N^{max}_{\mathcal{A}}$ then $(1, 0)$ must belong to $O^{max}_{\mathcal{A}} \setminus \{(x, y)\}$ in order to get an* **mbC**-*structure.*

Proposition 51 *Let $\mathcal{E}^{max*}_{\mathcal{A}} \stackrel{def}{=} \langle \mathcal{A}, N^{max}_{\mathcal{A}} \setminus \{(1, 1)\}, O^{max}_{\mathcal{A}} \setminus \{(1, 0)\}\rangle$. Then, $\mathcal{E}^{max*}_{\mathcal{A}}$ is an* **mbC**-*structure which is subdirectly irreducible in* $\mathbb{F}$**mbC**.

Proof. It is easy to see that $\mathcal{E}^{max*}_{\mathcal{A}}$ is indeed an **mbC**-structure. We will use Theorem 42 in order to show that $\mathcal{E} = \mathcal{E}^{max*}_{\mathcal{A}}$ is subdirectly irreducible in $\mathbb{F}$**mbC**. Thus, consider the predicate symbol O and the point $(1, 0) \in A^2$. Clearly, $(1, 0) \notin O^{\mathcal{E}}$. It will be shown that, for every onto **mbC**-homomorphism $h : \mathcal{E} \to \mathcal{E}'$ which is not an isomorphism, $(1, 0) = (h(1), h(0)) \in O^{\mathcal{E}'}$. Thus, let $h : \mathcal{E} \to \mathcal{E}'$ be an onto **mbC**-homomorphism which is not an isomorphism. There are two cases:

Case 1: h is an isomorphism of Boolean algebras. By adapting the argument used in the **Case 1** of the proof of Proposition 49, and taking into account that $\mathcal{E}$ is obtained from $\mathcal{E}^{max}_{\mathcal{A}}$ by removing just one point from $N^{max}_{\mathcal{A}}$ and just one point from $O^{max}_{\mathcal{A}}$, there are two subcases to analyze:

Case 1.1: $\mathcal{E}'$ is isomorphic to $\mathcal{E}^{max}_{\mathcal{A}}$ in $\mathbb{F}$**mbC** via h. That is, $\mathcal{E}' = \mathcal{E}^{max}_{\mathcal{A}'}$.

Case 1.2: $\mathcal{E}'$ is isomorphic to $\mathcal{E}^{max}_{\mathcal{A},N(1,1)}$ in $\mathbb{F}$**mbC** via h (recall Proposition 49). That is, $\mathcal{E}' = \mathcal{E}^{max}_{\mathcal{A}',N(h(1),h(1))} = \mathcal{E}^{max}_{\mathcal{A}',N(1,1)}$.

In both cases, $(h(1), h(0)) \in O^{\mathcal{E}'}$. Observe that the case that $\mathcal{E}'$ is isomorphic to $\mathcal{E}^{max}_{\mathcal{A},O(1,0)}$ is not allowed, given that $(1, 1) \in N^{\mathcal{E}'}$ and so $(1, 0)$ must be in $O^{\mathcal{E}'}$ (see Remark 5).

Case 2: h is not an isomorphism of Boolean algebras. Then, h is not injective. By Proposition 47(4), there exists $z \in A$ such that $z \neq 1$ but $h(z) = 1$. From this, $(z, 0) \in O^{\mathcal{E}}$ (since $(z, 0) \neq (1, 0)$) and so $(h(z), h(0)) = (1, 0) \in O^{\mathcal{E}'}$, since h is an homomorphism in $\mathbb{F}$**mbC**. That is, $(h(1), h(0)) \in O^{\mathcal{E}'}$.

By Theorem 42, $\mathcal{E}$ is s.i. in $\mathbb{F}$**mbC**. □

Proposition 52 *Let $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}}\rangle$ be an* **mbC**-*structure over $\mathcal{A}$ such that $card\big(N^{max}_{\mathcal{A}} \setminus N^{\mathcal{E}}\big) \geq 1$ and $card\big(O^{max}_{\mathcal{A}} \setminus O^{\mathcal{E}}\big) \geq 1$, and $\mathcal{E} \neq \mathcal{E}^{max*}_{\mathcal{A}}$ (see Proposition 51). Then $\mathcal{E}$ is not subdirectly irreducible in* $\mathbb{F}$**mbC**.

