$\mathbf{G}'_3$ as the logic of modal 3-valued Heyting algebras

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Abstract

In 2001, W. Carnielli and Marcos considered a 3-valued logic in order to prove that the schema $\varphi \vee (\varphi \rightarrow \psi)$ is not a theorem of da Costa's logic C_ω. In 2006, this logic was studied (and baptized) as $\mathbf{G}'_3$ by Osorio *et al.* as a tool to define semantics of logic programming. It is known that the truth-tables of $\mathbf{G}'_3$ have the same expressive power than the one of Łukasiewicz 3-valued logic as well as the one of Gödel 3-valued logic $\mathbf{G_3}$. From this, the three logics coincide up-to language, taking into acccount that 1 is the only designated truth-value in these logics.

From the algebraic point of view, Canals-Frau and Figallo have studied the 3-valued modal implicative semilattices, where the modal operator is the well-known Moisil-Monteiro-Baaz Δ operator, and the supremum is definable from this. We prove that the subvariety obtained from this by adding a bottom element 0 is term-equivalent to the variety generated by the 3-valued algebra of $\mathbf{G}'_3$. The algebras of that variety are called $\mathbf{G}'_3$-algebras. From this result, we obtain the equations which axiomatize the variety of $\mathbf{G_3'}$-algebras. Moreover, we prove that this variety is semisimple, and the 3-element and the 2-element chains are the unique simple algebras of the variety. Finally an extension of $\mathbf{G_3'}$ to first-order languages is presented, with an algebraic semantics based on complete $\mathbf{G_3'}$-algebras. The corresponding soundness and completeness theorems are obtained.

1 Introduction and preliminaries

In 2001, W. Carnielli *et al.* [4] considered a 3-valued logic in order to prove that the schema $\varphi \vee (\varphi \to \psi)$ is not a theorem of da Costa's logic C_ω. In 2006 this logic was studied (and baptized as $\mathbf{G_3'}$) by Osorio *et al.* [7] as a tool to define semantics of logic programming. They define the connectives $\to$ and $\neg$ of $\mathbf{G_3'}$ logic in terms of some connectives of the three-valued logic of Łukasiewicz $Ł_3$. Conjunction and disjunction, $\wedge$ and $\vee$ respectively, are defined as minimum and maximum. It is known that the truth-tables of $\mathbf{G_3'}$ have the same expressive power than the ones of Łukasiewicz 3-valued logic $Ł_3$ –hence, to the ones of Gödel 3-valued logic **G3**. From this, the three logics coincide up-to language, taking into account that **1** is the only designated truth-value in these logics.

The three-valued Gödel logic **G3**, which is also equivalent to $\mathbf{G_3'}$ and $Ł_3$, is well-suited to express the Stable Model Semantics. $\mathbf{G_3'}$, besides being very close to **G3**, can be used to express non-monotonic reasoning. It is worth mentioning that the negation of **G3** can be reconstructed from connectives of $\mathbf{G_3'}$ by virtue of the formula:

$$\neg_{\mathbf{G3}}\, \varphi = \varphi \to_{\mathbf{G_3'}} (\neg_{\mathbf{G_3'}}\, \varphi \wedge_{\mathbf{G_3'}} \neg_{\mathbf{G_3'}} \neg_{\mathbf{G_3'}}\, \varphi)$$

where the subscripts indicate the underlying logic.

Two different Hilbert-style systems for $\mathbf{G_3'}$ were introduced in [11] and [10], respectively. However, in both approaches it was assumed the validity of the *Deduction Theorem* in the proposed Hilbert calculi for $\mathbf{G_3'}$. As it will discussed in Remark 2.3 below, the Deduction Theorem does not hold in $\mathbf{G_3'}$. This issue in the previous axiomatic approaches to $\mathbf{G_3'}$ justifies proposing a new Hilbert calculus for $\mathbf{G_3'}$, as it will be done in Section 2. Taking into account that $\mathbf{G_3'}$ was introduced as a model of da Costa's logic C_ω, it seems reasonable to define a Hilbert calculus for $\mathbf{G_3'}$ which contains the calculus C_ω.

The paper is organized as follows. In the next section, we present a new Hilbert calculus for $\mathbf{G_3'}$ called $\mathbf{G_{3h}'}$, as an extension of C_ω. In Section 3 we consider the class of $\mathbf{G_3'}$-algebras, proving the soundness and completeness theorem of $\mathbf{G_{3h}'}$ w.r.t the class of $\mathbf{G_3'}$-algebras. After this, in Subsection 3.1 we connect the class of $\mathbf{G_3'}$-algebras with the variety of 3-valued modal implicative semilattices studied by Canals-Frau and Figallo. It will be proved that the subvariety of 3-valued modal implicative semilattices with bottom is term-equivalent to the class of $\mathbf{G_3'}$-algebras. From the latter, we obtain the equations that characterize the class of $\mathbf{G_3'}$-algebras as a variety. From this algebraic analysis, we prove in Subsection 3.2 a second adequacy theorem $\mathbf{G_{3h}'}$ w.r.t. the class of $\mathbf{G_3'}$-algebras. Finally, in Section 4 we present first-order version of $\mathbf{G_{3h}'}$

logic using algebraic tools developed in [5] (see, also, [6]) and our algebraic results of the class of $\mathbf{G'_3}$-algebra presented in the above section.

2 A new Hilbert-style axiomatization of $\mathbf{G'_3}$

Consider from now on the propositional signature $\Sigma = \{\wedge, \vee, \to, \neg\}$. First of all, let us recall the 3-valued semantics for $\mathbf{G'_3}$ logic. It is obtained from the logical matrix $\mathcal{M} = \langle \mathcal{D}, \mathcal{A}_3 \rangle$, where $\mathcal{D} = \{1\}$ and $\mathcal{A}_3 = \langle \mathcal{V}, \sigma \rangle$ is the 3-valued algebra over Σ with domain $\mathcal{V} = \{0, \frac{1}{2}, 1\}$ such that σ interprets the connectives of Σ as follows:

$\wedge$	$\mathbf{0}$	$\mathbf{\frac{1}{2}}$	$\mathbf{1}$
$\mathbf{0}$	0	0	0
$\mathbf{\frac{1}{2}}$	0	$\frac{1}{2}$	$\frac{1}{2}$
$\mathbf{1}$	0	$\frac{1}{2}$	1

$\vee$	$\mathbf{0}$	$\mathbf{\frac{1}{2}}$	$\mathbf{1}$
$\mathbf{0}$	0	$\frac{1}{2}$	1
$\mathbf{\frac{1}{2}}$	$\frac{1}{2}$	$\frac{1}{2}$	1
$\mathbf{1}$	1	1	1

$\to$	$\mathbf{0}$	$\mathbf{\frac{1}{2}}$	$\mathbf{1}$
$\mathbf{0}$	1	1	1
$\mathbf{\frac{1}{2}}$	0	1	1
$\mathbf{1}$	0	$\frac{1}{2}$	1

x	$\neg x$
$\mathbf{0}$	1
$\mathbf{\frac{1}{2}}$	1
$\mathbf{1}$	0

The set of well-formed formulas, denoted by $\mathcal{L}_\Sigma$, is constructed as usual from a given denumerable set $Var = \{p_0, p_1, \ldots\}$ of propositional variables. As usual, the bi-implication $\leftrightarrow$ can be defined in $\mathbf{G'_3}$ by $\varphi \leftrightarrow \psi \stackrel{\text{def}}{=} (\varphi \to \psi) \wedge (\psi \to \varphi)$. Its truth-table is displayed below.

$\leftrightarrow$	$\mathbf{0}$	$\mathbf{\frac{1}{2}}$	$\mathbf{1}$
$\mathbf{0}$	1	0	0
$\mathbf{\frac{1}{2}}$	0	1	$\frac{1}{2}$
$\mathbf{1}$	0	$\frac{1}{2}$	1

The consequence relation of $\mathbf{G'_3}$ induced by the logical matrix $\mathcal{M}$ will be denoted by $\models_{\mathbf{G'_3}}$. Thus: $\Gamma \models_{\mathbf{G'_3}} \varphi$ if and only if, for every valuation h (that is, for every homomorphism $h : \mathcal{L}_\Sigma \to \mathcal{A}_3$ of algebras over Σ), if $h(\psi) = 1$ for every $\psi \in \Gamma$ then $h(\varphi) = 1$.

A formal axiomatic system for $\mathbf{G'_3}$ called $\mathbf{G'_{3h}}$ over the signature Σ will be defined below (see Definition 2.1). Previous to this, some motivations will be given. The implication above is a particular case ($n = 3$) of the family of implicative systems LC_n proposed by Thomas in [12]. This implication, together with Thomas's axiom for 3-valued systems

(Tho) $((\varphi \to \psi) \to \gamma) \to (((\gamma \to \varphi) \to \gamma) \to \gamma)$

was used by L. Monteiro in [8] to introduce the class of 3-valued Heyting algebras. As we shall see, the logic $\mathbf{G'_3}$ is closely related to L. Monteiro's 3-valued Heyting algebras. Because of this, axiom **(Tho)** for 3-valued systems will be considered in $\mathbf{G'_{3h}}$. In addition, axiom

(CF) $(((\psi \to \neg\neg\psi) \to (\varphi \to \neg\neg\varphi)) \to \neg\neg(\varphi \to \psi)) \leftrightarrow (\neg\neg\varphi \to \neg\neg\psi)$

which is adapted from an axiom introduced by Canals-Frau and Figallo in [2] to axiomatize the variety of 3-valued implicative semilattices, will be also considered by reasons which will be clear in Section 3.1 below.

