Topic Transparency and Variable Sharing in Weak Relevant Logics

Abstract

In this paper, we examine a number of relevant logics' variable sharing properties from the perspective of theories of topic or subject-matter. We take cues from Franz Berto's recent work on topic to show an alignment between families of variable sharing properties and responses to the topic transparency of relevant implication and negation. We then introduce and defend novel variable sharing properties stronger than strong depth relevance—which we call *cn-relevance* and *lossless cn-relevance*—showing that the properties are satisfied by the weak relevant logics **B** and **BM**, respectively. We argue that such properties address a sort of semantic lossiness of strong depth relevance.

Keywords: relevant logic, topic transparency, variable sharing

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1 Variable Sharing in Relevant Logics

The feature of *relevance* that nominally unifies the family of *relevant logics* is a slippery creature, its slipperiness made worse for the tangle of disparate intuitions motivating its primary architects. One is unlikely, *e.g.*, to draw a clear understanding of relevance by triangulating from Anderson and Belnap's *use criterion*, Sylvan's *sociativity*, and Meyer's *lawful connections*. This exposes one clear drawback to such a vibrant collection of brilliant-but-inchoate pictures, namely, that there is little guarantee that any two individuals' portraits will cohere. A further drawback, of course, is that such inchoate portraits do not easily lend themselves over to mathematical analysis.

Conveniently, several proxies for the informal notion of relevance—precise enough to allow the formal logician to express and prove related theorems—have been offered. Most famed among these is the so-called *variable sharing property*, introduced and shown to hold of **E** in [1].

Definition 1. A sentence $A \to B$ exhibits the *variable sharing property* if some atomic variable appears in both A and B.

From this we arrive at a definition of a logic's enjoying such variable sharing.

Definition 2. A logic L enjoys the variable sharing property if every L-theorem $A \to B$ exhibits variable sharing.

That a logic enjoys the variable sharing property is importantly understood as a necessary but not sufficient condition for relevance as it ensures that an entailment's antecedent and consequent share some "common content."

In Anderson and Belnap's words, this common content is guaranteed for the following reasons:

> A formal condition for "common meaning content" becomes almost obvious once we note that commonality of meaning in propositional logic is carried by commonality of propositional variables. So we propose as a *necessary*, but by no means sufficient, condition for the relevance of A to B in the pure calculus of entailment, that A and B must share a variable... If this property fails, then the variables in A and B may be assigned propositional values, in such a way that the resulting propositions have no meaning content in common and are totally irrelevant to each other.[2, p. 33]

The aforementioned *commonality of meaning* is unconcerned with truth conditions, *i.e.*, does not presuppose commonality of truth conditions. On topic-sensitive frameworks like a Yablo-style *two-component* understanding of meaning (see *e.g.* [3]), then, one common component must be one of *topic*.[1] This leads to an intuitive reading of relevance in this sense as *topic overlap*.

In spite of its formal elegance, the variable sharing property is arguably coarse in other ways, namely in its indiscriminately welcoming *any* shared variable as an authentic certificate of relevance.[2] There may be reasons to think this attitude to be overly permissive—some of which we will survey shortly—which can be codified as *refinements* that require a shared variable to satisfy some further criteria if it is to certify relevance. Consequently, the variable sharing property has been complemented by a succession of refinements. Even if any of the variable sharing properties in the literature are not sufficient, clearly some fit more tightly than others.

In this paper we investigate the shape of this succession of variable sharing properties. In particular, we draw from the contemporary theory of topics to establish tight connections between particular variable sharing properties and the *topic-transparency* of particular connectives. We then follow these connections to the frontiers of variable sharing, leveraging them to diagnose deficiencies of the strongest property and to identify desiderata for improved

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[1] The applicability of two-component approaches to topic in mainstream relevant logics has recently seen an extraordinarily compelling defense and model theory in [4].

[2] It is worth mentioning how Meyer, Dunn, and Leblanc aptly refer to the variable sharing property as the "crudest but most memorable result"[5] concerning relevance with respect to the logic **R**.

properties. We conclude by arguing in support of a novel species of variable sharing—*strong cn-relevance*—and establishing that this property holds of the relevant logic **B**.

2 Variations on the Variable Sharing Property

In this section, we look at properties that refine the "crudeness" of stock variable sharing. We presumably wish for such criteria to be driven by philosophical intuitions about the informal notion of relevance. Thus, a reasonable first place to look to motivate these refinements—implicit in the foregoing quote from [2, p. 33]—is an understanding of overlapping of *topic* between two sentences. The discussion of topic and relevance in [6] provides a detailed catalogue of central figures in relevant logic aligning variable sharing properties with topic-theoretic concerns. Notable instances include:

- Ed Mares' [7] identification of the variable sharing property as "a formal principle... appl[ied] to force theorems and inferences to 'stay on topic'."
- Lloyd Humberstone's introduction of relevance as a "formaliz[ation of] the idea that for an antecedent to be relevant to its consequent, they must overlap as to subject-matter."[8, p. 336]
- Neil Tennant's description of relevance as a guarantor of "a proper 'connection of subject matter' between the premises and the conclusion of any proof."[9, p. 2]

With this as an anchor, advances in the development of the theory of topic should be fruitful for the analysis of relevance as well. We will observe that the canonical variable sharing properties indeed line up with distinctions in theories of topic, namely, whether or not particular connectives can contribute to the overall topic of a complex.[3]

In work on the theory of topic, it is generally acknowledged that there exist connectives that exhibit a topic-theoretically transformative function through which the connective itself may influence and shape the topic of a complex sentence. In such cases, the connective may nontrivially contribute to the overall topic of a complex sentence in which it appears. Clearly, such connectives are not *topic transparent*, *i.e.*, the topic of a sentence in which the connective appears need not coincide with the simple mereological fusion of the topics of its subformulae.

The extent of the connectives and operators of this kind is a much more contentious matter. That topic transparency should fail for *intensional* connectives—including relevant implication—is a relatively orthodox position. *E.g.*, Franz Berto takes this as more-or-less obvious:

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[3]The theory of topic is still in its relative infancy especially with respect to its ontology. We try to remain somewhat agnostic about the particular model of topic—favoring instead less formal intuitions—but note that many, if not all, of the historical approaches reviewed in *e.g.* [10] cohere with these informal intuitions.

> [I]t's uncertain whether all the vocabulary we may want to call 'logical' is topic-transparent. Surely the topic-sensitive intentional operators... are not [topic-transparent]... 'Necessarily, John is human' seems to address a different topic from 'John is human' in a number of natural conversational contexts.[3, p. 34]

In the case of relevant implication, then, we take it as unexceptional that relevant conditionals wield some influence over the ultimate topic of a complex in which they appear.

More controversial is whether *negation* can exert the same type of topic-theoretic influence as the intensional relevant implication connective. Independently of the arguments on both sides—some of which will be touched on in the sequel—is that such a principle has taken form in the foundations of relevant logic in many ways.

Over the years, Belnap's stock variable sharing has been complemented with a succession of refinements incorporating stricter requirements on the notion of relevance. A pattern can be recognized across these refined properties. Against the informal assumption that relevance is overlap of topic, the foregoing observation admits a reading as an acknowledgement that intensional conditionals and negations maintain a transformational power over the topics of their subformulae. In other words, the canonical revisions to the variable sharing property each encapsulate a particular regularity governing how a complex's topic is shaped by the implications or negations appearing in it.

2.1 The Transformative Nature of Conditionals

Relevant implication is, of course, an intensional connective, whence we should not be surprised to discover cases among the canonical sources hinting at a relevant implication's capacity for topic-theoretic influence, in one form or another.

A very important occasion in which this capacity begins to take shape is found in Ross Brady's work in which relevant implication is interpreted as *meaning containment*. In this setting, Brady places several constraints on the conditions under which the meaning of one sentence may be contained in another, *i.e.*, is relevant to another. In particular, such meaning containment between two sentences is regarded as possible only in case those sentences are "like objects," *i.e.*, express the same type of proposition. Where a "containment" describes a formula $A \to B$, Brady reasons:

> It is mandatory for meaning containments to apply between sentences that are about like objects, for otherwise the relationship between them would not be a meaning containment. So, taking meaning containment statements as objects, meaning containments should not apply between containments that are not alike or between containments and non-containments. [11, p. 29]

Brady identifies containments and non-containments as providing a salient illustration in which two sentences are not considered "like objects." That *e.g.* $A \to B$ and $A \wedge B$ are not "like objects" suggests a tacit assumption that relevant implication actively modifies the type—*i.e.* the meaning—of subformulae. The intensional connective's refashioning of subsentences' meanings is followed by corresponding modifications to the class of meaning containments in which the complex may participate. Insofar as Brady's meaning containment is a special case of relevance, the intensional implication determines the collection of formulae to which that sentence is relevant, *i.e.*, with which its topic overlaps.