Proof. Once again, Remark 43 will be used in order to prove that $\mathcal{E}$ is not subdirectly irreducible. Let $(x, y) \in N_{\mathcal{A}}^{max} \setminus N^{\mathcal{E}}$ and $(z, t) \in O_{\mathcal{A}}^{max} \setminus O^{\mathcal{E}}$. Let $(c, d) \notin N^{\mathcal{E}}$, and let $\mathcal{E}' = \langle \mathcal{A}, N^{\mathcal{E}}, O_{\mathcal{A}}^{max} \rangle$. It is easy to see that $\mathcal{E}'$ is an **mbC**-structure. Then, the identity map $h : A \to A$ induces an onto homomorphism $h : \mathcal{E} \to \mathcal{E}'$ which is not an isomorphism, since $(h(z), h(t)) = (z, t) \in O^{\mathcal{E}'}$ but $(z, t) \notin O^{\mathcal{E}}$. Observe that $(h(c), h(d)) = (c, d) \notin N^{\mathcal{E}'}$, since $N^{\mathcal{E}'} = N^{\mathcal{E}}$.

Now, let $(c, d) \notin O^{\mathcal{E}}$. We have two subcases to analyze:

Case 1: $N^{\mathcal{E}} = N_{\mathcal{A}}^{max} \setminus \{(1, 1)\}$. Then $O^{\mathcal{E}} \neq O_{\mathcal{A}}^{max} \setminus \{(1, 0)\}$ since, by hypothesis, $\mathcal{E} \neq \mathcal{E}_{\mathcal{A}}^{max*}$.

Case 1.1: $(c, d) = (1, 0)$. Then $(1, 0) \notin O^{\mathcal{E}}$ and so there exists $(z, t) \neq (1, 0)$ such that $(z, t) \notin O^{\mathcal{E}}$ (given that $\mathcal{E} \neq \mathcal{E}_{\mathcal{A}}^{max*}$). Thus, consider $\mathcal{E}' = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \cup \{(z, t)\} \rangle$. It is clear that $\mathcal{E}'$ is an **mbC**-structure such that the identity map $h : A \to A$ is an onto homomorphism $h : \mathcal{E} \to \mathcal{E}'$ which is not an isomorphism, since $(h(z), h(t)) = (z, t) \in O^{\mathcal{E}'}$ but $(z, t) \notin O^{\mathcal{E}}$. Observe that $(h(c), h(d)) = (c, d) = (1, 0) \notin O^{\mathcal{E}'}$.

Case 1.2: $(c, d) \neq (1, 0)$. Observe that $N_{\mathcal{A}}^{max} = N^{\mathcal{E}} \cup \{(1, 1)\}$. Let $\mathcal{E}' = \langle \mathcal{A}, N_{\mathcal{A}}^{max}, O^{\mathcal{E}} \cup \{(1, 0)\} \rangle$. It is easy to prove that $\mathcal{E}'$ is an **mbC**-structure. In addition, the identity map $h : A \to A$ is an onto homomorphism $h : \mathcal{E} \to \mathcal{E}'$ which is not an isomorphism, since $(h(1), h(1)) = (1, 1) \in N^{\mathcal{E}'}$ but $(1, 1) \notin N^{\mathcal{E}}$. Observe that $(h(c), h(d)) = (c, d) \notin O^{\mathcal{E}'}$ given that $(c, d) \neq (1, 0)$.