Definition 2.1. *The Hilbert calculus* $\mathbf{G'_{3h}}$ *over* Σ *is defined as follows:*

Axiom schemas:

(Ax1) $\varphi \rightarrow (\psi \rightarrow \varphi)$
(Ax2) $(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma))$
(Ax3) $(\varphi \wedge \psi) \rightarrow \varphi$
(Ax4) $(\varphi \wedge \psi) \rightarrow \psi$
(Ax5) $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$
(Ax6) $\varphi \rightarrow (\varphi \vee \psi)$
(Ax7) $\psi \rightarrow (\varphi \vee \psi)$
(Ax8) $(\varphi \rightarrow \gamma) \rightarrow ((\psi \rightarrow \gamma) \rightarrow ((\varphi \vee \psi) \rightarrow \gamma))$
(Ax9) $\varphi \vee \neg\varphi$
(Ax10) $\neg\neg\varphi \rightarrow \varphi$
(Ax11) $\neg\varphi \rightarrow (\neg\neg\varphi \rightarrow \psi)$
(Ax12) $\neg\neg(\varphi \vee \psi) \rightarrow (\neg\neg\varphi \vee \neg\neg\varphi)$
(Ax13) $\neg\neg(\varphi \rightarrow \psi) \leftrightarrow ((\varphi \rightarrow \psi) \wedge (\neg\neg\varphi \rightarrow \neg\neg\varphi))$
(Tho) $((\varphi \rightarrow \psi) \rightarrow \gamma) \rightarrow (((\gamma \rightarrow \varphi) \rightarrow \gamma) \rightarrow \gamma)$
(CF) $(((\psi \rightarrow \neg\neg\psi) \rightarrow (\varphi \rightarrow \neg\neg\varphi)) \rightarrow \neg\neg(\varphi \rightarrow \psi)) \leftrightarrow (\neg\neg\varphi \rightarrow \neg\neg\psi)$

Inference Rules:

(MP) $\dfrac{\varphi \quad \varphi \rightarrow \psi}{\psi}$ ***(imp)*** $\dfrac{\varphi \rightarrow \psi}{\neg\psi \rightarrow \neg\varphi}$

It is worth noting that axioms **(Ax1)-(Ax8)** plus **(MP)** constitute a Hilbert calculus sound and complete for *Positive Intuitionistic Propositional Logic* $\mathbf{IPL}^+$. This means that $\mathbf{IPL}^+$ is contained in $\mathbf{G'_{3h}}$ (this fact will be used later). In addition, the calculus formed by axioms **(Ax1)-(Ax10)** plus **(MP)** is exactly da Costa's logic C_ω. Thus, $\mathbf{G'_3}$ is an extension of C_ω, in accordance with the original intuitions mentioned in Section 1.

Definition 2.2. *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$ be a set of formulas. A derivation of φ from Γ in $\mathbf{G'_{3h}}$ is a finite sequence $\varphi_1 \cdots \varphi_n$ of formulas in $\mathcal{L}_\Sigma$ such that $\varphi_n = \varphi$ and for $1 \leq i \leq n$, it holds:*

1. *φ_i is an instance of some axiom in $\mathbf{G'_{3h}}$, or*
2. *$\varphi_i \in \Gamma$, or*
3. *there exist some $j, k < i$ such that φ_i follows from φ_j and φ_k by applying **MP**, or*
4. *there exist some $j < i$ such that φ_i follows from φ_j by applying **imp**.*

We say that φ is derivable from Γ in $\mathbf{G'_{3h}}$, *denoted as $\Gamma \vdash_{\mathbf{G'_{3h}}} \varphi$ (or simply $\Gamma \vdash \varphi$), if there exists a derivation of φ from Γ in $\mathbf{G'_{3h}}$.*

Remark 2.3. *It is easy to see that the* Deduction Theorem *does not hold* $\mathbf{G}'_{\mathbf{3}}$*: indeed,* $p, \neg p \vDash_{\mathbf{G}'_{\mathbf{3}}} q$ *for every propositional variables* p *and* q *with* $p \neq q$.[1] *However,* $\nvDash_{\mathbf{G}'_{\mathbf{3}}} p \to (\neg p \to q)$*: it is enough to consider a valuation* h *such that* $h(p) = \frac{1}{2}$ *and* $h(q) = 0$*. Alternatively, the failure of the Deduction Theorem in* $\mathbf{G}'_{\mathbf{3}}$ *can be seen by observing that* $p \vDash_{\mathbf{G}'_{\mathbf{3}}} \neg\neg p$ *for every propositional variable* p*, but* $\nvDash_{\mathbf{G}'_{\mathbf{3}}} p \to \neg\neg p$*: it suffices to consider a valuation* h *such that* $h(p) = \frac{1}{2}$*. From this, the Deduction Theorem should not be valid in* $\mathbf{G}'_{\mathbf{3h}}$ *(since* $\mathbf{G}'_{\mathbf{3h}}$ *is intended to be adequate to* $\mathbf{G}'_{\mathbf{3}}$*). This fact will be proven in Corollary 3.28.*

Despite this, a restricted version of the Deduction Theorem holds in $\mathbf{G}'_{\mathbf{3h}}$:

Proposition 2.4 (Restricted Deduction Theorem **(RDT)**)**.** *Let* $\Gamma \cup \{\varphi, \psi\}$ *be a set of formulas in* $\mathcal{L}_\Sigma$*. Assume that* $\Gamma, \varphi \vdash_{\mathbf{G}'_{\mathbf{3h}}} \psi$ *such that there is a derivation in* $\mathbf{G}'_{\mathbf{3h}}$ *of* ψ *from* $\Gamma \cup \{\varphi\}$ *in which the inference rule* ***imp*** *is not used. Then,* $\Gamma \vdash_{\mathbf{G}'_{\mathbf{3h}}} \varphi \to \psi$ *without using* ***imp****.*

Proof. It follows from the fact that, in such derivation of ψ from $\Gamma \cup \{\varphi\}$ in $\mathbf{G}'_{\mathbf{3h}}$, axioms **Ax1** and **Ax2** are available, and **MP** is the only inference rule used there. Under these circumstances, the Deduction Theorem holds (see, for instance, [9]). Hence, $\Gamma \vdash \varphi \to \psi$ without using **imp**. □

A different form of the Deduction Theorem will be obtained in Theorem 3.13 below. A direct consequence of Proposition 2.4 is the *Restricted Proof by Cases* property:

Proposition 2.5 (Restricted Proof by Cases **(RPC)**)**.** *Let* $\Gamma \cup \{\varphi, \psi\}$ *be a set of formulas in* $\mathcal{L}_\Sigma$*. Then, the following holds in* $\mathbf{G}'_{\mathbf{3h}}$*:*
If $\Gamma, \varphi \vdash_{\mathbf{G}'_{\mathbf{3h}}} \gamma$ *and* $\Gamma, \psi \vdash_{\mathbf{G}'_{\mathbf{3h}}} \gamma$ *without using* ***imp****, then* $\Gamma, \varphi \vee \psi \vdash_{\mathbf{G}'_{\mathbf{3h}}} \gamma$ *without using* ***imp****.*
In particular,
If $\Gamma, \varphi \vdash_{\mathbf{G}'_{\mathbf{3h}}} \gamma$ *and* $\Gamma, \neg\varphi \vdash_{\mathbf{G}'_{\mathbf{3h}}} \gamma$ *without using* ***imp****, then* $\Gamma \vdash_{\mathbf{G}'_{\mathbf{3h}}} \gamma$ *without using* ***imp****.*

Proof. The first part is a direct consequence of Proposition 2.4, **(Ax8)** and **(MP)**. The second part follows from the first one by using **(Ax9)** and **(MP)**. □

Proposition 2.6. *The following schemas are derivable in* $\mathbf{G}'_{\mathbf{3h}}$*:*
(1) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$*;*
(2) $(\gamma \to \alpha) \to ((\gamma \to \beta) \to (\gamma \to (\alpha \wedge \beta)))$*;*
(3) $(\alpha \to \alpha') \to ((\beta \to \beta') \to ((\alpha \wedge \beta) \to (\alpha' \wedge \beta')))$*;*
(4) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to \gamma)$*;*
(5) $(\alpha' \to \alpha) \to ((\beta \to \beta') \to ((\alpha \to \beta) \to (\alpha' \to \beta')))$*.*

Proof. It follows from the fact that all these schemas are provable in positive intuitionistic propositional logic $\mathbf{IPL}^+$, which is contained in $\mathbf{G}'_{\mathbf{3h}}$. □

[1] By structurality, $\varphi, \neg\varphi \vDash_{\mathbf{G}'_{\mathbf{3}}} \psi$ for every formulas φ and ψ. Hence, the negation $\neg$ is *explosive* in $\mathbf{G}'_{\mathbf{3}}$, so this logic is *not* paraconsistent w.r.t. $\neg$.

Proposition 2.7. *The following rules*

$$\textit{(Dneg)}\ \frac{\varphi}{\neg\neg\varphi} \qquad \textit{(exp)}\ \frac{\varphi \quad \neg\varphi}{\psi}$$

are derivable in $\mathbf{G}'_{\mathbf{3h}}$.

Proof. For **(Dneg)**, observe firstly that $(\varphi \to \varphi) \to \neg\neg(\varphi \to \varphi)$ is a theorem in $\mathbf{G}'_{\mathbf{3h}}$. Indeed, since both $(\varphi \to \varphi) \to (\neg\neg\varphi \to \neg\neg\varphi)$ and $(\varphi \to \varphi) \to (\varphi \to \varphi)$ are derivable in $\mathbf{G}'_{\mathbf{3h}}$ (by $\mathbf{IPL^+}$), so is $(\varphi \to \varphi) \to (\varphi \to \varphi) \wedge (\neg\neg\varphi \to \neg\neg\varphi)$. Hence $(\varphi \to \varphi) \to \neg\neg(\varphi \to \varphi)$ follows from this by using **(Ax13)**. Now, consider the following derivation in $\mathbf{G}'_{\mathbf{3h}}$:

1.	φ	Hyp.
2.	$(\varphi \to \varphi) \to \varphi$	$\mathbf{IPL^+}$
3.	$\neg\neg(\varphi \to \varphi) \to \neg\neg\varphi$	**(imp)**, 2 (two times)
4.	$(\varphi \to \varphi) \to \neg\neg(\varphi \to \varphi)$	Observation above
5.	$(\varphi \to \varphi) \to \neg\neg\varphi$	$\mathbf{IPL^+}$, 4,3
6.	$\neg\neg\varphi$	$\mathbf{IPL^+}$, 5

For **(exp)**, consider the following (meta)derivation in $\mathbf{G}'_{\mathbf{3h}}$:

1.	φ	Hyp.
2.	$\neg\varphi$	Hyp.
3.	$\neg\neg\varphi$	**(Dneg)**, 1
4.	$\neg\varphi \to (\neg\neg\varphi \to \psi)$	**(Ax11)**
5.	$\neg\neg\varphi \to \psi$	**(MP)**, 2,4
6.	ψ	**(MP)**, 3,5

□

Proposition 2.8. *The following schemas are derivable in* $\mathbf{G}'_{\mathbf{3h}}$*:*

(1) $\neg\neg(\varphi \wedge \psi) \leftrightarrow (\neg\neg\varphi \wedge \neg\neg\psi)$*;*

(2) $\neg\neg(\varphi \vee \psi) \leftrightarrow (\neg\neg\varphi \vee \neg\neg\psi)$

(3) $\neg\neg\neg\varphi \leftrightarrow \neg\varphi$*.*

Proof.