Understanding relevance as topic overlap, the systems considered by Brady incorporate formal reflections of an assumption that relevant implications manufacture novel topics from the raw material of the topics of subformulae. This is just to say that relevant implication is topic-theoretically transformative in the appropriate sense.

This directly invites us to introduce the modified variable sharing property that acts as a hallmark of Brady's program, that is, the *depth relevance property*. Given the role that Brady takes implications—and, by extension, nestings of implications—to play in the regulation of type (*i.e.*, manufacture of topic), it is to be expected that the definitions are driven by reference to relevant implications. This is clearly manifested in the necessary preliminary definition of the property of *depth*:

Definition 3. For an occurrence of B appearing in A, *depth* is defined so that:

- B appears at depth 0 in B
- if B appears at depth n in A, then:
  - B appears at depth n in $\neg A$
  - B appears at depth n in $A \wedge C$ and $C \wedge A$
  - B appears at depth n in $A \vee C$ and $C \vee A$
  - B appears at depth $n + 1$ in $A \rightarrow C$ and $C \rightarrow A$

Essentially, the depth of B in A is a discrete measure of the degree to which intensional connectives have operated on the topic of B within the complex A.

Definition 4. A sentence $A \rightarrow B$ exhibits the *depth relevance property* if some atomic variable appears in both A and B at the same depth.

This allows us to define depth relevance of a logic L

Definition 5. A logic L enjoys depth relevance if every L-theorem $A \rightarrow B$ exhibits the depth relevance property.

Depth relevance's concern for degree of the intensional conditional shows it to be a formal reflection of the topic-theoretically transformative character of intensional connectives and—more importantly—its influence over matters of relevance.[4]

This formal expression of the conditional's effect on topic is complemented by an analogous variable sharing property that follows from the transformative character of negations, to which we now turn.

2.2 The Transformative Nature of Negations

Although it is generally accepted that *topic transparency* should fail for intensional operators, the special case claim concerning negation has received a far cooler reception. Much contemporary work on topic and subject-matter endorses instead a thesis of *negation transparency*—that the topics of a sentence and its negation coincide—most prominently the recent work of *e.g.* Franz Berto and his collaborators. Berto offers some compelling justification for the topic transparency of negation:

> It is hard to come up with a discourse context where 'Jane is a lawyer' is on-topic, but 'Jane is not a lawyer' would be off-topic, or vice versa. Either seems on-topic with respect to the obvious topics the other is about: **whether Jane is a *lawyer***, **Jane's profession**, **Jane**, *etc.* One easily imagines contexts where only one is informative. But this kind of irrelevance is easily distinguished from being off-topic.[3, p. 33]

Despite the contemporary support for such negation transparency, opposing principles acknowledging the influence of negation over matters of relevance—and thus topic—have informed the heart of relevant logic since nearly the very inception of the field, arguably sharing equal priority with the variable sharing property itself.[5] The particular instantiation that we will be concerned with in this section is the *strong variable sharing property*—a refined version of variable sharing that perspicuously encapsulates an acknowledgement that negation plays a nontrivial role in determining topics. Before diving into strong variable sharing, we can consider reasons drawn from the constellation of relevant logics that speak *against* negation transparency.

Now, if we in general resist intensional transparency, then caution should be exercised concerning negations in contexts in which negation is cited as an intensional operator. In the mainstream relevant tradition, as it happens, the intensionality of relevant negation is defended from a number of perspectives. For example, Brady's [13], for example, argues that the intensionality of relevant negation follows from his understanding of a sentence's content (and, indeed, cites strong variable sharing as evidence of this). Furthermore, Restall's [14] sources the intensionality of relevant negation to the nature of the proof-theoretic rules governing its introduction and elimination. More recently, Berto and Jago in *e.g.* [15] and [16] appeal to the model theory of relevant

[4] To be sure, we do not offer the foregoing as, say, an exegesis of Brady's goals but rather as a possible explanation. We should note, however, that there is undoubtedly a topic-theoretic reading of Brady's interpretation of his model theory of *contents*.

[5] As noted in *e.g.*, [12], although Belnap's initial proof of variable sharing from [1] does not acknowledge the stronger form of variable sharing, the 1960 proof is reproduced in [2] as a proof of the stronger property.

logic to suggest that relevant negation is a modality and *a fortiori* is intensional. There are several dimensions, in other words, along which the relevant tradition resists a thesis of negation transparency.

Richard Angell's relevant logic of *analytic containment* (**AC**) of [17] exhibits features that are easily read as a rejection of negation transparency, including the unprovability of $A \to (A \vee \neg A)$ in **AC**. Angell's intended interpretation treats the relevant implication as a relation of *inclusion of content*. On this reading, the failure of the sentence's validity signals that $\neg A$ may include some subject-matter not found among that of A alone, a situation only possible in case *the negation is productive*, *i.e.*, transmutes the topic of A into a new topic thoroughly distinct from its origin.

Despite its proximity to mainstream relevant logics, **AC** still lies outside the core relevant systems; *e.g.* $A \to (A \vee \neg A)$ is provable in even the weakest of Sylvan's preferred systems. We cannot merely appeal to authority, in other words. Luckily, Angell complemented his formal work with informal illustrations suggesting that negation transparency is incompatible with reasonable assumptions about meaning. *E.g.*

> For example, '(Jo died and Jo did not die and Flo wept)' does not mean the same as '(Jo died and Flo did not weep and Flo wept)'; for the first contains a false and inconsistent statement about Flo though the second does not. How can two sentences mean the same thing if one contains a false and inconsistent statement about an individual while the other does not?[18, p. 121]

The context of a "two-component" approach to semantic content in the sense of [3] provides one perspective from which to examine Angell's illustration. A "two-component" approach represents a sentence's content as a pair including its *truth conditions* and its *topic*. As contradictions, the example's sentences have identical truth conditions. Any difference in meaning between the sentences therefore could only be produced by a difference in topic, a difference that would have to be placed at the feet of negation.

Now, Angell's illustration seems to assume that the two contradictions have the same truth conditions. As a reviewer has pointed out, this assumption is troublesome in a relevant context. On four-valued semantics for first-degree entailment, for example, the states that assign 'Jo died and Jo did not die and Flo wept' a designated value need not coincide with those that assign 'Jo died and Flo did not weep and Flo wept' a designated value.

Yet focusing too closely on the four-valued setting may obscure some fine structure in which distinct topics may be assigned to 'Flo wept' and 'Flo did not weep.' Fine's truthmaker semantics for **AC** includes the interpretation that

> the subject-matter of a statement is what it is about and it may be identified with the closure $\langle A \rangle$ of the set of verifiers of the statement under fusions and parts.[19]

As the states that verify 'Flo wept' may differ from the states that verify 'Flo did not weep,' the subject matters (*i.e.* topics) **Flo wept** and **Flo did not weep** need not coincide. The two subject-matters may share a *part* but *as* *topics*, they are in general distinct. As the **AC** model theory is shown to provide a semantics for first-degree entailment in [19] (and **R** in Jago's [20]), we can disambiguate again the matter of the truth-conditions of a sentence from the collection of states making up its subject-matter. Thus, in well-motivated models of subject-matter that have been shown to support "mainstream" relevant logics, negation—swapping out verifiers for falsifiers—need not preserve topic.

Having considered some evidence in the relevant tradition for the transformative character of negation, we can proceed to strong variable sharing. Just as our earlier emphasis on implication led to a definition of *depth*, our emphasis on negation leads to a definition of *sign*:

Definition 6. The *sign* of an occurrence of B appearing in A is defined so that:

- B appears positively in B
- if B appears positively (resp, negatively) in A, then:
  - B appears negatively (positively) in $\neg A$
  - B appears positively (negatively) in $A \wedge C$ and $C \wedge A$
  - B appears positively (negatively) in $A \vee C$ and $C \vee A$
  - B appears negatively (positively) in $A \to C$
  - B appears positively (negatively) in $C \to A$

One observation is worth quickly making. The above definition includes two distinct mechanisms through which the sign of an occurrence might change during the above procedure, *i.e.*, an update may be triggered either by appearing within the scope of a negation (a *pure update*) or by appearing in a particular location in a conditional (a *positional update*). Importantly, the sign of an occurrence of B neither records nor recognizes the particular operations through which it was determined.[6] From this notion of sign, the details of strong variable sharing can be defined:

Definition 7. A sentence $A \to B$ exhibits the *strong variable sharing property* if some atomic variable appears in both A and B with the same sign.

