How Category Theory Works:
The Elements & Distinctions Analysis of the Morphisms, Duality, and Universal Constructions in *Sets*

David Ellerman
University of Ljubljana, Slovenia

Abstract

The purpose of this paper is to show that the dual notions of elements & distinctions are the basic analytical concepts needed to unpack and analyze morphisms, duality, and universal constructions in the *Sets*, the category of sets and functions. The analysis extends directly to other concrete categories (groups, rings, vector spaces, etc.) where the objects are sets with a certain type of structure and the morphisms are functions that preserve that structure. Then the elements & distinctions-based definitions can be abstracted in purely arrow-theoretic way for abstract category theory. In short, the language of elements & distinctions is the conceptual language in which the category of sets is written, and abstract category theory gives the abstract arrows version of those definitions.

Contents

1 Elements & Distinctions Analysis

1.1 Introduction

The purpose of this paper is to show that the dual notions of "elements & distinctions" are the basic analytical concepts needed to unpack and analyze morphisms, duality, and universal constructions in the *Sets*, the category of sets and functions. The analysis extends directly to other concrete categories (groups, rings, vector spaces, etc.) where the objects are sets with a certain type of structure and the morphisms are functions that preserve or reflect that structure. Then the elements & distinctions-based definitions can be abstracted in purely arrow-theoretic way for abstract category theory.

One way to approach the concepts of "elements" and "distinctions" is to start with the category-theoretic duality between subsets and quotient sets (= partitions = equivalence relations): "The dual notion (obtained by reversing the arrows) of 'part' [subobject] is the notion of *partition*." [8, p. 85]. That leads to the two dual forms of mathematical logic: the Boolean logic of subsets and the logic of partitions ([3]; [4]). If partitions are dual to subsets, then what is the dual concept that corresponds to the notion of *elements of a subset*? The notion dual to the elements of a subset is the notion of the *distinctions of a partition* (pairs of elements in distinct blocks of the partition).

1.2 The elements & distinctions analysis of *canonical* maps

The canonical maps and the unique factor morphisms in the universal mapping properties in *Sets* are always constructed in the two ways that maps are canonically constructed with subsets and partitions. In the Boolean algebra of subsets $\wp(U)$ of U, the partial order is the inclusion relation $S \subseteq T$ for $S, T \subseteq U$, which induces the canonical injection $S \to T$. That's the way canonical injective maps are defined using subsets.

In the dual algebra of partitions $\Pi(U)$ on U, the partial order is the refinement relation between partitions and it induces a canonical map using refinement. A *partition* $\pi = \{B, B', ...\}$ on U is a set of subsets of U (called *blocks*, $B, B', ...$) that are mutually exclusive (i.e., disjoint) and jointly exhaustive (i.e., whose union is U). Given another partition $\sigma = \{C, C', ...\}$ on U, a partition π is said to *refine* σ (or σ is *refined by* π), written $\sigma \precsim \pi$, if for every block $B \in \pi$, there is a block $C \in \sigma$ (necessarily unique) such that $B \subseteq C$. If we denote the set of distinctions or dits of a partition (ordered pairs of elements in different blocks) by $\text{dit}(\pi)$, the ditset of π, then just as the partial order in $\wp(U)$ is the inclusion of elements, so the refinement partial order on $\Pi(U)$ is the inclusion of distinctions, i.e., $\sigma \precsim \pi$ iff (if and only if) $\text{dit}(\sigma) \subseteq \text{dit}(\pi)$. And just as the inclusion ordering on subsets induces a canonical map between subsets, so the refinement ordering on partitions induces a canonical surjection between partitions, namely $\pi \to \sigma$ where $B \in \pi$ is taken to the unique $C \in \sigma$ where $B \subseteq C$. That's the way canonical surjective maps are defined using partitions.

The claim that all "canonical" maps in *Sets* arise in these two ways (or by compositions of them) cannot be proven since "canonical" is an intuitive notion. But we will show that all the canonical maps and unique factor maps in the universal constructions (limits and colimits) in *Sets* arise in this way from the partial orders of the dual lattices (or algebras) of subsets and partitions.

