The Quantum Logic of Direct-Sum Decompositions

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Abstract

Since the pioneering work of Birkhoff and von Neumann, quantum logic has been interpreted as the logic of (closed) subspaces of a Hilbert space. There is a progression from the usual Boolean logic of subsets to the "quantum logic" of subspaces of a general vector space–which is then specialized to the closed subspaces of a Hilbert space. But there is a "dual" progression. The notion of a partition (or quotient set or equivalence relation) is dual (in a category-theoretic sense) to the notion of a subset. Hence the Boolean logic of subsets has a dual logic of partitions. Then the dual progression is from that logic of partitions to the quantum logic of direct-sum decompositions (i.e., the vector space version of a set partition) of a general vector space–which can then be specialized to the direct-sum decompositions of a Hilbert space. This allows the logic to express measurement by any self-adjoint operators rather than just the projection operators associated with subspaces. In this introductory paper, the focus is on the quantum logic of direct-sum decompositions of a finite-dimensional vector space (including such a Hilbert space). The primary special case examined is finite vector spaces over $\mathbb{Z}_2$ where the pedagogical model of quantum mechanics over sets (QM/Sets) is formulated. In the Appendix, the combinatorics of direct-sum decompositions of finite vector spaces over $GF(q)$ is analyzed with computations for the case of QM/Sets where $q = 2$.

Contents

1 Introduction

This paper is an introduction to quantum logic based on direct-sum decompositions rather than on subspaces of vector spaces. This allows the logic to express measurement by any self-adjoint operators rather than just the projection operators corresponding to subspaces. A *direct-sum decomposition* (DSD) of a vector space V over a base field $\mathbb{K}$ is a set of (nonzero) subspaces $\{V_i\}_{i\in I}$ that are *disjoint* (i.e., their pair-wise intersections are the zero space 0) and that span the space so that each vector $v \in V$ has a unique expression $v = \sum_{i\in I} v_i$ with each $v_i \in V_i$ (with only a finite number of v_i's nonzero). For introductory purposes, it is best to assume V is finite dimensional (although many of the proofs are more general).

Each self-adjoint operator, and in general diagonalizable operator, has eigenspaces that form a direct-sum decomposition of the vector space but the notion of a direct-sum decomposition makes sense over arbitrary vector spaces independently of an operator. For instance, in the pedagogical model of "quantum mechanics over sets" or QM/Sets ([12]; [14]),the vector space is $\mathbb{Z}_2^n$ so the only operators (always assumed diagonalizable) are projection operators $P : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$. But given a set $U = \{v_1, ..., v_n\}$ of basis vectors for $\mathbb{Z}_2^n$, any real-valued "random variable" or function $f : U \to \mathbb{R}$ determines a DSD $\left\{\wp\left(f^{-1}(r)\right)\right\}_{r\in f(U)}$ of $\mathbb{Z}_2^n$ (where $\wp()$ is the power-set and $f(U)$ is the image or "spectrum" of "eigenvalues" of the numerical attribute f). Thus the concept of a direct-sum decomposition of a vector space allows one to capture many of the relevant properties of such a real-valued "observable" even though it does not take values in the base field (which is only $\mathbb{Z}_2$ in QM/Sets). It is only as the base field is increased up to the complex numbers that all real-valued observables can be "internalized" as self-adjoint operators.

The genesis of the usual quantum logic of projection operators or subspaces can be seen as starting with the Boolean logic of subsets in the Boolean lattice $\wp(U)$ of subsets of a universe set U and taking the vector space version of a "subset"–which is a subspace. That yields the lattice of subspaces of an arbitrary vector space–which can then be specialized to the lattice of (closed) subspaces of a Hilbert space for the strictly quantum mechanical case. In category theory, the notion of a subobject, such as a subset or subspace, has a dual notion of a quotient object. Thus the dual concept to a subset is the concept of a quotient set, equivalence relation, or partition on a set U. That gives rise to the idea of the logic of partitions ([11]; [13]) instead of the Boolean logic of subsets. The origin and topic of this paper is the vector space version of a partition, namely a direct-sum decomposition (NB: not a quotient space). The analogue of a partial Boolean algebra [19] is then a

"partial partition algebra" of DSDs on an arbitrary vector space (our topic here)–which can then be specialized to a Hilbert space for the strictly quantum mechanical interpretation or specialized to a vector space over $\mathbb{Z}_2$ for pedagogical purposes.

Set concept	Linearizes to vector-space concept	Quantum logic case: (Hilbert space)	Pedagogical model of QM/Sets (space over $\mathbb{Z}_2$)
Boolean algebra of *subsets* of a set	Lattice of *subspaces* of a vector space or pBA of projection operators.	Orthomodular lattice of closed *subspaces* or pBA of projection operators	Partial BA of projection operators on vector space over $\mathbb{Z}_2$
Partition algebra of *partitions* on a set	Partial partition algebra of *direct-sum decompositions* of a vector space	Special quantum partition algebra of *DSDs* on a Hilbert space	Partial partition algebra of DSDs on vector space over $\mathbb{Z}_2$

Figure 1: Progressions from sets to vector spaces starting with dual concepts of subset and partition.

There is a natural partial order ("refinement" as with partitions on sets) on the DSDs of a vector space V and there is a minimum element $\mathbf{0} = \{V\}$, the *indiscrete* DSD (nicknamed the "blob") which consists of the whole space V. A DSD is "atomic" in the partial order if there is no DSD between it and the minimum DSD $\mathbf{0}$, and the atomic DSDs are the binary ones consisting of just two subspaces. Each atomic DSD determines a pair of projection operators, and the indiscrete DSD also determines a pair of projection operators, namely the zero operator $\hat{0}$ and the identity operator I. Conversely, each projection operator $P : V \to V$ on an arbitrary vector space V (other than the identity or zero operator) determines an atomic DSD consisting of the image of P and the image of $I - P$, while the identity and zero operators determine the indiscrete DSD. In that sense, the quantum logic of DSDs extends the quantum logic of projection operators associated with atomic DSDs. In the quantum logic of (closed) subspaces, only measurement of projection operators (associated with atomic DSDs and the blob) can be represented, so the quantum logic of direct-sum decompositions allows the representation of the measurement of any self-adjoint operators.

2 The partial partition algebra of direct-sum decompositions

2.1 Compatibility of DSDs

Let V be a finite dimensional vector space over a field $\mathbb{K}$. A *direct sum decomposition* (DSD) of V is a set of subspaces $\{V_i\}_{i\in I}$ such that $V_i \cap \sum_{i'\neq i} V_{i'} = 0$ (the zero space) for $i \in I$ and which span the space: written $\oplus_{i\in I} V_i = V$. Let $DSD\,(V)$ be the set of DSDs of V.

In the algebra of partitions on a set, the operations of join, meet, and implication are always defined, but in the context of "vector space partitions," i.e., DSDs, we need to define a notion of compatibility. If we were dealing with operators (and their associated DSDs of eigenspaces), then compatibility would be defined by commutativity. But we are dealing with DSDs directly with no assumption that they are the eigenspace DSDs of operators.

Given two DSDs $\pi = \{V_i\}_{i\in I}$ and $\sigma = \{W_j\}_{j\in J}$, their *proto-join* is the set of non-zero subspaces $\{V_i \cap W_j\}_{(i,j)\in I\times J}$. If the two DSDs π and σ were defined as the eigenspace DSDs of two operators, then the space spanned by the proto-join would be the space spanned by the simultaneous eigenvectors of the two operators, and that space would be the kernel of the commutator of the two operators. If the two operators commuted, then their commutator would be the zero operator whose kernel is the whole space so the proto-join would span the whole space. Hence the natural definition of compatibility without any mention of operators is:

π and σ are *compatible*, written $\pi \leftrightarrow \sigma$, if the proto-join spans the whole space V.

The *indiscrete DSD* $\mathbf{0} = \{V\}$ (the "blob") is compatible with all DSDs, i.e., $\mathbf{0} \leftrightarrow \pi$ for any π.

2.2 The join of compatible DSDs

When two DSDs are compatible, the proto-join is the *join:*

$$\pi \vee \sigma = \{V_i \cap W_j\}_{(i,j)\in I\times J}$$
Join of DSDs when $\pi \leftrightarrow \sigma$.

The binary relation of compatibility on DSDs is reflexive and symmetric. The indiscrete DSD acts as the identity for the join: $\mathbf{0} \vee \pi = \pi$ for any DSD π.

In a set of mutually compatible DSDs, we need to show that the join operation preserves compatibility. If $\pi \leftrightarrow \sigma$, it is trivial that $(\pi \vee \sigma) \leftrightarrow \pi$ and $(\pi \vee \sigma) \leftrightarrow \sigma$, but for a third DSD τ with $\pi \leftrightarrow \tau$ and $\sigma \leftrightarrow \tau$, does $(\pi \vee \sigma) \leftrightarrow \tau$?

Lemma 1 *Let the DSDs $\pi = \{V_i\}_{i\in I}$ and $\sigma = \{W_j\}_{j\in J}$ be compatible so that $\pi\vee\sigma = \{V_i \cap W_j\}_{(i,j)\in I\times J}$ is a DSD and thus any $v \in V$ has a unique expression $v = \sum_{(i,j)\in I\times J} v_{ij}$ where $v_{ij} \in V_i \cap W_j$. Let $v_i = \sum_{j\in J} v_{ij} \in V_i$ so that $v = \sum_{i\in I} v_i$. If $v \in V_i$, then $v = v_i$.*

Proof: Let $\widehat{v_i} = \sum_{i'\in I, i'\neq i} v_{i'}$ so that $v = v_i + \widehat{v_i}$. Then $v - v_i = \widehat{v_i} \in V_i$. If $\widehat{v_i} \neq 0$, then $\widehat{v_i}$ itself and $\sum_{i'\neq i,j} v_{i'j}$ are two different expressions for $\widehat{v_i}$ of vectors in a direct sum, so $\widehat{v_i} = 0$. □

Theorem 2 *Given three DSDs, $\pi = \{V_i\}_{i\in I}$, $\sigma = \{W_j\}_{j\in J}$, and $\tau = \{X_k\}_{k\in K}$ that are mutually compatible, i.e., $\pi \leftrightarrow \sigma$, $\pi \leftrightarrow \tau$, and $\sigma \leftrightarrow \tau$, then $(\pi \vee \sigma) \leftrightarrow \tau$.*

Proof: We need to prove $\pi \vee \sigma = \{V_i \cap W_j\}_{(i,j)\in I\times J}$ is compatible with $\tau = \{X_k\}_{k\in K}$, i.e., that $\oplus_{(i,j,k)\in I\times J\times K} (V_i \cap W_j \cap X_k) = V$. Consider any nonzero $v \in V$ where since $\pi \leftrightarrow \sigma$, $\exists v_{ij} \in V_i \cap W_j$ such that $v = \sum_{(i,j)\in I\times J} v_{ij}$.

Now since $\pi \leftrightarrow \tau$, for each nonzero $v_{ij} \in V_i \cap W_j$, $\exists v_{ij,i'k} \in V_{i'} \cap X_k$ such that $v_{ij} = \sum_{(i',k)\in I\times K} v_{ij,i'k}$. But since $v_{ij} \in V_i$, by the Lemma, only $v_{ij,ik}$ is nonzero, so $v_{ij} = \sum_{k\in K} v_{ij,ik}$.

Symmetrically, since $\sigma \leftrightarrow \tau$, for each $v_{ij} \in V_i \cap W_j$, $\exists v_{ij,j'k} \in W_{j'} \cap X_k$ such that $v_{ij} = \sum_{(j',k)\in J\times K} v_{ij,j'k}$. But since $v_{ij} \in W_j$, by the Lemma, only $v_{ij,jk}$ is nonzero, so $v_{ij} = \sum_{k\in K} v_{ij,jk}$.

Now since $\{X_k\}_{k\in K}$ is a DSD, there is a unique expression for each nonzero $v_{ij} = \sum_{k\in K} v_{ijk}$ where $v_{ijk} \in X_k$. Hence by uniqueness: $v_{ijk} = v_{ij,ik} = v_{ij,jk}$. But since $v_{ij,ik} \in V_i$ and $v_{ij,jk} \in W_j$ and $v_{ij,ik} = v_{ijk} = v_{ij,jk}$, we have $v_{ijk} \in V_i \cap W_j \cap X_k$. Thus

$v = \sum_{(i,j)\in I\times J} v_{ij} = \sum_{(i,j)\in I\times J} \sum_{k\in K} v_{ijk} = \sum_{(i,j,k)\in I\times J\times K} v_{ijk}$. Since v was arbitrary,

$$\oplus_{(i,j,k)\in I\times J\times K} (V_i \cap W_j \cap X_k) = V. \square$$

2.3 The meet of two DSDs

For any two DSDs π and σ, the *meet* $\pi \wedge \sigma$ is the DSD whose subspaces are direct sums of subspaces from π and the direct sum of subspaces from σ and are minimal subspaces in that regard. That is, $\{Y_l\}_{l\in L}$ is the meet if there is a set partition $\{I_l\}_{l\in L}$ on I and a set partition $\{J_l\}_{l\in L}$ such that:

$$Y_l = \oplus_{i\in I_l} V_i = \oplus_{j\in J_l} W_j$$

and that holds for no more refined partitions on the index sets. If $\pi \leftrightarrow \tau$ and $\sigma \leftrightarrow \tau$, then it is trivial that $(\pi \wedge \sigma) \leftrightarrow \tau$.

As in the movie of the same name,

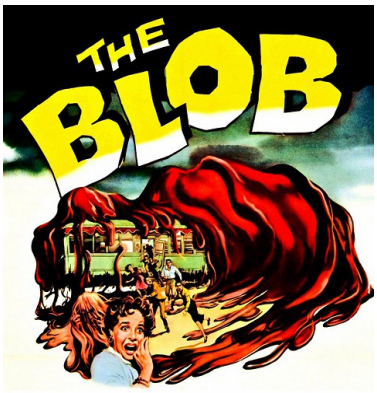

Figure 2: "The blob absorbs everything it meets": $\mathbf{0} \wedge \pi = \mathbf{0}$.

2.4 The refinement partial order on DSDs

The *partial order* on the DSDs of V is defined as for set partitions but with subspaces replacing subsets:

$$\pi \textit{ refines } \sigma, \text{ written } \sigma \preceq \pi, \text{ if for every } V_i \in \pi, \exists W_j \in \sigma \text{ such that } V_i \subseteq W_j.$$

If $\sigma \preceq \pi$ holds, then each $W_j = \oplus \{V_i : V_i \subseteq W_j\}$ so $\pi \leftrightarrow \sigma$ and $\pi \vee \sigma = \pi$ as well as $\pi \wedge \sigma = \sigma$ as expected.

Proposition 1 *Where it exists, the join $\pi \vee \sigma$ is the least upper bound of π and σ.*

Proof: If $\pi, \sigma \preceq \tau$, i.e., π and σ have a common upper bound $\tau = \{X_k\}_{k \in K}$, then $V_i = \oplus \{X_k : X_k \subseteq V_i\}$ and $W_j = \oplus \{X_k : X_k \subseteq W_j\}$. Given a nonzero $v \in V_i \cap W_j$, it can be expressed uniquely as $v = \sum_{X_k \subseteq V_i} v_{ik}$ where $v_{ik} \in X_k$ and as $v = \sum_{X_k \subseteq W_j} v_{jk}$ where $v_{jk} \in X_k$. But since τ is a DSD, there is a unique expression $v = \sum_{k \in K} v_k$ so $v_{ik} = v_{jk} = v_k$ where $v_k \in V_i \cap W_j \cap X_k$ where $X_k \subseteq V_i$ and $X_i \subseteq W_j$ so $X_k \subseteq V_i \cap W_j$ and thus $\pi \vee \sigma \preceq \tau$. Hence $\pi \vee \sigma$ is the least upper bound of π and σ. $\square$

Corollary 1 *If π and σ have a common upper bound, i.e., $\pi, \sigma \preceq \tau$, then $\pi \leftrightarrow \sigma$.*

Two DSDs π and σ need not have a common upper bound so $DSD\,(V)$ is not a join-semilattice.

Lemma 3 *Given a DSD $\pi = \{V_i\}_{i \in I}$, let $X = \oplus_{i \in I_X} V_i$ and $Y = \oplus_{i \in I_Y} V_i$ both be direct sums of some V_i's. If $X \cap Y$ is nonzero, then it is also a direct sum of some V_i's.*

Proof: Consider a nonzero $v \in X \cap Y$ so there is a unique expression $v = \sum_{i \in I_X} v_{i,X}$ where $v_{i,X} \in V_i \subseteq X$ and a unique expression $v = \sum_{i \in I_Y} v_{i,Y}$ where $v_{i,Y} \in V_i \subseteq Y$. Since π is a DSD, there is also a unique expression $v = \sum_{i \in I} v_i$ so for each nonzero v_i. $v_i = v_{i,X} = v_{i,Y} \in V_i \cap X \cap Y$. Thus for any such i, V_i is a common direct summand to X and Y, so $V_i \subseteq X \cap Y$. Thus every nonzero element $v \in X \cap Y$ is in a direct sum of V_i's for $V_i \subseteq X \cap Y$ and thus $X \cap Y$ is the direct sum of V_i that are common direct summands of X and Y. $\square$

Proposition 2 *The meet $\pi \wedge \sigma$ is the greatest lower bound of π and σ.*

Proof: If $\tau \preceq \pi, \sigma$ then each $X_k = \oplus \{V_i : V_i \subseteq X_k\} = \oplus \{W_j : W_j \subseteq X_k\}$. By the construction of $\pi \wedge \sigma$, there is a set partition $\{I_l\}_{l \in L}$ on I and a set partition $\{J_l\}_{l \in L}$ on J such that the subspaces in the meet $\pi \wedge \sigma = \{Y_l\}$ are:

$$Y_l = \oplus_{i \in I_l} V_i = \oplus_{j \in J_l} W_j,$$

and where no subsets of I smaller than I_l and subsets of J smaller than J_l have that property. Since each V_i is contained in some X_k, if $i \in I_l$, then $V_i \subseteq Y_l \cap X_k$. Since both Y_l and X_k are direct sums of some V_i, then by the Lemma the nonzero subspace $Y_l \cap X_k$ is also a direct sum of some V_i's. Symmetrically, since the same Y_l and X_k are direct sums of some W_j's, then by the Lemma the nonzero subspace $Y_l \cap X_k$ is also a direct sum of some W_j's. But since Y_l is the smallest direct sum of both V_i's and W_j's, $Y_l \cap X_k = Y_l$, i.e., $Y_l \subseteq X_k$, and thus $\pi \wedge \sigma$ is the greatest (in the refinement partial ordering) lower bound on π and σ. $\square$

As the blob is compatible with all DSDs, it is the minimum element in the ordering: $\mathbf{0} \preceq \pi$ for any π. Hence any two DSDs π and σ always have a common lower bound, so they always have a meet $\pi \wedge \sigma$, i.e., $DSD(V)$ is a meet-semilattice. Thus the *partial partition algebra* $DSD(V)$ could also be called the *meet-semi-lattice of DSDs* on a vector space V.