We will sometimes speak informally about two sentences enjoying strong variable sharing, *etc.*. This leads to the definition of the refined property in the case of a logic:

Definition 8. A logic L enjoys the strong variable sharing property if every L-theorem $A \to B$ exhibits strong variable sharing.

[6] Note that this characterization coincides with the definition of *antecedent* and *consequent parts* of a sentence, *i.e.*, that B appears positively (resp., negatively) in A corresponds to its being an antecedent part (resp., consequent part) of A.

Just as depth relevance can be read as an acknowledgement of the effects of intensional connectives' character on relevance, strong variable sharing can be read as an acknowledgement of the consequences for relevance flowing from negation's topic-theoretic influence.

Each refinement, however, is independent of the the contributions encapsulated by its complement, *e.g.*, strong variable sharing is unconcerned about any topic-theoretic side-effects contributed by the relevant conditional (besides alterations to sign). A more holistic picture would be offered by combining the two, clearly, which immediately leads to a more recent refinement of variable sharing introduced by Logan in [21].

2.3 Strong Depth Relevance

As Logan points out in [21], strong variable sharing and depth relevance can be "hybridized" in order to yield a stronger variable sharing property. This can be expressed as syntactic criteria reflected in the following definitions.

Definition 9. A sentence $A \to B$ exhibits the *strong depth relevance property* if some atomic variable appears in both A and B at the same depth and with the same sign.

We will describe two formulae A and B as sharing strong depth relevance if the conditional $A \to B$ exhibits the property.

Definition 10. A logic L enjoys strong depth relevance if every L-theorem $A \to B$ exhibits strong depth relevance.

[21] goes on to show that all subsystems of $\mathbf{DR}^-$ have this strong depth relevance property.

Associated with such a syntactic hybridization is an interpretative hybridization, namely, an acknowledgement that topic transparency should fail for negation *and* the intensional conditional. But that strong depth relevance reflects the *failure of transparency* is different than its adequately reflecting the subtleties of the *ways in which transparency fails*. That these connectives exert *some* degree of influence over topic is not to say that the *correct* type of influence is adequately represented in strong depth relevance.

3 Refining Relevance Criteria

In this section, we will look at whether strong depth relevance is as strong a variable sharing property as one can hope for, or whether there are cases in which relevance may find expression in a still more restrictive property. This requires that we have a sense of when relevance obtains and fails. Following our interpretation of Anderson and Belnap's "commonality of meaning" as topic overlap, we must therefore begin by describing some informal indicators that signal relevance or irrelevance.

Of course, variable sharing properties are by no means sufficient to ensure relevance. But if we can describe some common features of cases in which relevance informally holds while strong depth relevance fails, such regularities suggest a particular way in which strong depth relevance can be strengthened.

3.1 Informal Indicators of Relevance and Irrelevance

In considering natural language examples, we must determine some criteria for when we interpret a case as providing evidence of relevance or irrelevance.

To offer an informal criterion for irrelevance, we note that relevance as overlap of topic essentially ensures that when two sentences are relevant to one another, alternating between the two should never prompt a change in subject-matter. For example, if one has initiated the topic of an interaction with one sentence, one may segue to introduce the other without having pivoted entirely away from the topic at hand. Relevance, in other words, acts as insurance against veering off topic. Conversely, the possibility of topic-shifting serves as a mark of a lack of relevance. We will understand the possibility of such topic-shifting to act as a sufficient—if informal—indicator of the failure of relevance between two sentences.

Although our earlier survey of [7], [8], and [9] in Section 2 suggested that the interpretation of relevance as topic overlap is more-or-less standard, we note that the interpretation of topic overlap as a barrier against changes of topic leads to a stronger reading of relevance than is standard. To be sure, we do not intend to attribute this particular reading to any particular logician, *e.g.*, it is not meant as an exegesis of Anderson and Belnap, Brady, Sylvan and Plumwood, or any other member of the relevant pantheon.

Nevertheless, we see this interpretation as a natural sharpening of the inchoate-yet-widespread topic-theoretic glosses on relevance and one worth investigation in virtue of its sensitivity to recent trends concerning topic and transparency. Our question is then simply: On a strong reading of relevance as a guardrail against changes in topic, what consequences might follow from matters of topic transparency, what formal conditions might we expect to emerge, and are there any well-known relevant logics that satisfy such conditions?

Providing informal criteria confirming relevance requires some thought concerning the degree to which topics must overlap to establish this stricter form of relevance between sentences. It seems that the matter of staying on topic presupposes preservation of a topic *in toto*. However, the overlap of topic comes in degrees. Topics may decompose into finer parts that are not themselves topics. On the issue-based theory of topic described in Peter Hawke's [22], for example, the topic corresponding to an individual referring term includes a Salmonian *guise*—*i.e.* a mode of presentation—which is not itself a topic. That two topics include the same guise as a part, then, is insufficient to establish that the two have a common subtopic.

Topic overlap—that two sentences share a common topic—must thus be distinguished between a relation of overlap in *topic parts*. In particular, this distinction is clearly implicit in many relevant logics already insofar as mere overlap in topic parts is insufficient to guarantee relevance in even very strong first-order relevant logics. *E.g.*, neither of the first-order relevant logics **QR** and **RQ**[7] count the following sentences as theorems:

[T] $(Pt \to Pt) \to (Qt \to Qt)$
[P] $(Ps \to Ps) \to (Pt \to Pt)$

An overlap of nonlogical syntax is exhibited by both sentences—[T] in virtue of a shared term t and [P] in virtue of a common predicate P. In the case of [T], is reasonable to correlate a term t with a *part* of the topic of Pt, which suggests that the topics of the antecedents and consequents of [T] ought to share an analogous overlap of topic parts. Despite this, the antecedents and consequents of [T] and [P] are theorems, whence the failure of theoremhood can only be attributed to a failure of relevance, *i.e.*, overlap with respect to a topic. Overlap of topic parts is thus distinct from overlap of topic, whence the overlap of topic parts is insufficient to guarantee relevance between sentences.[8]

The importance of distinguishing between a whole topic and a topic part for the present project lies in the following matter. We state that in some contexts conditionals are topic-transformative, whence *e.g.*, the topics of formulae A and $A \to A$ need not coincide, *i.e.*, to transition from the topic of A to the topic of $A \to A$ may incur a change in subject. Yet one might with some justification intuit that the presence of a common subformula between A and $A \to A$ requires the presence of something in common between their topics. It is thus important to suggest that this commonality does not serve to always buttress against topic-shifting. To illustrate more concretely, consider an example familiar to readers of [3]:

[I] Jane is a lawyer.

[II] That Jane is a lawyer entails that Jane is a lawyer.

The topics of [I] and [II]—which Berto might represent as **whether Jane is a lawyer** and **whether that Jane is a lawyer bears an entailment relation to that Jane is a lawyer**—indeed seem to have some mutual part. **Jane**, for example, that appears to constitute a part of each. But the topics are nevertheless about different things; [I] is about worldly states of affairs while [II], in contrast, is about entailment between propositions. Accordingly, segues between one to the other would generally be jarring. To examine this

[7]For a clear discussion of the distinctions between these two expansions of **R**, see Mares and Goldblatt's [23].

[8]This is closely related to a problem in Meyer's relevant arithmetic $\mathbf{R}^\sharp$, described by Brady as the that in $\mathbf{R}^\sharp$, "$m = n \to m' = n'$ [is a theorem] which has the natural numbers m and n in common, leads to $0 = 0 \to 100 = 100$, where the two numbers involved can be as far apart as you like."[11, p. 11] That this theorem "smacks of irrelevance" (as Dunn says in [24]) involves a similar acknowledgement that the overlap of terms—in this case two instances of 0 followed by sequences of $'$—need not ensure relevance. Estrada-González and Tapia-Navarro's [25] takes up this matter in more detail.

more closely, we will introduce several further illustrations clustered around a single pattern.[9]

3.2 Contraposition and Relevance

Examining individual cases in which the relevance between two statements is strained despite their apparent satisfaction of strong depth relevance should help natural desiderata when considering stronger conditions. In this section, we offer several natural language examples in which strong depth relevance fails to guarantee relevance and draw lessons concerning how strong depth relevance might be improved.

In preparation, though, we briefly consider the relationship between strong depth relevance and *contraposition*. It is clear that this operation—transforming a sentence $(A \to B)$ to $(\neg B \to \neg A)$—respects strong depth relevance. Both the depth and sign of a subformula are preserved under the operation, ensuring that a formula and its contrapose—a *contraposed pair*—will share strong depth relevance. Alternately, one can note strong depth relevance is exhibited by the sentential form of contraposition, *i.e.*, $(A \to B) \to (\neg B \to \neg A)$. Given the clarity of the relationship, it is natural to let contraposition serve as a common theme for the following illustrations.