Case 2: $N^{\mathcal{E}} \neq N_{\mathcal{A}}^{max} \setminus \{(1, 1)\}$. Let $(x, y) \in N_{\mathcal{A}}^{max} \setminus N^{\mathcal{E}}$ such that $(x, y) \neq (1, 1)$ (such point must exists, by hypothesis). Then $x \sqcap y \neq 1$. Let $\mathcal{E}' = \langle \mathcal{A}, N^{\mathcal{E}'}, O^{\mathcal{E}'} \rangle$ such that $N^{\mathcal{E}'} \stackrel{\text{def}}{=} N^{\mathcal{E}} \cup \{(x, y)\}$ and $O^{\mathcal{E}'}$ is defined according to the following subcases:

Case 2.1: If $d = 0$ then $O^{\mathcal{E}'} \stackrel{\text{def}}{=} O^{\mathcal{E}} \cup \{(x, {\sim}(x \sqcap y))\}$. Observe that $(x, {\sim}(x \sqcap y)) \neq (c, d)$ since $x \sqcap y \neq 1$ (hence, ${\sim}(x \sqcap y) \neq 0 = d$).

Case 2.2: If $d \neq 0$ then $O^{\mathcal{E}'} \stackrel{\text{def}}{=} O^{\mathcal{E}} \cup \{(x, 0)\}$. Clearly $(x, 0) \neq (c, d)$.

Observe that $\mathcal{E}'$ is an **mbC**-structure: indeed, if (x, w) is the new point added to $O^{\mathcal{E}}$ in $O^{\mathcal{E}'}$ then $(x \sqcap y) \sqcap w = 0$. Thus, the property required in Definition 3(b)(iv) is satisfied for the new point (x, y) added to $N^{\mathcal{E}}$ in $N^{\mathcal{E}'}$. Moreover, the identity map $h : A \to A$ is an onto homomorphism $h : \mathcal{E} \to \mathcal{E}'$ which is not an isomorphism, since $(h(x), h(y)) = (x, y) \in N^{\mathcal{E}'}$ but $(x, y) \notin N^{\mathcal{E}}$. Note that $(h(c), h(d)) = (c, d) \notin O^{\mathcal{E}'}$ given that the point added to $O^{\mathcal{E}}$ in each case is different to (c, d).

Finally, if $(c, d) \in A^2$ such that $c \neq d$ then the identity map $h : A \to A$ is an onto homomorphism $h : \mathcal{E} \to \mathcal{E}_{\mathcal{A}}^{max}$ which is not an isomorphism, such that $h(c) \neq h(d)$.

By Remark 43, $\mathcal{E}$ is not subdirectly irreducible. □

Proposition 53 *Let $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ be an* **mbC***-structure over $\mathcal{A}$ such that $N^{\mathcal{E}} = N_{\mathcal{A}}^{max}$ and $card\big(O_{\mathcal{A}}^{max} \setminus O^{\mathcal{E}}\big) \geq 2$. Then, $\mathcal{E}$ is not subdirectly irreducible in $\mathbb{F}$***mbC***.*

Proof. By hypothesis, there exist $(x, y) \neq (z, t)$ such that $(x, y), (z, t) \in O_{\mathcal{A}}^{max} \setminus O^{\mathcal{E}}$. Let $(c, d) \notin N^{\mathcal{E}}$. Then, the identity map $h : A \to A$ is an onto homomorphism $h : \mathcal{E} \to \mathcal{E}_{\mathcal{A}}^{max}$ which is not an isomorphism, since $(h(z), h(t)) = (z, t) \in O_{\mathcal{A}}^{max}$ but $(z, t) \notin O^{\mathcal{E}}$. Observe that $(h(c), h(d)) = (c, d) \notin N^{\mathcal{E}}$.

Now, let $(c, d) \notin O^{\mathcal{E}}$. By hypothesis, there exists $(c', d') \neq (c, d)$ such that $(c', d') \notin O^{\mathcal{E}}$. Let $\mathcal{E}' = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \cup \{(c', d')\} \rangle$. Clearly $\mathcal{E}'$ is an **mbC**-structure such that the identity map h :

$A \to A$ is an onto homomorphism $h : \mathcal{E} \to \mathcal{E}'$ which is not an isomorphism, since $(h(c'), h(d')) = (c', d') \in O^{\mathcal{E}'}$ but $(c', d') \notin O^{\mathcal{E}}$. Observe that $(h(c), h(d)) = (c, d) \notin O^{\mathcal{E}'}$. The case for the identity predicate $\approx$ is treated as in the proof of Proposition 52. □

Proposition 54 *Let $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ be an* **mbC***-structure over $\mathcal{A} \neq \mathbb{A}_2$ such that $O^{\mathcal{E}} = O^{max}_{\mathcal{A}}$ and $card\big(N^{max}_{\mathcal{A}} \setminus N^{\mathcal{E}}\big) \geq 2$. Then $\mathcal{E}$ is an* **mbC***-structure which is not subdirectly irreducible in* $\mathbb{F}$**mbC**.