(1) From **(Ax5)** and **(Dneg)** it follows that $\neg\neg(\varphi \to (\psi \to (\varphi \wedge \psi)))$ is a theorem of $\mathbf{G}'_{\mathbf{3h}}$. Using **(Ax13)**, **(Ax3)**, **(Ax4)**, Proposition 2.6 items (1), (4) and **(MP)** it follows that $(\neg\neg\varphi \wedge \neg\neg\psi) \to \neg\neg(\varphi \wedge \psi)))$. The converse is proved analogously.

(2) Analogously to the proof of item (1) (but now using **(Ax6)** and **(Ax7)**) it is proved that $\neg\neg(\varphi \vee \psi) \to (\neg\neg\varphi \vee \neg\neg\psi)$ is a theorem of $\mathbf{G}'_{\mathbf{3h}}$. The converse is just **(Ax12)**.

(3) By **(Ax10)** it follows that $\neg\neg\neg\varphi \to \neg\varphi$ is a theorem of $\mathbf{G}'_{\mathbf{3h}}$. In addition, $\neg\varphi, \neg\neg\varphi \vdash_{\mathbf{G}'_{\mathbf{3h}}} \neg\neg\neg\varphi$ without using **imp** (just by using **(Ax10)** and **(MP)**), and also $\neg\varphi, \neg\neg\neg\varphi \vdash_{\mathbf{G}'_{\mathbf{3h}}} \neg\neg\neg\varphi$ without using **imp** (by Definition 2.2). Then, $\neg\varphi \vdash_{\mathbf{G}'_{\mathbf{3h}}} \neg\neg\neg\varphi$ without **imp**, by Proposition 2.5. Hence, $\vdash_{\mathbf{G}'_{\mathbf{3h}}} \neg\varphi \to \neg\neg\neg\varphi$ by Proposition 2.4. □

Instead of proving directly the soundness and completeness of $\mathbf{G'_{3h}}$ w.r.t. $\mathbf{G'_3}$, in the next section an algebraic semantics for $\mathbf{G'_{3h}}$ will be proposed, based on a new class of algebras called $\mathbf{G'_3}$-algebras. After proving the adequacy of $\mathbf{G'_{3h}}$ w.r.t. this algebraic semantics, in Section 3.1 it will be proved that the class of $\mathbf{G'_3}$-algebras is in fact a variety (that is, it can be axiomatized by means of equations) which is term-equivalent to a subvariety of a variety already studied in the literature ([2]). This allows us to show that the algebra underlying the 3-valued matrix of $\mathbf{G'_3}$ generates the variety of $\mathbf{G'_3}$-algebras (see Corollary 3.23). The completeness of $\mathbf{G'_{3h}}$ w.r.t. $\mathbf{G'_3}$ will be obtained easily from this (see Theorem 3.27).

3 The class of $\mathbf{G'_3}$-algebras

Recall that $\Sigma = \{\wedge, \vee, \to, \neg\}$ is the propositional signature for logic $\mathbf{G'_3}$, and that $\mathcal{L}_\Sigma$ is the algebra of formulas of $\mathbf{G'_3}$ generated over Σ by Var. Let $\Sigma_I = \{\wedge, \vee, \to, \mathbf{0}, \mathbf{1}\}$ be the signature of Heyting algebras and let $\Sigma_+ = \{\wedge, \vee, \to, \neg, \mathbf{0}, \mathbf{1}\}$.

Definition 3.1. *A $\mathbf{G'_3}$-algebra is an algebra $\mathcal{A} = \langle A, \wedge, \vee, \to, \neg, 0, 1\rangle$ of type $(2,2,2,1,0,0)$ such that*

(i) The reduct $\mathcal{H}_\mathcal{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$ is a 3-valued Heyting algebra (see [8]). That is, $\mathcal{H}_\mathcal{A}$ is a Heyting algebra such that, for every $x, y, z \in A$:

$$((x \to y) \to z) \to (((z \to x) \to z) \to z) = 1;$$

(ii) $x \vee \neg x = 1$, *for every* x;

(iii) $\neg x \wedge \neg\neg x = 0$, *for every* x;

(iv) $\neg\neg x \to x = 1$, *for every* x;

(v) $(\neg\neg(x \vee y) \to (\neg\neg x \vee \neg\neg y)) = 1$, *for every* x, y;

(vi) $\neg\neg(x \to y) = (x \to y) \wedge (\neg\neg x \to \neg\neg y)$, *for every* x, y;

(vii) $(((y \to \neg\neg y) \to (x \to \neg\neg x)) \to \neg\neg(x \to y)) = (\neg\neg x \to \neg\neg y)$, *for every* x, y;

(viii) for every x, y: *if* $x \to y = 1$ *then* $\neg y \to \neg x = 1$.

The class of $\mathbf{G'_3}$-algebras will be denoted by $\mathbb{A}\mathbf{G'_3}$.

Proposition 3.2. *Let $\mathcal{A}$ be a $\mathbf{G'_3}$-algebra. Then, for any $x, y \in A$:*
(1) $\neg\neg(x \wedge y) = (\neg\neg x \wedge \neg\neg y)$;
(2) $\neg\neg(x \vee y) = (\neg\neg x \vee \neg\neg y)$;
(3) $\neg\neg\neg x = \neg x$;

(4) $\neg\neg x \to 0 = \neg x$*;*

(5) $\neg 0 = 1$*,* $\neg\neg 0 = 0$*,* $\neg 1 = 0$ *and* $\neg\neg 1 = 1$*;*

(6) $(\neg\neg x \to \neg\neg y) \to \neg\neg x = \neg\neg x$*;*

(7) If $\neg\neg x \leq y \to z$ *then* $\neg\neg x \leq \neg\neg y \to \neg\neg z$*;*

(8) $\neg\neg x \to \neg\neg y = \neg y \to \neg x$*.*

Proof. Straightforward, taking into account that $\mathcal{A}$ is an implicative lattice, hence: $x \leq y$ iff $x \to y = 1$. □

Definition 3.3. *Let* $\mathcal{A}$ *be a* $\mathbf{G_3'}$*-algebra. The* logical matrix induced by $\mathcal{A}$ *is* $\mathcal{M}_\mathcal{A} \overset{def}{=} \langle \mathcal{A}, \{1\} \rangle$*.*

A *valuation over* $\mathcal{M}_\mathcal{A}$ is any homomorphism $h : \mathcal{L}_\Sigma \to \mathcal{A}$.[2] If $\Gamma \cup \{\varphi\}$ is a set of formulas in $\mathcal{L}_\Sigma$ we say that φ *is a consequence of* Γ *w.r.t. the logical matrix* $\mathcal{M}_\mathcal{A}$, written as $\Gamma \models_{\mathcal{M}_\mathcal{A}} \varphi$, if the following holds: for every valuation h over $\mathcal{M}_\mathcal{A}$, $h(\varphi) = 1$ whenever $h(\gamma) = 1$ for every $\gamma \in \Gamma$.

Definition 3.4. *Let* $\Gamma \cup \{\varphi\}$ *be a set of formulas in* $\mathcal{L}_\Sigma$*. Then* φ *is said to be a* consequence of Γ w.r.t. $\mathbf{G_3'}$-algebras*, denoted by* $\Gamma \models_{\mathbb{A}\mathbf{G_3'}} \varphi$*, if* $\Gamma \models_{\mathcal{M}_\mathcal{A}} \varphi$ *for every matrix* $\mathcal{M}_\mathcal{A}$ *and every* $\mathbf{G_3'}$*-algebra* $\mathcal{A}$*.*

Now, the adequacy of $\mathbf{G_{3h}'}$ with respect to the $\mathbf{G_3'}$-algebras semantics $\models_{\mathbb{A}\mathbf{G_3'}}$ will be proved.

Theorem 3.5 (Soundness of $\mathbf{G_{3h}'}$ w.r.t. $\mathbf{G_3'}$-algebras)**.** *Let* $\Gamma \cup \{\varphi\}$ *be a set of formulas in* $\mathcal{L}_\Sigma$*. Then,* $\Gamma \vdash_{\mathbf{G_{3h}'}} \varphi$ *implies that* $\Gamma \models_{\mathbb{A}\mathbf{G_3'}} \varphi$*.*

Proof. It is easy to see that every axiom in $\mathbf{G_{3h}'}$ is valid in any $\mathbf{G_3'}$-algebra, that is: for every $\mathcal{A} \in \mathbb{A}\mathbf{G_3'}$ and for every valuation h over $\mathcal{A}$, $h(\varphi) = 1$ for every instance φ of every axiom of $\mathbf{G_{3h}'}$. In addition, satisfaction is preserved by the inference rules. Indeed, suppose that h is a valuation over $\mathcal{A}$ such that $h(\varphi) = h(\varphi \to \psi) = 1$. Then $1 = h(\varphi) \to h(\psi) = 1 \to h(\psi) = h(\psi)$ (recall that, in any Heyting algebra, $1 \to x = x$ for every x). In addition, if h is a valuation such that $h(\varphi \to \psi) = h(\varphi) \to h(\psi) = 1$ then $h(\neg\psi \to \neg\varphi) = \neg h(\psi) \to \neg h(\varphi) = 1$, by Definition 3.1(viii). Using this, the result follows by induction on the length of derivations. □

In order to prove the completeness of $\mathbf{G_{3h}'}$ with respect to $\mathbf{G_3'}$-algebras, some previous definitions and results are needed.

Definition 3.6 (Tarskian Logic)**.** *A logic* $\mathcal{L}$ *is* Tarskian *if it satisfies the following properties, for every set of formulas* $\Gamma \cup \Upsilon \cup \{\alpha\}$*:*

(i) if $\alpha \in \Gamma$ *then* $\Gamma \vdash \alpha$*;*

(ii) if $\Gamma \vdash \alpha$ *and* $\Gamma \subseteq \Upsilon$ *then* $\Upsilon \vdash \alpha$*;*

[2] To be rigorous, h is a homomorphism from $\mathcal{L}_\Sigma$ to the Σ-reduct of $\mathcal{A}$.

(iii) if $\Upsilon \vdash \alpha$ *and* $\Gamma \vdash \beta$ *for every* $\beta \in \Upsilon$ *then* $\Gamma \vdash \alpha$.

The logic $\mathcal{L}$ *is* finitary *if it satisfies the following property:*

(iv) if $\Gamma \vdash \alpha$ *then there exists a finite subset* Γ_0 *of* Γ *such that* $\Gamma_0 \vdash \alpha$.

Definition 3.7. *Let* $\mathcal{L}$ *be a Tarskian logic. A set of formulas* Γ *is* closed in $\mathcal{L}$ *if, for every formula* ψ*:* $\Gamma \vdash \psi$ *iff* $\psi \in \Gamma$.