The binary DSDs $\alpha = \{A_1, A_2\}$ are the atoms of the meet-semi-lattice $DSD(V)$. A meet-semi-lattice is said to be *atomistic* if every element is the join of the atoms below it.

Proposition 3 *The meet-semi-lattice $DSD(V)$ is atomistic.*

Proof: Consider a non-blob DSD $\pi = \{V_i\}_{i \in I}$. If $\alpha = \{A_1, A_2\} \preceq \pi = \{V_i\}_{i \in I}$, then $A_k = \oplus \{V_i : V_i \subseteq A_k\}$ for $k = 1, 2$. Thus for any other atom $\alpha' = \{A'_1, A'_2\} \preceq \pi$, the join $\alpha \vee \alpha'$ is defined and $\alpha \vee \alpha' \preceq \pi$, and each nonzero subspace $A_k \cap A'_{k'}$ is the direct sum of some V_i's. If a join of atoms had a subspace $V_{i_1} \oplus V_{i_2}$, $i_1, i_2 \in I$, then the join with the atom $\{V_{i_1}, \oplus_{i' \neq i_1, i' \in I} V_{i'}\}$ would split apart $V_{i_1} \oplus V_{i_2}$, so the join of the atoms below π gives π.$\square$

$DSD(V)$ is the *quantum partition logic* determined by V. To be more specifically "quantum", V could be a finite-dimensional Hilbert space.

3 Partition logics in a partial partition algebra

3.1 The implication DSD in partition logics

Just as a partial Boolean algebra is made up of overlapping Boolean algebras, so a partial partition algebra is made up of overlapping partition logics or algebras. There is no maximum DSD, only maximal DSDs. Each maximal element in the partial ordering is a *discrete* (or "non-degenerate") DSD $\omega = \{U_k\}_{k \in K}$ of one-dimensional subspaces (rays) of V (so $|K|$ is the dimension of V). A *partition logic* is determined by the set of DSDs $\prod(\omega) = \{\pi : \pi \preceq \omega\} = [\mathbf{0}, \omega] \subseteq DSD(V)$ compatible with a maximal element ω with the induced ordering and operations (which is analogous to the way in a partial Boolean algebra, a complete set of one-dimensional subspaces determines a Boolean algebra).

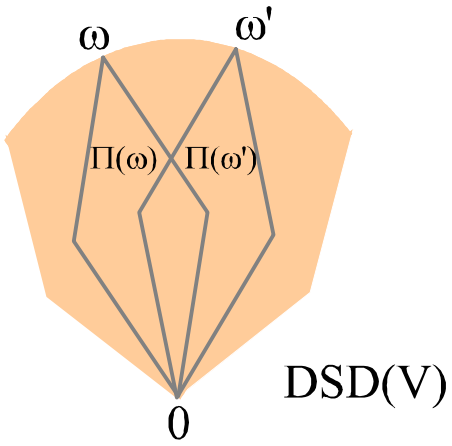

Figure 3: Partial Partition Logic or Meet-Semi-Lattice of DSDs of V with partition logics $\prod(\omega)$ and $\prod(\omega')$.

For any $\pi \in \prod(\omega)$, $\pi \preceq \omega$ so ω is (by construction) the maximum or *top* DSD in $\prod(\omega)$ and thus might be symbolized as the discrete DSD $\mathbf{1}_\omega$ $(= \omega)$. Each subspace $V_i \in \pi \preceq \omega$ has $V_i = \oplus\{U_k : U_k \subseteq V_i, k \in K\}$ so ω absorbs what it joins and is the unit element for meets within $\prod(\omega)$:

$$\pi \vee \omega = \omega \text{ and } \pi \wedge \omega = \pi.$$

All the DSDs π and σ compatible with ω, i.e., $\pi, \sigma \in \prod(\omega)$, are compatible with each other since they have a common upper bound. Explicitly, each $\pi = \{V_i\}_{i \in I}$ in $\prod(\omega)$ determines a set partition $\pi(\omega)$ on K (the index set for ω), and thus $|\prod(\omega)| = B(|K|) = B(\dim(V))$, the Bell number for the dimension of V. Each U_k would be contained in some block of the set-partition join $\pi(\omega) \vee \sigma(\omega)$ and thus those corresponding subspaces $V_i \cap W_j = \oplus\{U_k : U_k \subseteq V_i \cap W_j, k \in K\}$ would span V so $\pi \leftrightarrow \sigma$.

In order to be properly called a "logic," each partition lattice $\prod(\omega)$ has a naturally defined implication inherited from the logic of set partitions (so "partition logic" refers to a partition lattice plus the implication operation). For $\sigma, \pi \in \prod(\omega)$, and for each $V_i \in \pi$, the corresponding subspaces of the *implication* $\sigma \Rightarrow \pi$ are:

$$\begin{cases} \text{all } U_k \subseteq V_i \text{ if } \exists W_j \in \sigma \text{ such that } V_i \subseteq W_j \\ V_i \text{ if not.} \end{cases}.$$

Since each $V_i = \oplus\{U_k : U_k \subseteq V_i\}$, the implication $\sigma \Rightarrow \pi$ is still a DSD in spite of some of the $V_i \in \pi$ being "discretized" into the U_k contained in it. In the implication DSD $\sigma \Rightarrow \pi$, each $V_i \in \pi$ either remains whole like a mini-zero-blob $\mathbf{0}_{V_i} = \{V_i\}$ on the space V_i (if V_i is not contained in any $W_j \in \sigma$) or it is discretized into the "atoms" $U_k \subseteq V_i$ which in effect assigns a "$\mathbf{1}$" to V_i if $\exists W_j$ such that $V_i \subseteq W_j$. In other words, the implication $\sigma \Rightarrow \pi$ acts like an indicator or characteristic function assigning a 1 or 0 to each V_i depending on whether or not $\exists W_j$ such that $V_i \subseteq W_j$. Thus trivially:

$$\sigma \Rightarrow \pi = \mathbf{1}_\omega \text{ iff } \sigma \preceq \pi.$$

If we just take $\omega = \{U_k\}_{k \in K}$ as a set of entities (forgetting about any vector space structure), then each DSD $\sigma = \{W_j\}_{j \in J}$ in $\prod(\omega) = [\mathbf{0}, \omega]$ defines a set partition on $\omega = \{U_k\}_{k \in K}$ where each subspace W_j determines a block $\{U_k : U_k \subseteq W_j\}$.

Indeed, given any DSD $\pi = \{V_i\}_{i\in I}$, each subspace W_j of $\sigma \in [\mathbf{0}, \pi]$ determines a block $\{V_i : V_i \subseteq W_j\}$ so σ defines a set partition on π. Thus the interval $[\mathbf{0}, \pi]$ is isomorphic to the set-based partition logic (join, meet, and implication operations) on that set π [13]. As a partition lattice, $[\mathbf{0}, \pi]$ has many of well-known properties ([27]; [5]; [16, Chapter IV, section 4]). However, the late development of partition logic was in part retarded by the practice of referring to the lattice of equivalence relations as the "lattice of partitions" where the partial order however "corresponds to set inclusion for the corresponding equivalence relations" [16, p. 251] so instead of being refinement it is actually "reverse refinement" [21, p.30]. The partial order on the partition lattice $\prod(\omega)$ (as defined here) corresponds to set inclusion of the binary relations that are the complements of equivalence relations and are called *partition relations* [13] or *apartness relations*. In the lattice of equivalence relations, the top is the biggest (indiscrete) equivalence relation (where everything is identified) and the bottom is the smallest (discrete) equivalence relation where each element is identified only with itself–whereas the partition lattice $\prod(\omega)$ uses the opposite partial order.[1] Either way the lattice is complete and relatively complemented but not distributive. But the reversed order reverses the join and meet, the top and bottom, and the atoms and coatoms.

3.2 DSDs, CSCOs, and measurement

Given a self-adjoint operator L, the projections P_λ can be reconstructed from the DSD $\pi = \{V_\lambda\}_\lambda$of eigenspaces and then the operator can reconstructed–given the eigenvalues–from the spectral decomposition $L = \sum_\lambda \lambda P_\lambda$. What information about self-adjoint operators is lost by dealing only with their DSDs of eigenspaces? The information about which eigenvalues for eigenvectors are the same or different is retained by the distinct eigenspaces in the DSD. It is only the specific numerical values of the eigenvalues that is lost, and those numerical values are of little importance in QM. Any transformation into other real numbers that is one-to-one (thus avoiding "accidental" degeneracy) would do as well. Thus we can say that the essentials of the measurement process in QM can be translated into the language of the quantum logic of direct-sum decompositions. Kolmogorov referred to the set partition given by the inverse-image of a random variable as the "experiment" [20, p. 6], so it is natural to abstractly represent the direct-sum decomposition of eigenspaces given by a self-adjoint operator as the "measurement."

Thus unlike the quantum logic of subspaces, the quantum logic of direct-sum decompositions can directly represent the process of measurement for any self-adjoint operators (rather than just projection operators). Given a state ψ and a self-adjoint operator $L : V \to V$ on a finite dimensional Hilbert space, the operator determines the DSD $\pi = \{V_\lambda\}_\lambda$of eigenspaces for the eigenvalues λ. The measurement operation uses the eigenspace DSD to decompose ψ into the unique parts given by the projections $P_\lambda(\psi)$ into the eigenspaces V_λ, where $P_\lambda(\psi)$ is the outcome of the projective measurement with probability $\Pr(\lambda|\psi) = \|P_\lambda(\psi)\|^2 / \|\psi\|^2$.

The eigenspace DSD $\pi = \{V_\lambda\}_\lambda$ of L is refined by one or more maximal eigenvector DSDs, $\pi = \{V_\lambda\}_\lambda \preceq \omega = \{U_k\}_{k\in K}$. For each ω, there is a set partition $\{B_\lambda\}_\lambda$ on the index set K such that V_λ is the direct sum of the U_k for $k \in B_\lambda$, i.e., $V_\lambda = \oplus_{k\in B_\lambda} U_k$.

If some of the V_λ have dimension larger than one ("degeneracy"), then more measurements by commuting operators will be necessary to further decompose down to single eigenvectors. If two operators commute, that means that their eigenspace DSDs are compatible. Given another self-adjoint operator $M : V \to V$ commuting with L, its eigenspace DSD $\sigma = \{W_\mu\}_\mu$ (for eigenvalues μ of M) is compatible with $\pi = \{V_\lambda\}_\lambda$ and thus has a join DSD $\pi \vee \sigma$ in $DSD(V)$ which is also in $\prod(\omega)$ for one or more maximal ω each representing an orthonormal basis of simultaneous eigenvectors. The combined measurement by the two commuting operators is just the single measurement using the join DSD $\pi \vee \sigma$.

[1] Instead of the usual duality relation within a Boolean algebra, there is a duality relation between the logic of partitions and the "logic" of equivalence relations [13].

Dirac's notion of a Complete Set of Commuting Operators (CSCO) [9] translates into the language of the quantum logic of DSDs as a set of compatible DSDs whose join is a maximal DSD ω in $DSD(V)$ and thus is the maximum DSD $\mathbf{1}_\omega$ in $\prod(\omega)$. As the join, that DSD ω refines each of the compatible DSDs. The combined measurement of the CSCO of commuting operators is the single (non-degenerate) measurement by the maximal DSD that is the join of their eigenspace DSDs.

In addition to being able to naturally represent measurement, the quantum logic of DSDs in useful for quite different reasons. There is a pedagogical model of quantum mechanics using vector spaces over $\mathbb{Z}_2$, called "quantum mechanics over sets" (QM/Sets) ([12], [14]), whose probability calculus is a non-commutative version of the classical Laplace-Boole finite probability theory with real-valued random variables. Such real-valued random variables on a finite sample space U cannot be represented or "internalized" as operators on $\mathbb{Z}_2^{|U|}$–but they *can be represented* by DSDs on $\mathbb{Z}_2^{|U|}$. This allows the quantum logic of DSDs' treatment of measurement in QM to be reproduced in an appropriate form in the pedagogical model of QM/Sets, and that in turn allows simplified pedagogical versions of quantum results such as the two-slit experiment, the indeterminacy principle, Bell's Theorem, and so forth.

In the remainder of this introductory treatment of quantum partition logic, we will focus on this pedagogical model of QM/Sets using vector spaces over $\mathbb{Z}_2$–together with an Appendix on the combinatorics of DSDs over finite vector spaces over $GF(q)$ since QM/Sets uses the special case of $q = 2$.

4 Review of QM/Sets

4.1 Previous attempts to model QM over sets

QM/Sets is a pedagogical or "toy" model of quantum mechanics over sets where the quantum probability calculus is a non-commutative version of the ordinary Laplace-Boole finite logical probability theory ([22], [6]) and where the usual vector spaces over $\mathbb{C}$ for QM are replaced with vector spaces $\mathbb{Z}_2^n$ over $\mathbb{Z}_2$. Fix a basis for $\mathbb{Z}_2^n$ [i.e., a maximal DSD in $DSD(\mathbb{Z}_2^n)$] and that basis set is the sample space or outcome space for the Laplace-Boole finite probability calculus. But there are many incompatible basis sets for $\mathbb{Z}_2^n$ so, in that sense, the probability calculus of QM/Sets is a non-commutative version of the Laplace-Boole calculus.

Quantum mechanics over sets is a bare-bones (e.g., non-physical[2]) "logical skeleton" of QM with appropriate versions of spectral decomposition, the Dirac brackets, the norm, observable-attributes, the Born rule, commutators, and density matrices all in the simple setting of sets,[3] but that nevertheless provides models of characteristically quantum results (e.g., a QM/Sets version of the double-slit experiment [14]). In that manner, QM/Sets can serve not only as a pedagogical (or "toy") model of QM but perhaps as an engine to better elucidate QM itself by representing the quantum features in a simple setting.

There have been at least three previous attempts at developing a version of QM where the base field of $\mathbb{C}$ is replaced by $\mathbb{Z}_2$ ([28], [17], and [31]). Since there are no inner products in vector spaces over a finite field, the "trick" is how to define the brackets, the norm, and then the probability algorithm. All these previous attempts use the aspect of full QM that the bras are dual vectors so the brackets take their values in the base field of $\mathbb{Z}_2$. For instance, the Schumacher-Westmoreland model does "not make use of the idea of probability" [28, p. 919] and have instead only a modal interpretation (1 = possibility and 0 = impossibility). There is a fourth category-theoretic model where the objects are sets [1] but it also has the "brackets" taking only $0, 1$ values and thus has only a modal or "possibilistic" interpretation.

[2] In full QM, the DeBroglie relations connect mathematical notions such as frequency and wave-length to physical notions such as energy and momentum. QM/sets is "non-physical" in the sense that it is a sets-version of the pure mathematical framework of (finite-dimensional) QM without those direct physical connections.

[3] Given a basis set for $\mathbb{Z}_2^n$, each vector is expressed as a subset of the basis set.

4.2 The Yoga of transporting vector space structures

There is a method or "Yoga" to transport some structures from a vector space V over a field $\mathbb{K}$ to a vector space V' over a *different* field $\mathbb{K}'$. Select a basis set U for the source space V and then consider a structure on V that can be characterized in terms of the basis set U. Then apply the free vector space over the field $\mathbb{K}'$ construction to U to generate the target vector space V'. Since the source structure was defined in terms of the basis set U, it can be carried over or "transported" to V' via *its* basis set U.

This Yoga can be stated in rigorous terms using category theory ([24]; [3]). The construction of the free vector space over a field $\mathbb{K}$ is a functor from the category $Sets$ of sets and functions to the category $Vect_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$ and linear transformations. The functor will only be used here on finite sets where it takes a finite set U to the vector space $\mathbb{K}^U$. This paper is about direct-sum decompositions of a finite-dimensional vector space V. A DSD a set $\{V_i\}$ of disjoint subspaces so that the whole space V is their direct sum, or, in terms of category theory, V is the coproduct $V = \oplus V_i$ of the subspaces $\{V_i\}$. In the category $Sets$, a set $\{B_i\}$ of disjoint subsets of a set U is a set partition of U if $\cup B_i = U$, or, in terms of category theory, U is the coproduct of the disjoint subsets $\{B_i\}$. The free vector space over $\mathbb{K}$ functor is a left adjoint, "left adjoints preserve colimits" [3, p. 197], and coproducts are a special type of colimit. Hence the free vector space functor carries a set partition $\pi = \{B_i\}_{i=1,...,m}$ to the DSD $\left\{V_i = \mathbb{K}^{B_i}\right\}$ of $V = \mathbb{K}^U = \oplus \mathbb{K}^{B_i}$.