Proof. The proof is analogous to that of Proposition 53. □

Observe that, if $(x, y), (z, t) \in N^{max}_2$ such that $(x, y) \neq (z, t)$ then $\langle \mathbb{A}_2, N^{max}_2 \setminus \{(x, y), (z, t)\}, O^{max}_2 \rangle$ is *not* an **mbC**-structure. This is why is required that $\mathcal{A} \neq \mathbb{A}_2$ in the last proposition.

By combining the previous results, it is possible to determine, among all the **mbC**-structures, which of them are s.i. and which of them are not:

Theorem 55 *Let $\mathcal{E}$ be an* **mbC***-structure defined over a Boolean algebra $\mathcal{A}$. Then $\mathcal{E}$ is subdirectly irreducible in* $\mathbb{F}$**mbC** *if and only if exactly one of the following conditions holds:*

(1) $\mathcal{E} = \mathbf{1}_{\perp}$; or

(2) $\mathcal{E} = \mathcal{E}^{max}_2$; or

(3) $\mathcal{E} = \mathcal{E}^{max}_{\mathcal{A},N(x,y)} \stackrel{def}{=} \langle \mathcal{A}, N^{max}_{\mathcal{A}} \setminus \{(x, y)\}, O^{max}_{\mathcal{A}} \rangle$ for some $(x, y) \in N^{max}_{\mathcal{A}} \setminus \{(0, 1)\}$; or

(4) $\mathcal{E} = \mathcal{E}^{max}_{\mathcal{A},O(x,y)} \stackrel{def}{=} \langle \mathcal{A}, N^{max}_{\mathcal{A}}, O^{max}_{\mathcal{A}} \setminus \{(x, y)\} \rangle$ for some $(x, y) \in A^2 \setminus \{(1, 0)\}$; or

(5) $\mathcal{E} = \mathcal{E}^{max*}_{\mathcal{A}} \stackrel{def}{=} \langle \mathcal{A}, N^{max}_{\mathcal{A}} \setminus \{(1, 1)\}, O^{max}_{\mathcal{A}} \setminus \{(1, 0)\} \rangle$.

Proof. It is a direct consequence of propositions 44, 46, 48, 49, 51, 52, 53 and 54. □

8.3 Subdirect decomposition theorem for mbC-structures

Finally, we are ready to obtain a decomposition theorem for **mbC**-structures in terms of subdirectly irreducible structures. It is an instance of a general theorem obtained by Caicedo in [6].

Indeed, as it was observed at the end of Section 4, the class **FmbC** of **mbC**-structures can be axiomatized by sentences over signature Θ of the form $\forall x_1 \cdots \forall x_n(\sigma_1 \to \exists y_1 \cdots \exists y_k \sigma_2))$, where σ_1 and σ_2 are quantifier-free positive formulas over Θ. Thus, by combining Corollary 5 and Theorem 4 in [6], the following result is obtained:

Theorem 56 (Birkhoff-Caicedo's decomposition theorem for mbC-structures) *Any non-trivial* **mbC***-structure $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ is a subdirect product of at most $\aleph_0 + card(A)$ non-trivial subdirectly irreducible structures.*

The relationship between Birkhoff-Caicedo's decomposition theorem for **mbC**-structures and the usual Birkhoff's decomposition theorem for varieties of algebras is not immediate. In the case of **mbC**-structures, it seems that there are more than necessary s.i. structures. Indeed, the single **mbC**-structure $\mathcal{E}^{max}_2$ is enough in order to characterize the logic **mbC**.