The top of the partition algebra $\Pi(U)$ is the *discrete partition* $\mathbf{1} = \{\{u\}\}_{u \in U}$ of all singletons and the bottom is the *indiscrete partition* $\mathbf{0} = \{U\}$ with only one block U. Since every partition π is refined by $\mathbf{1}$, i.e., $\pi \precsim \mathbf{1}$, there is the canonical surjection $\mathbf{1} \to \pi$ that takes the singleton $\{u\}$ to the unique block B such that $u \in B$. And similarly the top of the Boolean algebra $\wp(U)$ is U, where each subset $S \subseteq U$ induces the canonical injection $S \to U$.

Dualities	Subset logic	Partition logic
'Elements'	Elements u of S	Dits (u, u') of π
Partial order	Inclusion $S \subseteq T$	$\sigma \precsim \pi$: $\mathrm{dit}(\sigma) \subseteq \mathrm{dit}(\pi)$
Canonical map	$S \to T$	$\pi \to \sigma$
Top of partial order	U all elements	$\mathrm{dit}(\mathbf{1}) = U^2 - \Delta$, all dits
Bottom of partial order	$\emptyset$ no elements	$\mathrm{dit}(\mathbf{0}) = \emptyset$, no dits

Table 1: Elements and distinctions in the dual logics

1.3 Quantitative measures of elements & distinctions

The quantitative (normalized) counting measure of the elements in a subset gives the classical Laplacian notion of 'logical' probability.

The quantitative (normalized) counting measure of the distinctions in a partition gives the notion of logical entropy that underlies the Shannon notion of entropy (which is not a measure in the sense of measure theory) ([2]; [5]; [6]).

That realizes the idea expressed in Gian-Carlo Rota's Fubini Lectures [9] (and in his lectures at MIT), where he noted that in view of duality between partitions and subsets, the "lattice of partitions plays for information the role that the Boolean algebra of subsets plays for size or probability" [7, p. 30] or symbolically:

$$\frac{\text{information}}{\text{partitions}} \approx \frac{\text{probability}}{\text{subsets.}}$$

Since "Probability is a measure on the Boolean algebra of events" that gives quantitatively the "intuitive idea of the size of a set", we may ask by "analogy" for some measure to capture a property for a partition like "what size is to a set." The answer is the number of distinctions. The logical entropy $h(\pi) = \frac{|\mathrm{dit}(\pi)|}{|U \times U|}$ of a partition π on a finite set U is that measure on the lattice of partitions on U, i.e., the normalized counting measure on the isomorphic lattice of partition relations (= ditsets), the binary relations that are the complements of the equivalence relations on $U \times U$. Since the logical entropy $h(\pi)$ is also a normalized measure, it has a probability interpretation, i.e., $h(\pi)$ is the probability that in two independent draws from U, one will get a distinction of π, i.e., π distinguishes, just as $\Pr(S)$ is interpreted as the probability that in one draw from U, one will get an element of S, i.e., S occurs.

Duality in quant. measures	Subset logic	Partition logic
'Outcomes'	Elements u of U	Distinctions $(u, u') \in U \times U$
'Events'	Subsets S of U	Partitions π of U
'Event occurs'	$u \in S$	$(u, u') \in \mathrm{dit}(\pi)$
Logical measure	$\Pr(S) = \frac{|S|}{|U|}$	$h(\pi) = \frac{|\mathrm{dit}(\pi)|}{|U \times U|}$
Interpretation	Prob. event S occurs	Prob. partition π distinguishes

Table 2: Logical measures on elements and distinctions

2 The Elements & Distinctions Characterization of the Morphisms in $Sets$

2.1 Set functions transmit elements and reflect distinctions

The duality between elements ("its") of a subset and distinctions ("dits") of a partition already appears in the very notion of a function between sets. The concepts of elements and distinctions provide the natural notions to specify the binary relations, i.e., subsets $R \subseteq X \times Y$, that define functions $f : X \to Y$.

