Classical Logic is Connexive

Camillo Fiore

Abstract

Connexive logics are based on two ideas: that no statement entails or is entailed by its own negation (this is *Aristotle's thesis*) and that no statement entails both something and the negation of this very thing (this is *Boethius' thesis*). Usually, connexive logics are contra-classical. In this note, I introduce a reading of the connexive theses that makes them compatible with classical logic. According to this reading, the theses in question do not talk about validity alone; rather, they talk in part about (a property related to) the *soundness* of arguments.

Keywords: connexive logics, classical logic, soundness, validity

1 Connexive Logics

Connexive logics are based on two related ideas. The first is that no statement entails or is entailed by its own negation. This is known as *Aristotle's thesis*, and can be formalized in various ways, the most popular being the requirement that the following schemas are valid:

$$\neg(A \to \neg A) \tag{A1}$$

$$\neg(\neg A \to A) \tag{A2}$$

where $\neg$ and $\to$ stand for the object-linguistic negation and conditional, respectively. The second idea is that no statement entails both something and the negation of this very thing. This is known as *Boethius' thesis*, and is usually formalized by means of the schemas

$$(A \to B) \to \neg(A \to \neg B) \tag{B1}$$

$$(A \to \neg B) \to \neg(A \to B) \tag{B2}$$

The nowadays standard definition of connexivity (see, e.g. Wansing [29]) stipulates that a logic is *connexive* just in case it validates the four principles mentioned and, moreover, invalidates the schema expressing the symmetry of the conditional:

$$(A \to B) \to (B \to A) \tag{$\leftrightarrow$}$$

The point of this last requirement is to to prevent the conditional from collapsing with a biconditional. The term 'connexive logic' was introduced by McCall [17] and makes reference to the presence of some connection between the antecedent and the succedent of a valid conditional and/or between the premises and conclusion of a valid argument.[1]

Principles (A1), (A2), (B1) and (B2) are all invalid in classical logic. This is why connexive logics are often taken as paradigmatic examples of 'contra-classical' systems, that is, systems that validate things that classical logic fails to validate.[2] It is well-known, however, that connexivity and classical logic can to some extent be reconciled. For instance, one can read Aristotle's and Boethius' theses as principles about the validity (or otherwise) of certain arguments with contingent premises.[3] Let $\Rightarrow$ stand for entailment and $\not\Rightarrow$ for lack thereof. Consider the following reformulations of the connexive theses:

$$A \not\Rightarrow \neg A \tag{A1$\Rightarrow$}$$

$$\neg A \not\Rightarrow A \tag{A2$\Rightarrow$}$$

$$\text{If} \quad A \Rightarrow B \quad \text{then} \quad A \not\Rightarrow \neg B \tag{B1$\Rightarrow$}$$

$$\text{If} \quad A \Rightarrow \neg B \quad \text{then} \quad A \not\Rightarrow B \tag{B2$\Rightarrow$}$$

They all hold in classical logic when restricted to As that are contingent.[4] And of course, symmetry of entailment

$$\text{If} \quad (A \Rightarrow B) \quad \text{then} \quad (B \Rightarrow A) \tag{$\Leftrightarrow$}$$

[1] Of course, there is more to say about the very notion of connexivity. See, e.g. Estrada-González and Ramírez-Cámara [8] for additional definitions and discussion.

[2] Humberstone [11] is the locus classicus for the notion of contra-classicality.

[3] Various authors have argued for reading the connexive theses as restricted to a certain class of 'normal' statements. Lenzen [16, 15] motivates such a reading from a historical standpoint. Kapsner [13] and Iacona [12] appeal to the semantics of conditionals in natural language.

[4] We can say something even stronger: they all hold in classical logic as long as the things on the left-hand side of the arrows are not contradictions. Wansing et. al. [30] argue that this is quite weak a requirement. The reason is that, if one accepts the view that all contradictions express the same proposition, then there are infinitely many propositions that satisfy the connexive principles, but only one that does not.

fails in classical logic. So, there is a sense in which classical logic vindicates the connexive theses. The vindication is only partial, though, since it relativizes the correctness of these theses to the logical status of certain statements they involve.