Now start with the structure of a DSD $\{V_i\}$ on $V \in Vect_{\mathbb{K}}$. What we previously called "characterizing the structure in terms of a basis set U" is rigorously interpreted to mean, in this case, finding a basis U and a partition $\{B_i\}$ on U so that the given DSD $\{V_i\}$ is the image of the free vector space functor, i.e., $V = \mathbb{K}^U = \oplus \mathbb{K}^{B_i} = \oplus V_i$. But then the free vector space functor over a different field $\mathbb{K}'$ can be applied to the same set partition $\{B_i\}$ of the set U to generate a DSD $\left\{V_i' = \mathbb{K}'^{B_i}\right\}$ of $V' = \mathbb{K}'^U$. That is how to rigorously describe "transporting" a set-based structure on a vector V over $\mathbb{K}$ to a vector space V' over a different field $\mathbb{K}'$.

To show that any given DSD $\{V_i\}$ of V is in the image of the free vector space over $\mathbb{K}$ functor, pick basis set B_i of V_i. The sets B_i are disjoint and since $\{V_i\}$ is a DSD, the union $U = \cup B_i$ is a basis for V so $V_i = \mathbb{K}^{B_i}$ and $V = \mathbb{K}^U = \oplus \mathbb{K}^{B_i}$.

This method is applied to the transporting of self-adjoint operators from $V = \mathbb{C}^n$ to $V' = \mathbb{Z}_2^n$ that motivates QM/Sets. A self-adjoint operator $F : \mathbb{C}^n \to \mathbb{C}^n$ has a basis $U = \{u_1, ..., u_n\}$ of orthonormal eigenvectors and it has real distinct eigenvalues $\{\phi_i\}_{j=1,...,m}$, so it defines the real eigenvalue function $f : U \to \mathbb{R}$ where for $u_j \in U$, $f(u_j)$ is one of the distinct eigenvalues $\{\phi_i\}_{i=1,...m}$. For each distinct eigenvalue ϕ_i, there is the eigenspace V_i of its eigenvectors and $\{V_i\}_{i=1,...,m}$ is a DSD on $V = \mathbb{C}^n$. The inverse-image $\pi = \left\{B_i = f^{-1}(\phi_i)\right\}_{i=1,...,m}$ of the eigenvalue function $f : U \to \mathbb{R}$ is a set partition on U.

Thus the set-based structure we have is the set U with a partition $\left\{B_i = f^{-1}(\phi_i)\right\}_i$ on U induced by a real-value function $f : U \to \mathbb{R}$ on U. That set-based structure is sufficient to reconstruct the DSD $\left\{V_i = \mathbb{C}^{B_i}\right\}_i$ on $V = \mathbb{C}^n \cong \mathbb{C}^U = \oplus \mathbb{C}^{B_i}$ as well as the original operator F. The operator F is defined on the basis U by $Fu_j = f(u_j) u_j$ for $j = 1, ..., n$. But it might be helpful to go through the categorical construction. Scalar multiplication in the vector space is given by the set function $\mathbb{C} \times \mathbb{C}^n \stackrel{\cdot}{\to} \mathbb{C}^n$. There is the injection of the generators function $\iota : U \to \mathbb{C}^U \cong \mathbb{C}^n$ and there is the function $f : U \to \mathbb{R} \subseteq \mathbb{C}$ so by the universal mapping property (UMP) of the product $\mathbb{C} \times \mathbb{C}^n$, we have the factor map $< f, \iota >: U \to \mathbb{C} \times \mathbb{C}^n$ and thus the composition $f \cdot \iota : U \stackrel{<f,\iota>}{\longrightarrow} \mathbb{C} \times \mathbb{C}^n \stackrel{\cdot}{\longrightarrow} \mathbb{C}^n$. Then we use the UMP of the *free* vector space over $\mathbb{C}$ functor.

$$\begin{array}{ccc} U & & \\ {\scriptstyle \iota}\downarrow & \searrow^{\varphi} & \\ \mathbb{C}^n & \stackrel{\exists! F}{\longrightarrow} & W \end{array}$$

UMP of free vector space functor

That UMP is that for any function $\varphi : U \to W$ from the set U to any vector space W over $\mathbb{C}$, there is a linear transformation $F : \mathbb{C}^n = \mathbb{C}^U \to W$ such that $F\iota = \varphi$. Taking $\varphi = f \cdot \iota$ and $W = \mathbb{C}^n$, there is a unique linear operator F on $\mathbb{C}^n$ such that for any $u_j \in U$, $F\iota(u_j)$ is the scalar multiple $f(u_j) \cdot \iota(u)_j$ or, where we write $u_j = \iota(u_j)$ (and scalar multiplication by juxtaposition), $Fu_j = f(u_j) u_j$ for $j = 1, ..., n$. That process of going from the function $f : U \to \mathbb{R}$ on a basis set U of $\mathbb{C}^U$ to an operator on $\mathbb{C}^U$ might be called *internalizing* the function $f : U \to \mathbb{R}$ in $\mathbb{C}^U$.

Given the set-based structure of a real-valued function $f : U \to \mathbb{R}$, which determines the set partition $\left\{f^{-1}(\phi_i)\right\}_{i=1,...,m}$ on U, we then apply the free vector space over $\mathbb{Z}_2$ functor to construct the vector space $\mathbb{Z}_2^U$. That vector space is more familiar in the form of the powerset $\wp(U) \cong \mathbb{Z}_2^U$ since each function $U \to \mathbb{Z}_2 = \{0, 1\}$ in $\mathbb{Z}_2^U$ is the characteristic function χ_S of a subset $S \in \wp(U)$. The free vector space functor $\mathbb{Z}_2^{(\,)}$ takes the coproduct $U = \cup_{i=1}^m f^{-1}(\phi_i)$ to the DSD $\left\{\wp\left(f^{-1}(\phi_i)\right)\right\}$ of $\wp(U)$. The attempt to internalize the real function $f : U \to \mathbb{R}$ would use the scalar multiplication function $\mathbb{Z}_2 \times \mathbb{Z}_2^U \stackrel{\cdot}{\to} \mathbb{Z}_2^U$ and would only work if f took values in $\mathbb{Z}_2 = \{0, 1\} \subseteq \mathbb{R}$ in which case f would be a characteristic function χ_S for some subset $S \in \wp(U)$. In that special case, the internalized operator would be the projection operator $P_S : \mathbb{Z}_2^U \to \mathbb{Z}_2^U$ which in terms of the basis U has the action $P_S(T) = S \cap T$ taking any subset $T \in \wp(U)$ to $S \cap T \in \wp(S)$.

Hence outside of characteristic functions, the real-valued functions $f : U \to \mathbb{R}$ cannot be internalized as operators on $\mathbb{Z}_2^U$. But that is fine since the idea of the model QM/Sets is that given a basis U of $\mathbb{Z}_2^n$, the quantum probability calculus will just be the classical finite probability calculus with the outcome set or sample space U where $f : U \to \mathbb{R}$ is a real-valued random variable. We have illustrated the transporting of set-based structures on $\mathbb{C}^n$ to $\mathbb{Z}_2^n$ using a basis set U, but in the stand-alone model QM/Sets, we cut the umbilical cord to $\mathbb{C}^n$ and work with any other basis U' of $\mathbb{Z}_2^n$ and real-valued random variables $g : U' \to \mathbb{R}$ on *that* sample space.

Other structures can be transported across the bridge from $\mathbb{C}^n$ to $\mathbb{Z}_2^n$. QM/Sets differs from the other four attempts to define some toy version of QM on sets by the treatment of the Dirac brackets. Starting with our orthonormal basis U on a finite-dimensional Hilbert space $\mathbb{C}^n$ (where the bracket is the inner product), we need to define the transported brackets applied to two subsets $S, T \subseteq U$ in $\wp(U)$. The two subsets define the vectors $\psi_S = \sum_{u \in S} |u\rangle$ and $\psi_T = \sum_{u \in T} |u\rangle$ in $\mathbb{C}^n$ which have the bracket value $\langle \psi_S | \psi_T \rangle = |S \cap T|$. Since that value is defined just in terms of the subsets $S, T \subseteq U$ as the cardinality of their overlap, that value can be transported to $\wp(U)$ as the real-valued basis-dependent brackets $\langle S|_U T\rangle = |S \cap T|$ (see below).

4.3 Laplace-Boole finite probability theory

Since our purpose is conceptual rather than mathematical, we will stick to the simplest case of finite probability theory with a finite sample space or outcome space $U = \{u_1, ..., u_n\}$ of n equiprobable outcomes and to finite dimensional QM.[4] The *events* are the subsets $S \subseteq U$, and the *probability* of an event S occurring in a trial is the ratio of the cardinalities: $\Pr(S) = \frac{|S|}{|U|}$. Given that a conditioning event $S \subseteq U$ occurs, the *conditional probability* that $T \subseteq U$ occurs is: $\Pr(T|S) = \frac{\Pr(T \cap S)}{\Pr(S)} = \frac{|T \cap S|}{|S|}$. The ordinary probability $\Pr(T)$ of an event T can be taken as the conditional probability with U as the conditioning event so all probabilities can be seen as conditional probabilities. Given a real-valued random variable $f : U \to \mathbb{R}$ on the elements of U, the *probability of observing a value r given an event S* is the conditional probability of the event $f^{-1}(r)$ given S:

$$\Pr(r|S) = \frac{\left|f^{-1}(r) \cap S\right|}{|S|}.$$

[4] The mathematics can be generalized to the case where each point u_j in the sample space has a probability p_j (when it is a basis set with point probabilities that is transported) but the simpler case of equiprobable points serves our conceptual purposes.

That is all the probability theory we will need here. Our first task is to show how the mathematics of finite probability theory can be recast using the mathematical notions of quantum mechanics with the base field of $\mathbb{Z}_2$.

4.4 Vector spaces over $\mathbb{Z}_2$

To show how classical Laplace-Boole finite probability theory can be recast as a quantum probability calculus, we use finite dimensional vector spaces over $\mathbb{Z}_2$. The power set $\wp(U)$ of $U = \{u_1, ..., u_n\}$ is a vector space over $\mathbb{Z}_2 = \{0, 1\}$, isomorphic to $\mathbb{Z}_2^n$, where the vector addition $S + T$ is the *symmetric difference* of subsets. That is, for $S, T \subseteq U$,

$$S + T = (S - T) \cup (T - S) = S \cup T - S \cap T$$

so the members of $S + T$ are the elements that are members of S or members of T but not members of both.

The U-*basis* in $\wp(U)$ is the set of singletons $\{u_1\}, \{u_2\}, ..., \{u_n\}$, i.e., the set $\{\{u_j\}\}_{j=1,...,n}$. In the context of $DSD(\mathbb{Z}_2^n)$, that basis set would correspond to the maximal element $\omega = \{U_j\}_{j=1,...,n}$ where U_j is the one-dimensional subspace $\{\emptyset, \{u_j\}\}$. A vector $S \in \wp(U)$ is specified in the U-basis as $S = \sum_{u_j \in S} \{u_j\}$ and it is characterized by its $\mathbb{Z}_2$-valued characteristic function $\chi_S : U \to \mathbb{Z}_2 \subseteq \mathbb{R}$ of coefficients since $S = \sum_{u_j \in U} \chi_S(u_j) \{u_j\}$.

Consider the simple case of $U = \{a, b, c\}$ where the U-basis is $\{a\}$, $\{b\}$, and $\{c\}$. The three subsets $\{a, b\}$, $\{b, c\}$, and $\{a, b, c\}$ also form a basis since:

$\{b, c\} + \{a, b, c\} = \{a\}$;

$\{b, c\} + \{a, b\} + \{a, b, c\} = \{b\}$; and

$\{a, b\} + \{a, b, c\} = \{c\}$.

These new basis vectors could be considered as the basis-singletons in another equicardinal sample space $U' = \{a', b', c'\}$ where $\{a'\}$, $\{b'\}$, and $\{c'\}$ refer to the same abstract vector as $\{a, b\}$, $\{b, c\}$, and $\{a, b, c\}$ respectively.

In the following *ket table*, each row is an abstract vector of $\mathbb{Z}_2^3$ expressed in the U-basis, the U'-basis, and a U''-basis.

$U = \{a, b, c\}$	$U' = \{a', b', c'\}$	$U'' = \{a'', b'', c''\}$
$\{a, b, c\}$	$\{c'\}$	$\{a'', b'', c''\}$
$\{a, b\}$	$\{a'\}$	$\{b''\}$
$\{b, c\}$	$\{b'\}$	$\{b'', c''\}$
$\{a, c\}$	$\{a', b'\}$	$\{c''\}$
$\{a\}$	$\{b', c'\}$	$\{a''\}$
$\{b\}$	$\{a', b', c'\}$	$\{a'', b''\}$
$\{c\}$	$\{a', c'\}$	$\{a'', c''\}$
$\emptyset$	$\emptyset$	$\emptyset$

Ket table giving a vector space isomorphism: $\mathbb{Z}_2^3 \cong \wp(U) \cong \wp(U') \cong \wp(U'')$ where row = ket.

In the Dirac notation [9], the *ket* $|\{a, c\}\rangle$ represents the abstract vector that is represented in the U-basis coordinates as $\{a, c\}$. A row of the ket table gives the different representations of the *same* ket in the different bases, e.g., $|\{a, c\}\rangle = |\{a', b'\}\rangle = |\{c''\}\rangle$.

4.5 The brackets and the norm

In a Hilbert space, the inner product is used to define the brackets $\langle v_i|v\rangle$ and the norm $\|v\| = \sqrt{\langle v|v\rangle}$ but there are no inner products in vector spaces over finite fields. The different attempts to develop a toy model of QM over a finite field ([28], [31], [17]) such as $\mathbb{Z}_2$ differ from this model in how they

address this problem. The treatment of the Dirac brackets and norm defined here is distinguished by the fact that the resulting probability calculus in QM/Sets is (a non-commutative version of) classical finite probability theory (instead of just a modal calculus with values 0 and 1).

For a singleton basis vector $\{u_j\} \subseteq U$, the (basis-dependent) *bra* $\langle \{u_j\}|_U : \wp(U) \to \mathbb{R}$ is defined by the *bracket*:

$$\langle \{u\}|_U S\rangle = \begin{cases} 1 \text{ if } u \in S \\ 0 \text{ if } u \notin S \end{cases} = |\{u_j\} \cap S| = \chi_S(u_j).$$

Note that the bra and the bracket is defined in terms of the U-basis and that is indicated by the U-subscript on the bra portion of the bracket. Then for $u_j, u_k \in U$, $\langle \{u_j\}|_U \{u_k\}\rangle = \chi_{\{u_k\}}(u_j) = \chi_{\{u_j\}}(u_k) = \delta_{jk}$ (the Kronecker delta function) which is the QM/Sets-version of $\langle v_j|v_k\rangle = \delta_{jk}$ for an orthonormal basis $\{|v_j\rangle\}$ of $\mathbb{C}^n$. The bracket linearly extends *in the natural numbers* $\mathbb{N} \subseteq \mathbb{R}$ to any two vectors $T, S \in \wp(U)$:[5]

$$\langle T|_U S\rangle = |T \cap S|.$$

This is the QM/Sets-version of the Dirac brackets in the mathematics of QM.

As noted above, this treatment of the brackets is motivated by the general heuristic for transporting basis-set-defined structures between vector spaces over different fields, e.g., from $\mathbb{C}^n$ to $\mathbb{Z}_2^n$. In both cases, the bracket gives a measure of the overlap or indistinctness of the two vectors.[6] The ket $|S\rangle$ is the same as the ket $|S'\rangle$ for some subset $S' \subseteq U'$ in another U'-basis, but when the bra $\langle \{u_j\}|_U$ is applied to the ket $|S\rangle = |S'\rangle$, then it is the subset $S \subseteq U$, not $S' \subseteq U'$, that comes outside the ket symbol $|\ \rangle$ in $\langle \{u_j\}|_U S\rangle = |\{u_j\} \cap S|$.[7] Heuristically, the bra $\langle T|_U$ can be thought of as a row-vector of zeros and ones expressed in the U-basis, and then the ket $|S\rangle$ is expressed as a column vector in the U-basis, and $\langle T|_U S\rangle$ is their dot product *computed in the reals*.

The *U-norm* $\|S\|_U : \wp(U) \to \mathbb{R}$ is defined, as usual, as the square root of the bracket:[8]

$$\|S\|_U = \sqrt{\langle S|_U S\rangle} = \sqrt{|S \cap S|} = \sqrt{|S|}$$

for $S \in \wp(U)$ which is the QM/Sets-version of the norm $\|\psi\| = \sqrt{\langle \psi|\psi\rangle}$ in ordinary QM. Hence $\|S\|_U^2 = |S|$ is the counting measure on $\wp(U)$. Note that a ket has to be expressed in the U-basis to apply the U-norm definition so, for example, $\|\{a'\}\|_U = \sqrt{2}$ since $|\{a'\}\rangle = |\{a, b\}\rangle$.