These examples offer cases in which a sentence's topic is sufficiently transformed under contraposition to require topic-shifting between that sentence and its contrapose. As contraposition respects strong depth relevance, this wins insight into features common to cases in which strong depth relevance is insufficient and will suggest criteria for stronger relevance properties.

It should be noted that the following are assumed to involve only an intensional conditional rather than a properly relevant conditional.

Example 1. Recall that in the *Naming and Necessity* lectures, Saul Kripke underscored a point concerning physical constitution by gesturing towards a wooden desk and suggesting that it could not have been made of ice. In this context, consider the following two sentences:

[III] If the desk towards which Kripke pointed were to still exist, then the desk towards which he pointed would not have been made of ice.

[IV] If the desk towards which Kripke pointed had been made of ice, then the desk towards which he pointed would no longer exist.

We intend to reveal irrelevance between the contraposed pair of [III] and [IV] by uncovering a topical incommensurability.

We begin with the modest observation that both [III] and [IV] are about desks or, more precisely, about desks' instantiating particular properties. If relevance is to hold between the two, [III] and [IV] must be about *the same*

[9] For a broader range of examples, one can consult the discussion of intensionality and subject-matter in [26].

desk; otherwise, a conversational segue from [III] to [IV] would involve a shift in topic. On standard metaphysical grounds, however, [III] and [IV] cannot be about the same desk. On natural theories of topic that consider individuals—like Hawke's issue-based theory of [22]—the referents of referring terms constitute a part of a sentence's subject-matter, guraanteeing a common constituent between the topics of two terms precisely in case they corefer. But the common referring term "the desk towards which Kripke pointed" cannot have the same referents in [III] and [IV].

As its antecedent indicates, [III] is about the *actual* desk towards which Kripke pointed, *i.e.*, the desk picked out by the definite description "the desk towards which Kripke pointed." In considering scenarios in which this desk exists at the present time, [III] posits no states beyond those in which Kripke's *actual* desk could exist. After all, the possibility that this desk should continue to exist in the present year is metaphysically unobjectionable.

In contrast, [IV] exclusively focuses on situations in which the desk picked out by the description had been made of ice, *i.e.*, any desk that [IV] could be about would be one for which its being made of ice is metaphysically possible. However, because Kripke's *actual* desk was constructed of wood, it is metaphysically impossible that it could have been made of ice. The *actual* desk—that which composes the topic of [III]—is therefore *ruled out* as a constituent of the topic of [IV].

As [III] and [IV] are *about* different desks, on an issue-based theory like Hawke's, the *focal topics* of the assertions dramatically part ways. *In principle*—if not in practice—a segue from [III] to [IV] should trigger a shift in topic as the sentences are about different things.

Example 2. For this case, consider two subjunctive conditionals:

[V] Were Bill to get promoted, Bill will go to dinner.

[VI] Were Bill to not go to dinner, then Bill will not get promoted.

As subjunctive conditionals, Kratzer's theory of selectors described in *e.g.* [27] provides a natural perspective from which to examine [V] and [VI].

According to Kratzer's framework, the antecedents of [V] and [VI] serve to pin down the collection of situations over which the conditionals are to be evaluated. *E.g.*, the antecedent of [V] identifies situations in which Bill receives a promotion as making up the setting within which one will investigate Bill's going to dinner.

In practice, the antecedent's role in determining this collection of situations is so prominent as to enmesh this collection with the conditional's overall topic. The primary or focal topic of a conversation initiated by a subjunctive conditional can convincingly be described as those contingencies selected by the conditional's antecedent. To introduce a subjunctive conditional *not* about those same contingencies in this setting would obviously trigger a shift in topic.

In other words, [V] is *about* the collection of situations in which Bill is promoted. To follow [V] with a conditional whose antecedent selects a *different* collection of situations amounts to a change in subject. Insofar as the primary topic of [VI] is the collection of situations in which Bill does not go to dinner, shifting topic must accompany any segue from [V] to [VI]. By our informal criteria, then, relevance fails to obtain between [V] and [VI] despite the two sharing strong depth relevance.

Example 3. The final example borrows a familiar case in Hempel's *raven paradox*, which centers around the following sentences:

[VII] All ravens are black.

[VIII] All nonblack things are nonravens.

Although [VII] and [VIII] are phrased in such a way to not explicitly include a conditional, in modernity it has become customary to assume that such universal assertions are representable in a form in which conditionals play the role of the main operator. Indeed, in the relevant tradition itself, a relevant conditional has been proposed to cover restricted quantification in *e.g.* [28].

The raven paradox has seen renewed interest in virtue of a topic-theoretic analysis, *i.e.*, the difference can be read in terms of a difference of topic. Yablo's [10] is particularly interested in this case, locating the topic-theoretic distinction between the two as follows:

> *All ravens are black* is about ravens and how they are colored, not how writing desks are colored or whether nonblack things fail to be ravens... [sentence [VII]] is true in a world because, or by way of, or in virtue of, what that world's ravens are like, not the properties of writing desks or nonblack things.[10, p. 21–22]

Although contraposed, [VII] and [VIII] are *about* different things entirely. Thus, this example provides further evidence that contraposition is not topic-preserving in all cases.

Before proceeding, let us briefly return to our earlier remarks about contraposition to consider what lessons we might draw. For the sake of concreteness we make a centerpiece of the sentence [A]:

[A] $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$

Restricting our attention to the role played by the atom p in establishing the strong depth relevance of [A], note that p appears negatively in its antecedent and consequent in slightly different senses. While the negativity of its sign in $p \rightarrow q$ is ascribable strictly to its location in an implication (*i.e.*, the result of a positional update), the negativity of p in $\neg q \rightarrow \neg p$ is the contribution of the negation sign with which which it is decorated (*i.e.*, the product of a pure update). We can think of pure updates and positional updates as reflections of the manners in which negations and implications process the signs of immediate subformulae, respectively.

In a sense, then, strong depth relevance cannot distinguish between pure and positional updates and, consequently, cannot help but conflate these two distinct aspects of intensional operators' respective topic-theoretic contributions.

What cause do we have for thinking these two contributions to be distinct? To answer, note that each of the pairs of examples we considered took advantage of a common topic-theoretic asymmetry between the antecedent and consequent. In the first example, we found it natural to allow the antecedent to actively dictate the referent of a definite description—and thus its topic—while denying a similar role to the consequent. In the second, the Kratzer-style picture that we had assumed is one in which an antecedent independently selects and constrains the collection of situations for evaluation. This again afforded the antecedent an outsized influence over the determination of topic. The final example, too, privileged the antecedent over the consequent in determining a conditional's overall topic.

In each case, some order of priority was assumed to hold between the antecedent and consequent.[10] Our examples suggest the general possibility of gross disparities in the topic-theoretic influence wielded by antecedent and consequent. Two instances of the same subformula appearing in different sides of an intensional implication may play markedly different roles in a complex's overall topic. As the examples illustrate, juggling a subformula between an antecedent and consequent is not topic-theoretically innocent as it may exert significant side-effects over the overall topic.

We might distill the difference as follows: Negation, while topic-theoretically transformative, executes its transformations cleanly and straightforwardly, *i.e.*, pure updates are indeed pure. The semantics of negation amounts to a simple toggle between two modes of assertion and rejection—neither of which is biased in favor of the other—whence the application of a negation symbol involves nothing more or less than effecting an update to a formula's sign. Consequently, perturbations in a subformula's sign for being placed within a negation symbol are self-contained and innocent of side-effects.

In contrast, changes in sign through positional updates—those changed for having been shuffled to one side of an implication—are invariably laden with the consequences of the shuffling itself. The consequences may generally be negligible but in the cases that we have surveyed, they can be come quite pronounced. Transporting a subformula from consequent to antecedent is more than a switch toggling its sign; the transformation in some contexts involves the assignment of a dramatically increased role to the subformula in the determination of an overall topic.

N.b. the "purity" of an update should not be construed as a commitment to the *emptiness* of a pure update, amounting only to a checkmark. Relevant negation in *e.g.* truth-maker semantics of [19] can radically transform the subject-matter of a the negatum to something wholly disjoint. What is important is that changes in sign via negation and via conditionals correspond *a priori* to distinct topic-theoretic operations. There is a clear illustration of

[10] This priority is not *universal*—Dov Gabbay considers several subtle means through which consequents can exert a similar role in [29]—but the priority is extremely *entrenched*.

this observation in the differences between a DeMorgan "toggle" negation (on the one hand) and intuitionistic negation (on the other), as the source of negativity arises via distinct pathways. As an example, in Nelson's constructive arithmetic, one distinguishes between a strong refutation $\neg A$ and an intuitionistic refutation $A \to \bot$. The types of constructions that these two sentences are *about*—their *topics*—are unequivocally distinct. One is about a direct way to construct witnesses providing a refutation of A while the other is about a method of converting purported proofs of A into absurdities. Different methods of altering sign affect the topics of negata in different ways; in order to preserve the full history of the topic-theoretic contribution made by a subformula, one must be able to distinguish between the two.