In this note, I introduce a reading of the connexive theses that makes them compatible with classical logic in an unrestricted way (that is, without relativization to statements of a certain class). According to this reading, the connexive theses do not only talk about the validity of certain arguments; they also, and crucially, talk about (a property related to) their soundness. For reasons that will become clear, my proposal is closely tied to the way in which Wansing's logic **C** [28] and its kin interpret the falsity conditions of conditional statements.

2 Antisoundness

Logic studies how to discriminate between the good and the bad arguments. Arguments are good or bad depending on various properties they exhibit. From a logical viewpoint, the most important of those properties is validity. One informal understanding of this notion runs as follows: an argument is valid just in case, whenever its premises are in good standing (they are true, assertable, provable, or whatever we ask them to be), the conclusion is in good standing as well. Invalidity is just the lack of validity. All other things being equal, validity is a good property and invalidity a bad one. But invalidity alone does not make an argument useless: there are many invalid arguments which serve perfectly well the purpose of supporting their conclusions. The study of such invalid but useful arguments pertains to the domain of inductive logic.

Invalidity is not the only bad property that arguments may have. It is not even the worst one. An argument can be valid and useless at the same time, and this can happen for logical (as opposed to rhetorical or dialectical) reasons. This is exemplified by what we call *antisound* arguments.[5] Intuitively, an argument from A to B is antisound, written $A \Downarrow B$, just in case, whenever A is in good standing, B is not. To make this a little bit more precise, let $\mathcal{L}$ be a sentential language with variables $p, q, r, ...$ and primitive constants $\neg$, $\rightarrow$ and $\wedge$ under their usual intended meanings. In the case of classical logic we have:

Definition 1. $A \Downarrow B$ in classical logic just in case, for every classical interpretation of $\mathcal{L}$, if A is true B is not.

So, whereas in invalid arguments there is no logical guarantee that the premises support the conclusion, in antisound arguments there *is* a logical guarantee that they *never* support it. We can also read the claim $A \Downarrow B$ as saying that A

[5]As far as I am aware of, the property was first explicitly studied by Cobreros et. al. [5], although under a different name. The label comes from Fiore et. al. [10].

and B are incompatible. Keep in mind that not all invalid arguments are antisound (e.g. $p \not\Rightarrow q$ but $p \not\Downarrow$ q) and not all antisound arguments are invalid (e.g. $p \wedge \neg p \Downarrow q$ but $p \wedge \neg p \Rightarrow q$). That is, antisoundness is not a proper subspecies of the better known properties of validity and invalidity.

Let us go back to connexivity. The standard reading of the connexive theses takes them to talk about the interplay between validity and invalidity. The reading I propose, in contrast, takes them to talk about the interplay between validity and antisoundness. I reformulate the thesis as follows:

$$A \Downarrow \neg A \tag{A1$\Downarrow$}$$

$$\neg A \Downarrow A \tag{A2$\Downarrow$}$$

$$\text{If} \quad A \Rightarrow B \quad \text{then} \quad A \Downarrow \neg B \tag{B1$\Downarrow$}$$

$$\text{If} \quad A \Rightarrow \neg B \quad \text{then} \quad A \Downarrow B \tag{B2$\Downarrow$}$$

So, intuitively, Aristotle's thesis says that a statement never gives support to its own negation, while Boethius' thesis says that if a statement entails something then it never gives support to the negation of this very thing. It is trivial to check that the four principles above hold in classical logic, with no restriction on the range of the schematic letters A and B. Hence, under this reading of connexivity, classical logic vindicates the connexive theses in an unrestricted way.[6]

Some comments to further qualify the proposal. First, the reading I am suggesting is admittedly non literal. Taken at face value, the connexive theses involve negated validity claims. (Remember how we introduced Aristotle's thesis: "no statement entails its own negation".) I do not suggest that a negative validity claim should be understood in terms of antisoundness; on the contrary, I take "not valid" to mean just "invalid". What I suggest, instead, is to read the connexive theses as talking about negated validity only on the surface, and to assume that antisoundness is what is really at stake. This move comes at a certain cost: all other things being equal, a more literal reading is preferable to a less literal one. The payoff is that we can reconcile classical logic with the connexive theses while arguably retaining the intuitive appeal of the latter.