4.6 Numerical attributes, linear operators, and DSDs

In classical physics, the observables are numerical attributes, e.g., the assignment of a position and momentum to particles in phase space. One of the differences between classical and quantum physics is the replacement of these observable numerical attributes by linear operators associated with the

[5] Here $\langle T|_U S\rangle = |T \cap S|$ takes values in the natural numbers $\mathbb{N}$ outside the base field of $\mathbb{Z}_2$ just like, say, the Hamming distance function $d_H(T, S) = |T + S|$ on vector spaces over $\mathbb{Z}_2$ in coding theory. [25] Thus the "size of overlap" bra $\langle T|_U : \wp(U) \to \mathbb{N}$ is not to be confused with the dual ("parity of overlap") functional $\varphi_T = \sum_{u_j \in T} \varphi_{u_j} : \wp(U) \to \mathbb{Z}_2$ where $\varphi_{u_j}(\{u_k\}) = \delta_{jk}$ for $U = \{u_1, ..., u_n\}$.

[6] One possible misinterpretation of QM/Sets is to misinterpret the transporting method as an embedding $\mathbb{Z}_2^n \to \mathbb{C}^n$ defined by $\{u_j\} \longmapsto |u_j\rangle$ using a basis for each space. But such an embedding from a vector space over a field of finite characteristic to a vector space of characteristic zero cannot be linear. The repeated sum of a nonzero element in the domain space will eventually be 0 but its repeated nonzero image in the codomain space can never be 0. Indeed in QM/Sets, the brackets $\langle T|_U S\rangle = |T \cap S|$ for $T, T', S \subseteq U$ should be thought of *only* as a measure of the overlap since they are not even linear, e.g., $\langle T + T'|_U S\rangle \neq \langle T|_U S\rangle + \langle T'|_U S\rangle$ whenever $T \cap T' \neq \emptyset$.

[7] The term "$\{u_j\} \cap S'$" is not even defined in general since it is the intersection of subsets $\{u_j\} \subseteq U$ and $S' \subseteq U'$ of two different universe sets U and U'.

[8] We use the double-line notation $\|S\|_U$ for the U-norm of a set to distinguish it from the single-line notation $|S|$ for the cardinality of a set. We also use the double-line notation $\|\psi\|$ for the norm in QM although sometimes the single line notation $|\psi|$ is used elsewhere.

observables where the values of the observables appear as eigenvalues of the operators. But this difference may be smaller than it would seem at first since a numerical attribute $f : U \to \mathbb{R}$ can be recast into an operator-like format in QM/Sets where it determines direct-sum decomposition of "eigenspaces," and there is even a QM/Sets-analogue of spectral decomposition.

An observable, i.e., a self-adjoint operator, on a finite-dimensional Hilbert space V has a "home" basis set of orthonormal eigenvectors. Using the transport method, a real-valued attribute $f : U \to \mathbb{R}$ defined on $U = \{u_1, ..., u_n\}$ has the U-basis for $\wp(U) \cong \mathbb{Z}_2^n$ as a "home" basis set. The connection between the numerical attributes $f : U \to \mathbb{R}$ of QM/Sets and the self-adjoint operators of full QM can also be established by seeing the function f as being *like* an "operator" $f \restriction ()$ on $\wp(U)$ in that it is used to define a sets-version of an "eigenvalue" equation [where $f \restriction S$ is the *restriction* of f to $S \in \wp(U)$]. For any subset $S \in \wp(U)$, the definition of the equation is:

$$f \restriction S = \phi_i S \text{ holds} \equiv_{df} f \text{ is constant on the subset } S \text{ with the value } \phi_i.$$

This is the QM/Sets-version of an *eigenvalue equation* for arbitrary functions on a set $f : U \to \mathbb{R}$. Whenever S satisfies $f \restriction S = \phi_i S$ for some ϕ_i, then S is said to be an *eigenvector* (= "level set") in the vector space $\wp(U)$ of the numerical attribute $f : U \to \mathbb{R}$, and $\phi_i \in \mathbb{R}$ is the associated *eigenvalue* (= constant value on a level set). Each eigenvalue ϕ_i determines as usual an *eigenspace* $\wp\left(f^{-1}(\phi_i)\right)$ of its eigenvectors which is a subspace of the vector space $\wp(U) \cong \mathbb{Z}_2^n$. The whole space $\wp(U)$ can be expressed as usual as the direct sum of the eigenspaces: $\wp(U) = \oplus_{\phi_i \in f(U)} \wp\left(f^{-1}(\phi_i)\right)$ so $\left\{\wp\left(f^{-1}(\phi_i)\right)\right\}_{\phi_i \in f(U)}$ is a DSD in $DSD(\mathbb{Z}_2^n)$. Since $f : U \to \mathbb{R}$ does not define an operator $\mathbb{Z}_2^n \to \mathbb{Z}_2^n$ (unless it is a characteristic function), we see one of the reasons for developing quantum partition logic using DSDs.

Moreover, for distinct eigenvalues $\phi_i \neq \phi_{i'}$, any corresponding eigenvectors $S \in \wp\left(f^{-1}(\phi_i)\right)$ and $T \in \wp\left(f^{-1}(\phi_{i'})\right)$ are *orthogonal* in the sense that $\langle T|_U S\rangle = 0$. In general, for vectors $S, T \in \wp(U)$, orthogonality means zero overlap, i.e., disjointness.

The characteristic function $\chi_S : U \to \mathbb{R}$ for $S \subseteq U$ has the eigenvalues of 0 and 1 so it is a numerical attribute that *can* be "internalized" as a linear operator $S \cap () : \wp(U) \to \wp(U)$. Hence in this case, the "eigenvalue equation" $f \restriction T = \phi_i T$ for $f = \chi_S$ internalizes as an actual eigenvalue equation $S \cap T = \phi_i T$ for a linear[9] operator $S \cap ()$ with the resulting eigenvalues of 1 and 0, and with the resulting eigenspaces $\wp(S)$ and $\wp(S^c)$ (where S^c is the complement of S) and atomic DSD $\{\wp(S), \wp(S^c)\}$.

The characteristic attributes $\chi_S : U \to \mathbb{R}$ are characterized by the property that their value-wise product, i.e., $(\chi_S \bullet \chi_S)(u_j) = \chi_S(u_j)\chi_S(u_j)$, is equal to the attribute value $\chi_S(u_j)$, and that is reflected in the idempotency of the corresponding operators:

$$\wp(U) \overset{S\cap()}{\longrightarrow} \wp(U) \overset{S\cap()}{\longrightarrow} \wp(U) = \wp(U) \overset{S\cap()}{\longrightarrow} \wp(U).$$

Thus the operators $S \cap ()$ corresponding to the characteristic functions χ_S are *projection operators*.

The (maximum) eigenvectors $f^{-1}(r)$ for f, with ϕ_i in the *image* or *spectrum* $f(U) \subseteq \mathbb{R}$, span the set U, i.e., $U = \cup_{\phi_i \in f(U)} f^{-1}(\phi_i)$. Hence the attribute $f : U \to \mathbb{R}$ has a spectral decomposition in terms of its (projection-defining) characteristic functions:

$$f = \textstyle\sum_{i=1}^m \phi_i \chi_{f^{-1}(\phi_i)} : U \to \mathbb{R}$$
$$\textit{Spectral decomposition} \text{ of real-valued function } f : U \to \mathbb{R}$$

which is the QM/Sets-version of the spectral decomposition $F = \sum_{i=1}^m \phi_i P_i$ of a self-adjoint operator F in terms of the projection operators P_i for its eigenvalues ϕ_i.

[9] It should be noted that the projection operator $S \cap () : \wp(U) \to \wp(U)$ is not only idempotent but linear, i.e., $(S \cap T_1) + (S \cap T_2) = S \cap (T_1 + T_2)$. Indeed, this is the distributive law when $\wp(U)$ is interpreted as a Boolean ring with intersection as multiplication.

4.7 The Born Rule for measurement in QM and QM/Sets

An orthogonal decomposition of a finite set U is just a partition $\pi = \{B, ...\}$ of U since the blocks $B, B', ...$ are orthogonal (i.e., disjoint) and their union, which is a disjoint union, is U. Given such a disjoint-union decomposition of U, we have the:

$$\|U\|_U^2 = \sum_{B\in\pi} \|B\|_U^2$$
Pythagorean Theorem
for disjoint-union decompositions of sets.

An old question is: "why the squaring in the Born rule of QM?" A superposition state between certain definite orthogonal alternatives A and B, where the latter are represented by vectors $\overrightarrow{A}$ and $\overrightarrow{B}$, is represented by the vector sum $\overrightarrow{C} = \overrightarrow{A} + \overrightarrow{B}$. But what is the "strength," "intensity," or relative importance of the vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ in the vector sum $\overrightarrow{C}$? That question requires a *scalar* measure of strength or intensity. The magnitude or "length" given by the norm $\|\|$ does not answer the question since $\left\|\overrightarrow{A}\right\| + \left\|\overrightarrow{B}\right\| \neq \left\|\overrightarrow{C}\right\|$. But the Pythagorean Theorem shows that the norm-squared gives the scalar measure of "intensity" that answers the question: $\left\|\overrightarrow{A}\right\|^2 + \left\|\overrightarrow{B}\right\|^2 = \left\|\overrightarrow{C}\right\|^2$ in vector spaces over $\mathbb{Z}_2$ or over $\mathbb{C}$. And when the superposition state is reduced by a measurement, then the *probability* that the indefinite state will reduce to one of the definite alternatives is given by that relative scalar measure of the eigen-alternative's "strength" or "intensity"–and that is the Born Rule. In a slogan, Born is the off-spring of Pythagoras.

Given an observable-operator F in ordinary QM/$\mathbb{C}$ and a numerical attribute in QM/Sets, the corresponding Pythagorean Theorems for the complete sets of orthogonal projection operators are:

$$\|\psi\|^2 = \sum_i \|P_i(\psi)\|^2 \text{ and}$$
$$\|S\|_U^2 = \sum_i \left\|f^{-1}(\phi_i) \cap S\right\|_U^2 = \sum_i \left|f^{-1}(\phi_i) \cap S\right| = |S|.$$

Normalizing gives:

$$\sum_i \frac{\|P_i(\psi)\|^2}{\|\psi\|^2} = 1 \text{ and}$$
$$\sum_i \frac{\left\|f^{-1}(\phi_i)\cap S\right\|_U^2}{\|S\|_U^2} = \sum_i \frac{\left|f^{-1}(\phi_i)\cap S\right|}{|S|} = 1$$

so the non-negative summands can be interpreted as probabilities–which is the Born rule in QM and in QM/Sets.[10]

Here $\frac{\|P_i(\psi)\|^2}{\|\psi\|^2}$ is the quantum probability of getting ϕ_i in an F-measurement of ψ, while $\frac{\left\|f^{-1}(\phi_i)\cap S\right\|_U^2}{\|S\|_U^2} = \frac{\left|f^{-1}(\phi_i)\cap S\right|}{|S|}$ has the classical interpretation as the probability $\Pr(\phi_i|S)$ of the numerical attribute $f : U \to \mathbb{R}$ having the eigenvalue ϕ_i when "measuring" $S \subseteq U$. Thus the QM/Sets-version of the Born Rule is the perfectly ordinary Laplace-Boole rule for the conditional probability $\Pr(\phi_i|S) = \frac{\left|f^{-1}(\phi_i)\cap S\right|}{|S|}$, that given an event S on the sample space U, a random variable $f : U \to \mathbb{R}$ takes the value ϕ_i.

In QM/Sets, when the state S is being "measured" using the observable f where the probability $\Pr(\phi_i|S)$ of getting the eigenvalue ϕ_i is $\frac{\left\|f^{-1}(\phi_i r)\cap S\right\|_U^2}{\|S\|_U^2} = \frac{\left|f^{-1}(\phi_i)\cap S\right|}{|S|}$, the "damned quantum jump" (Schrödinger) goes from S by the projection operator $f^{-1}(\phi_i) \cap ()$ to the projected resultant state

[10] Note that there is no notion of a normalized vector in a vector space over $\mathbb{Z}_2$ (another consequence of the lack of an inner product). The normalization is, as it were, postponed to the probability algorithm which is computed in the reals. This "external" probability algorithm is "internalized" when $\mathbb{Z}_2$ is strengthened to $\mathbb{C}$ in going from QM/sets to full QM.

$f^{-1}(\phi_i) \cap S$ which is in the eigenspace $\wp\left(f^{-1}(\phi_i)\right)$ for that eigenvalue ϕ_i. The state resulting from the measurement represents a more-definite state $f^{-1}(\phi_i) \cap S$ that now has the definite f-value of ϕ_i–so a second measurement would yield the same eigenvalue ϕ_i with probability:

$$\Pr\left(\phi_i | f^{-1}(\phi_i) \cap S\right) = \frac{\left|f^{-1}(\phi_i) \cap \left[f^{-1}(\phi_i) \cap S\right]\right|}{|f^{-1}(\phi_i) \cap S|} = \frac{\left|f^{-1}(\phi_i) \cap S\right|}{|f^{-1}(\phi_i) \cap S|} = 1$$

and the same resulting vector $f^{-1}(\phi_i) \cap \left[f^{-1}(\phi_i) \cap S\right] = f^{-1}(\phi_i) \cap S$ using the idempotency of the projection operators.

This treatment of measurement in QM/Sets is just the set-version of the treatment of measurement in standard Dirac-von-Neumann QM.

4.8 Summary of QM/Sets and QM

The QM/set-versions of the corresponding QM notions are summarized in the following table for the finite U-basis of the $\mathbb{Z}_2$-vector space $\wp(U) \cong \mathbb{Z}_2^n$ and for a finite dimensional Hilbert space V.

QM/Sets over $\mathbb{Z}_2$	Standard QM over $\mathbb{C}$
Projections: $S \cap (\,) : \wp(U) \to \wp(U)$	$P : V \to V$ where $P^2 = P$
Spectral Decomposition: $f = \sum_i \phi_i \chi_{f^{-1}(\phi_i)}$	$F = \sum_i \phi_i P_i$
Brackets: $\langle S|_U T\rangle = |S \cap T|$ = overlap of $S, T \subseteq U$	$\langle \psi|\varphi\rangle$ = "overlap" of ψ and φ
Norm: $\|S\|_U = \sqrt{\langle S|_U S\rangle} = \sqrt{|S|}$ where $S \subseteq U$	$\|\psi\| = \sqrt{\langle \psi|\psi\rangle}$
Pythagoras: $\|S\|_U^2 = \sum_i \left\|f^{-1}(\phi_i) \cap S\right\|_U^2$	$\|\psi\|^2 = \sum_i \|P_i(\psi)\|^2$
Normalized: $\sum_i \frac{\left\|f^{-1}(\phi_i) \cap S\right\|_U^2}{\|S\|_U^2} = \sum_i \frac{\left|f^{-1}(\phi_i) \cap S\right|}{|S|} = 1$	$\sum_i \frac{\|P_i(\psi)\|^2}{\|\psi\|^2} = 1$
Born rule: $\Pr(\phi_i|S) = \frac{\left\|f^{-1}(\phi_i) \cap S\right\|_U^2}{\|S\|_U^2} = \frac{\left|f^{-1}(\phi_i) \cap S\right|}{|S|}$	$\Pr(\phi_i|\psi) = \frac{\|P_i(\psi)\|^2}{\|\psi\|^2}$

Probability calculus for QM/Sets over $\mathbb{Z}_2$ and for standard QM over $\mathbb{C}$

The word "logic" is thrown around quite loosely as in constant references to "the logic" of this and that. But there is a clear sense in which $\mathbb{Z}_2^n$ is "logical" since given a basis set U, it is isomorphic to the powerset $\wp(U)$ with the symmetric difference as the vector sum, and each ket in $\mathbb{Z}_2^n$ represented in the U-coordinates is just a subset of U. Hence the transporting of the structures of the observables and the probability algorithm from a Hilbert space $\mathbb{C}^n$ to $\mathbb{Z}_2^n$ can be seen as extracting the *logical skeleton* of QM in QM/Sets, and, in that sense, QM/Sets can also be seen as a "logic of QM."

5 Measurement in QM/Sets

5.1 Set partitions and DSDs in vector spaces over $\mathbb{Z}_2$

The language of sets, e.g., $S \subseteq U$, can be translated into the language of vector spaces over $\mathbb{Z}_2$, e.g. $S \in \wp(U) \cong \mathbb{Z}_2^{|U|}$. QM/Sets uses either language depending on the context.

- A set partition $\pi = \{B, ...\}$ on set U is a disjoint-union decomposition of $U = \cup_{B \in \pi} B$, and;
- the "vector space partition" $\{\wp(B)\}_{B \in \pi}$ is a direct-sum decomposition of the vector space $\wp(U) = \oplus_{B \in \pi} \wp(B)$.

In particular, a real-valued numerical attribute $f : U \to \mathbb{Z}$ defines both:

- a set partition of constant sets $\left\{f^{-1}(r)\right\}_{r \in f(U)}$, and

- a DSD of eigenspaces $\phi = \left\{\wp\left(f^{-1}(r)\right)\right\}_{r\in f(U)}$.

Given an attribute $f : U \to \mathbb{R}$ on a basis set U and an attribute $g : U' \to \mathbb{R}$ on a different (equicardinal) basis set U' for $\mathbb{Z}_2^n$, the two attributes (or "observables") are *compatible* if their corresponding DSDs $\phi = \left\{\wp\left(f^{-1}(r)\right)\right\}_{r\in f(U)}$ and $\gamma = \left\{\wp\left(g^{-1}(s)\right)\right\}_{s\in g(U')}$ are compatible, i.e., if the proto-join $\left\{\wp\left(f^{-1}(r)\right) \cap \wp\left(g^{-1}(s)\right)\right\}_{r\in f(U), s\in g(U')}$ spans the whole space. Then choosing a basis set for the nonzero subspaces $\wp\left(f^{-1}(r)\right) \cap \wp\left(g^{-1}(s)\right)$ yields a basis set U'' of "simultaneous eigenvectors" so that the two attributes can each be redefined on U'' so as to determine the same DSDs ϕ and γ. Hence we can always consider compatible attributes as being defined on the same basis set, which we can take as U.