Definition 3.8. *Let* $\mathcal{L}$ *be a Tarskian logic, and let* $\Gamma \cup \{\varphi\}$ *be a set of formulas. The set* Γ *is* maximal non-trivial w.r.t. φ *in* $\mathcal{L}$*, or* φ-saturated *in* $\mathcal{L}$*, if* $\Gamma \nvdash \varphi$ *but* $\Gamma, \psi \vdash \varphi$ *for any* $\psi \notin \Gamma$.

It is easy to prove that any φ-saturated set of formulas in a Tarskian logic is closed. Recall now the following classical result (see [13, Theorem 22.2]):

Theorem 3.9 (Lindenbaum-Łoś)**.** *Let* $\mathcal{L}$ *be a Tarskian and finitary logic, and let* $\Gamma \cup \{\varphi\}$ *be a set of formulas such that* $\Gamma \nvdash \varphi$*. Then, there exists a set of formulas* Υ *such that* Υ *is* φ*-saturated in* $\mathcal{L}$ *and* $\Gamma \subseteq \Upsilon$.

Remark 3.10. *Clearly* $\mathbf{G'_{3h}}$ *is Tarskian and finitary, then Theorem 3.9 applies to it. Observe that, if* Υ *is a* φ*-saturated set in* $\mathbf{G'_{3h}}$ *then, for every formula* β*:* $\beta \in \Upsilon$ *iff* $\neg\neg\beta \in \Upsilon$.

Theorem 3.11 (Completeness of $\mathbf{G'_{3h}}$ w.r.t. $\mathbf{G'_3}$-algebras)**.** *Let* $\Gamma \cup \{\varphi\}$ *be a set of formulas in* $\mathcal{L}_\Sigma$*. Then,* $\Gamma \models_{\mathbb{A}\mathbf{G'_3}} \varphi$ *implies that* $\Gamma \vdash_{\mathbf{G'_{3h}}} \varphi$.

Proof. Suppose that $\Gamma \nvdash_{\mathbf{G'_{3h}}} \varphi$. By Theorem 3.9 and Remark 3.10 there exists a set Υ which is φ-saturated in $\mathbf{G'_{3h}}$ such that $\Gamma \subseteq \Upsilon$. Define the following relation in $\mathcal{L}_\Sigma$: $\beta \equiv_\Upsilon \gamma$ iff $\Upsilon \vdash_{\mathbf{G'_{3h}}} \beta \leftrightarrow \gamma$. By the properties of $\mathbf{IPL}^+$, including the ones listed in Proposition 2.6, it is easy to prove that $\equiv_\Upsilon$ is a congruence over $\mathcal{L}_\Sigma$ with respect to the connectives $\wedge$, $\vee$ and $\to$. Moreover, $\mathcal{A}_\Upsilon \stackrel{\text{def}}{=} \mathcal{L}_\Sigma/_{\equiv_\Upsilon}$ is an implicative lattice with such operations. In addition, it has a bottom element given by $0 = [\neg\beta \wedge \neg\neg\beta]_\Upsilon$ for any formula β, where $[\psi]_\Upsilon$ denotes the equivalence class of the formula ψ w.r.t. $\equiv_\Upsilon$. This means that $\mathcal{A}_\Upsilon$ is a 3-valued Heyting algebra, by virtue of **(Tho)**. Note also that $\neg[\beta]_\Upsilon \stackrel{\text{def}}{=} [\neg\beta]_\Upsilon$ is a well-defined operation in $\mathcal{A}_\Upsilon$, because of **(imp)**. It is immediate to see that $\mathcal{A}_\Upsilon$ satisfies properties (ii)-(viii) of Definition 3.1. Hence, $\mathcal{A}$ is a $\mathbf{G'_3}$-algebra such that $[\gamma]_\Upsilon = 1$ iff $\Upsilon \vdash_{\mathbf{G'_{3h}}} \gamma$ iff $\gamma \in \Upsilon$. Consider now the function $h_\Upsilon : \mathcal{L}_\Sigma \to \mathcal{A}_\Upsilon$ given by $h_\Upsilon(\gamma) = [\gamma]_\Upsilon$. It is easy to see that h_Υ is a valuation over $\mathcal{M}_{\mathcal{A}_\Upsilon}$ such that $h_\Upsilon(\gamma) = 1$ iff $\gamma \in \Upsilon$. Therefore, h_Υ is a valuation over $\mathcal{M}_{\mathcal{A}_\Upsilon}$ such that $h_\Upsilon(\gamma) = 1$ for every $\gamma \in \Gamma$ but $h_\Upsilon(\varphi) \neq 1$, since $\varphi \notin \Upsilon$. This shows that $\Gamma \not\models_{\mathbb{A}\mathbf{G'_3}} \varphi$. □

As a corollary of the completeness theorem above, a special and useful form of the Deduction Theorem can be obtained (see Theorem 3.13 below). Previously, some results must be stated.

Lemma 3.12. *Let* $\Gamma \cup \{\varphi, \psi, \gamma\}$ *be a set of formulas in* $\mathcal{L}_\Sigma$.

(1) If $\Gamma \vdash_{\mathbf{G'_{3h}}} \neg\neg\varphi \to \psi$ *and* $\Gamma \vdash_{\mathbf{G'_{3h}}} \neg\neg\varphi \to (\psi \to \gamma)$ *then* $\Gamma \vdash_{\mathbf{G'_{3h}}} \neg\neg\varphi \to \gamma$.

(2) If $\Gamma \vdash_{\mathbf{G'_{3h}}} \neg\neg\varphi \to (\psi \to \gamma)$ *then* $\Gamma \vdash_{\mathbf{G'_{3h}}} \neg\neg\varphi \to (\neg\gamma \to \neg\psi)$.

Proof. (1) It follows by using the hypothesis together with the theorems $(\neg\neg\varphi \to \psi) \to ((\neg\neg\varphi \to (\psi \to \gamma)) \to (\neg\neg\varphi \to (\psi \wedge (\psi \to \gamma)))$ and $(\psi \wedge (\psi \to \gamma)) \to \gamma$ of $\mathbf{G'_{3h}}$, taking also into account Proposition 2.6(1).

(2) Suppose that $\Gamma \vdash_{\mathbf{G'_{3h}}} \neg\neg\varphi \to (\psi \to \gamma)$. Then $\Gamma \models_{\mathbb{A}\mathbf{G'_3}} \neg\neg\varphi \to (\psi \to \gamma)$, by Theorem 3.5. Let h be a valuation over a matrix $\mathcal{M}_{\mathcal{A}}$, for a given $\mathbf{G'_3}$-algebra $\mathcal{A}$, such that $h(\delta) = 1$ for every $\delta \in \Gamma$. Then, $h(\neg\neg\varphi \to (\psi \to \gamma)) = 1$. Let $x = h(\varphi)$, $y = h(\psi)$ and $z = h(\gamma)$. Then $\neg\neg x \leq y \to z$ and so $\neg\neg x \leq \neg\neg y \to \neg\neg z = \neg z \to \neg y$, by Proposition 3.2 items (7) and (8). That is, $h(\neg\neg\varphi \to (\neg\gamma \to \neg\psi)) = 1$. This shows that $\Gamma \models_{\mathbb{A}\mathbf{G'_3}} \neg\neg\varphi \to (\neg\gamma \to \neg\psi)$. By Theorem 3.11, $\Gamma \vdash_{\mathbf{G'_{3h}}} \neg\neg\varphi \to (\neg\gamma \to \neg\psi)$. □

Theorem 3.13 (Special Deduction Theorem (SDT))**.** *Let* $\Gamma \cup \{\varphi, \psi\}$ *be a set of formulas in* $\mathcal{L}_\Sigma$*. Then,* $\Gamma, \varphi \vdash_{\mathbf{G'_{3h}}} \psi$ *if and only if* $\Gamma \vdash_{\mathbf{G'_{3h}}} \neg\neg\varphi \to \psi$*.*

Proof.
(*Only if* part). Suppose that $\Gamma, \varphi \vdash_{\mathbf{G'_{3h}}} \psi$. By induction on the length n of a derivation $\varphi_1 \cdots \varphi_n$ of ψ from $\Gamma \cup \{\varphi\}$ in $\mathbf{G'_{3h}}$, it can be proven that $\Gamma \vdash_{\mathbf{G'_{3h}}} \neg\neg\varphi \to \varphi_i$ for every $1 \leq i \leq n$, and so $\Gamma \vdash_{\mathbf{G'_{3h}}} \neg\neg\varphi \to \psi$ (for $i = n$). To do this, it must taken into account the fact that $\Gamma \vdash_{\mathbf{G'_{3h}}} \neg\neg\varphi \to \psi$ if either $\psi \in \Gamma \cup \{\varphi\}$ or ψ is an instance of an axiom (for the base step), and Lemma 3.12 (to deal with the inference rules from the induction hypothesis). The details of the proof are left to the reader.

(*If* part). Suppose that $\Gamma \vdash_{\mathbf{G'_{3h}}} \neg\neg\varphi \to \psi$, and let $\varphi_1 \cdots \varphi_n = \neg\neg\varphi \to \psi$ be a derivation of $\neg\neg\varphi \to \psi$ from Γ in $\mathbf{G'_{3h}}$. Consider now the following (meta)derivation of ψ from $\Gamma \cup \{\varphi\}$ in $\mathbf{G'_{3h}}$:

1.	φ_1	
$\vdots$	$\vdots$	
n.	$\varphi_n = \neg\neg\varphi \to \psi$	
$n+1$.	φ	Hyp.
$n+2$.	$\neg\neg\varphi$	**(Dneg)**, n+1
$n+3$.	ψ	**(MP)**, n, n+2

This shows that $\Gamma, \varphi \vdash_{\mathbf{G'_{3h}}} \psi$. □

3.1 $\mathbf{G'_3}$-algebras as a variety

The aim of this section is proving that $\mathbf{G'_3}$-algebras are three-valued modal implicative semilattices (see [2]) with a bottom element. From this, and from the results obtained in [2], together with Theorem 3.11, the completeness of $\mathbf{G'_{3h}}$ w.r.t. the matrix $\mathbf{G'_3}$ will follow easily (see Theorem 3.27 below).