Second, it pays to notice that the reading is not trivial, in the sense that it doesn't make every logic connexive. On the contrary, it allows distinctions between various well-known systems. To exemplify this we will consider the paraconsistent logic **LP** [2, 26], the paracomplete **K3** [14], and the non-transitive

[6] In this note we focus on connexivity of entailment. If we focused on conditionals, we could reformulate the above observation as follows. Take any normal modal logic. Define $A \strictif B$ as $\Box(A \to B)$ and $A \mathrel{\rotatebox[origin=c]{-90}{$\strictif$}} B$ as $A \strictif \neg B$. Then, the system will validate the schemas (i) $A \mathrel{\rotatebox[origin=c]{-90}{$\strictif$}} \neg A$, (ii) $\neg A \mathrel{\rotatebox[origin=c]{-90}{$\strictif$}} A$, (iii) $(A \strictif B) \strictif (A \mathrel{\rotatebox[origin=c]{-90}{$\strictif$}} \neg B)$, and (iv) $(A \strictif \neg B) \strictif (A \mathrel{\rotatebox[origin=c]{-90}{$\strictif$}} B)$.

ST [4].[7] The typical semantics of these systems is given by the strong Kleene evaluation schema, which for the reader's comfort I display below:

$\neg$	A
1	0
$1/2$	$1/2$
0	1

$\wedge$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	$1/2$	$1/2$	0
0	0	0	0

$\rightarrow$	1	$1/2$	0
1	1	$1/2$	0
$1/2$	1	$1/2$	$1/2$
0	1	1	1

Validity is defined as follows: $A \Rightarrow B$ in **LP** (**K3**) [**ST**] just in case, for every strong Kleene interpretation of $\mathcal{L}$, if A has value 1 or $1/2$ (1) [1], then B has value 1 or $1/2$ (1) [1 or $1/2$]. Then, the obvious definitions of antisoundness are:

Definition 2. An argument from A to B is *antisound* in **LP** (**K3**) [**ST**] just in case, for every strong Kleene interpretation of the language, if A has value 1 or $1/2$ (1) [1], then B has value 0 (0 or $1/2$) [0].

It is easy to check that **K3** satisfies all our connexive principles. **LP**, in contrast, satisfies none of them: the interpretation that assigns $1/2$ to p shows that $p \not\Downarrow \neg p$ and $\neg p \not\Downarrow p$, which already falsifies (A1$\Downarrow$) and (A2$\Downarrow$); since we also have that $p \Rightarrow p$ and $\neg p \Rightarrow \neg p$, (B1$\Downarrow$) and (B2$\Downarrow$) are falsified as well. Lastly, the case of **ST** is interesting, because (in our language $\mathcal{L}$) the system is *coextensive* with classical logic (viz. it validates the same arguments). However, it does not satisfy all of our connexive principles. While it does satisfy (A1$\Downarrow$) and (A2$\Downarrow$), it violates (B1$\Downarrow$) and (B2$\Downarrow$): for instance, $p \Rightarrow \neg\neg(q \vee \neg q)$ but $p \not\Downarrow \neg\neg\neg(q \vee \neg q)$ and $p \not\Downarrow \neg(q \vee \neg q)$. Hence, our reading is able to discriminate between systems that are coextensive to one another: classical logic is fully connexive, whereas **ST** is at most partially so.[8]

Third, the reading has some clear antecedents in the connexive literature. In fact, it is closely related to Wansing's well known system **C** [28] and its extensions [3, 9, 22, 23, 24]. In all these systems, the falsity of a conditional $A \rightarrow B$ is equated with the truth of $A \rightarrow \neg B$. The parallel becomes transparent when we notice that, in classical logic, $A \Downarrow B$ just in case $A \Rightarrow \neg B$. We could say Wansing's approach and ours agree in the diagnose of *why* the connexive theses get things right: they do so because, when they say that A does not entail/imply B,

[7] When we say that **ST** is non-transitive we are assuming the *local* understanding of metainferential validity (see [6, 7]). Basically, in **ST** it is not the case that whenever an interpretation satisfies $A \Rightarrow B$ and $B \Rightarrow C$ it also satisfies $A \Rightarrow C$.