Measurement will later be treated using density matrices, but we can begin with a simpler framework. In the correspondences between QM/Sets and QM, a block S in a partition on U [i.e., a vector $S \in \wp(U)$] corresponds to *pure* state in QM, and a partition $\pi = \{B, ...\}$ on U is the *mixed state* of orthogonal pure states B with the probabilities $p_B = \frac{|B|}{|U|}$. In QM, a measurement makes distinctions, i.e., makes alternatives distinguishable, and that turns a pure state into a mixture of probabilistic outcomes.

A measurement of an attribute $f : U \to \mathbb{R}$ in QM/Sets is the distinction-creating process that uses the set partition $\left\{f^{-1}(r)\right\}_{r\in f(U)}$ of U to decompose (or "decohere") a "pure state" $S \subseteq U$ into a "mixed state" set partition $\left\{f^{-1}(r) \cap S\right\}_{r\in f(U)}$ on S obtained by applying the decomposing operations $f^{-1}(r) \cap ()$ to S with the block $f^{-1}(r) \cap S$ having the probability $\Pr(r|S) = \frac{\left|f^{-1}(r)\cap S\right|}{|S|}$.

Alternatively this could be described in vector space terms. A measurement of an attribute $f : U \to \mathbb{R}$ in QM/Sets is the distinction-creating process that uses the DSD $\phi = \left\{\wp\left(f^{-1}(r)\right)\right\}_{r\in f(U)}$ to decompose a vector $S \in \wp(U)$ into the parts obtained by applying the projection operators $f^{-1}(r) \cap () : \wp(U) \to \wp(U)$ to the vector S with each part having the probability $\Pr(r|S) = \frac{\left\|f^{-1}(r)\cap S\right\|_U^2}{\|S\|_U^2} = \frac{\left|f^{-1}(r)\cap S\right|}{|S|}$.

Either way, this is just the pedagogical QM/Sets version of the full QM measurement of a given state ψ using an observable operator $F = \sum_\lambda \lambda P_\lambda$ where the state ψ projects to the parts $P_\lambda(\psi)$ which occur with the probabilities $\Pr(\lambda|\psi) = \frac{\|P_\lambda(\psi)\|^2}{\|\psi\|^2}$.

5.2 Weyl's anticipation of QM/Sets

The pedagogical model, QM/Sets, could be seen as a development of some of the hints in Hermann Weyl's expository writings about quantum mechanics. He called a partition a "grating" or "sieve"[11], and then considered *both* set partitions and vector space partitions (direct sum decompositions) as the respective types of gratings.[32, pp. 255-257] He started with a numerical attribute on a set, e.g., $f : U \to \mathbb{R}$ (in my notation), which defined the set partition or "grating" [32, p. 255] with blocks having the same attribute-value, e.g., $\left\{f^{-1}(r)\right\}_{r\in f(U)}$. Then he moved to the QM case where the universe set, e.g., $U = \{u_1, ..., u_n\}$, or "aggregate of n states has to be replaced by an n-dimensional Euclidean vector space" [32, p. 256].[12] The appropriate notion of a vector space partition or "grating" is a "splitting of the total vector space into mutually orthogonal subspaces" so that "each vector $\overrightarrow{x}$ splits into r component vectors lying in the several subspaces" [32, p. 256], i.e., a direct sum decomposition of the space. After referring to a partition as a "grating" or "sieve,"

[11] Arthur Eddington made a very early use of the sieve idea:

> In Einstein's theory of relativity the observer is a man who sets out in quest of truth armed with a measuring-rod. In quantum theory he sets out armed with a sieve. [10, p. 267]

This passage was quoted by Weyl [32, p. 255] in his treatment of gratings.

[12] In his expository popular writings, Weyl used a Euclidean space instead of a Hilbert space.

Weyl notes that "Measurement means application of a sieve or grating" [32, p. 259]. In QM/Sets, this "application" of the set-grating or set partition $\left\{f^{-1}(r)\right\}_{r\in f(U)}$ to the "pure" state S gives the "mixed state" set partition $\left\{f^{-1}(r)\cap S\right\}_{r\in f(U)}$of S. In terms of the DSD $\phi=\left\{\wp\left(f^{-1}(r)\right)\right\}_{r\in f(U)}$ determined by the attribute $f:U\to\mathbb{R}$, this is just decomposing the vector S into the disjoint parts $S=\sum_{r\in f(U)}f^{-1}(r)\cap S$ where $f^{-1}(r)\cap S\in\wp\left(f^{-1}(r)\right)$. By the "projection postulate," the state S projects to one of its parts $f^{-1}(r)\cap S$ with the probability $\Pr(r|S)=\left|f^{-1}(r)\cap S\right|/|S|$.

For some visual imagery of measurement, we might think of a grating or sieve as a series of regular-polygonal-shaped holes that might shape an indefinite blob of dough. For illustrative purposes, we say the blob of dough is the sum of the more definite shapes: $\bigcirc=\triangle+\square+....$ In a measurement, the blob of dough falls through one of the polygonal holes in the grating with equal probability and then takes on or "projects to" that shape.

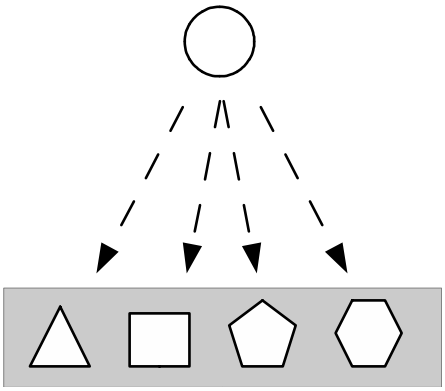

Figure 4: Measurement as randomly giving an indefinite blob of dough a definite polygonal shape.

5.3 Example of measurements

In the simple example illustrated below, we start at the one block or state of the indiscrete partition or blob which is $\{a,b,c\}$. A measurement uses some attribute that defines an inverse-image partition on $U=\{a,b,c\}$. In the case at hand, there are "essentially" four possible attributes that could be used to "measure" the state $\{a,b,c\}$ (since there are four partitions that refine the indiscrete partition).

For an example of a degenerate measurement, we choose an attribute with a non-discrete inverse-image partition such as the partition $\pi=\{\{a\},\{b,c\}\}$ which determines a non-maximal DSD $\{\wp(\{a\}),\wp(\{b,c\})\}$. Hence the attribute could just be the characteristic function $\chi_{\{b,c\}}$ with the two eigenspaces $\wp(\{a\})$ and $\wp(\{b,c\})$ and the two eigenvalues 0 and 1 respectively. Since the eigenspace $\wp\left(\chi^{-1}_{\{b,c\}}(1)\right)=\wp(\{b,c\})$ is not one dimensional, the eigenvalue of 1 is a QM/Sets-version of a *degenerate* eigenvalue. This attribute $\chi_{\{b,c\}}$ has four (non-zero) eigenvectors:

$$\chi_{\{b,c\}}\upharpoonright\{b,c\}=1\{b,c\},\ \chi_{\{b,c\}}\upharpoonright\{b\}=1\{b\},\ \chi_{\{b,c\}}\upharpoonright\{c\}=1\{c\},\text{ and }\chi_{\{b,c\}}\upharpoonright\{a\}=0\{a\}.$$

The "measuring apparatus" makes distinctions by joining the attribute's inverse-image partition

$$\chi^{-1}_{\{b,c\}}=\left\{\chi^{-1}_{\{b,c\}}(1),\chi^{-1}_{\{b,c\}}(0)\right\}=\{\{b,c\},\{a\}\}$$

with the pure state representing the indefinite entity $U=\{a,b,c\}$. The action on the pure state is:

$$U\to\{U\}\vee\chi^{-1}_{\{b,c\}}=\chi^{-1}_{\{b,c\}}=\{\{b,c\},\{a\}\}.$$

The measurement of that attribute returns one of the eigenvalues with the probabilities:

$$\Pr(0|U) = \frac{|\{a\} \cap \{a,b,c\}|}{|\{a,b,c\}|} = \frac{1}{3} \text{ and } \Pr(1|U) = \frac{|\{b,c\} \cap \{a,b,c\}|}{|\{a,b,c\}|} = \frac{2}{3}.$$

Suppose it returns the eigenvalue 1. Then the indefinite entity $\{a,b,c\}$ reduces to the projected eigenstate $\chi^{-1}_{\{b,c\}}(1) \cap \{a,b,c\} = \{b,c\}$ for that eigenvalue [7, p. 221].

Since this is a degenerate result (i.e., the eigenspace $\wp\left(\chi^{-1}_{\{b,c\}}(1)\right) = \wp(\{b,c\})$ doesn't have dimension one), another measurement is needed to make more distinctions. Measurements by attributes, such as $\chi_{\{a,b\}}$ or $\chi_{\{a,c\}}$, that give either of the other two partitions, $\{\{a,b\},\{c\}\}$ or $\{\{b\},\{a,c\}\}$ as inverse images, would suffice to distinguish $\{b,c\}$ into $\{b\}$ or $\{c\}$. Then either attribute together with the attribute $\chi_{\{b,c\}}$ would form a *Complete Set of Compatible Attributes* or CSCA (i.e., the QM/Sets-version of Dirac's Complete Set of Commuting Operators or CSCO), where *complete* means that the join of the attributes' inverse-image partitions gives the discrete partition and where *compatible* means that all the attributes can be taken as defined on the same set of (simultaneous) basis eigenvectors, e.g., the U-basis.

Taking, for example, the other attribute as $\chi_{\{a,b\}}$, the join of the two attributes' partitions is discrete:

$$\chi^{-1}_{\{b,c\}} \vee \chi^{-1}_{\{a,b\}} = \{\{a\},\{b,c\}\} \vee \{\{a,b\},\{c\}\} = \{\{a\},\{b\},\{c\}\} = \mathbf{1}.$$

Hence all the eigenstate singletons can be characterized by the ordered pairs of the eigenvalues of these two attributes: $\{a\} = |0,1\rangle$, $\{b\} = |1,1\rangle$, and $\{c\} = |1,0\rangle$ (using Dirac's ket-notation to give the ordered pairs and listing the eigenvalues of $\chi_{\{b,c\}}$ first on the left).

The second projective measurement of the indefinite entity $\{b,c\}$ using the attribute $\chi_{\{a,b\}}$ with the inverse-image partition $\chi^{-1}_{\{a,b\}} = \{\{a,b\},\{c\}\}$ would have the pure-to-mixed state action:

$$\{b,c\} \to \left\{\{b,c\} \cap \chi_{\{a,b\}}(1), \{b,c\} \cap \chi_{\{a,b\}}(0)\right\} = \{\{b\},\{c\}\}.$$

The distinction-making measurement would cause the indefinite entity $\{b,c\}$ to turn into one of the definite entities of $\{b\}$ or $\{c\}$ with the probabilities:

$$\Pr(1|\{b,c\}) = \frac{|\{a,b\} \cap \{b,c\}|}{|\{b,c\}|} = \frac{1}{2} \text{ and } \Pr(0|\{b,c\}) = \frac{|\{c\} \cap \{b,c\}|}{|\{b,c\}|} = \frac{1}{2}.$$

If the measured eigenvalue is 0, then the state $\{b,c\}$ projects to $\chi^{-1}_{\{a,b\}}(0) \cap \{b,c\} = \{c\}$ as pictured below.

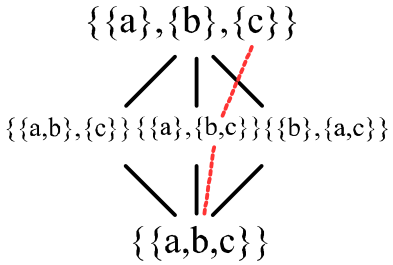

Figure 5: Degenerate measurement

The two projective measurements of $\{a,b,c\}$ using the complete set of compatible (e.g., both defined on U) attributes $\chi_{\{b,c\}}$ and $\chi_{\{a,b\}}$ produced the respective eigenvalues 1 and 0 so the resulting eigenstate was characterized by the eigenket $|1,0\rangle = \{c\}$.

Again, this is all analogous to standard Dirac-von-Neumann quantum mechanics.

5.4 Density matrices in QM/Sets

The previous treatment of the role of partitions in measurement can be restated using density matrices over the reals. Given a partition $\pi = \{B, ...\}$ on $U = \{u_1, ..., u_n\}$, the blocks $B \in \pi$ can be thought of as (nonoverlapping or "orthogonal") "pure states" where the "state" B occurs with the probability $p_B = \frac{|B|}{|U|}$. Then we can transport the usual procedure for forming the density matrix $\rho(\pi)$ for the "orthogonal pure states" B with the probabilities p_B. The "pure state" B normalized in the reals to length 1 is represented by the column vector $|B\rangle_1 = \frac{1}{\sqrt{|B|}} [\chi_B(u_1), ..., \chi_B(u_n)]^t$(where $[\,]^t$ indicates the transpose). Then the *density matrix* $\rho(B)$ *for the pure state* $B \subseteq U$ is then (calculating in the reals):

$$\rho(B) = |B\rangle_1 (|B\rangle_1)^t = \frac{1}{|B|} \begin{bmatrix} \chi_B(u_1) \\ \chi_B(u_2) \\ \vdots \\ \chi_B(u_n) \end{bmatrix} [\chi_B(u_1), ..., \chi_B(u_n)]$$

$$= \frac{1}{|B|} \begin{bmatrix} \chi_B(u_1) & \chi_B(u_1)\chi_B(u_2) & \cdots & \chi_B(u_1)\chi_B(u_n) \\ \chi_B(u_2)\chi_B(u_1) & \chi_B(u_2) & \cdots & \chi_B(u_2)\chi_B(u_n) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_B(u_n)\chi_B(u_1) & \chi_B(u_n)\chi_B(u_2) & \cdots & \chi_B(u_n) \end{bmatrix}.$$

For instance if $U = \{u_1, u_2, u_3\}$, then for the blocks in the partition $\pi = \{\{u_1, u_2\}, \{u_3\}\}$:

$$\rho(\{u_1, u_2\}) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \rho(\{u_3\}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then the "mixed state" *density matrix* $\rho(\pi)$ *of the partition* π is the weighted sum:

$$\rho(\pi) = \textstyle\sum_{B \in \pi} p_B \rho(B).$$

In the example, this is:

$$\rho(\pi) = \frac{2}{3} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

In partition logic [13], given a set partition $\pi = \{B, B', ...\}$ on a universe set U, an ordered pair $(u, u') \in U \times U$ is called a *distinction* or *dit* of π if the elements are in different blocks of π, and the set of all distinctions is the *dit set* $\mathrm{dit}(\pi)$. An ordered pair (u, u') is called an *indistinction* or *indit* of π if the two elements are in the same block of π, and the set of all indistinctions is the indit set $\mathrm{indit}(\pi)$. A partition π has an associated binary equivalence relation which is its indit set $\mathrm{indit}(\pi) \subseteq U \times U$, and an associated partition relation or apartness relation which is the complementary dit set $\mathrm{dit}(\pi) = U \times U - \mathrm{indit}(\pi)$. The density matrix $\rho(\pi)$ of the partition can then be directly interpreted in terms of its indit set:

$$\rho_{jk}(\pi) = \begin{cases} \frac{1}{|U|} \text{ if } (u_j, u_k) \in \mathrm{indit}(\pi) \\ 0 \text{ if } (u_j, u_k) \notin \mathrm{indit}(\pi) \end{cases}.$$

All the entries are real "amplitudes" whose squares are the two-draw probabilities of drawing a pair of elements from U (with replacement) that is an indistinction of π. Like in the full quantum case, the non-zero entries of the density matrix $\rho_{jk}(\pi) = \sqrt{\frac{1}{|U|}\frac{1}{|U|}} = \frac{1}{|U|}$ are the "coherences" [7, p. 302]

which indicate that u_j and u_k "cohere" together in a block or "pure state" of the partition, i.e., for some block $B \in \pi$, $u_j, u_k \in B$. Since the ordered pairs (u_j, u_j) in the diagonal $\Delta \subseteq U \times U$ are always indits of any partition, the diagonal entries in $\rho(\pi)$ are always $\frac{1}{|U|}$.

Combinatorial theory gives a natural way to define the same density matrix $\rho(\pi)$ of a partition π. A binary relation $R \subseteq U \times U$ on $U = \{u_1, ..., u_n\}$ can be represented by an $n \times n$ *incidence matrix* $I(R)$ where

$$I(R)_{jk} = \begin{cases} 1 \text{ if } (u_j, u_k) \in R \\ 0 \text{ if } (u_j, u_k) \notin R. \end{cases}$$

Taking R as the equivalence relation $\text{indit}(\pi)$ associated with a partition π, the density matrix $\rho(\pi)$ defined above is just the incidence matrix $I(\text{indit}(\pi))$ rescaled to be of trace 1 (i.e., sum of diagonal entries is 1):

$$\rho(\pi) = \tfrac{1}{|U|} I(\text{indit}(\pi)).$$

5.5 Measurement in QM/Sets using density matrices

If the subsets $T \in \wp(U)$ are represented by the n-ary column vectors $[\chi_T(u_1), ..., \chi_T(u_n)]^t$, then the action of the projection operator $B \cap (\,) : \wp(U) \to \wp(U)$ is represented in the U-basis by the $n \times n$ diagonal matrix P_B where the diagonal entries are:

$$(P_B)_{jj} = \begin{cases} 1 \text{ if } u_j \in B \\ 0 \text{ if } u_j \notin B \end{cases} = \chi_B(u_j)$$

which is idempotent, $P_B^2 = P_B$, and symmetric, $P_B^t = P_B$. For any state $S \in \wp(U)$, the trace (sum of diagonal entries) of $P_B \rho(S)$ is:

$$\text{tr}[P_B \rho(S)] = \tfrac{1}{|S|} \sum_{j=1}^n \chi_S(u_j) \chi_B(u_j) = \tfrac{|B \cap S|}{|S|} = \Pr(B|S)$$

so given $f : U \to \mathbb{R}$,

$$\Pr(r|S) = \tfrac{\left|f^{-1}(r) \cap S\right|}{|S|} = \text{tr}\left[P_{f^{-1}(r)} \rho(S)\right]$$

This is the QM/Sets version of the usual result: $\Pr(\lambda|\psi) = \frac{\|P_\lambda(\psi)\|^2}{\|\psi\|^2} = \text{tr}[P_\lambda \rho(\psi)]$.