As mentioned in the Introduction, Canals-Frau and Figallo have studied in [2] the reduct $\{\wedge, \to, \triangle, 1\}$ of the three-valued MV-algebras, where $\to$ is a three-valued Heyting implication, and $\triangle$ is a Moisil operator from the three-valued Łukasiewicz-Moisil algebras (or, equivalently,

$\triangle$ is a Monteiro-Baaz Delta-operator). They also consider the operator $\nabla x = (x \to \Delta x) \to \Delta x$. This reduct can be defined as follows:

Definition 3.14 (See [2])**.** *An algebra $\mathcal{A} = \langle A, \wedge, \to, \triangle, 1\rangle$ of type* $(2, 2, 1, 0)$ *is a* three-valued modal implicative semilattice *(a MIS_3-*algebra*, for short) if it satisfies the following identities, for every $x, y, z \in A$:*

(IS1) $(x \to x) = 1$,

(IS2) $((x \to y) \wedge y) = y$,

(IS3) $(x \to (y \wedge z)) = ((x \to z) \wedge (x \to y))$,

(IS4) $(x \wedge (x \to y)) = (x \wedge y)$,

(T) $((x \to y) \to z) \to (((z \to x) \to z) \to z) = 1$,

(M1) $(\triangle x \to x) = 1$,

(M2) $(((y \to \triangle y) \to (x \to \triangle\triangle x)) \to \triangle(x \to y)) = (\triangle x \to \triangle\triangle y)$,

(M3) $((\triangle x \to \triangle y) \to \triangle x) = \triangle x$.

It is worth mentioning that any MIS_3-algebra is an ordered structure if we consider $x \leq y$ if and only if $x \to y = 1$ (if and only if $x \wedge y = x$). Moreover, in [5] it was proved that any MIS_3-algebra $\mathcal{A}$ is a distributive lattice where the supremum is given by $x \vee y \stackrel{\text{def}}{=} ((x \to y) \to y) \wedge ((y \to x) \to x)$ for $x, y \in A$.

Recall (see the beginning of Section 2) that $\mathcal{A}_3 = \langle \mathcal{V}, \sigma\rangle$ is the 3-valued algebra of $\mathbf{G_3'}$ with domain $\mathcal{V} = \{0, \frac{1}{2}, 1\}$. Let $\mathcal{B}_3 = \langle \mathcal{V}, \sigma'\rangle$ be the 3-valued algebra over $\{\wedge, \to, \triangle, \mathbf{1}\}$ with domain $\mathcal{V}$ such that σ' interprets the connectives as follows: $\sigma'(\mathbf{1}) = 1$; $\sigma'(\wedge)$ and $\sigma'(\to)$ coincide with the corresponding operators of $\mathcal{A}_3$; and $\sigma'(\triangle)$ is defined by the truth-table below.[3]

x	$\triangle x$
$\mathbf{0}$	0
$\mathbf{\frac{1}{2}}$	0
$\mathbf{1}$	1

It is easy to see that the induced operator $x \vee y \stackrel{\text{def}}{=} ((x \to y) \to y) \wedge ((y \to x) \to x)$ coincides with the $\vee$-operator of $\mathcal{A}_3$. In addition, $0 \leq \frac{1}{2} \leq 1$. On the other hand, $\nabla x = 0$ if $x = 0$, and 1 otherwise.

The following fundamental results can be found in [2] (see also [5]).

Definition 3.15. *Let $\mathcal{A}$ be a MIS_3-algebra and let $D \subseteq A$. D is said to be deductive system if $1 \in D$, and if $x, x \to y \in D$ imply $y \in D$. Also, we say that D is modal, if $x \in D$ implies $\triangle x \in D$. Besides, we denote by $D_m(\mathcal{A})$ the set of modal deductive systems and by $Con(\mathcal{A})$ the set of congruence relations.*

[3] As usual, we identify $\sigma'(c)$ with c, for any connective c.

Lemma 3.16. *([2]) For a given MIS_3-algebra $\mathcal{A}$, the poset $D_m(\mathcal{A})$ is lattice-isomorphic to $Con(\mathcal{A})$.*

Now, for a given MIS_3-algebra $\mathcal{A}$, a deductive systems D of $\mathcal{A}$ is said to be a maximal if for every deductive system M such that $D \subseteq M$ implies $M = A$ or $M = D$.

Theorem 3.17. *([2]) Let M be a non-trivial maximal modal deductive system of MIS_3-algebras $\mathcal{A}$. Let us consider the sets $M_0 = \{x \in A \ : \ \nabla x \notin M\}$ and $M_{1/2} = \{x \in A \ : \ x \notin M, \nabla x \in M\}$, and the map $h : A \longrightarrow \mathcal{V}$ defined by*

$$h(x) = \begin{cases} 0 & \text{if } x \in M_0 \\ 1/2 & \text{if } x \in M_{1/2} \\ 1 & \text{if } x \in M. \end{cases}$$

Then, h is a MIS_3-homomorphism $h : \mathcal{A} \to \mathcal{B}_3$ such that $h^{-1}(\{1\}) = M$.

Theorem 3.18. *([2]) The variety of MIS_3-algebras is semisimple and it is generated by the 3-valued algebra $\mathcal{B}_3$.*

Remark 3.19. *The above theorem states that an equation $s = t$ holds in every MIS_3-algebra iff it holds in $\mathcal{B}_3$. For instance, since $\triangle\triangle x = \triangle x$ holds in $\mathcal{B}_3$ for every $x \in \mathcal{V}$, it follows that, for every MIS_3-algebra $\mathcal{A}$, $\triangle\triangle x = \triangle x$ for every $x \in A$.*

The next step is to connect the variety of MIS_3-algebras with the class of $\mathbf{G_3'}$-algebras introduced in Definition 3.1. Firstly, observe that any $\mathbf{G_3'}$-algebra has a bottom element 0. This suggest the following definition:

Definition 3.20. *An algebra $\mathcal{A} = \langle A, \wedge, \to, \triangle, 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ is a* three-valued modal Heyting algebra *(a MIS_3^0-*algebra*, for short) if its reduct $\langle A, \wedge, \to, \triangle, 1\rangle$ is a MIS_3-algebra and, for every $x \in A$:*

(IS5) $(0 \to x) = 1$.

Observe that the expansion $\mathcal{B}_3^0 = \langle \mathcal{V}, \sigma' \rangle$ of $\mathcal{B}_3$ to the signature $\{\wedge, \to, \triangle, \mathbf{0}, \mathbf{1}\}$ such that $\sigma'(\mathbf{0}) = 0$ is a MIS_3^0-algebra. Moreover, the following result holds:

Theorem 3.21. *The variety of MIS_3^0-algebras is generated by the 3-valued algebra $\mathcal{B}_3^0$.*

Proof. According to Theorem 3.18, there is a non-empty set X and a homomorphism $h : \mathcal{A} \to \mathcal{B}_3^X$ for every MIS_3-algebra $\mathcal{A}$. Besides, it is clear that h verify $h(x \vee y) = h(x) \vee h(y)$. Thus, in particular, this representation holds for every MIS_3^0-algebra $\mathcal{A}$. Thus, taking into account axiom (IS5), it is clear that $h(0) \leq h(x)$ for every $x \in A$. But $\mathcal{A}_3$ is a subdirectly irreducible algebra of the variety of MIS_3-algebras. Therefore, every canonical projection $q_i : h(\mathcal{A}) \to \mathcal{B}_3$ is onto and so, $q_i(h(0)) \leq q_i(h(x))$ for every $x \in A$, in particular $q_i(h(0)) \leq 0$. Therefore, $q_i(h(0)) = 0$ for every $i \in X$ and thus, $h(0) = 0$, which completes the proof. ☐

Theorem 3.22. *The class* $\mathbb{A}\mathbf{G}'_3$ *of* $\mathbf{G}'_3$*-algebras and the variety of* MIS_3^0*-algebras are term-equivalent via* $\triangle x \stackrel{def}{=} \neg\neg x$*, on the one hand; and* $\neg x \stackrel{def}{=} \triangle x \to 0$ *and* $x \vee y \stackrel{def}{=} ((x \to y) \to y) \wedge ((y \to x) \to x)$*, on the other.*

Proof. Let $\mathcal{A} = \langle A, \wedge, \to, \triangle, 0, 1\rangle$ be a MIS_3^0-algebra, and define the following operators:

$\neg x \stackrel{def}{=} \triangle x \to 0$ and $x \vee y \stackrel{def}{=} ((x \to y) \to y) \wedge ((y \to x) \to x)$, for every $x, y \in A$.

Let $\mathcal{A}^{\neg} \stackrel{def}{=} \langle A, \wedge, \vee, \to, \neg, 0, 1\rangle$. It is immediate to see that $\langle A, \wedge, \vee, \to, 0, 1\rangle$ is a 3-valued Heyting algebra. This follows from the fact that $\mathcal{B}_3^0$ satisfies the equations characterizing such class of algebras, and then by using Theorem 3.21. Moreover, by using the same argument it can be proven that $\mathcal{A}^{\neg}$ satisfies properties (ii)-(vii) of Definition 3.1. Finally, suppose that $x \leq y$. Then, $\triangle x \leq \triangle y$ (see [2] and [5]). From this, $\neg y = \triangle y \to 0 \leq \triangle x \to 0 = \neg x$. Hence, $\mathcal{A}^{\neg}$ satisfies property (viii) of Definition 3.1. This means that any MIS_3^0-algebra can be transformed into a $\mathbf{G}'_3$-algebra by using appropriate terms.

Conversely, let $\mathcal{A} = \langle A, \wedge, \vee, \to, \neg, 0, 1\rangle$ be a $\mathbf{G}'_3$-algebra and define the following operation: $\triangle x \stackrel{def}{=} \neg\neg x$, for every $x \in A$. Consider the algebra $\mathcal{A}^{\triangle} \stackrel{def}{=} \langle A, \wedge, \to, \triangle, 0, 1\rangle$. We shall prove that $\mathcal{A}^{\triangle}$ is a MIS_3^0-algebra. Observe that $\mathcal{A}^{\triangle}$ satisfies properties (IS1)-(IS4) and (T) of Definition 3.14, as well as property (IS5) of Definition 3.20. The algebra $\mathcal{A}^{\triangle}$ satisfies properties (M1) and (M2) since $\mathcal{A}$ satisfies properties (iv) and (vii) of Definition 3.1, and since $\neg\neg\neg\neg x = \neg\neg x$, by Proposition 3.2(3). Finally, $\mathcal{A}^{\triangle}$ satisfies property (M3), by Proposition 3.2(6). This means that any $\mathbf{G}'_3$-algebra can be transformed into a MIS_3^0-algebra by means of a suitable term. This shows that the class $\mathbb{A}\mathbf{G}'_3$ of $\mathbf{G}'_3$-algebras and the variety of MIS_3^0-algebras are term-equivalent via the proposed terms. □

Let $\mathcal{A}_3^0 = \langle \mathcal{V}, \sigma\rangle$ be the expansion of $\mathcal{A}_3$ (recall the beginning of Section 2) to the signature Σ_+ such that $\sigma(\mathbf{0}) = 0$

Corollary 3.23. *The class* $\mathbb{A}\mathbf{G}'_3$ *of* $\mathbf{G}'_3$*-algebras is generated by the 3-valued algebra* $\mathcal{A}_3^0$*.*

Proof. If follows immediately form theorems 3.22 and 3.21.