[8] A related observation is that some but not all of the systems that have a connexive conditional—viz. a conditional satisfying (A1)–(B2)—are connexive in our sense. For instance, the four-valued logic introduced by Angell [1] and later axiomatized by McCall [18] is. But Wansing's logic **C** and its extensions are not. (I let the interested reader fill in the details.)

they mean that A entails/implies $\neg B$. Where the approaches part ways is in the question of whether the connexive theses can be read in a non-literal way. If the answer is negative, then a non-classical logic is called for. If the answer is positive, classical logic and connexivity can peacefully coexist.

As an anonymous reviewer rightly notes, **C** and its relatives have been criticized by McCall [20] for validating the schema $\neg(A \to B) \to (A \to \neg B)$. From a connexive standpoint, the schema says that if there is no connection between A and B then there must be one between A and $\neg B$. But this is highly counterintuitive, since it clashes with the natural thought that A may not be connected to either B or $\neg B$. Now, I take it that a similar objection does not affect our proposal. Here, the troublesome implication would be the one going from $A \Downarrow B$ to $A \Rightarrow \neg B$. But $A \Downarrow B$ cannot be read as saying that there is no connection between A and B, because that is the informal reading of $A \not\Rightarrow B$, and the two are not equivalent. Instead, $A \Downarrow B$ may be read as saying that there is a connection between A and the *falsity* of B. Assuming that a negation is true when the thing being negated is false, a connection between A and the truth of $\neg B$ follows.

My fourth and last comment is that the proposal explored here is not the only way of using the notion of soundness (or some variation thereof) to reconcile classical logic and connexivity. Indeed, an even simpler approach suggests itself. Let $A \Rrightarrow B$ mean that the argument from A to B is sound—that is, valid and with a true premise. Then, we can reformulate the connexive theses as follows:

$$A \not\Rrightarrow \neg A \tag{A1$\Rrightarrow$}$$

$$\neg A \not\Rrightarrow A \tag{A2$\Rrightarrow$}$$

$$\text{If} \quad A \Rrightarrow B \quad \text{then} \quad A \not\Rrightarrow \neg B \tag{B1$\Rrightarrow$}$$

$$\text{If} \quad A \Rrightarrow \neg B \quad \text{then} \quad A \not\Rrightarrow B \tag{B2$\Rrightarrow$}$$

Under this guise, the connexive theses just talk about the interplay between soundness and unsoundness. These four principles hold for classical logic, in the sense that for any classical interpretation of the language, they hold relative to that interpretation.[9,10] I think that this alternative reading is attractive for its simplicity. The reason why I have focused on the antisoundness based approach is that, in my view, it more interesting from a logical viewpoint. Soundness is not a purely

[9]Take, for instance, (A1$\Rrightarrow$): any classical interpretation will make A false or true; if the former, then the argument from A to $\neg A$ has a false premise; if the latter, then the argument in question has a counterexample; in either case, $A \not\Rrightarrow \neg A$ relative to this interpretation.