Given a state S, the measurement by the f-attribute DSD $\left\{\wp\left(f^{-1}(r)\right)\right\}_{r \in f(U)}$ projects S to the state $f^{-1}(r) \cap S$ with the probability $\text{tr}[P_{f^{-1}(r)} \rho(S)] = \frac{|f^{-1}(r) \cap S|}{|S|} = \Pr(r|S)$. We need to convert this into the language of density matrices. Starting with the pure state S as a normalized column vector $|S\rangle_1$, the subset $f^{-1}(r) \cap S$ resulting from that projection is the column vector $P_{f^{-1}(r)} |S\rangle$. To calculate the corresponding density matrix we must first normalize the column vector $P_{f^{-1}(r)} |S\rangle$ by dividing through by $\sqrt{|f^{-1}(r) \cap S|}$ (where nonzero). But the normalizing factor to compute $\rho(S)$ was $\sqrt{|S|}$, i.e., $|S\rangle_1 = \frac{1}{\sqrt{|S|}} |S\rangle$. Since $\text{tr}\left[P_{f^{-1}(r)} \rho(S)\right] = \frac{\left|f^{-1}(r) \cap S\right|}{|S|}$, the normalized version of $P_{f^{-1}(r)} |S\rangle$ is:

$$\tfrac{1}{\sqrt{|f^{-1}(r) \cap S|}} P_{f^{-1}(r)} |S\rangle = \tfrac{1}{\sqrt{|f^{-1}(r) \cap S|}} P_{f^{-1}(r)} \sqrt{|S|}\, |S\rangle_1 = \tfrac{1}{\sqrt{\text{tr}\left[P_{f^{-1}(r)} \rho(S)\right]}} P_{f^{-1}(r)} |S\rangle_1.$$

Hence the density matrix corresponding to the projected state $P_{f^{-1}(r)} |S\rangle$ is:

$$\tfrac{1}{\text{tr}\left[P_{f^{-1}(r)} \rho(S)\right]} \left(P_{f^{-1}(r)} |S\rangle_1\right) \left(P_{f^{-1}(r)} |S\rangle_1\right)^t$$
$$= \tfrac{1}{\text{tr}\left[P_{f^{-1}(r)} \rho(S)\right]} P_{f^{-1}(r)} |S\rangle_1 \left(|S\rangle_1\right)^t \left(P_{f^{-1}(r)}\right)^t = \tfrac{P_{f^{-1}(r)} \rho(S) P_{f^{-1}(r)}}{\text{tr}\left[P_{f^{-1}(r)} \rho(S)\right]}.$$

This might be illustrated by using the second part of the above degenerate measurement where $f = \chi_{\{a,b\}}$ and $S = \{b, c\}$. Then the density matrix is:

$$\rho(\{b,c\}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \text{ and } \chi_{\{a,b\}}^{-1}(1) = f^{-1}(1) = \{a,b\} \text{ so } P_{f^{-1}(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$P_{f^{-1}(1)}\rho(\{b,c\}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_{f^{-1}(1)}\rho(\{b,c\})P_{f^{-1}(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $\text{tr}\left[P_{f^{-1}(1)}\rho(\{b,c\})\right] = \frac{1}{2}$, the resultant state from that projection is:

$$\frac{P_{f^{-1}(1)}\rho(\{b,c\})P_{f^{-1}(1)}}{\text{tr}\left[P_{f^{-1}(1)}\rho(\{b,c\})\right]} = \frac{1}{1/2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with the density matrix $\rho(\{b\})$ where $\{b\} = f^{-1}(1) \cap \{b,c\} = \{a,b\} \cap \{b,c\}$. For the other eigenvalue of 0, we have

$$P_{f^{-1}(0)}\rho(\{b,c\})P_{f^{-1}(0)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

and $\text{tr}\left[P_{f^{-1}(0)}\rho(\{b,c\})\right] = \frac{1}{2}$ so

$$\frac{P_{f^{-1}(0)}\rho(\{b,c\})P_{f^{-1}(0)}}{\text{tr}\left[P_{f^{-1}(0)}\rho(\{b,c\})\right]} = \frac{1}{1/2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is the density matrix for the pure state $\{c\} = f^{-1}(0) \cap \{b,c\} = \{c\} \cap \{b,c\}$.

The final formula for the post-measurement mixed state $\hat{\rho}(S)$ would weigh the projected states by their probability, so we have:

$$\hat{\rho}(S) = \sum_{r \in f(U)} \Pr(r|S) \frac{P_{f^{-1}(r)}\rho(S)P_{f^{-1}(r)}}{\text{tr}\left[P_{f^{-1}(r)}\rho(S)\right]}$$

$$= \sum_{r \in f(U)} \text{tr}\left[P_{f^{-1}(r)}\rho(S)\right] \frac{P_{f^{-1}(r)}\rho(S)P_{f^{-1}(r)}}{\text{tr}\left[P_{f^{-1}(r)}\rho(S)\right]} = \sum_{r \in f(U)} P_{f^{-1}(r)}\rho(S)P_{f^{-1}(r)}.$$

Thus the action of the measurement is:

$$\rho(S) \longrightarrow \hat{\rho}(S) = \sum_{r \in f(U)} P_{f^{-1}(r)}\rho(S)P_{f^{-1}(r)}$$

Measurement of S using f-attribute in density matrix form.

This result is just the "transported" QM/Sets version of the description of measurement in full QM. Consider the projective measurement using a self-adjoint operator F on V with the DSD $\{V_\lambda\}$ of eigenspaces and the projections to the eigenspaces $P_\lambda : V \to V_\lambda$. The measurement of a normalized pure state $|\psi\rangle$ results in the state $P_\lambda|\psi\rangle$ with the probability $p_\lambda = \text{tr}\left[P_\lambda\rho(\psi)\right] = \Pr(\lambda|\psi)$ where $\rho(\psi) = |\psi\rangle\langle\psi|$. The projected resultant state $P_\lambda|\psi\rangle$ has the density matrix $\frac{P_\lambda|\psi\rangle\langle\psi|P_\lambda}{\text{tr}[P_\lambda\rho(\psi)]} = \frac{P_\lambda\rho(\psi)P_\lambda}{\text{tr}[P_\lambda\rho(\psi)]}$ so the mixed state describing the probabilistic results of the measurement is [26, p. 101 or p. 515]:

$$\hat{\rho}(\psi) = \sum_\lambda p_\lambda \frac{P_\lambda \rho(\psi) P_\lambda}{\mathrm{tr}[P_\lambda \rho(\psi)]} = \sum_\lambda \mathrm{tr}\left[P_\lambda \rho(\psi)\right] \frac{P_\lambda \rho(\psi) P_\lambda}{\mathrm{tr}[P_\lambda \rho(\psi)]} = \sum_\lambda P_\lambda \rho(\psi) P_\lambda.$$

Thus we see how the density matrix treatment of measurement in QM/Sets is just a sets-version of the density matrix treatment of projective measurement in standard Dirac-von-Neumann QM:

$$\rho(\psi) \longmapsto \hat{\rho}(\psi) = \sum_\lambda P_\lambda \rho(\psi) P_\lambda.$$

6 Final remarks

The usual version of quantum logic can be viewed as the extension of the Boolean logic of subsets to the logic of subspaces of a vector space (specifically, closed subspaces of a Hilbert space). Since the notion of a set partition (or equivalence relation or quotient set) is the category-theoretic dual to the notion of a subset, the logic of set partitions is, in that sense, dual to the Boolean logic of subsets. Hence there is a dual form of quantum logic that can be viewed as the extension of the logic of set partitions to the logic of direct-sum decompositions of a vector space (specifically, a Hilbert space).

The usual quantum logic of subspaces can be viewed as focusing on propositions, i.e., the proposition that a given state vector is in a subspace, and projection operators. Since a self-adjoint operator (observable) determines a direct-sum decomposition (losing only the specific numerical eigenvalues), the quantum logic of DSDs can be viewed as focusing on observables or self-adjoint operators (abstracted from specific eigenvalues)–with the two projection operators associated with a proposition and its negation included in the form of the atomic DSDs (and the blob). Unlike the quantum logic of subspaces, the logic of DSDs (vector space partitions) provides the natural setting to model measurement since, as Weyl put it: "Measurement means application of a sieve or grating" [32, p. 259].

In this introductory treatment, we have focused on first developing that quantum logic of direct-sum decompositions of a general finite-dimensional vector space V over a field $\mathbb{K}$. Then we turned to the special case of the quantum logic of direct-sum decompositions of a finite vector space over $\mathbb{Z}_2$ which applies to the pedagogical model of quantum mechanics over sets, QM/Sets, the logical skeleton of QM. In the Appendix, we give an elementary treatment of combinatorics of DSDs of finite vector spaces over finite fields with q elements, with some numerical calculations and examples for the special case of QM/Sets where $q = 2$.

7 Appendix: Counting DSDs of finite vector spaces

7.1 Reviewing q-analogs: From sets to vector spaces

The theory of q-analogs shows how many "classical" combinatorial formulas for finite sets can be extended to finite vector spaces where q is the cardinality of the finite base field $GF(q)$, i.e., $q = p^n$, a power of a prime.

The natural number n is replaced by:

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + ... + q^{n-1}$$

so as $q \to 1$, then $[n]_q \to n$ in the passage from vector spaces to sets. The factorial $n!$ is replaced, in the q-analog

$$[n]_q! = [n]_q [n-1]_q ... [1]_q$$

where $[1]_q = [0]_q = 1$.

To obtain the Gaussian binomial coefficients we calculate with ordered bases of a k-dimensional subspace of an n-dimensional vector space over the finite field $GF(q)$ with q elements. There are q^n

elements in the space so the first choice for a basis vector has $(q^n - 1)$ (excluding 0) possibilities, and since that vector generated a subspace of dimension q, the choice of the second basis vector is limited to $(q^n - q)$ elements, and so forth. Thus:

$$\begin{gathered}(q^n-1)(q^n-q)\left(q^n-q^2\right)...\left(q^n-q^{k-1}\right)\\ =(q^n-1)q^1\left(q^{n-1}-1\right)q^2\left(q^{n-1}-1\right)...q^{k-1}\left(q^{n-k+1}-1\right)\\ =\frac{[n]_q!}{[n-k]_q!}q^{(1+2+...+(k-1))}=\frac{[n]_q!}{[n-k]_q!}q^{k(k-1)/2}=\frac{[n]_q!}{[n-k]_q!}q^{\binom{k}{2}}.\end{gathered}$$

Number of ordered bases for a k-dimensional subspace in an n-dimensional space.

But for a space of dimension k, the number of ordered bases are:

$$\begin{gathered}\left(q^k-1\right)\left(q^k-q\right)\left(q^k-q^2\right)...\left(q^k-q^{k-1}\right)\\ =\left(q^k-1\right)q^1\left(q^{k-1}-1\right)q^2\left(q^{k-1}-1\right)...q^{k-1}\left(q^{k-k+1}-1\right)\\ =[k]_q!q^{k(k-1)/2}=[k]_q!q^{\binom{k}{2}}\end{gathered}$$

Number of ordered bases for a k-dimensional space.

Thus the number of subspaces of dimension k is the ratio:

$$\binom{n}{k}_q=\frac{[n]_q!q^{k(k-1)/2}}{[n-k]_q![k]_q!q^{k(k-1)/2}}=\frac{[n]_q!}{[n-k]_q![k]_q!}$$

Gaussian binomial coefficient

where $\binom{n}{k}_q \to \binom{n}{k}$ as $q \to 1$, i.e., the number of k-dimensional subspaces $\to$ number of k-element subsets. Many classical identities for binomial coefficients generalize to Gaussian binomial coefficients [15].

7.2 Counting partitions of finite sets and vector spaces

7.2.1 The direct formulas for counting partitions of finite sets

Using sophisticated techniques, the direct-sum decompositions of a finite vector space over $GF(q)$ have been enumerated in the sense of giving the exponential generating function for the numbers ([4]; [29]). Our goal is to derive, by elementary methods, the formulas to enumerate these and some related direct-sum decompositions.

Two subspaces of a vector space are said to be *disjoint* if their intersection is the zero subspace 0. A *direct-sum decomposition* (DSD) of a finite-dimensional vector space V over a base field $\mathbb{K}$ is a set of (nonzero) pair-wise disjoint subspaces, called *blocks* (as with partitions), $\{V_i\}_{i=1,...,m}$ that span the space. Then each vector $v \in V$ has a unique expression $v = \sum_{i=1}^{m} v_i$ with each $v_i \in V_i$. Since a direct-sum decomposition can be seen as the vector-space version of a set partition, we begin with counting the number of partitions on a set.

Each set partition $\{B_1, ..., B_m\}$ of an n-element set has a "type" or "signature" number partition giving the cardinality of the blocks where they might be presented in nondecreasing order which we can assume to be: $(|B_1|, |B_2|, ..., |B_m|)$ which is a number partition of n. For our purposes, there is another way to present number partitions, the *part-count representation*, where a_k is the number of times the integer k occurs in the number partition (and $a_k = 0$ if k does not appear) so that:

$$a_1 1 + a_2 2 + ... + a_n n = \sum_{k=1}^{n} a_k k = n.$$

Part-count representation of number partitions keeping track of repetitions.

Each set partition $\{B_1, ..., B_m\}$ of an n-element set has a part-count signature $a_1, ..., a_n$, and then there is a "classical" formula for the number of partitions with that signature ([2, p. 215]; [18, p. 427]).

Proposition 4 *The number of set partitions for the given signature:* $a_1, ..., a_n$ *where* $\sum_{k=1}^{n} a_k k = n$ *is:*

$$\frac{n!}{a_1!a_2!...a_n!(1!)^{a_1}(2!)^{a_2}...(n!)^{a_n}}.$$

.

Proof: Suppose we count the number of set partitions $\{B_1, ..., B_m\}$ of an n-element set when the blocks have the given cardinalities: $n_j = |B_j|$ for $j = 1, ..., m$ so $\sum_{j=1}^{m} n_j = n$. The first block B_1 can be chosen in $\binom{n}{n_1}$ ways, the second block in $\binom{n-n_1}{n_2}$ ways and so forth, so the total number of ways is:

$$\begin{aligned}\binom{n}{n_1}\binom{n-n_1}{n_2}...\binom{n-n_1-...-n_{m-1}}{n_m} &= \frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}...\frac{(n-n_1-...-n_{m-1})!}{n_m!(n-n_1-...-n_m)!} \\ &= \frac{n!}{n_1!...n_m!} = \binom{n}{n_1,...,n_m}\end{aligned}$$

the multinomial coefficient. This formula can then be restated in terms of the part-count signature $a_1, ..., a_n$ where $\sum_{k=1}^{n} a_k k = n$ as: $\frac{n!}{(1!)^{a_1}(2!)^{a_2}...(n!)^{a_n}}$. But that overcounts since the a_k blocks of size k can be permuted without changing the partition's signature so one needs to divide by $a_k!$ for $k = 1, ..., n$ which yields the formula for the number of partitions with that signature. □

The *Stirling numbers* $S(n, m)$ *of the second kind* are the number of partitions of an n-element set with m blocks. Since $\sum_{k=1}^{n} a_k = m$ is the number of blocks, the direct formula (as opposed to a recurrence formula) is:

$$S(n, m) = \sum_{\substack{1a_1+2a_2+...+na_n=n \\ a_1+a_2+...+a_n=m}} \frac{n!}{a_1!a_2!...a_n!(1!)^{a_1}(2!)^{a_2}...(n!)^{a_n}}$$

Direct formula for Stirling numbers of the second kind.

The *Bell numbers* $B(n)$ are the total number of partitions on an n-element set so the direct formula is:

$$B(n) = \sum_{m=1}^{n} S(n, m) = \sum_{1a_1+2a_2+...+na_n=n} \frac{n!}{a_1!a_2!...a_n!(1!)^{a_1}(2!)^{a_2}...(n!)^{a_n}}$$

Direct formula for total number of partitions of an n-element set.

7.2.2 The direct formulas for counting DSDs of finite vector spaces

Each DSD $\pi = \{V_i\}_{i=1,...,m}$ of a finite vector space of dimension n also determines a number partition of n using the dimensions $n_i = \dim(V_i)$ in place of the set cardinalities, and thus each DSD also has a signature $a_1, ..., a_n$ where the subspaces are ordered by nondecreasing dimension and where $\sum_{k=1}^{n} a_k k = n$ and $\sum_{k=1}^{n} a_k = m$.

Proposition 5 *The number of DSDs of a vector space* V *of dimension* n *over* $GF(q)$ *with the part-count signature* $a_1, ..., a_n$ *is:*

$$\frac{1}{a_1!a_2!...a_n!}\frac{[n]_q!}{\left([1]_q!\right)^{a_1}...\left([n]_q!\right)^{a_n}} q^{\frac{1}{2}\left(n^2-\sum_k a_k k^2\right)}$$

Number of DSDs for the given signature $a_1, ..., a_n$ where $\sum_{k=1}^{n} a_k k = n$.