□

Corollary 3.24. *Let* φ *be a formula. Then* $\models_{\mathbb{A}\mathbf{G}'_3} \varphi$ *if and only if* $\models_{\mathbf{G}'_3} \varphi$*.*

Corollary 3.25. *The class* $\mathbb{A}\mathbf{G}'_3$ *of* $\mathbf{G}'_3$*-algebras is a variety defined by the following equations:*

($\mathbf{G}'_3$1) $(x \to x) = 1$,

($\mathbf{G}'_3$2) $((x \to y) \wedge y) = y$,

($\mathbf{G}'_3$3) $(x \to (y \wedge z)) = ((x \to z) \wedge (x \to y))$,

($\mathbf{G}'_3$4) $(x \wedge (x \to y)) = (x \wedge y)$,

($\mathbf{G}'_3$5) $((x \to y) \to z) \to (((z \to x) \to z) \to z) = 1$,

($\mathbf{G}'_3$6) $(\neg\neg x \to x) = 1$,

($\mathbf{G}'_3$7) $(\neg x \to \neg\neg\neg x) = 1$,

($\mathbf{G_3'}$8) $(((y \to \neg\neg y) \to (x \to \neg\neg x)) \to \neg\neg(x \to y)) = (\neg\neg x \to \neg\neg y)$,

($\mathbf{G_3'}$9) $((\neg\neg x \to \neg\neg y) \to \neg\neg x) = \neg\neg x$,

($\mathbf{G_3'}$10) $(x \vee y) = (((x \to y) \to y) \wedge ((y \to x) \to x))$,

($\mathbf{G_3'}$11) $(x \vee \neg x) = 1$,

($\mathbf{G_3'}$12) $(\neg x \wedge \neg\neg x) = 0$,

($\mathbf{G_3'}$13) $(0 \to x) = 1$.

Proof. By Definition 3.1 and by Corollary 3.23, it follows that any $\mathbf{G_3'}$-algebra satisfies the equations ($\mathbf{G_3'}$1)-($\mathbf{G_3'}$13). Conversely, let $\mathcal{A}$ be an algebra satisfying ($\mathbf{G_3'}$1)-($\mathbf{G_3'}$13), and define $\triangle x \stackrel{\text{def}}{=} \neg\neg x$. Then $\triangle\triangle x = \triangle x$ for every $x \in A$, and so $\mathcal{A}^{\triangle} \stackrel{\text{def}}{=} \langle A, \wedge, \to, \triangle, 0, 1\rangle$ is a MIS_3^0-algebra. By the proof of Theorem 3.22, the algebra $(\mathcal{A}^{\triangle})^{\neg}$ is a $\mathbf{G_3'}$-algebra. It will be shown that $(\mathcal{A}^{\triangle})^{\neg} = \mathcal{A}$. Indeed, the negation in $(\mathcal{A}^{\triangle})^{\neg}$ is given by $\neg' x \stackrel{\text{def}}{=} \triangle x \to 0 = \neg\neg x \to 0$. But $\neg\neg x \to 0 = \neg x$. The proof of this fact is analogous to the one given for Proposition 3.2(4) above. Using this, it follows that $\neg' x = \neg x$. In addition, the disjunction $x \vee' y \stackrel{\text{def}}{=} ((x \to y) \to y) \wedge ((y \to x) \to x)$ in $(\mathcal{A}^{\triangle})^{\neg}$ coincides with $x \vee y$, by ($\mathbf{G_3'}$10). This shows that $(\mathcal{A}^{\triangle})^{\neg} = \mathcal{A}$, hence $\mathcal{A}$ is a $\mathbf{G_3'}$-algebra, by Theorem 3.22. □

3.2 Adequacy of $\mathbf{G_{3h}'}$ w.r.t. $\mathbf{G_3'}$

Finally, we can prove the adequacy of the Hilbert calculus $\mathbf{G_{3h}'}$ with respect to the intended 3-valued semantics $\mathbf{G_3'}$. Firstly, a technical result will be stated:

Proposition 3.26. *If $\varphi \models_{\mathbf{G_3'}} \psi$ then $\models_{\mathbf{G_3'}} \neg\neg\varphi \to \psi$.*

Proof. Supose that $\varphi \models_{\mathbf{G_3'}} \psi$, and let h be a valuation over $\mathbf{G_3'}$. If $h(\varphi) = 1$ then $h(\psi) = 1$, by hypothesis, hence $h(\neg\neg\varphi \to \psi) = 1 \to 1 = 1$. Otherwise, if $h(\varphi) \neq 1$ then $h(\neg\neg\varphi) = 0$ and so $h(\neg\neg\varphi \to \psi) = 0 \to h(\psi) = 1$. In any case, $h(\neg\neg\varphi \to \psi) = 1$. This shows that $\models_{\mathbf{G_3'}} \neg\neg\varphi \to \psi$. □

Theorem 3.27 (Soundness and completeness of $\mathbf{G_{3h}'}$ w.r.t. $\mathbf{G_3'}$)**.** *For every finite set $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}_\Sigma$:*

$$\Gamma \vdash_{\mathbf{G_{3h}'}} \varphi \ \textit{ if and only if } \ \Gamma \models_{\mathbf{G_3'}} \varphi.$$

Proof.

Only if part (Soundness): It follows from Theorem 3.5 and the fact that $\mathcal{A}_3^0$ is a $\mathbf{G_3'}$-algebra.

If part (Completeness): Suppose that $\Gamma \models_{\mathbf{G_3'}} \varphi$. If $\Gamma = \emptyset$ then $\models_{\mathbb{A}\mathbf{G_3'}} \varphi$, by Corollary 3.24. From this, $\vdash_{\mathbf{G_{3h}'}} \varphi$, by Theorem 3.11. Otherwise, if $\Gamma = \{\psi_1, \ldots, \psi_n\}$ for $n \geq 1$ let $\psi = (\ldots((\psi_1 \wedge \psi_2) \wedge \psi_3) \wedge \ldots) \wedge \psi_n$ if $n > 1$, and $\psi = \psi_1$ if $n = 1$. Since $\psi \models_{\mathbf{G_3'}} \varphi$ then $\models_{\mathbf{G_3'}} \neg\neg\psi \to \varphi$, by Proposition 3.26. From this $\models_{\mathbb{A}\mathbf{G_3'}} \neg\neg\psi \to \varphi$, by Corollary 3.24. Then $\vdash_{\mathbf{G_{3h}'}} \neg\neg\psi \to \varphi$, by Theorem 3.11. By Theorem 3.13, $\psi \vdash_{\mathbf{G_{3h}'}} \varphi$. By using the properties of the conjunction in $\mathbf{G_{3h}'}$ it follows from here that $\Gamma \vdash_{\mathbf{G_{3h}'}} \varphi$. □

Corollary 3.28. *The Deduction Theorem does not hold in* $\mathbf{G'_{3h}}$.

Proof. It is an immediate consequence of Remark 2.3 and Theorem 3.27. □

4 The first-order $\mathbf{G_3'}$-logic

In this section, a first-order version of $\mathbf{G_3'}$, called $\mathcal{Q}\mathbf{G3'}$, will be proposed. The semantics will be given by structures defined over $\mathbf{G_3'}$-algebras which are complete (as lattices), in order to interpret the quantifiers.

Recall that Σ denotes the propositional signature $\{\wedge, \vee, \to, \neg\}$ for $\mathbf{G_3'}$.

Definition 4.1. *Consider the symbols* $\forall$ *(universal quantifier) and* $\exists$ *(existential quantifier), together with commas and parentesis as the punctuation marks. Let* $IVar = \{v_1, v_2, \ldots\}$ *be a denumerable set of* individual *variables. A first-order signature is a triple* $\Theta = \langle \mathcal{C}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \{\mathcal{P}_n\}_{n\in\mathbb{N}}\rangle$ *such that:*

- *$\mathcal{C}$ is a set of individual constants;*
- *for each $n \geq 1$, $\mathcal{F}_n$ is a set of function symbols of arity n,*
- *for each $n \geq 1$, $\mathcal{P}_n$ is a set of predicate symbols of arity n.*[4]

The notions of bound and free variables inside a formula, closed terms, closed formulas (or sentences), and of term free for a variable in a formula are defined as usual (see, for instance, [9]). We denote by Ter_Θ and $\mathfrak{Fm}_\Theta$ the set of terms and the set of first-order formulas over Θ (by using the connectives in Σ), respectively. Given a formula φ, the formula obtained from φ by substituting every free occurrence of a variable x by a term t will be denoted by $\varphi(x/t)$.

Definition 4.2. *Let* Θ *be a first-order signature. The logic* $\mathcal{Q}\mathbf{G3'}$ *over* Θ *is defined by the Hilbert calculus obtained by extending* $\mathbf{G_3'}$ *expressed in the language* $\mathfrak{Fm}_\Theta$ *by adding the following:*

Axiom schemas:

(Ax14) $\varphi(x/t) \to \exists x\varphi$ *if t is a term free for x in φ*

(Ax15) $\forall x\varphi \to \varphi(x/t)$ *if t is a term free for x in φ*

Inference Rules:

$(\exists - In)$ $\dfrac{\varphi \to \psi}{\exists x\varphi \to \psi}$ *where x does not occur free in ψ*

$(\forall - In)$ $\dfrac{\varphi \to \psi}{\varphi \to \forall x\psi}$ *where x does not occur free in φ*

[4]It will be assumed, as usual, that Θ has at least one predicate symbol, in order to have a non-empty set of formulas.