A binary relation $R \subseteq X \times Y$ *transmits elements* if for each element $x \in X$, there is an ordered pair $(x, y) \in R$ for some $y \in Y$.

A binary relation $R \subseteq X \times Y$ *reflects elements* if for each element $y \in Y$, there is an ordered pair $(x, y) \in R$ for some $x \in X$.

A binary relation $R \subseteq X \times Y$ *transmits distinctions* if for any pairs (x, y) and (x', y') in R, if $x \neq x'$, then $y \neq y'$.

A binary relation $R \subseteq X \times Y$ *reflects distinctions* if for any pairs (x, y) and (x', y') in R, if $y \neq y'$, then $x \neq x'$.

The dual role of elements and distinctions can be seen if we translate the usual characterization of the binary relations that define functions into the elements-and-distinctions language. In the usual treatment, a binary relation $R \subseteq X \times Y$ defines a *function* $X \to Y$ if it is defined everywhere on X and is single-valued. But "being defined everywhere" is the same as transmitting (or "preserving" elements, and being single-valued is the same as reflecting distinctions so the more natural definition is:

a binary relation R is a *function* if it transmits elements and reflects distinctions.

What about the other two special types of relations, i.e., those which transmit (or preserve) distinctions or reflect elements? The two important special types of functions are the injections and surjections, and they are defined by the other two notions:

a function is *injective* if it transmits distinctions, and
a function is *surjective* if it reflects elements.

Given a set function $f : X \to Y$ with *domain* X and *codomain* Y, a subset of the codomain is determined as the *image* $f(X)$ of an injective function f, and a partition on the domain is determined as the *coimage* (or inverse-image) $\{f^{-1}(y)\}_{y \in Y}$ of a surjective function f.

2.2   Abstracting to arrow-theoretic definitions

One of our themes is that the concepts of elements and distinctions unpack and analyze the basic category theoretic concepts in the basic category $Sets$, and they are abstracted into purely arrow-theoretic definitions in abstract category theory. For instance, the elements & distinctions definitions of injections and surjections yield "arrow-theoretic" characterizations which can then be applied in any category to provide the usual category-theoretic dual definitions of monomorphisms (injections for set functions) and epimorphisms (surjections for set functions).

Two set functions $f, g : X \rightrightarrows Y$ are different, i.e., $f \neq g$, if there is an element x of X such that their values $f(x)$ and $g(x)$ are a distinction of Y, i.e., $f(x) \neq g(x)$. Hence if f and g are followed by a function $h : Y \to Z$, then the compositions $hf, hg : X \to Y \to Z$ must be different if h *preserves distinctions* (so that the distinction $f(x) \neq g(x)$ is preserved as $hf(x) \neq hg(x)$), i.e., if h is injective. Thus in the category of sets, h being injective is characterized by: for any $f, g : X \rightrightarrows Y$, "$f \neq g$ implies $hf \neq hg$" or equivalently, "$hf = hg$ implies $f = g$" which is the general category-theoretic definition of a *monomorphism*.

In a similar manner, if we have functions $f, g : X \to Y$ where $f \neq g$, i.e., where there is an element x of X such that their values $f(x)$ and $g(x)$ are a distinction of Y, then suppose the functions are preceded by a function $h : W \to X$. Then the compositions $fh, gh : W \to X \to Y$ must be different if h *reflects elements* (so that the element x where f and g differ is sure to be in the image of h), i.e., if h is surjective. Thus in the category of sets, h being surjective is characterized by: for any $f, g : X \rightrightarrows Y$, "$f \neq g$ implies $fh \neq fg$" or "$fh = gh$ implies $f = g$" which is the general category-theoretic definition of an *epimorphism*.

Hence the dual interplay of the notions of elements & distinctions can be seen as yielding the arrow-theoretic characterizations of injections and surjections which are lifted into the general categorical dual definitions of monomorphisms and epimorphisms.