Theorem 57 (Soundness and completeness of mbC w.r.t. $\mathcal{E}_2^{max}$)
Let $\Gamma \cup \{\alpha\}$ be a finite set of formulas in $For(\Sigma)$. Then: $\Gamma \vdash_{\mathbf{mbC}} \alpha$ *iff* $\Gamma \Vdash^{\mathbf{mbC}}_{\mathcal{E}_2^{max}} \alpha$.

Proof. The 'only if' part is a consequence of Theorem 8. Conversely, suppose that $\Gamma \nvdash_{\mathbf{mbC}} \alpha$. By adapting the proof of [9, Theorem 6.2.16], there exists an **mbC**-structure $\mathcal{E}$ over $\mathbb{A}_2$ and a valuation v over $\mathcal{E}$ such that $v[\Gamma] \subseteq \{1\}$ but $v(\alpha) = 0$. Given that $\mathcal{E}$ is a weak substructure of $\mathcal{E}_2^{max}$, it follows that v is also a valuation over $\mathcal{E}_2^{max}$, by the last observation in Remark 20. This shows that $\Gamma \nVdash^{\mathbf{mbC}}_{\mathcal{E}_2^{max}} \alpha$. $\square$

By a similar argument, from Theorem 22 it can be proven the adequacy of **mbC** w.r.t. the **mbC**-structure $\mathcal{E}_{\mathcal{A}}^{max}$, for any Boolean algebra $\mathcal{A}$ with more than two elements.

Theorem 58 (Soundness and completeness of mbC w.r.t. $\mathcal{E}_{\mathcal{A}}^{max}$)
Let $\mathcal{A}$ be a Boolean algebra with more than two elements, and let $\Gamma \cup \{\alpha\}$ be a finite set of formulas in $For(\Sigma)$. Then: $\Gamma \vdash_{\mathbf{mbC}} \alpha$ *iff* $\Gamma \Vdash^{\mathbf{mbC}}_{\mathcal{E}_{\mathcal{A}}^{max}} \alpha$.

Proof. The 'only if' part is a consequence of Theorem 8. Now, assume that $\Gamma \nvdash_{\mathbf{mbC}} \alpha$. By Theorem 22, there exists an **mbC**-structure $\mathcal{E}$ over $\mathcal{A}$ and a valuation v over $\mathcal{E}$ such that $v[\Gamma] \subseteq \{1\}$ but $v(\alpha) = 0$. Since $\mathcal{E}$ is a weak substructure of $\mathcal{E}_{\mathcal{A}}^{max}$, it follows that v is also a valuation over $\mathcal{E}_{\mathcal{A}}^{max}$, by the last observation in Remark 20. Therefore $\Gamma \nVdash^{\mathbf{mbC}}_{\mathcal{E}_{\mathcal{A}}^{max}} \alpha$. $\square$

In the next subsection an alternative decomposition theorem for **mbC**-structures will be presented, by adapting a decomposition theorem due to Fidel. As it will be argued below, this alternative decomposition theorem is closer to the traditional Birkhoff's decomposition theorem of Universal Algebra.

8.4 Fidel's decomposition theorem for mbC-structures

In 1977, Fidel (see [19]) obtained for the first time the decidability of the hierarchy of paraconsistent calculi C_n of da Costa (see [15]) in terms of certain **F**-structures called C_n-structures. In that paper, Fidel showed that every C_n-structure is weakly isomorphic to a weak substructure of a product of a special C_n-structure defined over the two-element Boolean algebra $\mathbb{A}_2$ called **C**, see [19, Theorem 8]. By a weak isomorphism we mean a homomorphism which is bijective as a mapping.[13]

By adopting the notion of weak substructure in $\mathbb{F}$**mbC** (recall Definition 19), it is possible to obtain another decomposition theorem for **mbC**-structures, alternative to the one given in Theorem 56. Thus, it will be shown in Theorem 60 that each **mbC**-structure is weakly isomorphic to a weak substructure in $\mathbb{F}$**mbC** of a product of **mbC**-structures over $\mathbb{A}_2$. The same result can be rephrased without using the notion of weak isomorphism (see Theorem 65 below).

