Inferential Constants

Anonymized

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Abstract

A *metainference* is usually understood as a pair consisting of a collection of inferences, called *premises*, and a single inference, called *conclusion*. In the last few years, much attention has been paid to the study of metainferences—and, in particular, to the question of what are the *valid* metainferences of a given logic. So far, however, this study has been done in quite a poor language. Our usual sequent calculi have no way to represent, e.g. negations, disjunctions or conjunctions of inferences. In this paper we tackle this expressive issue. We assume some background sentential language as given and define what we call an *inferential* language, that is, a language whose atomic formulas are inferences. We provide a model-theoretic characterization of *validity* for this language—relative to some given characterization of validity for the background sentential language—and provide a proof-theoretic analysis of validity. We argue that our novel language has fruitful philosophical applications. Lastly, we generalize some of our definitions and results to arbitrary metainferential levels.

Key words: Logical Constants, Metainferences, Metainferential Validity, Non-Classical Logic

1 Introduction

A *metainference* is usually understood as a pair consisting of a collection of inferences, called *premises*, and a single inference, called *conclusion*. Those who have been tracking the literature on metainferences know how much it has grown in the last few years.[1] Some might say that what is being studied is 'the logic of inferences', that is, what (collections of) inferences follow from what.[2] Unfortunately, if we are really trying to untangle the logic behind inferences, we are doing it in quite a poor language. Our usual sequent calculi

[1] The recent interest in metainferences emerged within studies in truth, vagueness and other paradoxical phenomena. At first, they were used as a technical tool to characterize logics **ST**, **TS** and theories based upon them [13, 11, 39, 38, 23]. As the debate progressed, metainferences started to attract philosophical attention. Among other things, they have been used to define infinite hierarchies of 'increasingly classical' theories of naive truth [33, 2, 4], to argue for or against various criteria of identity between logics [47, 34, 5], to show similarities between various *prima facie* very different logical systems [20, 3, 12] and to design refined versions of the collapse argument against logical pluralism [6]. Also, efforts are being made to develop the proof-theory of metainferences of arbitrary levels [25, 15, 43].

[2] An anonymous reviewer rightly notes that the usual reading of a sequent $\Gamma \Rightarrow \Delta$ is that Δ follows from Γ; thus, everyday sequent rules can be plausibly understood as metainferences. (To be more precise, metainferential *schemas*; see [17] for the details.) There is a sense, then, in which the study of metainferences can be traced back to Gentzen's [24] seminal works on sequent calculi.

have no way to represent, e.g. negations of inferences, or a disjunction in the premises of a metainference, or a conjunction in the conclusion. To explain why, and thus provide an intuitive motivation for our work, we will make an informal analogy with what we know best: good old sentential logic.

In the language of any sentential logic, there are lots of arguments that do not involve any logical connective: "p therefore p", "p, q, therefore q", if we allow multiple conclusions also "p, q, therefore p, q", and so on. The validity or otherwise of these arguments does not depend on the meaning of the logical connectives; rather, it is determined by the meaning of logical consequence and, in particular, by the structural properties that logical consequence displays. For example, the argument "p therefore p" is valid in classical logic, for classical consequence is reflexive; likewise, "p, q, therefore q" is valid because classical consequence is monotone. Now, it seems uncontroversial to claim that, when we study sentential logic—or 'the logic of sentences'—we are mostly interested in arguments whose validity or invalidity *does* depend on the meaning of logical connectives: "p *or* q, therefore q", "p, q; therefore p *and* q", and so on. Clearly, all such arguments contain the connectives in question.[3] Thus, suppose we gave a course on sentential logic and restricted ourselves to a language without any logical connective; then, we could be rightly accused of using a far too poor language, and thus giving a blatantly incomplete course.

We suggest that, at the present state of art, 'the logic of inferences' is studied in an analogously poor language. Let φ, ψ, χ and π be inferences. The current approach only considers metainferences such as "φ is valid, therefore ψ is valid", "φ is valid, ψ is valid, therefore χ is valid", if we allow multiples conclusions also "φ is valid, ψ is valid, therefore χ is valid, π is valid", and so on. The validity or otherwise of all these metainferences is determined solely by the meaning of logical consequence. The language lacks the resources to compose or negate validity claims, and so the approach restricts its analysis of metainferential validity to a 'structural level', so to say.