2.3 Duality interchanges elements & distinctions

The reverse-the-arrows duality of category theory is the abstraction from the reversing of the roles of elements & distinctions in dualizing $Sets$ to $Sets^{op}$. That is, a concrete morphism in $Sets^{op}$ is a binary relation, which might be called a *cofunction*, that preserves distinctions and reflects elements–instead of preserving elements and reflecting distinctions. Thus with every binary relation $f \subseteq X \times Y$ that is a function $f : X \to Y$, there is a binary relation $f^{op} \subseteq Y \times X$ that is a cofunction $f^{op} : Y \to X$.

The interchange of elements and distinctions means that the coimage of a function becomes the image of a cofunction and the image of a function becomes the coimage of a cofunction. For the universal constructions in $Sets$, the interchange in the roles of elements and distinctions interchanges each construction and its dual: products and coproducts, equalizers and coequalizers, and in general limits and colimits. That is then abstracted to make the reverse-the-arrows duality in abstract category theory.

This begins to illustrate our theme that the language of elements & distinctions is the conceptual language in which the category of sets and functions is written, and abstract category theory gives the abstract-arrows version of those definitions. Hence we turn to universal constructions for further analysis.

3 The Elements & Distinctions Analysis of Products and Coproducts

3.1 The coproduct in $Sets$

Given two sets X and Y in $Sets$, the idea of the *coproduct* is to create the set with the maximum number of elements starting with X and Y. Since X and Y may overlap, we must make two copies of the elements in the intersection. Hence the relevant operation is not the union of sets $X \cup Y$ but the disjoint union $X \sqcup Y$. To take the disjoint union of a set X with itself, a copy $X^* = \{x^* : x \in X\}$ of X is made so that $X \sqcup X$ can be constructed as $X \cup X^*$. In a similar manner, if X and Y overlap, then $X \sqcup Y = X \cup Y^*$. Then the inclusions $X, Y \subseteq X \sqcup Y$, give the canonical injections $i_X : X \to X \sqcup Y$ and $i_Y : Y \to X \sqcup Y$.

The universal mapping property for the coproduct in $Sets$ is that given any other maps $f : X \to Z$ and $g : Y \to Z$, there is a unique map $f \sqcup g : X \sqcup Y \to Z$ such that $X \overset{i_X}{\to} X \sqcup Y \overset{f \sqcup g}{\to} Z = X \overset{f}{\to} Z$ and $Y \overset{i_Y}{\to} X \sqcup Y \overset{f \sqcup g}{\to} Z = Y \overset{g}{\to} Z$.

$$
\begin{array}{ccccc}
X & \overset{i_X}{\longrightarrow} & X \sqcup Y & \overset{i_Y}{\longleftarrow} & Y \\
& \searrow^{f} & \exists! \downarrow^{f \sqcup g} & g \nearrow & \\
& & Z & &
\end{array}
$$

Coproduct diagram

For the universal constructions with the prefix "co-" (the "co-constructions") as in coproduct or coequalizer, the unique factor maps in $Sets$ are generated by partition refinements. From the data $f : X \to Z$ and $g : Y \to Z$, we need to construct the unique factor map $X \sqcup Y \to Z$. The map f contributes the coimage (or inverse-image) f^{-1} partition on X and g contributes g^{-1} on Y so the blocks of f^{-1} together with the blocks of g^{-1} make up a partition $f^{-1} \sqcup g^{-1}$ on $X \sqcup Y$. To define the unique factor map $f \sqcup g : X \sqcup Y \to Z$, we use the refinement relation $f^{-1} \sqcup g^{-1} \precsim \mathbf{1}_{X \sqcup Y}$. For each element singleton in the discrete partition $\mathbf{1}_{X \sqcup Y}$ on $X \sqcup Y$, there is a unique block $f^{-1}(z)$ or $g^{-1}(z)$ in $f^{-1} \sqcup g^{-1}$ containing it, so the element in the singleton gets mapped to the corresponding z, and the factor mapping property automatically holds.