Compressing the sequence of pure and positional updates to a subformula's sign into a binary value is therefore incredibly lossy. It thus seems insufficient to expect a feature as coarse as *sign* to record an accurate picture of a subformula's topic-theoretic contribution to a complex. A notion of relevance based on a property *from which this information may be recovered* will be necessary to adequately capture relevance. Additionally, such a notion of relevance can only be expressed given a way to talk about variables that tracks sufficient information to cover these cases.

4 Beyond Strong Depth Relevance

We have outlined some apparent deficiencies with respect to strong depth relevance, *i.e.*, types of cases for which the property is insufficient to ensure relevance. By emphasizing a single logical procedure—contraposition—we were able to propose a narrative explaining the inadequacy.

This narrative was distilled into a simple observation that strong depth relevance is indifferent to distinctions between a subformula's sign changing due to a pure update (*i.e.* due to negation) or a positional update (*i.e.* due to appearing in an antecedent position). Our illustrations involved contexts in the role of an antecedent is heavily amplified—overshadowing the contribution of the consequent—in determining the topic of a conditional.

On a very strict—possibly severe—interpretation of relevance as a guarantor that proceeding from one formula to the next will not incur a change of topic, the foregoing examples suggest that contraposition fails to preserve relevance. Contraposition is an inference, as we have seen, that preserves both sign and depth, meaning that the interruption in this strict notion of relevance occurs in light of such asymmetries between antecedent and consequent. Strong depth relevance's insensitivity to distinctions between pure and positional updates mean that ensuring such a conception of relevance will require a yet stronger condition. We'll now propose an alternative variable sharing property that can address these deficiencies, show that it holds of the weak relevant logic **B** while not holding of all logics enjoying strong depth relevance.[11]

[11] It's worthwhile to note that there is in the literature another way in which depth relevance has been 'expanded'; namely in the exploration of depth relevant logics that aren't typically included in the class of relevant logics. This project has been taken up in e.g. [30].

4.1 Definitions

Definition 11. A cn-sequence is a finite sequence of c's and n's. Where A is a formula and B is a subformula of A, write $A[B]$ for A with a specified occurrence of B as a subformula highlighted. Given any such, we can associate with it a cn-sequence as follows:

- If $B = A$, then $\mathsf{cn}(A[B]) = \langle - \rangle$ is the empty sequence.
- If $\mathsf{cn}(A[B]) = \overline{x}$, $\mathsf{cn}(A[B] \wedge C) = \mathsf{cn}(C \wedge A[B]) = \mathsf{cn}(A[B] \vee C) = \mathsf{cn}(C \vee A[B]) = \overline{x}$.
- If $\mathsf{cn}(A[B]) = \overline{x}$, then $\mathsf{cn}(\neg(A[B])) = n\overline{x}$.
- If $\mathsf{cn}(A[B]) = \overline{x}$, then $\mathsf{cn}(A[B] \to C)) = \mathsf{cn}(C \to A[B]) = c\overline{x}$.

An alternative definition that perhaps sheds more light on what exactly cn-sequences are: produce a parse tree for A. Trace the path from the node of the parse tree labeled by the highlighted occurrence of B to the root of the parse tree. The sequence of **c**onditionals and **n**egations traversed on this path is $\mathsf{cn}(A[B])$.

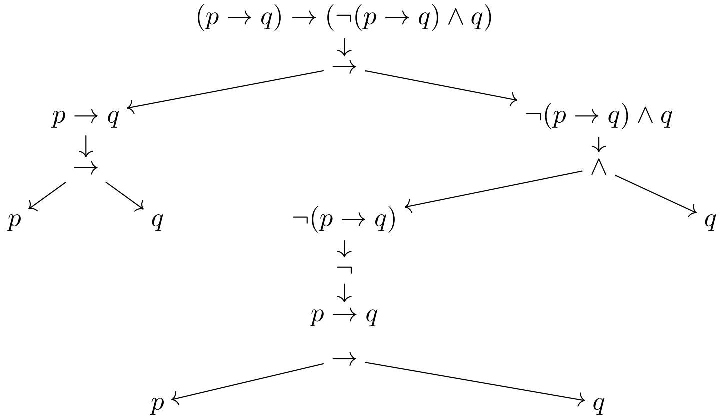
Example 4. Consider $(\underline{p \to q}) \to (\neg(\underline{p \to q}) \wedge q)$, with the underlined occurrence of $p \to q$ highlighted. The cn-sequence of this occurrence of $p \to q$ can be computed from the definition as follows:

- According to the first clause, $\mathsf{cn}(\underline{p \to q}) = \langle - \rangle$.
- Thus, according to the second clause, $\mathsf{cn}((\underline{p \to q}) \to (\neg(p \to q) \wedge q)) = c$.

If we instead consider the second occurrence of $p \to q$, we compute the corresponding cn-sequence as follows:

- According to the first clause, $\mathsf{cn}(\underline{p \to q}) = \langle - \rangle$.
- Thus, according to the third clause, $\mathsf{cn}(\neg(\underline{p \to q})) = n$.
- So, from the second clause, we see that $\mathsf{cn}(\neg(\underline{p \to q}) \wedge q) = n$ as well.
- Thus, from the third clause, we see that $\mathsf{cn}((p \to q) \to (\neg(\underline{p \to q}) \wedge q)) = cn$.

Example 5. Now consider the same formula, but with its parse tree written out:



From this tree, we can compute the cn-sequence of, e.g. $(p \to q) \to (\neg(p \to \underline{q}) \wedge q)$ by simply tracing the path from node labeled by the underlined occurrence of q up to the root of the tree and recording the sequence of **c**onditionals and **n**egations we encounter; in this case the result is *cnc*.

Definition 12. We define an equivalence relation on cn-sequences by saying that $\overline{x}nn\overline{y}$ is immediately similar to $\overline{xy}$ and taking similarity to be the reflexive, symmetric, transitive closure of immediate similarity. We write cns for the set of cn-sequences and take E to be the set of equivalence classes of cn-sequences. We write $[\overline{x}]$ for the equivalence class containing $\overline{x}$.

Since the members of cns are finite sequences from a finite alphabet, cns is countable. And there are clearly no more members of E than there are members of cns. Thus E is also countable.

Definition 13. A function $\sigma : (\mathsf{cns} \times \mathsf{At}) \to \mathcal{L}$ is called a cn-substitution. Given any such function, we extend it to a function $\sigma^+ : (\mathsf{cns} \times \mathcal{L}) \to \mathcal{L}$ as follows:

- For $p \in \mathsf{At}$, $\sigma^+(\overline{x}, p) = \sigma(\overline{x}, p)$.
- $\sigma^+(\overline{x}, A \wedge B) = \sigma^+(\overline{x}, A) \wedge \sigma^+(\overline{x}, B)$.
- $\sigma^+(\overline{x}, A \vee B) = \sigma^+(\overline{x}, A) \vee \sigma^+(\overline{x}, B)$.
- $\sigma^+(\overline{x}, A \to B) = \sigma^+(c\overline{x}, A) \to \sigma^+(c\overline{x}, B)$.
- $\sigma^+(\overline{x}, \neg A) = \neg\sigma^+(n\overline{x}, A)$.

We can describe some important properties that may hold of a particular cn-substitution:

Definition 14. We say that a cn-substitution σ is atomic when its range is a subset of At. We say σ is faithful if $\sigma(\overline{x}, A) = \sigma(\overline{y}, A)$ whenever $[\overline{x}] = [\overline{y}]$. We say σ is essentially injective when $\sigma(\overline{x}, A) = \sigma(\overline{y}, B)$ only if $[\overline{x}] = [\overline{y}]$ and $A = B$. We say a cn-substitution is strong when it is atomic, faithful, and essentially injective.

Example 6. Let π_i be the ith prime number $ln(\overline{x})$ be the length of the sequence $\overline{x}$, and define $\varepsilon(c) = 2$ and $\varepsilon(n) = 1$. We can then give a few useful examples of cn-substitution.