[13] Such homomorphisms are called *isomorphisms* in [19, Definition 6].

Definition 59 *Let $h : \mathcal{E} \to \mathcal{E}'$ be an* **mbC**-*homomorphism. Then h is said to be a* weak isomorphism *if h is a bijective mapping.*

Theorem 60 (Weak subdirect decomposition theorem for mbC-structures) *Let $\mathcal{E}$ be an* **mbC**-*structure. Then, there exists a set I such that $\mathcal{E}$ is weakly isomorphic to a weak substructure of $\prod_{i\in I} \mathcal{E}_i$, where each $\mathcal{E}_i$ is defined over $\mathbb{A}_2$ for every $i \in I$.*

Proof. Let $\mathcal{E} = \langle \mathcal{A}, N^{\mathcal{E}}, O^{\mathcal{E}} \rangle$ be an **mbC**-structure. By Birkhoff's representation theorem for Boolean algebras [3], there exists a set I and a monomorphism of Boolean algebras $h : \mathcal{A} \to \prod_{i\in I} \mathcal{A}_i$ such that $\mathcal{A}_i = \mathbb{A}_2$ for every $i \in I$. If $I = \emptyset$ then $\mathcal{A} \simeq \mathbb{A}_\perp$ and so $\mathcal{E}$ is the terminal **mbC**-structure $\mathbf{1}_\perp$, hence the result holds with $I = \emptyset$. Now, suppose that $I \neq \emptyset$. For each $i \in I$ consider the structure $\mathcal{E}_i = \langle \mathbb{A}_2, N^{\mathcal{E}_i}, O^{\mathcal{E}_i} \rangle$ such that $N^{\mathcal{E}_i} = \{((\pi_i \circ h)(a), (\pi_i \circ h)(b)) \ : \ (a, b) \in N^{\mathcal{E}}\}$ and $O^{\mathcal{E}_i} = \{((\pi_i \circ h)(a), (\pi_i \circ h)(b)) \ : \ (a, b) \in O^{\mathcal{E}}\}$. Since $(0, 1) \in N^{\mathcal{E}}$ and $(1, b) \in N^{\mathcal{E}}$ for some $b \in A$ then $(0, 1) \in N^{\mathcal{E}_i}$ and $(1, (\pi_i \circ h)(b)) \in N^{\mathcal{E}_i}$. Analogously, it is proved that $(0, x) \in O^{\mathcal{E}_i}$ and $(1, y) \in O^{\mathcal{E}_i}$ for some $x, y \in TWO$. Suppose now that $(x, y) \in N^{\mathcal{E}_i}$. Then, there exists $(a, b) \in N^{\mathcal{E}}$ such that $x = (\pi_i \circ h)(a)$ and $y = (\pi_i \circ h)(b)$. Since $\mathcal{E}$ is an **mbC**-structure, there exists $c \in A$ such that $(a, c) \in O^{\mathcal{E}}$ and $a \sqcap^{\mathcal{E}} b \sqcap^{\mathcal{E}} c = 0$. From this, $(x, (\pi_i \circ h)(c)) \in O^{\mathcal{E}_i}$ such that $0 = (\pi_i \circ h)(0) = x \sqcap^{\mathbb{A}_2} y \sqcap^{\mathbb{A}_2} (\pi_i \circ h)(c))$. This shows that $\mathcal{E}_i = \langle \mathbb{A}_2, N^{\mathcal{E}_i}, O^{\mathcal{E}_i} \rangle$ is an **mbC**-structure, for every $i \in I$. Now, let $\mathcal{E}' = \prod_{i\in I} \mathcal{E}_i$ be the direct product in $\mathbb{F}$**mbC** of the family $\{\mathcal{E}_i\}_{i\in I}$, see Definition 38. Clearly, h is an **mbC**-homomorphism from $\mathcal{E}$ to $\mathcal{E}'$: if $(a, b) \in N^{\mathcal{E}}$ then $((\pi_i \circ h)(a), (\pi_i \circ h)(b)) \in N^{\mathcal{E}_i}$ for every $i \in I$. Hence, $(h(a), h(b)) \in N^{\mathcal{E}'}$ by Definition 38. Analogously, if $(a, b) \in O^{\mathcal{E}}$ then $(h(a), h(b)) \in O^{\mathcal{E}'}$. Consider now the structure $\mathcal{E}'' = \langle h(\mathcal{A}), N^{\mathcal{E}''}, O^{\mathcal{E}''} \rangle$ defined as follows: $N^{\mathcal{E}''} = \{(h(a), h(b)) \ : \ (a, b) \in N^{\mathcal{E}}\}$ and $O^{\mathcal{E}''} = \{(h(a), h(b)) \ : \ (a, b) \in O^{\mathcal{E}}\}$. It is easy to see that $\mathcal{E}''$ is an **mbC**-structure: in order to prove that property (b)(iv) of Definition 3 holds, suppose that $N^{\mathcal{E}''}(x, y)$. Then, there exists a unique $(a, b) \in A^2$ such that $x = h(a)$, $y = h(b)$ and $N^{\mathcal{E}}(a, b)$. From this, there exists $c \in A$ such that $O^{\mathcal{E}}(a, c)$ and $a \sqcap^{\mathcal{E}} b \sqcap^{\mathcal{E}} c = 0$. This means that $O^{\mathcal{E}''}(x, h(c))$ such that $x \sqcap^{h(\mathcal{A})} y \sqcap^{h(\mathcal{A})} h(c) = h(0) = 0$. The properties (b)(i)-(iii) are proved analogously. Since h is injective, it follows that $\mathcal{E}$ is weakly isomorphic to $\mathcal{E}''$ such that $\mathcal{E}'' \subseteq_W \prod_{i\in I} \mathcal{E}_i$, recalling Definition 19 of weak substructure in $\mathbb{F}$**mbC**. $\square$