3.2 The product in $Sets$

Given two sets X and Y in $Sets$, the idea of the *product* is to create the set with the maximum number of distinctions starting with X and Y. The product in $Sets$ is usually constructed as the set of ordered pairs in the Cartesian product $X \times Y$. But to emphasize the point about distinctions, we might employ the same trick of 'marking' the elements of Y, particularly when $Y = X$, with an asterisk. Then an alternative construction of the product in $Sets$ is the set of *unordered* pairs $X \boxtimes Y = \{\{x, y^*\} : x \in X; y^* \in Y^*\}$ which in the case of $Y = X$ would be $X \boxtimes X = \{\{x, x^*\} : x \in X; x^* \in X^*\}$. This alternative construction of the product (isomorphic to the Cartesian product) emphasizes the distinctions formed from X and Y so the ordering in the ordered pairs of the usual construction $X \times Y$ is only a way to make the same distinctions.

The set X defines a partition π_X on $X \times Y$ whose blocks are $B_x = \{(x, y) : y \in Y\} = \{x\} \times Y$ for each $x \in X$, and Y defines a partition π_Y whose blocks are $B_y = \{(x, y) : x \in X\} = X \times \{y\}$ for each $y \in Y$. Since $\pi_X, \pi_Y \precsim \mathbf{1}_{X \times Y}$, the induced surjections are the canonical projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$.

The universal mapping property for the product in $Sets$ is that given any other maps $f : Z \to X$ and $g : Z \to Y$, there is a unique map $\langle f, g \rangle : Z \to X \times Y$ such that $Z \overset{\langle f,g \rangle}{\to} X \times Y \overset{p_X}{\to} X = Z \overset{f}{\to} X$ and $Z \overset{\langle f,g \rangle}{\to} X \times Y \overset{p_Y}{\to} Y = Z \overset{g}{\to} Y$.

$$
\begin{array}{ccccc}
 & & Z & & \\
 & {}^{f}\swarrow & {}^{\exists !}\downarrow{}^{\langle f,g\rangle} & {}^{g}\searrow & \\
X & \overset{p_X}{\longleftarrow} & X \times Y & \overset{p_Y}{\longrightarrow} & Y
\end{array}
$$
Product diagram

For the universal constructions without the prefix "co-" (the "non-co-constructions") as in product or equalizer, the unique factor maps in $Sets$ are generated by set inclusions. From the data $f : Z \to X$ and $g : Z \to Y$, we need to construct the unique factor map $Z \to X \times Y$. The map f contributes the image $f(U)$ subset of X and g contributes the image $g(U)$ subset of Y so we have the subset $f(U) \times g(U)$ of the product $X \times Y$ where for each $z \in Z$, gives the pair $(f(z), g(z)) \in f(U) \times g(U)$. To define the unique factor map $\langle f, g \rangle : Z \to X \times Y$, we use the inclusion relation $f(U) \times g(U) \subseteq X \times Y$. Each element $z \in Z$ determines the element $(f(z), g(z)) \in f(U) \times g(U)$ so by the inclusion, we have the map $\langle f, g \rangle : Z \to X \times Y$ where $z \longmapsto (f(z), g(z)) \in X \times Y$, and the factor mapping property automatically holds.

4 The Elements & Distinctions Analysis of Equalizers and Coequalizers

4.1 The coequalizer in $Sets$

For the equalizer and coequalizer, the data is not just two sets but two parallel maps $f, g : X \rightrightarrows Y$. Then each element $x \in X$, gives us a pair $f(x)$ and $g(x)$ so we take the equivalence relation $\sim$ defined on Y that is generated by $f(x) \sim g(x)$ for any $x \in X$. Then the coequalizer is the quotient set $C = Y/\sim$. When $\sim$ is represented as a partition on Y, then it is refined by the discrete partition $\mathbf{1}_Y$ on Y, and that refinement defines the canonical map $can. : Y \to Y/\sim$.