First: let $\chi(\overline{x}, p_k) = 2^k \prod_{i=1}^{ln(\overline{x})} p_{i+1}^{\varepsilon(x_i)}$. We then define $s(\overline{x}, p_k) = p_{\chi(\overline{x}, p_k)}$. Since by construction $\chi(\overline{x}, p_k) = \chi(\overline{y}, p_l)$ just if $\overline{x} = \overline{y}$ and $k = l$, s is essentially (and in fact *actually*) injective. It's also clearly atomic. But s is not faithful because $\chi(\langle - \rangle, p_1) = 2$ but $\chi(nn, p_1) = 30$, so $s(\langle - \rangle, p_1) = p_2 \neq p_{30} = s(nn, p_1)$. Thus s is not strong.

Second: say a cn-sequence $\overline{x}$ is *fully reduced* if it contains no 'nn' subsequences. We take it to be clear that there is a unique fully reduced representative of each cn-equivalence class. So we set $\chi'(\overline{x}, p_k) = \chi(\overline{y}, p_k)$ where $\overline{y}$ is the fully reduced representative of $[\overline{x}]$, and let $t(\overline{x}, p_k) = p_{\chi'(\overline{x}, p_k)}$. t is then an example of a strong cn-substitution.

Definition 15. Given a cn-substitution σ and a cn-sequences $\overline{x}$ and $\overline{y}$, we define the cn-substitution $\sigma^{\overline{x}/\overline{y}}$ as follows:

$$\sigma^{\overline{x}/\overline{y}}(\overline{w}, p) = \begin{cases} \sigma(\overline{zy}, p) & \text{if } \overline{w} = \overline{zx} \text{ for some } \overline{z} \\ \sigma(\overline{w}, p) & \text{otherwise} \end{cases}$$

Put otherwise, whenever it makes sense, $\sigma^{\overline{x}/\overline{y}}(\overline{zx}, p) = \sigma(\overline{zy}, p)$ and otherwise $\sigma^{\overline{x}/\overline{y}}$ is just σ.

Lemma 1. For all formulas A, $\sigma^{\overline{x}/\overline{y}}(\overline{w}, A) = \begin{cases} \sigma(\overline{zy}, A) & \text{if } \overline{w} = \overline{zx} \text{ for some } \overline{z} \\ \sigma(\overline{w}, A) & \text{otherwise} \end{cases}$

Proof By induction on the complexity of A. If A is atomic, the result is immediate from the definition of $\sigma^{\overline{x}/\overline{y}}$.
Where $\odot \in \{\wedge, \vee\}$, $\sigma^{\overline{x}/\overline{y}}(\overline{w}, A_1 \odot A_2) = \sigma^{\overline{x}/\overline{y}}(\overline{w}, A_1) \odot \sigma^{\overline{x}/\overline{y}}(\overline{w}, A_2)$. And by the inductive hypothesis, for $i \in \{1, 2\}$, $\sigma^{\overline{x}/\overline{y}}(\overline{w}, A_i) = \begin{cases} \sigma(\overline{zy}, A_i) & \text{if } \overline{w} = \overline{zx} \text{ for some } \overline{z} \\ \sigma(\overline{w}, A_i) & \text{otherwise} \end{cases}$
Thus

$$\begin{aligned} \sigma^{\overline{x}/\overline{y}}(\overline{w}, A_1 \odot A_2) &= \begin{cases} \sigma(\overline{zy}, A_1) \odot \sigma(\overline{zy}, A_2) & \text{if } \overline{w} = \overline{zx} \text{ for some } \overline{z} \\ \sigma(\overline{w}, A_1) \odot \sigma(\overline{w}, A_1) & \text{otherwise} \end{cases} \\ &= \begin{cases} \sigma(\overline{zy}, A_1 \odot A_2) & \text{if } \overline{w} = \overline{zx} \text{ for some } \overline{z} \\ \sigma(\overline{w}, A_1 \odot A_2) & \text{otherwise} \end{cases} \end{aligned}$$

For negations, note that $\sigma^{\overline{x}/\overline{y}}(\overline{w}, \neg B) = \neg\sigma^{\overline{x}/\overline{y}}(n\overline{w}, B)$. So by the inductive hypothesis,

$$\sigma^{\overline{x}/\overline{y}}(\overline{w}, \neg B) = \begin{cases} \neg\sigma(n\overline{zy}, B) & \text{if } \overline{w} = \overline{zx} \text{ for some } \overline{z} \\ \neg\sigma(n\overline{w}, B) & \text{otherwise} \end{cases}$$

$$= \begin{cases} \sigma(\overline{zy}, \neg B) & \text{if } \overline{w} = \overline{zx} \text{ for some } \overline{z} \\ \sigma(\overline{w}, \neg B) & \text{otherwise} \end{cases}$$

Finally, note that $\sigma^{\overline{x}/\overline{y}}(\overline{w}, A_1 \to A_2) = \sigma^{\overline{x}/\overline{y}}(c\overline{w}, A_1) \to \sigma^{\overline{x}/\overline{y}}(c\overline{w}, A_2)$. By the inductive hypothesis, for $i \in \{1, 2\}$, $\sigma^{\overline{x}/\overline{y}}(c\overline{w}, A_i) = \begin{cases} \sigma(c\overline{zy}, A_i) & \text{if } \overline{w} = \overline{zx} \text{ for some } \overline{z} \\ \sigma(c\overline{w}, A_i) & \text{otherwise} \end{cases}$
Thus

$$\sigma^{\overline{x}/\overline{y}}(\overline{w}, A_1 \to A_2) = \begin{cases} \sigma(c\overline{zy}, A_1) \to \sigma(c\overline{zy}, A_2) & \text{if } \overline{w} = \overline{zx} \text{ for some } \overline{z} \\ \sigma(c\overline{w}, A_1) \to \sigma(c\overline{w}, A_1) & \text{otherwise} \end{cases}$$
$$= \begin{cases} \sigma(\overline{zy}, A_1 \to A_2) & \text{if } \overline{w} = \overline{zx} \text{ for some } \overline{z} \\ \sigma(\overline{w}, A_1 \to A_2) & \text{otherwise} \end{cases}$$

□

We note as special cases the functions $\sigma^{c/\langle - \rangle}$ and $\sigma^{\langle - \rangle/c}$ that are essentially just σ, but modified so as to have (respectively) either one less or one more 'c' at the end of its first argument.

4.2 Strong cn-Relevance

These definitions allow us to formulate the stronger variable sharing property. We note that $\mathsf{cn}(A[p])$ essentially provides a sort of genealogy of the appearance of p in A by providing a record of the applications of pure and positional updates influencing its sign.

Identity between these records would have prevented the identification between the contraposed pairs described in our examples. It thus seems as though insisting on identity between both sign and cn-sequences will induce a reasonable strengthening of strong depth relevance.

Definition 16. A sentence $A \to B$ exhibits the *strong cn-relevance property* if there exist occurrences $A[p]$ of p in A and $B[p]$ of p in B for which:

- $A[p]$ and $B[p]$ have identical sign, and
- $\mathsf{cn}(A[p])$ and $\mathsf{cn}(B[p])$ are similar

Definition 17. A logic L enjoys the strong cn-relevance property if every L-theorem $A \to B$ exhibits strong cn-relevance.

The property has access to the information encoded in the variables' cn-sequences and can apply it when judging whether the "common content" of two variable instances sufficiently overlap.

This thus improves on strong depth relevance insofar as this record is lost in the latter property. In other words, the locus of the inadequacy of the strong depth relevance property is its inability to record the *sequence of updates in sign* accompanying a subformula's path through a complex's parse tree. This inadequacy can be attributed to strong depth relevance property's inability to discern any distinctions between the pure and positional updates, *i.e.*, changes in sign effected by negations and implications, respectively.

The strong cn-relevance property, on the other hand, is sufficiently fine-grained to register many such distinctions. A sequence $\mathsf{cn}(A[B])$ can be understood as providing as a sort of provenance detailing the acquisition of B's sign in A. Recalling our sentence [A], the appearance of n in $\mathsf{cn}((\neg q \to \neg p)[p])$ records that p's sign follows from its appearance within a negation; the absence of n in $\mathsf{cn}((p \to q)[p])$, on the other hand, records that its position within a conditional was the determining feature in its sign.

Thus, strong cn-relevance can be seen to discriminate between cases to which strong depth relevance is indifferent.

4.3 Results

Recall that the logic **B** is axiomatized as follows:

A1. $A \to A$

A2. $(A \wedge B) \to A$

A3. $(A \wedge B) \to B$

A4. $A \to (A \vee B)$

A5. $B \to (A \vee B)$

A6. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

A7. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

A8. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$

A9. $\neg\neg A \to A$

R1. $\dfrac{A \qquad B}{A \wedge B}$

R2. $\dfrac{A \qquad A \to B}{B}$

R3. $\dfrac{A \to B \qquad C \to D}{(B \to C) \to (A \to D)}$

R4. $\dfrac{A \to \neg B}{B \to \neg A}$

B is a very weak relevant logic, referred to by Sylvan *et al.* as the "natural minimal system" [31, p. 285] of relevant logic.