In order to better understand the real significance of the latter decomoposition result, it is convenient to introduce some definitins in the general framework of Model Theory. Firstly, observe that the notion of weak isomorphism of Definition 59 makes sense in any class of first-order structures:

Definition 61 *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two first-order structures over a signature Ξ. A homomorphism $h : \mathfrak{A} \to \mathfrak{A}'$ in Ξ-**str** is said to be a* weak isomorphism *of Ξ-structures if it a bijective mapping.*

The notion of direct image of a structure by a homomorphism can be weakened in a suitable way:

Definition 62 *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two first-order structures over a signature Ξ, and let $h : \mathfrak{A} \to \mathfrak{A}'$ be a homomorphism in Ξ-**str**. The* weak image *of $\mathfrak{A}$ by h is the Ξ-structure $h(\mathfrak{A})_w$ with domain $h[|\mathfrak{A}|]$; for each function symbol f of arity n, f is interpreted in $h(\mathfrak{A})_w$ by restricting the domain and image of $f^{\mathfrak{A}'}$ to $h[|\mathfrak{A}|]$; the constants are interpreted as in $\mathfrak{A}'$;*[14]*; and for every n-ary predicate symbol P and every $(b_1, \ldots, b_n) \in \big(h[|\mathfrak{A}|)\big)^n$, $(b_1, \ldots, b_n) \in P^{h(\mathfrak{A})_w}$ if and only if there exists $(a_1, \ldots, a_n) \in |\mathfrak{A}|^n$ such that $(h(a_1), \ldots, h(a_n)) = (b_1, \ldots, b_n)$ and $(a_1, \ldots, a_n) \in P^{\mathfrak{A}}$.*

Recall the notion of weak substructure in Ξ-**str** considered right before Definition 19. The proof of the following results is straightforward.