[10]When the underlying consequence relation of $\Rrightarrow$ is the classical one, $\Rrightarrow$ comes very close to a system defined by Priest [27]. We have $A \Rightarrow B$ in Priest's system just in case we have both $A \Rightarrow B$ and $A \not\Rightarrow \bot$ in classical logic (where $\bot$ is any contradiction you like). The parallel, then, is that $A \Rightarrow B$ in Priest's system just in case $A \Rrightarrow B$ in classical logic relative to at least one interpretation v. (I thank an anonymous reviewer for pointing this out to me.)

logical property: it partly depends on whatever factors make the premises of an argument true. In contrast, antisoundness seems like a logical property, since it is formal and modal in any sense that one may think that validity is. Hence, it would seem that, by reading the connexive theses in terms of antisoundness, one retains their logical character to a higher degree.[11]

3 Takeaway

Antisoundness is a bad property of arguments. When an argument is antisound, we know by logic alone that the premises never support the conclusion. This fact enables a particular reading of the connexive theses, which makes them compatible with classical logic. According to this reading, the connexive theses say that arguments of certain form are antisound—sometimes on the condition that some other arguments are valid. The reading provides yet another explanation of why the classical logician can find the connexive thesis appealing: they are true statements about certain logically relevant properties that arguments may have.

To finish, let me emphasize that I by no means meant to provide *the* correct approach to connexivity. In particular, everything I said is compatible with the interest and fruitfulness of the non-classical approaches. The goal of this note was just to deepen our understanding of the relation between connexivity, classicality, validity and soundness.

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[11] In this note we mentioned some of the ways to reconcile classical logic and connexivity. To be sure, there are several others. For instance, Pizzi [25] puts forward a system with a connexive conditional that can be defined in the modal logic **T** as $\Box(A \to B) \wedge (\Box A \leftrightarrow \Box B) \wedge (\Diamond A \leftrightarrow \Diamond B)$. For another case, McCall [19] puts forward a system with a connexive conditional that, as shown by Meyer [21], can be defined in the first-degree fragment of **S5** as $\Box(A \to B) \wedge (A \leftrightarrow B)$. Of course, definability results of this kind could also be restated in terms of certain facts holding of classical entailment. However, a systematic study of all the ways in which classical logic and connexivity can come to agree is beyond the aims of this note—it is an interesting subject for future work.

References

[1] R. B. Angell. A propositional logic with subjunctive conditionals. *The Journal of Symbolic Logic*, 27(3):327–343, 1962. https://doi.org/10.2307/2964651.

[2] F. G. Asenjo. A Calculus of Antinomies. *Notre Dame Journal of Formal Logic*, 7(1):103–105, 1966. https://doi.org/10.1305/ndjfl/1093958482.

[3] J. Cantwell. The Logic of Conditional Negation. *Notre Dame Journal of Formal Logic*, 49:245–260, 2008. https://doi.org/10.1215/00294527-2008-010.

[4] P. Cobreros, P. Égré, D. Ripley, and R. Van Rooij. Reaching Transparent Truth. *Mind*, 122(488):841–866, 2013. https://doi.org/10.1093/mind/fzt110.

[5] P. Cobreros, E. La Rosa, and L. Tranchini. (I Can't Get No) Antisatisfaction. *Synthese*, 198(9):8251–8265, 2021. https://doi.org/10.1007/s11229-020-02570-x.

[6] B. Da Ré, D. Szmuc, and P. Teijeiro. Derivability and Metainferential Validity. *Journal of Philosophical Logic*, 51:1521–1547, 2021. https://doi.org/10.1007/s10992-021-09619-3.

[7] B. Dicher and F. Paoli. ST, LP and Tolerant Metainferences. In C. Başkent and T. M. Ferguson, editors, *Graham Priest on Dialetheism and Paraconsistency*, pages 111–134. Springer, 2019. https://doi.org/10.1007/978-3-030-25365-3_18.

[8] L. Estrada-González and E. Ramírez-Cámara. A Comparison of Connexive Logics. *IfCoLog Journal of Logics and their Applications*, 3(3):341–355, 2016.

[9] T. M. Ferguson. Inconsistent Models (and Infinite Models) for Arithmetics With Constructible Falsity. *Logic and Logical Philosophy*, 28(3):389–407, 2019. https://doi.org/10.12775/LLP.2018.011.