For the universality property, let $h : Y \to Z$ be such that $hf = hg$. Then we need to show there is a unique refinement-defined map $h^* : Y/\sim \to Z$ such that $h^* can. = h$.

$$
\begin{array}{ccccc}
X & \overset{f}{\underset{g}{\rightrightarrows}} & Y & \overset{can.}{\to} & Y/\sim \\
 & & & {}^{h}\searrow & {}^{\exists !}\downarrow{}^{h^*} \\
 & & & & Z
\end{array}
$$

We already have one partition $\sim$ on Y which was generated by $f(x) \sim g(x)$. Since $hf = hg$, we know that $hf(x) = hg(x)$ so the coimage or inverse-image h^{-1} has to *at least* identify $f(x)$ and $g(x)$ (and perhaps identify other elements) which means that $h^{-1} \precsim \sim$. Hence for each element of $Y/\sim$, i.e., each block b in the partition $\sim$, there is a unique block $h^{-1}(z)$ containing that block, so induced map is $h^*(b) = z$. Commutativity requires $h^{-1} \precsim \sim$ so the map h^* thereby defined $Y/\sim \to Z$ is thereby unique to make the triangle commute.

4.2 The equalizer in $Sets$

The data for the equalizer construction is the same two parallel maps $f, g : X \rightrightarrows Y$. The equalizer is the $E = \{x \in X : f(x) = g(x)\} \subseteq X$ so the map induced by that inclusion is the canonical map $can. : E \to X$.

The universal property is that for any other map $h : Z \to X$ such that $fh = gh$, then $\exists! h_* : Z \to E$ such that $h_* can. = h$.

$$
\begin{array}{ccccc}
E & \xrightarrow{can.} & X & \underset{g}{\overset{f}{\rightrightarrows}} & Y \\
\exists! \uparrow h_* & h \nearrow & & & \\
Z & & & &
\end{array}
$$

Equalizer diagram

The image of $h(Z)$ must satisfy $fh(z) = gh(z)$ for all $z \in Z$, so all the elements $h(z) \in X$ where f and g agree give $h(Z) \subseteq E$, and thus the factor map h_* is induced by that inclusion. That inclusion is implied by the $fh = gh$ condition so the factor map is unique.

5 The Elements & Distinctions Analysis of Cartesian and Co-Cartesian Squares

5.1 The pushout or co-Cartesian square in $Sets$

It is a standard theorem of category theory that if a category has products and equalizers, then it has all limits, and if it has coproducts and coequalizers, then it has all colimits. Since we have presented the elements & distinctions analysis of products and coproducts, and of equalizers and coequalizers, the analysis extends to all limits and colimits. However, the theme would be better illustrated by considering some more complicated limits and colimits such as Cartesian and co-Cartesian squares, i.e., pullbacks and pushouts.

For the pushout or co-Cartesian square, the data are two maps $f : Z \to X$ and $g : Z \to Y$ so we have the two parallel maps $Z \xrightarrow{f} X \xrightarrow{i_X} X \sqcup Y$ and $Z \xrightarrow{g} Y \xrightarrow{i_Y} X \sqcup Y$ and then we can take their coequalizer C formed by the equivalence relation $\sim$ on the common codomain $X \sqcup Y$ which is the equivalence relation generated by $x \sim y$ if there is a $z \in Z$ such that $f(z) = x$ and $g(z) = y$. The canonical maps $X \to X \sqcup Y/\sim$ and $Y \to X \sqcup Y/\sim$ are just the canonical injections into the disjoint union followed by the canonical map of the coequalizer construction analyzed above. As the composition of a canonical injection with a canonical surjection, those canonical maps need not be either injective or surjective.

$$
\begin{array}{ccccc}
Z & \xrightarrow{f} & X & = & X \\
g \downarrow & \searrow & \downarrow can.. & & \\
Y & \xrightarrow{can.} & C = X \sqcup Y/\sim & & \downarrow h \\
\| & & & \exists! \searrow h^* & \\
Y & \xrightarrow{h'} & & & U
\end{array}
$$