In this paper, we tackle the above expressive issue. We assume some background sentential language as given and define what we call an *inferential language*: a language whose atomic formulas are inferences. The language can express metainferences such as "φ is valid *or* ψ is valid, therefore χ is valid", "φ is valid, therefore ψ is valid *and* χ is valid", and so on. We propose a semantic characterization of validity for this language—relative to some given characterization of validity for the background sentential language. We provide a recipe for building proof systems for these logics. We also give reasons to think that the additional expressive power of our new language is of philosophical interest. Lastly, we extend the framework to arbitrary metainferential levels.

Before we proceed, we would like to address in advance a potential objection. In the literature, metainferential validity is typically defined in such a way that it strongly motivates a certain reading of premises and conclusions in metainferences. According to this reading, premises are to be read conjunctively, and conclusions disjunctively. For instance, the metainference "φ is valid, ψ is valid; therefore χ is valid, π is valid", would amount to "φ is valid *and* ψ is valid; therefore χ is valid *or* π is valid". Hence—the objection goes—the

[3] Notice that the converse is not true: the argument "p and q; therefore p and q" contains logical connectives, but its validity does not depend on their meaning, but on the reflexivity of logical consequence.

current literature already has some resources to express compounds of validity claims.[4] Also, we could perhaps express the idea that an inference φ is not valid with the metainference "φ is valid, therefore $\varnothing$", where '$\varnothing$' is the empty set. All this seems to undermine the intuitive motivation that we gave for our project.

But there are several responses to this objection. For simplicity, we shall focus on conjunctions, but all of our remarks can be transposed to the case of disjunctions. First of all, a collection of inferences and the conjunction of its members may display relevant differences in their logical properties. Consider the metainferences "ϕ is valid, χ is valid; therefore..." and "ϕ is valid *and* χ is valid; therefore...". They may or may not be equivalent to one another, and may or may not be equivalent to the metainferences that result by changing the order and/or amount of occurrences of the conjuncts ("χ is valid *and* ϕ is valid; therefore...", etc.). Whether these equivalences hold is a *logical* fact, and it will depend on the details of our framework (the behavior of the *and* particle, the properties of the collections at stake, and so on). Thus, using collections and using conjunctions cannot be a priori taken to be expressively equivalent things.

Secondly, there are many alternative definitions of metainferential validity. Some of them may lead us away from the standard reading of premises and conclusions in metainferences. For instance, one could define metainferential validity in such a way that both premises and conclusions are read conjunctively. Thus, for example, a metainference of the form "φ is valid, ψ is valid; therefore χ is valid, π is valid" would amount to "φ is valid *and* ψ is valid; therefore χ is valid *and* π is valid". In such a framework, the current approach to metainferences would not be able to express any kind of broadly conceived 'disjunctions' in the conclusions of a metainference.

Thirdly, it must be conceded that, even if we stick to the standard reading, the current suffers severe expressive limitations. It cannot express a conjunction in the conclusions of a metainference, or a disjunction in the premises. Also, even if we assume that the expression "φ is valid; therefore $\varnothing$" expresses the negation of φ, being this expression a metainference itself, it cannot feature among the premises or conclusions of another metainference. Thus, to talk about negated inferences, we would have to retort to so-called *meta-metainferences*, which are much more complicated syntactic objects (see, e.g. [5]). So, our project does undoubtedly help to increase the expressive power of the framework.

Fourth and last. Even if the above answers were not entirely convincing, we do not think that the objection can go too far. The typical definition of multiple-conclusion validity for sentential languages induces an analogous reading of inferences, namely, the set of premises is to be read conjunctively, and the set of conclusions disjunctively. As far as we know, however, nobody takes this fact as evidence that sentential connectives such as conjunction, disjunction and negation can be dispensed with. But we see no reasons why the inferential case would differ, in a relevant sense, from the metainferential one. So, we conclude that, whoever admits the need for logical connectives in standard sentential languages, has good reasons to admit them in what we call 'inferential languages' as well.