Proof: Reasoning first in terms of the dimensions n_i, we calculate the number of ordered bases in a subspace of dimension n_1 of a vector space of dimension n over the finite field $GF(q)$ with q elements. There are q^n elements in the space so the first choice for a basis vector is $(q^n - 1)$ (excluding 0), and since that vector generated a subspace of dimension q, the choice of the second basis vector is limited to $(q^n - q)$ elements, and so forth. Thus:

$$\begin{gathered}(q^n-1)(q^n-q)\left(q^n-q^2\right)...\left(q^n-q^{n_1-1}\right)\\ =(q^n-1)\,q^1\left(q^{n-1}-1\right)q^2\left(q^{n-1}-1\right)...q^{n_1-1}\left(q^{n-n_1+1}-1\right)\\ =(q^n-1)\left(q^{n-1}-1\right)...\left(q^{n-n_1-1}-1\right)q^{(1+2+...+(n_1-1))}\\ =(q^n-1)\left(q^{n-1}-1\right)...\left(q^{n-n_1-1}-1\right)q^{\binom{n_1}{2}}\end{gathered}$$

Number of ordered bases for an n_1-dimensional subspace of an n-dimensional space.

If we then divide by the number of ordered bases for an n_1-dimension space:

$$(q^{n_1}-1)(q^{n_1}-q)...\left(q^{n_1}-q^{n_1-1}\right)=(q^{n_1}-1)\left(q^{n_1-1}-1\right)...(q-1)\,q^{(1+2+...+(n_1-1))}$$

we could cancel the $q^{n_1(n_1-1)/2}=q^{\binom{n_1}{2}}$ terms to obtain the Gaussian binomial coefficient

$$\frac{(q^n-1)\left(q^{n-1}-1\right)...\left(q^{n-n_1-1}-1\right)q^{(1+2+...+(n_1-1))}}{(q^{n_1}-1)(q^{n_1-1}-1)...(q-1)q^{(1+2+...+(n_1-1))}}=\binom{n}{n_1}_q=\frac{[n]_q!}{[n-n_1]_q![n_1]_q!}$$

Number of different n_1-dimensional subspaces of an n-dimensional space.

If instead we continue to develop the numerator by multiplying by the number of ordered bases for an n_2-dimensional space that could be chosen from the remaining space of dimension $n-n_1$ to obtain:

$$\begin{gathered}(q^n-1)(q^n-q)\left(q^n-q^2\right)...\left(q^n-q^{n_1-1}\right)\times\left(q^n-q^{n_1}\right)\left(q^n-q^{n_1+1}\right)...\left(q^n-q^{n_1+n_2-1}\right)\\ =(q^n-1)\left(q^{n-1}-1\right)...\left(q^{n-n_1-n_2+1}-1\right)q^{(1+2+...+(n_1+n_2-1))}.\end{gathered}$$

Then dividing by the number of ordered bases of an n_1-dimensional space times the number of ordered bases of an n_2-dimensional space gives the number of different "disjoint" (i.e., only overlap is zero subspace) subspaces of n_1-dimensional and n_2-dimensional subspaces.

$$=\frac{(q^n-1)\left(q^{n-1}-1\right)...\left(q^{n-n_1-n_2+1}-1\right)q^{(1+2+...+(n_1+n_2-1))}}{(q^{n_1}-1)(q^{n_1-1}-1)...(q-1)q^{(1+2+...+(n_1-1))}\times(q^{n_2}-1)(q^{n_2-1}-1)...(q-1)q^{1+2+...+(n_2-1)}}.$$

Continuing in this fashion we arrive at the number of disjoint subspaces of dimensions $n_1,n_2,...,n_m$ where $\sum_{i=1}^m n_i=n$:

$$\begin{gathered}\frac{(q^n-1)\left(q^{n-1}-1\right)...(q-1)q^{(1+2+...+(n-1))}}{\prod\limits_{i=1,...,m}(q^{n_i}-1)\left(q^{n_i-1}-1\right)...(q-1)q^{(1+2+...+(n_i-1))}}=\frac{[n]_q!q^{n(n-1)/2}}{[n_1]_q!q^{n_1(n_1-1)/2}\times...\times[n_m]_q!q^{n_m(n_m-1)/2}}\\ =\frac{[n]_q!}{[n_1]_q!...[n_m]_q!}q^{\frac{1}{2}\left[n(n-1)-\sum_{i=1}^m n_i(n_i-1)\right]}.\end{gathered}$$

There may be a number a_k of subspaces with the same dimension, e.g., if $n_j=n_{j+1}=k$, then $a_k=2$ so the term $[n_j]_q!q^{n_j(n_j-1)/2}\times[n_{j+1}]_q!q^{n_{j+1}(n_{j+1}-1)/2}$ in the denominator could be replaced by $\left([k]_q!\right)^{a_k}q^{a_kk(k-1)/2}$. Hence the previous result could be rewritten in the part-count representation:

$$\frac{[n]_q!}{\left([1]_q!\right)^{a_1}...\left([n]_q!\right)^{a_n}}q^{\frac{1}{2}\left[n(n-1)-\sum_k a_kk(k-1)\right]}.$$

And permuting subspaces of the same dimension k yields a DSD with the same signature, so we need to divide by $a_k!$ to obtain the formula:

$$\frac{[n]_q!}{a_1!...a_n!\left([1]_q!\right)^{a_1}...\left([n]_q!\right)^{a_n}}q^{\frac{1}{2}\left[n(n-1)-\sum_k a_kk(k-1)\right]}.$$

The exponent on the q term can be simplified since $\sum_k a_kk=n$:

$$\begin{gathered}\tfrac{1}{2}\left[n\,(n-1)-\left(\textstyle\sum_k a_kk\,(k-1)\right)\right]=\tfrac{1}{2}\left[n^2-n-\left(\textstyle\sum_k a_kk^2-\sum_k a_kk\right)\right]\\ =\tfrac{1}{2}\left[n^2-n-\left(\textstyle\sum_k a_kk^2-n\right)\right]=\tfrac{1}{2}\left(n^2-\textstyle\sum_k a_kk^2\right).\end{gathered}$$

This yields the final formula for the number of DSDs with the part-count signature $a_1, ..., a_n$:

$$\frac{[n]_q!}{a_1!...a_n!\left([1]_q!\right)^{a_1}...\left([n]_q!\right)^{a_n}} q^{\frac{1}{2}\left(n^2-\sum_k a_k k^2\right)}. \square$$

Note that the formula is not obtained by a simple substitution of $[k]_q!$ for $k!$ in the set partition formula due to the extra term $q^{\frac{1}{2}\left(n^2-\sum_k a_k k^2\right)}$, but that it still reduces to the classical formula for set partitions with that signature as $q \to 1$. This formula leads directly to the vector space version of the Stirling numbers of the second kind to count the DSDs with m parts and to the vector space version of the Bell numbers to count the total number of DSDs.

Before giving those formulas, it should be noted that there is another q-analog formula called "generalized Stirling numbers" (of the second kind)–but it generalizes only one of the recurrence formulas for $S(n,m)$. It does not generalize the *interpretation* "number of set partitions on an n-element set with m parts" to count the vector space partitions (DSDs) of finite vector spaces of dimension n with m parts. The Stirling numbers satisfy the recurrence formula:

$$S(n+1,m) = mS(n,m) + S(n-1,m) \text{ with } S(0,m) = \delta_{0m}.$$

Donald Knuth uses the braces notation for the Stirling numbers, $\left\{ {n \atop m} \right\} = S(n,m)$, and then he defines the "generalized Stirling number" [18, p. 436] $\left\{ {n \atop m} \right\}_q$ by the q-analog recurrence relation:

$$\left\{ {n+1 \atop m} \right\}_q = \left(1 + q + ... + q^{m-1}\right) \left\{ {n \atop m} \right\}_q + \left\{ {n \atop m-1} \right\}_q; \left\{ {0 \atop m} \right\}_q = \delta_{0m}.$$

It is easy to generalize the direct formula for the Stirling numbers and it generalizes the partition interpretation:

$$D_q(n,m) = \sum_{\substack{1a_1+2a_2+...+na_n=n \\ a_1+a_2+...+a_n=m}} \frac{[n]_q!}{a_1!...a_n!\left([1]_q!\right)^{a_1}...\left([n]_q!\right)^{a_n}} q^{\frac{1}{2}\left(n^2-\sum_k a_k k^2\right)}$$

Number of DSDs of a finite vector space of dimension n over $GF(q)$ with m parts.

The number $D_q(n,m)$ is S_{nm} in [29]. Taking $q \to 1$ yields the Stirling numbers of the second kind, i.e., $D_1(n,m) = S(n,m)$. Knuth's generalized Stirling numbers $\left\{ {n \atop m} \right\}_q$ and $D_q(n,m)$ start off the same, e.g., $\left\{ {0 \atop 0} \right\}_q = 1 = D_q(0,0)$ and $\left\{ {1 \atop 1} \right\}_q = 1 = D_q(1,1)$, but then quickly diverge. For instance, all $\left\{ {n \atop n} \right\}_q = 1$ for all n, whereas the special case of $D_q(n,n)$ is the number of DSDs of 1-dimensional subspaces in a finite vector space of dimension n over $GF(q)$ (see table below for $q = 2$). The formula $D_q(n,n)$ is $M(n)$ in [30, Example 5.5.2(b), pp. 45-6] or [29, Example 2.2, p. 75].

The number $D_q(n,n)$ of DSDs of 1-dimensional subspaces is closely related to the number of basis sets. The old formula for that number of bases is [23, p. 71]:

$$\begin{aligned} &\frac{1}{n!}\left(q^n-1\right)\left(q^n-q\right)...\left(q^n-q^{n-1}\right) \\ &= \frac{1}{n!}\left(q^n-1\right)\left(q^{n-1}-1\right)...(q^1-1)q^{(1+2+...+(n-1))} \\ &= \frac{1}{n!}[n]_q!q^{\binom{n}{2}}(q-1)^n \end{aligned}$$

since $[k]_q = \frac{q^k-1}{q-1}$ for $k = 1, ..., n$.

In the formula for $D_q(n,n)$, there is only one signature $a_1 = n$ and $a_k = 0$ for $k = 2, ..., n$ which immediately gives the formula for the number of DSDs with n 1-dimensional blocks and each 1-dimensional block has $q-1$ choices for a basis vector so the total number of sets of basis vectors is given by the same formula:

$$D_q(n,n)(q-1)^n = \frac{[n]_q!}{a_1!}q^{\frac{1}{2}\left(n^2-a_1 1^2\right)}(q-1)^n = \frac{1}{n!}[n]_q!q^{\binom{n}{2}}(q-1)^n.$$

Note that for $q=2$, $(q-1)^n=1$ so $D_2(n,n)$ is the number of different basis sets.

Summing the $D_q(n,m)$ for all m gives the vector space version of the Bell numbers $B(n)$:

$$D_q(n)=\sum_{m=1}^{n} D_q(n,m)=\sum_{1a_1+2a_2+\ldots+na_n=n} \frac{1}{a_1!a_2!\ldots a_n!}\frac{[n]_q!}{\left([1]_q!\right)^{a_1}\ldots\left([n]_q!\right)^{a_n}} q^{\frac{1}{2}\left(n^2-\sum_k a_k k^2\right)}$$

Number of DSDs of a vector space of dimension n over $GF(q)$.

Our notation $D_q(n)$ is $D_n(q)$ in Bender and Goldman [4] and $|Q_n|$ in Stanley ([29], [30]). Setting $q=1$ gives the Bell numbers, i.e., $D_1(n)=B(n)$.

7.3 Counting DSDs with a block containing a designated vector v^*

Set partitions have a property not shared by vector space partitions, i.e., DSDs. Given a designated element u^* of the universe set U, the element is contained in some block of every partition on U. But given a nonzero vector v^* in a space V, it is not necessarily contained in a block of any given DSD of V. Some proofs of formulas use this property of set partitions so the proofs do not generalize to DSDs.

Consider one of the formulas for the Stirling numbers of the second kind:

$$S(n,m)=\sum_{k=0}^{n-1}\binom{n-1}{k}S(k,m-1)$$

Summation formula for $S(n,m)$.

The proof using the designated element u^* reasoning starts with the fact that any partition of U with $|U|=n$ with m blocks will have one block containing u^* so we then only need to count the number of $m-1$ blocks on the subset disjoint from the block containing u^*. If the block containing u^* had $n-k$ elements, there are $\binom{n-1}{k}$ blocks that could be complementary to an $(n-k)$-element block containing u^* and each of those k-element blocks had $S(k,m-1)$ partitions on it with $m-1$ blocks. Hence the total number of partitions on an n-element set with m blocks is that sum.

This reasoning can be extended to DSDs over finite vector spaces, but it only counts the number of DSDs with a block containing a designated nonzero vector v^* (it doesn't matter which one), not all DSDs. Furthermore, it is not a simple matter of substituting $\binom{n-1}{k}_q$ for $\binom{n-1}{k}$. Each $(n-k)$-element subset has a unique k-element subset disjoint from it (its complement), but the same does not hold in general vector spaces. Thus given a subspace with $(n-k)$-dimensions, we must compute the number of k-dimensional subspaces disjoint from it.

Let V be an n-dimensional vector space over $GF(q)$ and let v^* be a specific nonzero vector in V. In a DSD with an $(n-k)$-dimensional block containing v^*, how many k-dimensional subspaces are there disjoint from the $(n-k)$-dimensional subspace containing v^*? The number of ordered basis sets for a k-dimensional subspace disjoint from the given $(n-k)$-dimensional space is:

$$\begin{aligned}\left(q^n-q^{n-k}\right)\left(q^n-q^{n-k+1}\right)\ldots\left(q^n-q^{n-1}\right)&=\left(q^k-1\right)q^{n-k}\left(q^{k-1}-1\right)q^{n-k+1}\ldots(q-1)\,q^{n-1}\\&=\left(q^k-1\right)\left(q^{k-1}-1\right)\ldots(q-1)\,q^{(n-k)+(n-k+1)+\ldots+(n-1)}\\&=\left(q^k-1\right)\left(q^{k-1}-1\right)\ldots(q-1)\,q^{k(n-k)+\frac{1}{2}k(k-1)}\end{aligned}$$

since we use the usual trick to evaluate twice the exponent:

$$\begin{aligned}&(n-k)+(n-k+1)+\ldots+(n-1)\\&+\overline{(n-1)+(n-2)+\ldots+(n-k)}\\&=\overline{(2n-k-1)+\ldots+(2n-k-1)}\\=k(2n-k-1)&=2k(k+(n-k))-k^2-k=2k(n-k)+k^2-k.\end{aligned}$$

Now the number of ordered basis set of a k-dimensional space is:

$$\left(q^k-1\right)\left(q^{k-1}-1\right)\ldots(q-1)\,q^{\frac{1}{2}k(k-1)}$$

so dividing by that gives:

$$q^{k(n-k)}$$

The number of k-dimensional subspaces disjoint from any $(n-k)$-dimensional subspace.[13]

Note that taking $q \to 1$ yields the fact that an $(n-k)$-element subset of an n-element set has a unique k-element subset disjoint from it.

Hence in the q-analog formula, the binomial coefficient $\binom{n-1}{k}$ is replaced by the Gaussian binomial coefficient $\binom{n-1}{k}_q$ times $q^{k(n-k)}$. Then the rest of the proof proceeds as usual. Let $D_q^*(n,m)$ denote the number of DSDs of V with m blocks with one block containing a designated v^*. Then we can mimic the proof of the formula $S(n,m) = \sum_{k=0}^{n-1} \binom{n-1}{k} S(k,m-1)$ to derive the following:

Proposition 6 *Given a designated nonzero vector $v^* \in V$, the number of DSDs of V with m blocks one of which contains v^* is:*

$$D_q^*(n,m) = \sum_{k=0}^{n-1} \binom{n-1}{k}_q q^{k(n-k)} D_q(k,m-1). \square$$

Note that taking $q = 1$ gives the right-hand side of: $\sum_{k=0}^{n-1} \binom{n-1}{k} S(k,m-1)$ since $D_1(k,m-1) = S(k,m-1)$, and the left-hand side is the same as $S(n,m)$ since *every* set partition of an n-element with m blocks has to have a block containing some designated element u^*.

Since the Bell numbers can be obtained from the Stirling numbers of the second time as: $B(n) = \sum_{m=1}^{n} S(n,m)$, there is clearly a similar formula for the Bell numbers:

$$B(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} B(k)$$

Summation formula for $B(n)$.

This formula can also be directly proven using the designated element u^* reasoning, so it can be similarly be extended to computing $D_q^*(n)$, the number of DSDs of V with a block containing a designated nonzero vector v^*.

Proposition 7 *Given an designated nonzero vector $v^* \in V$, the number of DSDs of V with a block containing v^* is:*

$$D_q^*(n) = \sum_{k=0}^{n-1} \binom{n-1}{k}_q q^{k(n-k)} D_q(k). \square$$

In the same manner, taking $q = 1$ yields the classical summation formula for $B(n)$ since $D_1(k) = B(k)$, and every partition has to have a block containing a designated element u^*.

Furthermore the D^* numbers have the expected relation:

Corollary 2 $D_q^*(n) = \sum_{m=1}^{n} D_q^*(n,m). \square$

Note that both $D_q(n,m)$ and $D_q^*(n,m)$ are q-analogs of the Stirling numbers of the second kind $S(n,m)$, and that both $D_q(n)$ and $D_q^*(n)$ are q-analogs of the Bell numbers $B(n)$.