Pushout or co-Cartesian square diagram

For the universal mapping property, consider any $h : X \to U$ and $h' : Y \to U$ such that $hf = h'g$. Then h^{-1} is a partition on X and h'^{-1} is a partition on Y so let $h^{-1} \sqcup h'^{-1}$ be the partition on $X \sqcup Y$ with the blocks $h^{-1}(u) \sqcup h'^{-1}(u)$ for some $u \in U$. The condition that for any $z \in Z$, $hf(z) = h'g(z) = u$ for some $u \in U$ means that $h^{-1} \sqcup h'^{-1}$ must make at least the identifications of the coequalizer (and perhaps more) so that $h^{-1} \sqcup h'^{-1}$ is refined by $\sim$ as partitions on $X \sqcup Y$. Each block in $h^{-1} \sqcup h'^{-1}$ has the form $h^{-1}(u) \sqcup h'^{-1}(u)$ for some u and $h^{-1} \sqcup h'^{-1} \precsim \sim$ so each block b in $\sim$ is contained in a block of the form $h^{-1}(u) \sqcup h'^{-1}(u)$ of $h^{-1} \sqcup h'^{-1}$. Hence that block b of $\sim$ is mapped by h^* to u, and the commutativity properties follow automatically.

5.2 The pullback or Cartesian square in $Sets$

For the Cartesian square or pullback, the data are two maps $f : X \to Z$ and $g : Y \to Z$. We then have two parallel maps $X \times Y \rightrightarrows Z$ (the projections followed by f or g) so we take the pullback as their equalizer E. The canonical maps $E \to X$ and $E \to Y$ are the compositions of the canonical injective map $E \to X \times Y$ followed by the canonical projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$. As the composition of a canonical injection with a canonical surjection, those canonical maps need not be either injective or surjective.

$$
\begin{array}{ccccc}
U & & \overset{h}{\to} & & X \\
& \exists! \searrow h_* & & & \| \\
\downarrow h' & & E \subseteq X \times Y & \overset{can.}{\to} & X \\
& & \downarrow can. & & \downarrow f \\
Y & = & Y & \overset{g}{\to} & Z
\end{array}
$$

Pullback or Cartesian square diagram

For the universality property, consider any other maps $h : U \to X$ and $h' : U \to Y$ such that $fh = gh'$. We know that $h'(u)$ and $h(u)$ are elements such that $(h(u), h'(u)) \in E$ so for the images, there is the inclusion $h(U) \times h'(U) \subseteq E$. Hence that inclusion gives the uniquely defined map $h_* : U \to h(U) \times h'(U) \subseteq E$ where $u \longmapsto (h(u), h'(u)) \in E$ so that the commutativity property automatically holds.

6 Speculative Concluding Remarks

We shown how the dual concepts of elements & distinctions can be used to account for the notion of morphism, duality, and for the universal constructions in $Sets$–which are then abstracted in abstract category theory. This suggests that the notions have some broader significance. One possibility is they are respectively mathematical versions of the old metaphysical concepts of matter (or substance) and form (as in in-form-ation). The matter versus form idea [1] can be illustrated by comparing the two lattices of subsets and partitions on a set–the two lattices that we saw defined the canonical morphisms and unique factor maps in the universal constructions of $Sets$.

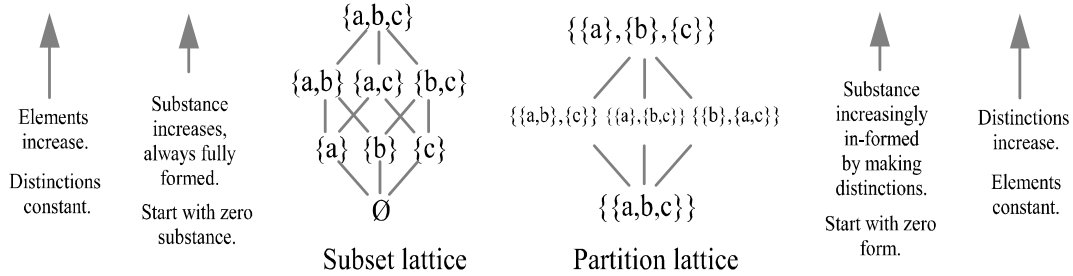For $U = \{a, b, c\}$, start at the bottom and move towards the top of each lattice.