Proposition 63 *Let $h : \mathfrak{A} \to \mathfrak{A}'$ be a homomorphism in Ξ-**str**. Then $h(\mathfrak{A})_w$ is a weak substructure of the direct image $h(\mathfrak{A})$ of $\mathfrak{A}$ by h, hence it is a weak substructure of $\mathfrak{A}'$.*

Proposition 64 *Let $h : \mathfrak{A} \to \mathfrak{A}'$ be a weak isomorphism in Ξ-**str**. Then $\mathfrak{A}$ is isomorphic (via h) to $h(\mathfrak{A})_w$, a weak substructure of $\mathfrak{A}'$.*

The last result relates weak isomorphisms with embeddings (recall Definition 16) in a clear way. Using the previous notions and results, Theorem 60 can be recast as follows (here, $\mathfrak{A} \simeq \mathfrak{A}'$ denotes that the Ξ-structures $\mathfrak{A}$ and $\mathfrak{A}'$ are isomorphic in Ξ-**str**):

Theorem 65 (Weak subdirect decomposition theorem for mbC-structures, version 2) *Let $\mathcal{E}$ be an **mbC**-structure. Then, there exists a set I and an **mbC**-structure $\mathcal{E}$' such that $\mathcal{E} \simeq \mathcal{E}' \subseteq_W \prod_{i \in I} \mathcal{E}_i$, where $\mathcal{E}_i \subseteq_W \mathcal{E}_2^{max}$ for every $i \in I$.*

From Theorem 9 we known that the **mbC**-structures defined over the two-element Boolean algebra are enough to semantically characterize the logic **mbC**. Then, Theorem 65 reflects in a precise way the relationship between the semantical structures and the logic in a similar way to the traditional algebraic approach to logic. In that sense, Theorem 65 is more informative than Theorem 56.

9 Concluding remarks

In this paper, the class of **F**-structures for **mbC** was analyzed under the perspective of Model Theory. This approach is based on the observation that **F**-structures are nothing more than first-order structures satisfying specific Horn sentences of its underlying language, as it was seen in Section 4. Under this broad perspective, the present study could be adapted to the study of

[14] Observe that the interpretation of function symbols and constants is well-defined since f is a homomorphism of first-order structures, hence it is an algebraic homomorphism.

the class of **F**-structures for another **LFI**s as the ones proposed in [8] and [9, Chapter 6]. The fact that all these **LFI**s are axiomatic extensions of **mbC** imposes additional restrictions to the corresponding class of **F**-structures (which are still axiomatized by Horn sentences of the same kind), and so it would be expected that the class of irreducible structures should be reduced. Being so, Caicedo's version of Birkhoff's decomposition theorem (recall Theorem 56) would keep closer to Fidel's one (recall theorems 60 and 65). As observed at the end of Section 8, it can be argued that Fidel's result reflects the meaning of Birkhoff's decomposition theorem for algebras in a more faithful way than the theorem obtained by using the notions from Model Theory, which reveals some limitations of the model-theoretic approach to **F**-structures. The fact that the notion of weak substructures play a fundamental role in Fidel's decomposition theorem suggests that new concepts and tools should be developed for standard Model Theory in order to analyze the (meta)theory of propositional non-algebraizable logics under the perspective of **F**-structures.

Related to this, there is another interesting topic of future research, now concerning the development of a new approach to Model Theory for first-order **LFI**s (and non-algebraizable logics in general) by means of Fidel-structures. To fix ideas, consider the first-order version of **mbC**, namely the logic **QmbC** (see [11]). This logic can be semantically characterized by the so-called *paraconsistent Tarskian structures*, which are (standard) first-order structures together with a paraconsistent two-valued **mbC**-valuation extended naturally to first-order languages. Such valuations could be replaced by valuations over a given **F**-structure for **mbC** (in particular, an **mbC**-structure over the two-element Boolean algebra). This perspective, which generalizes the standard approach to Model Theory over ordered algebras, open interesting lines of research. Thus, the notions and results obtained here for **F**-structures for **mbC** could be adapted for the development of the new Model Theory based on **F**-structures for **QmbC**, as well as for other quantified non-algebraizable logics. In particular, interesting results on Model Theory for quantified **LFI**s such as the Keisler-Shelah Theorem for **QmbC** obtained by T. Ferguson in [18] can be adapted to the proposed framework.

Finally, the connections between Priest's plurivalent semantics (recall Section 1) and other non-deterministic semantics, specifically Nmatrices and Fidel-structures, deserves future research. Moreover, a formal study of plurivalent semantics from the pespective of Model theory and Universal Algebra is an interesting task to be done.

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