[10] C. Fiore, F. Pailos, and M. Rubin. Inferential Constants. *Journal of Philosophical Logic*, 52(3):767–796, 2023. https://doi.org/10.1007/s10992-022-09687-z.

[11] L. Humberstone. Contra-Classical Logics. *Australasian Journal of Philosophy*, 78(4):438–474, 2000. https://doi.org/10.1080/00048400012349741.

[12] A. Iacona. Strictness and connexivity. *Inquiry*, 64(10):1024–1037, 2021. https://doi.org/10.1080/0020174X.2019.1680428.

[13] A. Kapsner. Humble connexivity. *Logic and Logical Philosophy*, 28(3):513–536, 2019. https://doi.org/10.12775/LLP.2019.001.

[14] S. C. Kleene. *Introduction to Metamathematics*. North-Holland, Amsterdam, 1952.

[15] W. Lenzen. Leibniz's Laws of Consistency and the Philosophical Foundations of Connexive Logic. *Logic and Logical Philosophy*, 28(3):537–551, 2019.

https://doi.org/10.12775/LLP.2019.004.

[16] W. Lenzen. A Critical Examination of the Historical Origins of Connexive Logic. *History and Philosophy of Logic*, 41(1):16–35, 2020. https://doi.org/10.1080/01445340.2019.1650610.

[17] S. McCall. *Non-classical Propositional Calculi*. PhD thesis, University of Oxford, 1964.

[18] S. McCall. Connexive Implication. *The Journal of Symbolic Logic*, 31(3):415–433, 1966. https://doi.org/10.2307/2270458.

[19] S. McCall. Connexive Implication. In A. Anderson and N. Belnap, editors, *Entailment: The Logic of Relevance and Necessity* (Volume 1), pages 434–446. Princeton University Press, 1975.

[20] S. McCall. A history of connexivity. In D. G. *et. al.*, editor, *Handbook of the History of Logic* (Volume 11), pages 415–449. Elsevier, 2012.

[21] R. Meyer. S5 - the poor man's connexive implication. *The Relevance Logic Newsletter*, 2(2):117–125, 1977.

[22] G. K. Olkhovikov. On a New Three-Valued Paraconsistent Logic (translated by T. M. Ferguson). *IfCoLog Journal of Logics and their Applications*, 3:317–334, 2016.

[23] H. Omori. From Paraconsistent logic to Dialetheic Logic. In H. Andreas and P. Verdée, editors, *Logical Studies of Paraconsistent Reasoning in Science and Mathematics*, pages 111–134. Springer, 2016. https://doi.org/10.1007/978-3-319-40220-8_8.

[24] H. Omori and H. Wansing. Connexive Logics. An Overview and Current Trends. *Logic and Logical Philosophy*, 28(3):371–387, 2019. https://doi.org/10.12775/LLP.2019.026.

[25] C. Pizzi. Decision procedures for logics of consequential implication. *Notre Dame Journal of Formal Logic*, 32(4):618–636, 1991. https://doi.org/10.1305/ndjfl/1093635934.

[26] G. Priest. The Logic of Paradox. *Journal of Philosophical Logic*, 8(1):219–241, 1979. https://www.jstor.org/stable/30227165.

[27] G. Priest. Negation as cancellation, and connexive logic. *Topoi*, 18(2):141–148, 1999. https://doi.org/10.1023/A:1006294205280.

[28] H. Wansing. Connexive Modal Logic. In R. Schmidt, I. Pratt-Hartmann, and M. Reynolds, editors, *Advances in Modal Logic, Volume 5*, pages 387–399. College Publications, 2004.

[29] H. Wansing. Connexive Logic. In E. N. Zalta and U. Nodelman, editors, *The Stanford Encyclopedia of Philosophy*. Stanford University, 2023. https://plato.stanford.edu/archives/sum2023/entries/logic-connexive/.

[30] H. Wansing, H. Omori, and T. M. Ferguson. The Tenacity of Connexive Logic: Preface to the Special Issue. *IfCoLog Journal of Logics and their Applications*, 3(3):279–296, 2016.