The structure of the paper will be as follows. In Section 2, we present a modicum amount

[4] The point could also be raised by appealing to hypersequents. One could argue that, since hypersequents are subject to a disjunctive reading, the current literature already has the means to express disjunctions of sequents. We think that the considerations to be found below also do justice to this variant of the objection.

of technical preliminaries. Then, in Section 3 we define our inferential language and provide the semantic characterization of a metainferential logic for it as well as a recipe for building calculi for these logics. In Section 4, we tackle the expressive power and philosophical relevance of our proposal. In Section 5 we show how to extend the framework for logics of arbitrary metainferential levels. We end with some final remarks in Section 6.

2 Technical Preliminaries

In this section we briefly lay down our starting technical toolkit.

Throughout the paper, we use capital Greek letters $\Gamma, \Delta, \Sigma, ...$ for sets of formulas, and lowercase Greek letters $\varphi, \psi, \chi, \pi, ...$ for schematic formulas of a given language. We hope that the context will indicate the relevant language in each case. That being said, let $\mathcal{L}_0$ be a sentential language with the connectives $\wedge_0, \vee_0$, and $\neg_0$, of arities 2, 2, and 1, intended as conjunction, disjunction and negation, respectively. Let Var be a countable set of sentential variables $\{p, q, r, ...\}$. By $FOR(\mathcal{L}_0)$ we denote the set of well formed formulas of $\mathcal{L}_0$, defined as the carrier set of the absolutely free $\mathcal{L}_0$-algebra generated by Var. We define an expression of the form $\varphi \to \psi$ as $\neg\varphi \vee \psi$.

Since we want our approach to be as general as possible, we will work in a multiple-conclusion framework, though none of our arguments hinges on this.

Definition 1. An inference on $\mathcal{L}_0$ (or inference *simpliciter*) is any element of $\mathcal{P}(FOR(\mathcal{L}_0)) \times \mathcal{P}(FOR(\mathcal{L}_0))$.

So, an inference is any pair $\langle \Gamma, \Delta \rangle$ such that $\Gamma, \Delta \subseteq FOR(\mathcal{L}_0)$. We let $\mathrm{INF}(\mathcal{L}_0)$ denote the set of all inferences. To improve readability, we will often replace the central comma of an inference with the arrow '$\Rightarrow$'; also, we will often drop the curly brackets '{' and '}'. Thus, the inference $\langle \{p \wedge q\}, \{p, q\} \rangle$ will be more easily displayed as $\langle p \wedge q \Rightarrow p, q \rangle$.

In this article we will work with five different logics for our base language $\mathcal{L}_0$: classical logic **CL**, 'Logic of Paradox' **LP** (see [35]), 'Strong Kleene' logic **K3** (see [29]), Strict-Tolerant logic **ST** (see [38]) and 'Tolerant-Strict' logic **TS** (see [23]). There is no strong reason to restrict ourselves to this particular collection of systems. We choose them as a test case of our approach, mainly because they have been intensively studied in the recent literature on metainferences (e.g. [5, 4, 2, 34, 42, 12]). It should be stressed, however, that the framework we offer in this paper can be justifiably applied to a wide range of logical systems, including most of the classical and non-classical systems we usually find in the literature on philosophical logic.[5]

We can characterize all five consequence relations using so-called $K3$ interpretations.

Definition 2. The Strong Kleene algebra $\mathcal{K}3$ is the set $\{0, 1/2, 1\}$ together with the following operations $\dot{\neg}$, $\dot{\wedge}$ and $\dot{\vee}$, of arities 1, 2 and 2, respectively:

$$\dot{\neg}x = 1 - x$$
$$x \dot{\wedge} y = \min(x, y)$$

[5]Indeed, we think that the inferential constants we define are sound for all logical systems where the validity of inferences is a bivalent property, that is, any given inference is either valid or invalid and never both. This includes all systems that we (the authors) know off. (More on this later.)

$$x\dot{\vee}y = \max(x, y)$$

A $K3$ interpretation of $\mathcal{L}_0$ is a homomorphism $v : FOR(\mathcal{L}_0) \to \mathcal{K}3$. We let Val be the set of all $K3$ interpretations of $\mathcal{L}_0$, and use $v[\Gamma]$ to denote the set $\{v(\gamma) : \gamma \in \Gamma\}$.