Definition 4.3. *A* Θ*-structure for* $\mathcal{Q}\mathbf{G3}'$ *is a triple* $\mathfrak{A} = \langle U, \mathcal{A}, \cdot^{\mathfrak{A}}\rangle$ *such that* U *is a non-empty set,* $\mathcal{A}$ *is a complete* $\mathbf{G_3'}$*-algebra and* $\cdot^{\mathfrak{A}}$ *is an interpretation map which assigns:*

- *to each individual constant* $c \in \mathcal{C}$*, an element* $c^{\mathfrak{A}}$ *of* U*;*
- *to each function symbol* f *of arity* n*, a function* $f^{\mathfrak{A}} : U^n \to U$*;*
- *to each predicate symbol* P *of arity* n*, a function* $P^{\mathfrak{A}} : U^n \to A$*.*

Given a Θ-structure $\mathfrak{A}$ for $\mathcal{Q}\mathbf{G3}'$, an *assignment* over $\mathfrak{A}$ is a function $s : IVar \to U$. Given s and $a \in U$ let $s[x \to a]$ be the assignment such that $s[x \to a](x) = a$ and $s[x \to a](y) = s(y)$ for every $x \neq y$. A Θ-structure $\mathfrak{A}$ and an assignment s induce an interpretation map $[\![\cdot]\!]^{\mathfrak{A}}_s$ for terms and formulas defined as follows:

$[\![x]\!]^{\mathfrak{A}}_s = s(x)$ if $x \in IVar$,
$[\![c]\!]^{\mathfrak{A}}_s = c^{\mathfrak{A}}$ if $c \in \mathcal{C}$,
$[\![f(t_1, \ldots, t_n)]\!]^{\mathfrak{A}}_s = f^{\mathfrak{A}}([\![t_1]\!]^{\mathfrak{A}}_s, \ldots, [\![t_n]\!]^{\mathfrak{A}}_s)$, if $f \in \mathcal{F}_n$,
$[\![P(t_1, \ldots, t_n)]\!]^{\mathfrak{A}}_s = P^{\mathfrak{A}}([\![t_1]\!]^{\mathfrak{A}}_s, \ldots, [\![t_n]\!]^{\mathfrak{A}}_s)$, if $P \in \mathcal{P}_n$,
$[\![\phi \# \varphi]\!]^{\mathfrak{A}}_s = [\![\phi]\!]^{\mathfrak{A}}_s \# [\![\varphi]\!]^{\mathfrak{A}}_s$ for $\# \in \{\wedge, \vee, \to\}$,
$[\![\neg\varphi]\!]^{\mathfrak{A}}_s = \neg[\![\varphi]\!]^{\mathfrak{A}}_s$,
$[\![\forall x\varphi]\!]^{\mathfrak{A}}_s = \bigwedge_{a \in U} [\![\varphi]\!]^{\mathfrak{A}}_{s[x \to a]}$,
$[\![\exists x\varphi]\!]^{\mathfrak{A}}_s = \bigvee_{a \in U} [\![\varphi]\!]^{\mathfrak{A}}_{s[x \to a]}$.

We say that $\mathfrak{A}$ and s *satisfy* a formula φ, denoted by $\mathfrak{A} \vDash \varphi[s]$, if $[\![\varphi]\!]^{\mathfrak{A}}_s = 1$. On the other hand, φ is *true in* $\mathfrak{A}$ if $\mathfrak{A} \vDash \varphi[s]$ for every s. We say that φ is a *semantical consequence of* Γ *in* $\mathcal{Q}\mathbf{G3}'$, denoted by $\Gamma \vDash_{\mathcal{Q}\mathbf{G3}'} \alpha$, if, for any structure $\mathfrak{A}$: if every $\psi \in \Gamma$ is true in $\mathfrak{A}$ then α is true in $\mathfrak{A}$. Observe that, if $\mathfrak{A}$ is a structure and φ is a closed formula, then $[\![\varphi]\!]^{\mathfrak{A}}_s = [\![\varphi]\!]^{\mathfrak{A}}_{s'}$, for every assignments s and s'. This being so, either $\mathfrak{A} \vDash \varphi[s]$ for every s or $\mathfrak{A} \nvDash \varphi[s]$ for every s. Thus, if $\Gamma \cup \{\varphi\}$ is a set of sentences then: $\Gamma \nvDash_{\mathcal{Q}\mathbf{G3}'} \varphi$ iff there is a structure $\mathfrak{A}$ such that every $\psi \in \Gamma$ is true in $\mathfrak{A}$ but $\mathfrak{A} \nvDash \varphi[s]$ for any assignment s.

In order to prove the soundness of $\mathcal{Q}\mathbf{G3}'$ w.r.t. the given semantics, an important technical result, the *substitution lemma*, must be established:

Proposition 4.4 (Substitution lemma)**.** *Let* φ *be a formula,* t *a term free for* x *in* φ*,* $\mathfrak{A}$ *an structure and* s *and assignment. Then:* $[\![\alpha]\!]^{\mathfrak{A}}_{s[x \to [\![t]\!]^{\mathfrak{A}}_s]} = [\![\alpha(x/t)]\!]^{\mathfrak{A}}_s$.

Proof. It is easily proved by induction on the complexity of the formula α. □

Theorem 4.5 (Soundness of $\mathcal{Q}\mathbf{G3}'$)**.** *Let* $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Theta$*. If* $\Gamma \vdash_{\mathcal{Q}\mathbf{G3}'} \varphi$ *then* $\Gamma \vDash_{\mathcal{Q}\mathbf{G3}'} \varphi$*.*

Proof: Consider a given structure $\mathfrak{A} = \langle U, \mathcal{A}, \cdot^{\mathfrak{A}}\rangle$. It is enough to prove the following facts: the new axioms **(Ax14)** and **(Ax15)** are true in $\mathfrak{A}$, and the new inference rules $(\exists - In)$ and

$(\forall - In)$ preserve trueness in $\mathfrak{A}$.

(Ax14) and **(Ax15)**: Suppose that φ is $\alpha(x/t) \to \exists x\alpha$, and let s be an assignment. Then, by Proposition 4.4, $[\![\varphi]\!]^{\mathfrak{A}}_s = [\![\alpha]\!]^{\mathfrak{A}}_{s[x\to[\![t]\!]^{\mathfrak{A}}_s]} \to [\![\exists x\alpha]\!]^{\mathfrak{A}}_s$. It is clear that $[\![\alpha]\!]^{\mathfrak{A}}_{s[x\to[\![t]\!]^{\mathfrak{A}}_s]} \leq \bigvee_{a\in U} [\![\alpha]\!]^{\mathfrak{A}}_{s[x\to a]}$, hence $[\![\alpha(x/t)]\!]^{\mathfrak{A}}_s \leq [\![\exists x\alpha]\!]^{\mathfrak{A}}_s$. Therefore $[\![\alpha(x/t) \to \exists x\alpha]\!]^{\mathfrak{A}}_s = 1$. The validity of **(Ax15)** is proved analogously.

$(\exists - In)$ and $(\forall - In)$: Let $\alpha \to \beta$ such that x is not free in β, and let $\varphi = \exists x\alpha \to \beta$. Suppose that that $[\![\alpha \to \beta]\!]^{\mathfrak{A}}_s = 1$ for every s, and fix an assignment s. By definition, $[\![\varphi]\!]^{\mathfrak{A}}_s = [\![\exists x\alpha]\!]^{\mathfrak{A}}_s \to [\![\beta]\!]^{\mathfrak{A}}_s = \bigvee_{a\in U} [\![\alpha]\!]^{\mathfrak{A}}_{s[x\to a]} \to [\![\beta]\!]^{\mathfrak{A}}_s$. By hypothesis, $[\![\alpha]\!]^{\mathfrak{A}}_{s'} \leq [\![\beta]\!]^{\mathfrak{A}}_{s'}$ for every s'. In particular, $[\![\alpha]\!]^{\mathfrak{A}}_{s[x\to a]} \leq [\![\beta]\!]^{\mathfrak{A}}_{s[x\to a]} = [\![\beta]\!]^{\mathfrak{A}}_s$ for every $a \in U$, since x is not free in β. So, $\bigvee_{a\in U} [\![\alpha]\!]^{\mathfrak{A}}_{s[x\to a]} \to [\![\beta]\!]^{\mathfrak{A}}_s = [\![\exists x\alpha \to \beta]\!]^{\mathfrak{A}}_s = [\![\varphi]\!]^{\mathfrak{A}}_s = 1$. The preservation of trueness by the rule $(\forall - In)$ is proved analogously. ☐

Now, let us consider the relation $\equiv$ defined by $\alpha \equiv \beta$ iff $\vdash_{\mathcal{Q}\mathbf{G3}'} \alpha \to \beta$ and $\vdash_{\mathcal{Q}\mathbf{G3}'} \alpha \to \beta$. Then, we have that the algebra $\mathfrak{Fm}_\Theta/_\equiv$ is a $\mathbf{G}_3'$-algebra (the proof is exactly the same as in the propositional case). It is clear that the algebra of formulas is an absolutely free algebra generated by the atomic formulas. The equivalence class of a formula α w.r.t. $\equiv$ will be denoted by $\overline{\alpha}$.

It is clear that $\mathcal{Q}\mathbf{G3}'$ is a Tarskian logic, see Definition 3.6. Besides, it is possible to consider the notion of set of formulas maximal non-trivial w.r.t to some formula φ (see Definition 3.8) and the notion of closed theories is defined in the same way as the propositional case, see Definition 3.7. Therefore, we have that the Lindenbaum- Łoś's Theorem holds for $\mathcal{Q}\mathbf{G3}'$. Then, we have the following

Lemma 4.6. *Let $\Gamma \cup \{\varphi\}$ be a set of formulas with Γ maximal non-trivial w.r.t. φ in* $\mathbf{QG3}'$. *Let $\Gamma/_\equiv = \{\overline{\alpha} \ : \ \alpha \in \Gamma\}$ be a subset of $\mathbf{G}_3'$-algebra $\mathfrak{Fm}_\Theta/_\equiv$, then:*

1. *If $\alpha \in \Gamma$ and $\overline{\alpha} = \overline{\beta}$, then $\beta \in \Gamma$. If $\overline{\alpha} \in \Gamma/_\equiv$, then $\overline{\forall x\alpha}, \overline{\exists x\alpha} \in \Gamma/_\equiv$.*

2. *$\Gamma/_\equiv$ is a modal deductive system of $\mathfrak{Fm}_\Theta/_\equiv$. Also, if $\overline{\varphi} \notin \Gamma/_\equiv$ then, for any closed modal deductive system $\overline{D}$ containing properly to $\Gamma/_\equiv$, it is the case that $\overline{\varphi} \in \overline{D}$.*

Proof. Suppose that $\alpha \in \Gamma$ and $\alpha \equiv \beta$. Then, $\vdash_{\mathcal{Q}\mathbf{G3}'} \alpha \to \beta$ and $\vdash_{\mathcal{Q}\mathbf{G3}'} \beta \to \alpha$. Therefore, $\beta \in \Gamma$. It is not hard to see that the conditions of Definition 3.15 are verified by $\Gamma/_\equiv$.