Our goal now is to prove that if $A \to B$ is a theorem of **B**, then there is a variable p that occurs 'under' similar cn-sequences in both A and B. More carefully, we have the following:

Theorem 1 *If $A \to B$ is a theorem of* **B**, *then there are occurrences $A[p]$ of p in A and $B[p]$ of p in B for which $A[p]$ and $B[p]$ are of the same sign and $\mathsf{cn}(A[p])$ and $\mathsf{cn}(B[p])$ are similar.*

As in [12], we will prove this by first showing that **B** is closed under a certain family of substitutions, then showing that the inclusion of **B** into the strong relevant logic **R** turns the usual variable sharing results in **R** into the above (very very strong!) variable sharing result essentially immediately once we apply an injective atomic substitution.

Thus, our first step is to prove the following lemma:

Lemma 2. If A is a theorem of **B** and σ is a faithful cn-substitution then $\sigma(\langle - \rangle, A)$ is a theorem of **B** as well.

Proof By induction on the derivation of A. We first check that each cn-substitution instance of an axiom is an instance of the same axiom. Many cases are obvious, for those that aren't we argue as follows:

A6:

$$\begin{aligned}&\sigma(\langle - \rangle, ((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))) = \\ &\sigma(c, (A \to B) \wedge (A \to C)) \to \sigma(c, A \to (B \wedge C)) = \\ &(\sigma(c, A \to B) \wedge \sigma(c, A \to C)) \to (\sigma(cc, A) \to \sigma(cc, B \wedge C)) = \\ &((\sigma(cc, A) \to \sigma(cc, B)) \wedge (\sigma(cc, A) \to \sigma(cc, C))) \to \\ &\qquad\qquad (\sigma(cc, A) \to (\sigma(cc, B) \wedge \sigma(cc, C)))\end{aligned}$$

Which is again an instance of A6

A9:

$$\begin{aligned}\sigma(\langle - \rangle, \neg\neg A \to A) &= \sigma(c, \neg\neg A) \to \sigma(c, A) \\ &= \neg\neg\sigma(nnc, A) \to \sigma(c, A)\end{aligned}$$

But since nnc and c are in the same equivalence class and σ is faithful, this is again an instance of A9.

For the rules, we argue as follows:

R1: If the last rule applied was R1, then for all σ, $\sigma(\langle - \rangle, A)$ is in $\mathbf{B}$ and $\sigma(\langle - \rangle, B)$ is in $\mathbf{B}$. But then clearly $\sigma(\langle - \rangle, A) \wedge \sigma(\langle - \rangle, B) = \sigma(\langle - \rangle, A \wedge B)$ is in $\mathbf{B}$ as well, as needed.

R2: If the last rule applied was R2, then for all σ, $\sigma(\langle - \rangle, A)$ is in $\mathbf{B}$ and $\sigma(\langle - \rangle, A \to B)$ is in $\mathbf{B}$. Thus in particular $\sigma^{c/\langle - \rangle}(\langle - \rangle, A \to B)$ is in $\mathbf{B}$. But this means that $\sigma^{c/\langle - \rangle}(c, A) \to \sigma^{c/\langle - \rangle}(c, B)$ is in $\mathbf{B}$. So $\sigma(\langle - \rangle, A) \to \sigma(\langle - \rangle, B)$ is in $\mathbf{B}$. So $\sigma(\langle - \rangle, B)$ is in $\mathbf{B}$.

R3: If the last rule applied was R3, then for all σ, $\sigma(\langle - \rangle, A \to B)$ is in $\mathbf{B}$ and $\sigma(\langle - \rangle, C \to D)$ is in $\mathbf{B}$. Thus in particular $\sigma^{\langle - \rangle/c}(\langle - \rangle, A \to B) = \sigma(cc, A) \to \sigma(cc, B)$ and $\sigma^{c/\langle - \rangle}(\langle - \rangle, C \to D) = \sigma(cc, C) \to \sigma(cc, D)$ are in $\mathbf{B}$. Thus $(\sigma(cc, B) \to \sigma(cc, C)) \to (\sigma(cc, A) \to \sigma(cc, D) = \sigma(\langle - \rangle, (B \to C) \to (A \to D)$ is in $\mathbf{B}$.

R4: If the last rule applied was R4, then for all σ, $\sigma(\langle - \rangle, A \to \neg B)$ is in $\mathbf{B}$. But then in particular, $\sigma^{c/nc}(\langle - \rangle, A \to \neg B)$ is in $\mathbf{B}$. Thus, $\sigma^{c/nc}(c, A) \to \neg\sigma^{c/nc}(nc, B)$ is in $\mathbf{B}$. But $\sigma^{c/nc}(c, A) = \sigma(nc, A)$ and $\sigma^{c/nc}(nc, B) = \sigma(nnc, B) = \sigma(c, B)$. So $\sigma(nc, A) \to \neg\sigma(c, B)$ is in $\mathbf{B}$. It follows that $\sigma(c, B) \to \neg\sigma(nc, A) = \sigma(\langle - \rangle, B \to \neg A)$ is in $\mathbf{B}$.

□

From here our strong variable sharing result is in reach. Let's first restate the result, then prove it:

Theorem 1. If $A \to B$ is a theorem of **B**, then there are occurrences $A[p]$ of p in A and $B[p]$ of p in B for which $A[p]$ and $B[p]$ are of the same sign and $\mathsf{cn}(A[p])$ and $\mathsf{cn}(B[p])$ are similar.

Proof Choose a strong cn-substution σ. Since $A \to B$ is a theorem of **B** and σ is strong (and thus faithful), so also is $\sigma(\langle - \rangle, A \to B) = \sigma(c, A) \to \sigma(c, B)$. But then since **B** is a sublogic of **R**, the strong variable sharing property guarantees that there is some variable p that occurs with the same sign in both $\sigma(c, A)$ and $\sigma(c, B)$. Since σ is strong (and thus atomic) every variable that occurs in $\sigma(c, A)$ has the form $\sigma(c\overline{x}, q_A)$ where $\mathsf{cn}(A[q_A]) = \overline{x}$ and every variable that occurs in $\sigma(c, B)$ has the form $\sigma(c\overline{y}, q_B)$ where $\mathsf{cn}(B[q_B]) = \overline{y}$. From the features of variable p, it follows that there are occurrences $A[q_A]$ of q_A in A and $B[q_B]$ of q_B in B whose signs are the same so that $\sigma(c\overline{x}, q_A) = \sigma(c\overline{y}, q_B)$. But since σ is strong (and thus essentially injective) it follows that $q_A = q_B$ and $c\overline{x}$ and $c\overline{y}$ are similar. But then $\overline{x}$ and $\overline{y}$ are similar as required.[12] □

One further item to note is that although every strong depth relevant logic is clearly strongly cn-relevant, the converse does not hold, *i.e.*, strong cn-relevance is a properly stronger species of variable sharing. (Intriguingly, this also allows us to return to the status of the axiom form of contraposition as a coda.) In [21], $\mathbf{DR}^-$—a fragment of Brady's logic **DR** of [32]—is shown to enjoy strong depth relevance. But we can see that strong cn-relevance fails for $\mathbf{DR}^-$, allowing us to infer:

Theorem 2 *Strong cn-relevance is strictly stronger than strong depth relevance*

Proof $\mathbf{DR}^-$ counts the sentential form of contraposition among its axioms. But if one considers the instance $(p \to \neg q) \to (q \to \neg p)$, one notes:

- $\mathsf{cn}(p \to \neg q[p]) = c \nsim cn = \mathsf{cn}(q \to \neg p[p])$
- $\mathsf{cn}(p \to \neg q[q]) = cn \nsim c = \mathsf{cn}(q \to \neg p[q])$

where $\nsim$ indicates that the two sequences are not similar. Thus, $\mathbf{DR}^-$ is not strongly cn-relevant.

□

Note that there are logics intermediate between **B** and $\mathbf{DR}^-$ that enjoy strong cn-relevance as well. One can add, *e.g.*, the axiom

A10. $((A \to B) \wedge (B \to C)) \to (A \to C)$

to **B** and show that the theorems of the resulting relevant logic are closed under similar substitutions as well. One needs merely to replicate the steps of Lemma 2 and add a step in the inductive proof for A10.

[12] As a reviewer points out, one can extract from this proof a proof of the claim that weak-enough relevant logics are depth relevant in the sense of [32]. In fact, the proof in [12] (which was corrected in [33]) is of exactly this form, though for a slightly broader class of logics.