In QM/Sets, the "observables" or attributes are defined by real-valued functions on basis sets. Given a basis set $U = \{u_1, ..., u_n\}$ for $V = \mathbb{Z}_2^n \cong \wp(U)$, a real-valued attribute $f : U \to \mathbb{R}$ determines a set partition $\left\{f^{-1}(r)\right\}_{r \in f(U)}$ on U and a DSD $\left\{\wp\left(f^{-1}(r)\right)\right\}_{r \in f(U)}$ on $\wp(U)$. In full QM, the important thing about an "observable" is not the specific numerical eigenvalues, but its eigenspaces for distinct eigenvalues, and that information is in the DSD of its eigenspaces. The attribute $f : U \to \mathbb{R}$ cannot be internalized as an operator on $\wp(U) \cong \mathbb{Z}_2^n$ (unless its values are $0, 1$), but it nevertheless determines the DSD $\left\{\wp\left(f^{-1}(r)\right)\right\}_{r \in f(U)}$ which is sufficient to pedagogically model many quantum results. Hence a DSD can be thought of an "abstract attribute" (without the eigenvalues) with its blocks serving as "eigenspaces." Then a natural question to ask is given any nonzero vector $v^* \in V = \mathbb{Z}_2^n$, how many "abstract attributes" are there where v^* is an "eigenvector"–and the answer is $D_2^*(n)$. And $D_2^*(n,m)$ is the number of "abstract attributes" with m distinct "eigenvalues" where v^* is an "eigenvector."

[13] This was proven using Möbius inversion on the lattice of subspaces by Crapo [8].

7.4 Atoms, maximal DSDs, and segments

For a finite n-dimensional vector space V over $GF(q)$, the partially ordered set $DSD(V)$ of DSDs is denoted as Q_n in Stanley [30, Example 5.5.2(b), p. 45] where, as usual in the combinatorial theory literature, the "unrefinement" ordering is used on partitions and DSDs–although there are some exceptions as in Andrews [2, p. 217]. Hence $DSD(V)$ is the opposite partial order Q_n^{op} of Stanley's Q_n which reverses maximal and minimal DSDs so our number of maximal DSDs $D_q(n,n))$ is Stanley's number of minimal DSDs $M(n)$.

To compute the number of atoms below a given maximal element $\omega = \{U_k\}_{k=1,...,n}$ of $DSD(V)$, note that there are $\sum_{m=1}^{n-1} \binom{n}{m} = 2^n - 2$ proper subsets of the set ω and each determines (by direct sum) a subspace of dimension 1, 2,..., or $n-1$ and the proper subsets occur in complementary pairs in the atoms or binary DSDs so there are:

$$2^{n-1} - 1$$

Number of atoms below each maximal DSD ω.

Each atom defines two projection operators and the blob $\mathbf{0}$ determines two more projection operators for a total of $2\left(2^{n-1}-1\right)+2 = 2^n$ projection operators which gives the Boolean algebra of subsets of the given n-element set $\omega = \{U_k\}_{k=1,...,n}$. Note that once a maximal DSD ω is picked in $DSD(V)$ and the DSDs are restricted to those below ω, i.e., to $\prod(\omega)$, then that is a "classical" partition lattice on an n-element set where $2^{n-1}-1$ is indeed the number of atoms.

Conversely the number of maximal elements ω above a given atom $\{V_1, V_2\}$ in $DSD(V)$ will depend on the positive dimensions $k = \dim(V_1)$ and $n-k-\dim(V_2)$ of the blocks $V_1$and V_2 in the atom. There are $D_q(k,k)$ maximal DSDs in $DSD(V_1)$ and $D_q(n-k,n-k)$ maximal DSDs in $DSD(V_2)$. Picking one maximal DSD of each set will give a maximal DSD for the whole space that is above the given atom and a maximal DSD above the atom can be partitioned into two such subsets, so we have:

$$D_q(k,k)\, D_q(n-k,n-k) = \frac{[k]_q!}{k!} q^{\frac{1}{2}\left(k^2-k\right)} \frac{[n-k]_q!}{(n-k)!} q^{\frac{1}{2}\left((n-k)^2-(n-k)\right)}$$
$$\frac{[k]_q![n-k]_q!}{k!(n-k)!} q^{\frac{1}{2}\left[\left(k^2-k\right)+\left((n-k)^2-(n-k)\right)\right]} = \frac{[k]_q![n-k]_q!}{k!(n-k)!} q^{\binom{k}{2}+\binom{n-k}{2}}.$$

Number of maximal DSDs above a given atom.

The reasoning clearly generalizes so for any given DSD $\sigma = \{W_j\}_{j=1,...,m}$ where $\dim(W_j) = n_j$, we have:

$$\prod_{j=1}^{m} D_q(n_j, n_j)$$

Number of maximal DSDs in $DSD(V)$ refining σ.

Moreover, the reasoning can be generalized to arbitrary DSDs above σ, i.e., to the upper segment $\{\pi \in DSD(V) : \sigma \preceq \pi\}$:

$$\prod_{j=1}^{m} D_q(\dim(W_j))$$

Number of DSDs in $DSD(V)$ refining σ.

As $q \to 1$, $D_q(n) \to B(n)$, the Bell numbers, and thus we get the classical formula for the number of set partitions $\prod_{i=1}^{m} B(|C_i|)$ that refine a given set partition $\sigma = \{C_1, ..., C_m\}$.

7.5 Computing initial values for $q = 2$

In the case of $n = 1, 2, 3$, the DSDs can be enumerated "by hand" to check the formulas, and then the formulas can be used to compute higher values of $D_2(n, m)$ or $D_2(n)$.

Since all subspaces contain the zero element which is the empty set $\emptyset$, it will be usually suppressed when listing the elements of a subspace. And subsets like $\{a\}$ or $\{a, b\}$ will be denoted as just a and ab. Thus the subspace $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ is denoted for brevity as $\{a, b, ab\}$. A k-dimensional subspace has 2^k elements so only $2^k - 1$ are listed.

For $n = 1$, there is only one nonzero subspace $\{a\}$, i.e., $\{\emptyset, \{a\}\}$, and $D_2(1, 1) = D_2(1) = 1$.

For $n = 2$, the whole subspace is $\{a, b, ab\}$ and it has three bases $\{a, b\}$, $\{a, ab\}$, and $\{b, ab\}$. The formula for the number of bases gives $D_2(2, 2) = 3$. The only $D_2(2, 1) = 1$ DSD is the whole space.

For $n = 3$, the whole space $\{a, b, c, ab, ac, bc, abc\}$ is the only $D_2(3, 1) = 1$ and indeed for any n and q, $D_q(n, 1) = 1$. For $n = 3$ and $m = 3$, $D_2(3, 3)$ is the number of maximal DSDs of $DSD(\mathbb{Z}_2^3)$ which is the number (unordered) bases of $\mathbb{Z}_2^3$ (recall $\begin{Bmatrix} n \\ n \end{Bmatrix}_q = 1$ for all q). Since we know the signature, i.e., $a_1 = 3$ and otherwise $a_k = 0$, we can easily compute $D_2(3, 3)$:

$$\frac{1}{a_1!a_2!\ldots a_n!}\frac{[n]_q!}{\left([1]_q!\right)^{a_1}\ldots\left([n]_q!\right)^{a_n}}q^{\frac{1}{2}\left(n^2-\sum_k a_k k^2\right)}$$
$$= \frac{1}{3!}\frac{[3]_2!}{\left([1]_2\right)^3}2^{\frac{1}{2}\left(3^2-3\right)} = \frac{1}{6}\frac{7\times 3}{1}2^{\frac{1}{2}(6)} = 28 = D_2(3, 3).$$

And here they are.

$\{a, b, c\}$	$\{a, b, ac\}$	$\{a, b, bc\}$	$\{a, b, abc\}$
$\{a, c, ab\}$	$\{a, c, bc\}$	$\{a, c, abc\}$	$\{a, ab, ac\}$
$\{a, ab, bc\}$	$\{a, ab, abc\}$	$\{a, ac, bc\}$	$\{a, ac, abc\}$
$\{b, c, ab\}$	$\{b, c, ac\}$	$\{b, c, abc\}$	$\{b, ab, ac\}$
$\{b, ab, bc\}$	$\{b, ab, abc\}$	$\{b, ac, bc\}$	$\{b, bc, abc\}$
$\{c, ab, ac\}$	$\{c, ab, bc\}$	$\{c, ac, bc\}$	$\{c, ac, abc\}$
$\{ab, ac, abc\}$	$\{ab, bc, abc\}$	$\{ac, bc, abc\}$	$\{bc, ab, abc\}$

All maximal DSDs in $DSD\left(\mathbb{Z}_2^3\right)$ = all bases of $\mathbb{Z}_2^3$.

For $n = 3$ and $m = 2$, $D_2(3, 2)$ is the number of atomic (i.e., binary) DSDs, each of which has the signature $a_1 = a_2 = 1$ so the total number of atomic DSDs is:

$$D_2(3, 2) = \frac{1}{1!1!}\frac{[3]_2!}{([1]!)^1([2]!)^1}2^{\frac{1}{2}\left(3^2-1-2^2\right)} = \frac{7\times 3}{3}2^{\frac{1}{2}(4)} = 7 \times 4 = 28.$$

And here they are:

$\{\{a\}, \{b, c, bc\}\}$	$\{\{a\}, \{ab, ac, bc\}\}$	$\{\{a\}, \{c, ab, abc\}\}$	$\{\{a\}, \{b, ac, abc\}\}$
$\{\{b\}, \{a, c, ac\}\}$	$\{\{b\}, \{ab, ac, bc\}\}$	$\{\{b\}, \{c, ab, abc\}\}$	$\{\{b\}, \{a, bc, abc\}\}$
$\{\{ab\}, \{b, c, bc\}\}$	$\{\{ab\}, \{a, bc, abc\}\}$	$\{\{ab\}, \{b, ac, abc\}\}$	$\{\{ab\}, \{a, c, ac\}\}$
$\{\{c\}, \{a, b, ab\}\}$	$\{\{c\}, \{ab, ac, bc\}\}$	$\{\{c\}, \{a, bc, abc\}\}$	$\{\{c\}, \{b, ac, abc\}\}$
$\{\{ac\}, \{a, b, ab\}\}$	$\{\{ac\}, \{a, bc, abc\}\}$	$\{\{ac\}, \{c, ab, abc\}\}$	$\{\{ac\}, \{b, c, bc\}\}$
$\{\{bc\}, \{a, b, ab\}\}$	$\{\{bc\}, \{b, ac, abc\}\}$	$\{\{bc\}, \{c, ab, abc\}\}$	$\{\{bc\}, \{a, c, ac\}\}$
$\{\{abc\}, \{a, b, ab\}\}$	$\{\{abc\}, \{b, c, bc\}\}$	$\{\{abc\}, \{a, c, ac\}\}$	$\{\{abc\}, \{ab, ac, bc\}\}$

All atomic DSDs in $DSD\left(\mathbb{Z}_2^3\right)$ = all binary DSDs for $\mathbb{Z}_2^3$.

The above table has been arranged to illustrate the result that any given k-dimensional subspaces has $q^{k(n-k)}$ subspaces disjoint from it. For $n = 3$ and $k = 1$, each row gives the $2^2 = 4$ subspaces disjoint from any given 1-dimensional subspace represented by $\{a\}$, $\{b\}$,..., $\{abc\}$. For instance, the four subspaces disjoint from the subspace $\{ab\}$ (shorthand for $\{\emptyset, \{a, b\}\}$) are given in the third row since those are the "complementary" subspaces that together with $\{ab\}$ form a DSD.

To illustrate the number of atoms below a maximal element ω, recall that a maximal DSD and a basis set are the "same thing" for $q = 2$. The basis set $\{a, ac, bc\}$ for $\mathbb{Z}_2^3$ has $2^2 - 1 = 3$ atoms below it, namely $\{\{a\}, \{ab, ac, bc\}\}$, $\{\{bc\}, \{a, c, ac\}\}$, and $\{\{ac\}, \{a, bc, abc\}\}$, and those three atoms determine the eight element BA $\wp(\{a, ac, bc\})$ of subsets of the basis set.

To illustrate the number of maximal DSDs above a given atom for $q = 2$ (where maximal DSD = basis set) and $n = 3$, each atom has $k = 1$ and $n - k = 2$, so the formula gives 3 which is correct. For instance, the atom $\{\{ac\}, \{a, bc, abc\}\}$ has three maximal DSDs above it, namely $\{a, ac, bc\}$, $\{a, ac, abc\}$, and $\{ac, bc, abc\}$.

Note that for $q = 1$, the formula gives the number of maximal partitions above a given binary partition on a set, namely 1, the discrete partition on the set.

For $q = 2$, the initial values up to $n = 6$ of $D_2(n, m)$are given the following table.

$n\backslash m$	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	1	3				
3	0	1	28	28			
4	0	1	400	1,680	840		
5	0	1	10,416	168,640	277,760	83,328	
6	0	1	525,792	36,053,248	159,989,760	139,991,040	27,998,208

$D_2(n, m)$ with $n, m = 1, 2, ..., 6$.

The seventh row $D_2(7, m)$ for $m = 0, 1, ..., 7$ is: 0, 1, 51116992, 17811244032, 209056841728, 419919790080, 227569434624, and 32509919232 which sum to $D_2(7)$.

The row sums give the values of $D_2(n)$ for $n = 0, 1, 2, ..., 7$.

n	$D_2(n)$
0	1
1	1
2	4
3	57
4	2,921
5	540,145
6	364,558,049
7	906,918,346,689

$D_2(n)$ for $n = 0, 1, ..., 7$.

We can also compute the D^* examples of DSDs with a block containing a designated element. For $q = 2$, the $D_2^*(n, m)$ numbers for $n, m = 0, 1, ..., 7$ are given in the following table.

$n\backslash m$	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	1	2					
3	0	1	16	12				
4	0	1	176	560	224			
5	0	1	3456	40000	53760	13440		
6	0	1	128000	5848832	20951040	15554560	2666496	
7	0	1	9115648	1934195712	17826414592	30398054400	14335082496	1791885312

Number of DSDs $D_2^*(n, m)$ containing any given nonzero vector v^*

For $n = 3$ and $m = 2$, the table says there are $D_2^*(3,2) = 16$ DSDs with 2 blocks one of which contains a given vector, say $v^* = ab$ which represents $\{a, b\}$, and here they are.

$\{\{ab\}, \{b, c, bc\}\}$	$\{\{ab\}, \{a, bc, abc\}\}$	$\{\{ab\}, \{b, ac, abc\}\}$	$\{\{ab\}, \{a, c, ac\}\}$
$\{\{c\}, \{a, b, ab\}\}$	$\{\{c\}, \{ab, ac, bc\}\}$	$\{\{ac\}, \{c, ab, abc\}\}$	$\{\{bc\}, \{c, ab, abc\}\}$
$\{\{ac\}, \{a, b, ab\}\}$	$\{\{a\}, \{ab, ac, bc\}\}$	$\{\{a\}, \{c, ab, abc\}\}$	$\{\{abc\}, \{a, b, ab\}\}$
$\{\{bc\}, \{a, b, ab\}\}$	$\{\{b\}, \{ab, ac, bc\}\}$	$\{\{b\}, \{c, ab, abc\}\}$	$\{\{abc\}, \{ab, ac, bc\}\}$

Two-block DSDs of $\wp(\{a, b, c\})$ with a block containing $ab = \{a, b\}$.

The table also says there are $D_2^*(3,3) = 12$ basis sets containing *any* given element which we could take to be $v^* = abc = \{a, b, c\}$, and here they are.

$\{a, b, abc\}$	$\{b, ab, abc\}$	$\{a, c, abc\}$	$\{b, bc, abc\}$
$\{a, ab, abc\}$	$\{a, ac, abc\}$	$\{b, c, abc\}$	$\{c, ac, abc\}$
$\{ab, ac, abc\}$	$\{ab, bc, abc\}$	$\{ac, bc, abc\}$	$\{bc, ab, abc\}$

Three-block DSDs (basis sets) of $\wp(\{a, b, c\})$ with a basis element $abc = \{a, b, c\}$.

Summing the rows in the $D_2^*(n, m)$ table gives the values for $D_2^*(n)$ for $n = 0, 1, ..., 7$.

n	$D_2^*(n)$
0	1
1	1
2	3
3	29
4	961
5	110,657
6	45,148,929
7	66,294,748,161

$D_2^*(n)$ for $n = 0, 1, ..., 7$.

In QM/Sets, a DSD on $\mathbb{Z}_2^n$ is an "abstract observable" that abstracts from the specific eigenvalues and gives only the DSD of eigenspaces. $D_2^*(n)$ counts the number of abstract observables that will have any given nonzero vector $v^* \in \mathbb{Z}_2^n$ as an eigenvector. For instance, for $n = 3$, any given nonzero vector v^* will be an eigenvector for $D_2^*(3) = 29$ abstract observables, i.e., 12 three-block DSDs, 16 two-block DSDs, and 1 single-block DSD (the blob).

The integer sequence $D_2(n, n)$ for $n = 0, 1, 2, ...$ is known as: $A053601$"Number of bases of an n-dimensional vector space over $GF(2)$" in the *On-Line Encyclopedia of Integer Sequences* (https://oeis.org/). The sequences defined and tabulated here for $q = 2$ have been added to the *Encyclopedia* as: $A270880$ $[D_2(n, m)]$, $A270881$ $[D_2(n)]$, $A270882$ $[D_2^*(n, m)]$, $A270883$ $[D_2^*(n)]$.

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