On the other hand, let $\overline{D} \subseteq \mathfrak{Fm}_\Sigma/_\equiv$ be a closed modal deductive system that properly contains $\Gamma/_\equiv$ and so, there is $\overline{\gamma} \in \overline{D}$ such that $\overline{\gamma} \notin \Gamma/_\equiv$. Now, we have that $\gamma \notin \Gamma$ and therefore, $\Gamma \cup \{\gamma\} \vdash_{\mathcal{Q}\mathbf{G3}'} \varphi$. From the latter and taking $D = \{\alpha \ : \ \overline{\alpha} \in \overline{D}\}$, we can infer that $D \vdash_{\mathcal{Q}\mathbf{G3}'} \varphi$. Now, since D is closed we obtain that $\overline{\varphi} \in \overline{D}$. ☐

It is worth mentioning that item 2. of last lemma states that $\Gamma/_{\equiv}$ is a maximal modal deductive system. Besides, we know that $\mathfrak{Fm}_\Theta/_{\equiv}$ is a $\mathbf{G_3'}$-algebra, and for every Γ maximal non-trivial w.r.t. φ we have that $\Gamma/_{\equiv}$ is a maximal modal deductive system of $\mathfrak{Fm}_\Sigma/_{\equiv}$. Then, from Theorems 3.17 and 3.21, there is a homomorphism $h : \mathfrak{Fm}_\Theta/_{\equiv} \to \mathcal{B}_3^0$ such that $h^{-1}(\{1\}) = \Gamma/_{\equiv}$. Thus, if we consider the canonical projection $\pi : \mathfrak{Fm}_\Theta \to \mathfrak{Fm}_\Theta/_{\equiv}$, there is a homomorphism $f : \mathfrak{Fm}_\Theta \to \mathcal{B}_3^0$ defined by $f = h \circ \pi$ such that $f^{-1}(\{1\}) = \Gamma$. Observe that $f(\alpha) = h(\overline{\alpha})$.

Proposition 4.7. *Let Γ be a set of formulas which is maximal non-trivial w.r.t. φ in $\mathbf{G_3'}$. Let $\mathfrak{Fm}_\Theta/_{\equiv_\Gamma}$ be the quotient algebra obtained by the following congruence: $\alpha \equiv_\Gamma \beta$ iff $(\alpha \leftrightarrow \beta) \in \Gamma$. Then $\mathfrak{Fm}_\Theta/_{\equiv_\Gamma}$ is isomorphic to a subalgebra of $\mathcal{B}_3^0$ and so is a simple $\mathbf{G_3'}$-algebra.*

Proof. Let $\bar{\pi} : \mathfrak{Fm}_\Theta \to \mathfrak{Fm}_\Theta/_{\equiv_\Gamma}$ be the canonical projection. By considering the homorphism $f : \mathfrak{Fm}_\Theta \to \mathcal{B}_3^0$ defined above and by adapting the first isomorphism Theorem from Universal Algebra (see [1]), there is a monomorphism $\bar{f} : \mathfrak{Fm}_\Theta/_{\equiv_\Gamma} \to \mathcal{B}_3^0$ such that $\bar{f} \circ \bar{\pi} = f$. This means that $\mathfrak{Fm}_\Theta/_{\equiv_\Gamma}$ is isomorphic to a subalgebra of $\mathcal{B}_3^0$; that is to say, $\mathfrak{Fm}_\Theta/_{\equiv_\Gamma}$ is a simple algebra. □

Observe that $\bar{f}([\alpha]_\Gamma) = f(\alpha) = h(\overline{\alpha})$, where $[\alpha]_\Gamma$ denotes the equivalence class of α in $\mathfrak{Fm}_\Theta/_{\equiv_\Gamma}$. Since $\mathcal{B}_3^0$ is finite, we have the following:

Corollary 4.8. *Let Γ be a set of formulas which is maximal non-trivial w.r.t. φ in $\mathbf{G_3'}$. Then, $\mathfrak{Fm}_\Theta/_{\equiv_\Gamma}$ is finite, hence it is a complete lattice.*

In fact, $\mathfrak{Fm}_\Theta/_{\equiv_\Gamma}$ is isomorphic to either the 2-element chain $\{0, 1\}$ or the 3-element chain $\mathcal{B}_3^0$.

Theorem 4.9 (Completeness (for sentences) of $\mathcal{Q}\mathbf{G3'}$)**.** *Let $\Gamma \cup \{\varphi\}$ be a set of closed formulas over Θ. Then: $\Gamma \vDash_{\mathcal{Q}\mathbf{G3'}} \varphi$ implies that $\Gamma \vdash_{\mathcal{Q}\mathbf{G3'}} \varphi$.*

Proof. Let us suppose that $\Gamma \nvdash_{\mathcal{Q}\mathbf{G3'}} \varphi$. Then, there is M maximal non-trivial w.r.t. φ such that $\Gamma \subseteq M$. Hence, $\alpha \in M$ for every $\alpha \in \Gamma$ and $\varphi \notin M$. Now, let us consider the algebra $\mathcal{A} := \mathfrak{Fm}_\Theta/_{\equiv_M}$ defined by the congruence $\alpha \equiv_M \beta$ iff $(\alpha \leftrightarrow \beta) \in M$. By Corollary 4.8, $\mathcal{A}$ is a complete $\mathbf{G_3'}$-algebra. It is easy to see that $[\alpha]_M \leq [\beta]_M$ iff $\alpha \to \beta \in M$.

Now, let us consider the canonical structure $\mathfrak{A} = \langle U, \mathcal{A}, \cdot^{\mathfrak{A}} \rangle$ such that U is the set Ter_Θ of terms over Θ and $\mathcal{A}$ is as above, for every term t we consider its name $\hat{t}$ as a constant of Θ. Assume that, if $\hat{t}$ is a constant then $\hat{t}^{\mathfrak{A}} := t$, and if $f \in \mathcal{F}_n$ then $f^{\mathfrak{A}}(t_1, \ldots, t_n) := f(t_1, \ldots, t_n)$. From this, it follows that, for any $t \in U$ and any assignment s, $[\![t]\!]_s^{\mathfrak{A}} = t$. On the other hand, if $P \in \mathcal{P}_n$, assume that the mapping $P^{\mathfrak{A}}$ is defined as follows: $P^{\mathfrak{A}}(t_1, \ldots, t_n) = [P(t_1, \ldots, t_n)]_M$. By induction on the complexity of the formula, it can be proven that, for every closed formula α and every s, $[\![\alpha]\!]_s^{\mathfrak{A}} = [\alpha]_M$. Indeed, the case for α atomic holds by definition of $\mathfrak{A}$. The cases $\alpha = \beta \# \psi$ and $\alpha = \neg\beta$ hold by induction hypothesis, the definition of $[\![\cdot]\!]_s^{\mathfrak{A}}$ and the definition of the operations in the $\mathbf{G_3'}$-algebra $\mathcal{A}$; moreover, it is not hard to see that for every formula $\psi(x)$ and every term t we have that $[\![\psi(x/\hat{t})]\!]_s^{\mathfrak{A}} = [\![\psi(x/t)]\!]_s^{\mathfrak{A}}$.

Suppose now that α is $\exists x\beta$. By axiom **(Ax14)**, for every $t \in U$, $\beta(x/\hat{t}) \to \alpha \in M$ and so $[\beta(x/\hat{t})]_M \leq [\alpha]_M$. By induction hypothesis, we have that $[\beta(x/\hat{t})]_M = [\![\beta(x/\hat{t})]\!]_s^{\mathfrak{A}} = [\![\beta(x/t)]\!]_s^{\mathfrak{A}}$

(by Proposition 4.4). Thus, $[\beta(x/t)]_M \leq [\alpha]_M$, for every $t \in U$. Now, let ψ be a sentence such that $[\beta(x/t)]_M \leq [\psi]_M$ for every term $t \in U$ and so $[\beta(x/\hat{t})]_M \leq [\psi]_M$ for every term $t \in U$. In particular, $[\beta(x/\hat{x})]_M \leq [\psi]_M$ and then, $[\beta(x)]_M \leq [\psi]_M$. This means that $\beta(x) \to \psi \in M$. Since x does not occur free in ψ then, by $(\exists - In)$, $\alpha \to \psi \in M$. This means that $[\alpha]_M \leq [\psi]_M$ and so $[\alpha]_M = \bigvee_{t \in U} [\beta(x/t)]_M$. Analogously, but now by using **(Ax15)** and $(\forall - In)$, it is proved that $[\![\alpha]\!]_s^{\mathfrak{A}} = [\alpha]_M$ for $\alpha = \forall x \beta$. This shows that $[\![\alpha]\!]_s^{\mathfrak{A}} = [\alpha]_M$ for every closed formula α and every s.

Thus, $\mathfrak{A}$ is a Θ-structure for $\mathcal{Q}\mathbf{G3'}$ such that, for every closed formula α, α is true in $\mathfrak{A}$ iff $\alpha \in M$. From this we have that $\Gamma \not\models_{\mathcal{Q}\mathbf{G3'}} \varphi$.

□

Given a formula α such that the set of variables occurring free in α is $\{x_1, \ldots, x_n\}$. The *universal closure* of α is the closed formula $(\forall \alpha)$ given by α (if $n = 0$) or $\forall x_1 \ldots \forall x_n \alpha$ otherwise. Then, the completeness theorem of $\mathcal{Q}\mathbf{G3'}$ for arbitrary formulas can now be easily obtained from the last result:

Theorem 4.10 (Completeness of $\mathcal{Q}\mathbf{G3'}$). *Let $\Gamma \cup \{\varphi\}$ be a set of formulas over Θ. Then: $\Gamma \models_{\mathcal{Q}\mathbf{G3'}} \varphi$ implies that $\Gamma \vdash_{\mathcal{Q}\mathbf{G3'}} \varphi$.*

Proof. By **(Ax15)** and $(\forall - In)$ it is easy to prove that $\alpha \vdash_{\mathcal{Q}\mathbf{G3'}} (\forall)\alpha$ and $(\forall)\alpha \vdash_{\mathcal{Q}\mathbf{G3'}} \alpha$, for every formula α. On the other hand, by definition of $\models_{\mathcal{Q}\mathbf{G3'}}$, it is immediate to see that $\alpha \models_{\mathcal{Q}\mathbf{G3'}} (\forall)\alpha$ and $(\forall)\alpha \models_{\mathcal{Q}\mathbf{G3'}} \alpha$, for every formula α. Then, for every $\Gamma \cup \{\varphi\}$: $\Gamma \vdash_{\mathcal{Q}\mathbf{G3'}} \varphi$ iff $(\forall)\Gamma \vdash_{\mathcal{Q}\mathbf{G3'}} (\forall)\varphi$, and $\Gamma \models_{\mathcal{Q}\mathbf{G3'}} \varphi$ iff $(\forall)\Gamma \models_{\mathcal{Q}\mathbf{G3'}} (\forall)\varphi$, where $(\forall)\Gamma = \{(\forall)\beta \ : \ \beta \in \Gamma\}$. From this, the desired result follows immediately from Theorem 4.9. □

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