{a,b,c}

{a,b} {a,c} {b,c}

{a} {b} {c}

∅

Subset lattice

{{a},{b},{c}}

{{a,b},{c}} {{a},{b,c}} {{b},{a,c}}

{{a,b,c}}

Partition lattice

Elements increase.

Distinctions constant.

Substance increases, always fully formed.

Start with zero substance.

Substance increasingly in-formed by making distinctions.

Start with zero form.

Distinctions increase.

Elements constant.

Figure 1: Moving up the subset and partition lattices.

At the bottom of the Boolean subset lattice is the empty set ∅ which represents no substance (no 'its'). As one moves up the lattice, new elements of substance are created that are always fully formed until finally one reaches the top, the universe U. Thus new substance is created in moving up the lattice but each element is fully formed and thus distinguished from the other elements.

At the bottom of the partition lattice is the indiscrete partition or "blob" $\mathbf{0} = \{U\}$ (where the universe set U makes one block) which represents all the substance or matter but with no distinctions to in-form the substance (no 'dits'). As one moves up the lattice, no new substance is created but distinctions are created that in-form the indistinct elements as they become more and more distinct. Finally one reaches the top, the discrete partition $\mathbf{1}$, where all the elements of U have been fully formed. A partition combines indefiniteness (within blocks) and definiteness (between blocks). At the top of the partition lattice, the discrete partition $\mathbf{1} = \{\{u\} : \{u\} \subseteq U\}$ is the result making all the distinctions to eliminate any indefiniteness. Thus one ends up at the "same place" (universe U of fully formed entities) either way, but by two totally different but dual 'creation stories':

- creating elements (as in creating fully-formed matter out of nothing) versus

- creating distinctions (as in starting with a totally undifferentiated matter and then in a 'big bang' start making distinctions, e.g., breaking symmetries, to give form to the matter).

Moreover, we have seen that:

- the quantitative increase in substance (normalized number of elements) moving up in the subset lattice is measured by logical probability, and

- the quantitative increase in form (normalized number of distinctions) moving up in the partition lattice is measured by logical information ([2]; [5]).

References

[1] Ainsworth, Thomas. 2020. "Form vs. Matter." In *The Stanford Encyclopedia of Philosophy (Summer 2020 Edition)*, edited by Edward N. Zalta. https://plato.stanford.edu/archives/sum2020/entries/form-matter/.

[2] Ellerman, David. 2009. Counting Distinctions: On the Conceptual Foundations of Shannon's Information Theory. *Synthese* 168 (1 May): 119–49.

[3] Ellerman, David 2010. The Logic of Partitions: Introduction to the Dual of the Logic of Subsets. *Review of Symbolic Logic*. 3 (2 June): 287-350.

[4] Ellerman, David. 2014. An Introduction of Partition Logic. *Logic Journal of the IGPL*. 22 (1): 94–125.

[5] Ellerman, David. 2017. Logical Information Theory: New Foundations for Information Theory. *Logic Journal of the IGPL* 25 (5 Oct.): 806–35. https://doi.org/10.1093/jigpal/jzx022.

[6] Ellerman, David. 2018. Logical Entropy: Introduction to Classical and Quantum Logical Information Theory. *Entropy* 20 (9): Article ID 679. https://doi.org/10.3390/e20090679.

[7] Kung, Joseph P. S., Gian-Carlo Rota, and Catherine H. Yan. 2009. *Combinatorics: The Rota Way*. New York: Cambridge University Press.

[8] Lawvere, F. William and Robert Rosebrugh 2003. *Sets for Mathematics*. Cambridge: Cambridge University Press.

[9] Rota, Gian-Carlo. 2001. Twelve Problems in Probability No One Likes to Bring up. In *Algebraic Combinatorics and Computer Science*, edited by Henry Crapo and Domenico Senato, 57–93. Milano: Springer.