4.4 Lossless cn-Relevance

Before concluding, we will take advantage of an opportunity to address a further way in which cn-relevance can be restricted. One might look at the property of *faithfulness*—that $\sigma(\bar{x}, A) = \sigma(\bar{y}, A)$ in case $\bar{x} \sim \bar{y}$—and note that this is a strong assumption, presupposing *e.g.* that $\bar{x}nn\bar{y}$ and $\bar{x}\bar{y}$ are interchangeable.[13] But this itself constitutes a sort of lossiness of information and an asymmetry with the influence of a conditional—one cannot recover the number of times a negation was applied to a subformula. If one insisted on preserving this information, one would be led to the following variable sharing properties:

Definition 18. A sentence $A \to B$ exhibits the *strong lossless cn-relevance property* if there exist occurrences $A[p]$ of p in A and $B[p]$ of p in B for which:

- $A[p]$ and $B[p]$ have identical sign, and
- $\mathsf{cn}(A[p]) = \mathsf{cn}(B[p])$

Definition 19. A logic L enjoys the strong lossless cn-relevance property if every L-theorem $A \to B$ exhibits strong cn-relevance.

We conclude with a brief investigation into weak relevant logics with this property, giving us a new vantage point from which to distinguish **B** from even weaker relevant logics and to draw a further notion of variable sharing explored in [34] into the discussion.

First, we can start looking for the subsystems of **B** that exhibit this property. Clearly, the inclusion of axiom A9 and rule R4 prevents **B** from enjoying lossless cn-relevance. That A9 and R4 are responsible coheres with the observation that faithfulness encodes a type of lossiness of information concerning negation as both A9 and R4 involve eliminating negation signs. We might then take a cue from setting aside these components of **B**.

We find such a subsystem in the logic **BM** discussed in [35] and [36]. **BM** can be axiomatized by removing A9 from the foregoing axiomatization of **B** and replacing the rule R4 with the following version of rule contraposition:

R4⋆. $\dfrac{A \to B}{\neg B \to \neg A}$

As a subsystem of **B**, that **BM** enjoys strong cn-relevance is trivial. But **BM** enjoys this even stronger property of lossless cn-relevance. To see this, first note the following lemma:

Lemma 3. If A is a theorem of **BM** and σ is *any* cn-substitution, then $\sigma(\langle - \rangle, A)$ is a theorem of **BM** as well.

[13] We thank both reviewers of this paper for drawing our attention to this assumption.

Proof By induction on the length of proofs of A. Since faithfulness of σ featured only in the particular case of R4, all steps in the induction are covered by Lemma 2. This leaves only the case of R4*, which we cover now.[14]

R4*: If the last rule in the proof of the formula was R4* then for all σ, $\sigma(\langle - \rangle, A \to B)$ is provable in **BM**. Fixing σ, we thus have that $\sigma^{c/nc}(\langle - \rangle, A \to B)$ is provable in **BM**. But $\sigma^{c/nc}(\langle - \rangle, A \to B) = \sigma^{c/nc}(c, A) \to \sigma^{c/nc}(c, B) = \sigma(nc, A) \to \sigma(nc, B)$. So $\sigma(nc, A) \to \sigma(nc, B)$ is provable. Thus by R4*, $\neg\sigma(nc, B) \to \neg\sigma(nc, A)$ is in **BM**. But this is $\sigma(c, \neg B) \to \sigma(c, \neg A)$, which is $\sigma(\langle - \rangle, \neg B \to \neg A)$, as required.

□

Following the consequences of Lemma 3 through an identical chain of reasoning as that establishing Theorem 1 ensures that **BM** enjoys strong lossless cn-relevance:

Theorem 3 *If $A \to B$ is a theorem of* **BM**, *then there are occurrences $A[p]$ of p in A and $B[p]$ of p in B for which $A[p]$ and $B[p]$ are of the same sign and $\mathsf{cn}(A[p]) = \mathsf{cn}(B[p])$.*

This also ties into a further restriction of the variable sharing property introduced in [34], in which the Routley star was generalized to examine first degree entailments in which Sylvan's mate function was assumed only to be *cyclical.* This led to a statement of the following scheme for strong variable sharing properties:

Definition 20. A logic has the cyclical variable sharing property with modulus n if a formula $A \to B$ is a theorem only if there exists an atom p and natural numbers j and k such that $j \equiv k \mod n$ for which p appears within the scope of j many negation signs in A and k many negation signs in B.

One can note (along with [34]) that strong variable sharing is equivalent cyclical variable sharing mod 2 in the first-degree case.

The restrictions on the variable sharing property in [34] were largely artificial, arising from a purely model-theoretic investigation of the consequences associated with the parity of the cycles of mate functions. Despite this, the considerations of lossiness that motivate lossless cn-relevance offer a point of intersection that seems far more natural.

The calculus $L\mathbf{E}^{[\![\omega]\!]}_{\mathtt{fde2}}$ enjoys an even stronger property as a consequence of being a subsystem of all the logics $L\mathbf{E}^{[\![n]\!]}_{\mathtt{fde2}}$ introduced in [34], namely, that it exhibits the following:

[14] It's interesting that the same modification of σ—that is, the change from σ to $\sigma^{c/nc}$—does the job in both cases.

Definition 21. A logic has the limiting cyclical variable sharing property if a formula $A \to B$ is a theorem only if there exists an atom p appearing within the scope of the same number of negation signs in A and in B.

In other words, the limit to the cyclical variable sharing with modulus n is insisting on identity between the number of appearances of the negation sign. This is, of course, a consequence of lossless cn-relevance.

Lemma 4. If a logic has the strong lossless cn-relevance property then it has the limiting cyclical variable sharing property

Proof As entries of the symbol n entering into $\mathsf{cn}(A[p])$ correspond to the number of negation signs within which the occurrence of p appears, that $\mathsf{cn}(A[p]) = \mathsf{cn}(B[p])$ entails that p appears within the same number of negation signs in A and B. □

This lemma allows us to bring in the variable sharing properties of [34] and show that they apply to natural higher-degree relevant logics like **BM**:

Corollary 1. **BM** has the limiting cyclical variable sharing property

5 Concluding Remarks

Despite strengthening strong depth relevance, there remain several potentially important distinctions that strong cn-relevance does not appear to capture. For example, the sentences $(p \to q) \to (q \to q)$ and $(q \to q) \to (q \to p)$ are strongly cn-relevant to one another. The formula p has the same sign (positive) in each sentence while the identity between cn-sequences $\mathsf{cn}(((p \to q) \to (q \to q))[p])$ and $\mathsf{cn}(((q \to q) \to (q \to p))[p])$ (*i.e.*, cc) ensures their similarity *a fortiori*. But if we had lauded strong cn-relevance for its general capacity to encode and store precise histories of a subformula's sign, something about this case appears to be lacking.

The history of the sign of the occurrence of p in $(p \to q) \to (q \to q)$ is *dynamic*, *i.e.*, its sign switches from positive to negative and back again as it traces a path across the parse tree. In contrast, the sign of the occurrence of p in $(q \to q) \to (q \to p)$ is *static*, *i.e.*, p remains positive in every subformula of $(q \to q) \to (q \to p)$ in which it appears.

Given our considerations on the topic-theoretic asymmetries that can unbalance the contributions of antecedent and consequent, one might worry about analogous asymmetries with respect to topic-theoretic contributions made by p in the respective complexes. On a Kratzer-style account, the part played by p in $(p \to q) \to (q \to q)$—in which it appears as the antecedent of an antecedent—must likely be pivotal in picking out the states of evaluation, *i.e.*, the overall topic of the complex. In contrast, the appearance of p in $(q \to q) \to (q \to p)$ plays no role in selecting these situations. Despite strong cn-relevance, there is no assurance of parity with respect to the roles p plays in determining the overall topic of the complexes. In the absence such parity, a risk grows that increases in depth are paired with increased risks of topic-theoretic side-effects. It is intriguing that the formulae used above to illustrate such concerns are among those failing to satisfy the "no loose pieces property" identified in Anderson and Belnap's [2] and made precise by Robles and Méndez in [37]. The connection between the no loose pieces property and topic transparency is worth exploring.

Essentially, as suggested in [12], the evolution from variable sharing to strong depth relevance (and now to strong cn-relevance and strong lossless cn-relevance) expresses a sequence of increasingly subtle criteria for relevance in which more and more information from formulas' parse trees is leveraged. The utility of such information is only as good as the tools for its recording and retrieval; indeed, the technological leap from the depth substitutions introduced in [12] to the present cn-substitutions can be viewed as an advance in informational transparency. Whether the technique can admit further refinements to the the representation of such information, whether such improvements will correspond to elegant relevance properties, and whether such properties characterize natural classes of relevant logics are promising questions.

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