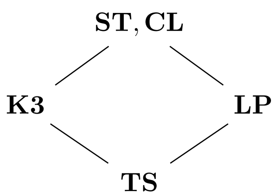
Definition 3. A *standard* is any subset of $\{0, 1/2, 1\}$. If x and y are standards, an interpretation v is said to *xy-satisfy* an inference $\langle\Gamma \Rightarrow \Delta\rangle$, in symbols $v \Vdash_{xy} \langle\Gamma \Rightarrow \Delta\rangle$, just in case, if $v[\Gamma] \subseteq x$, then $v[\Delta] \cap y \neq \varnothing$. If $V \subseteq Val$, an inference $\langle\Gamma \Rightarrow \Delta\rangle$ is *xy-valid* in the valuation space V, in symbols $\vDash^V_{xy} \langle\Gamma \Rightarrow \Delta\rangle$, just in case, for every $v \in V$, $v \Vdash_{xy} \langle\Gamma \Rightarrow \Delta\rangle$. An interpretation $v \in V$ is a *counterexample* to the claim $\vDash^V_{xy} \langle\Gamma \Rightarrow \Delta\rangle$ just in case $v \nVdash_{xy} \langle\Gamma \Rightarrow \Delta\rangle$. Lastly, we understand $\vDash^V_{xy}$ as denoting a consequence relation, and define that, for any $\Gamma, \Delta \subseteq FOR(\mathcal{L}_0)$, $\Gamma \vDash^V_{xy} \Delta$ if and only if $\vDash^V_{xy} \langle\Gamma \Rightarrow \Delta\rangle$.

Two standards will be particularly relevant for us: the 'strict' standard $S = \{1\}$, and the 'tolerant' standard $T = \{1/2, 1\}$. With these, we can define our systems:

Definition 4. Let $V_{\mathbf{CL}}$ be the set $\{v \in Val \,|\, v : \mathcal{L}_0 \to \{0, 1\}\}$.

- Logic **CL** is associated with the relation $\vDash_{\mathbf{CL}} = \vDash^{V_{\mathbf{CL}}}_{TT} = \vDash^{V_{\mathbf{CL}}}_{SS} = \vDash^{V_{\mathbf{CL}}}_{ST} = \vDash^{V_{\mathbf{CL}}}_{TS}$
- Logic **LP** is associated with the relation $\vDash_{\mathbf{LP}} = \vDash^{Val}_{TT}$
- Logic **K3** is associated with the relation $\vDash_{\mathbf{K3}} = \vDash^{Val}_{SS}$
- Logic **ST** is associated with the relation $\vDash_{\mathbf{ST}} = \vDash^{Val}_{ST}$
- Logic **TS** is associated with the relation $\vDash_{\mathbf{TS}} = \vDash^{Val}_{TS}$

To finish this section, we say a few words about the above systems—in case the reader is not familiar with them. They are ordered as per the following diagram, where valid inferences in the logic below are included in valid inferences of the logic above:



LP is a paraconsistent logic that invalidates, among other principles, Explosion, Modus Ponens and Disjunctive Syllogism; **K3** is a paracomplete logic that invalidates the Law of Excluded Middle, Contraposition and Hypothetical Proof. **LP** and **K3** are negation-dual, this meaning that $\varphi_1, ..., \varphi_n \vDash_{\mathbf{LP}} \psi_1, ..., \psi_m$ just in case $\neg\psi_1, ..., \neg\psi_m \vDash_{\mathbf{K3}} \neg\varphi_1, ..., \neg\varphi_n$. **ST** is non-transitive; though it has the same validities as **CL**, it invalidates many classically valid metainferences,[6] such as

$$(Cut)\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \qquad (mMP)\frac{\Rightarrow \varphi \to \psi \quad \Rightarrow \varphi}{\Rightarrow \psi} \qquad (mExp)\frac{\Rightarrow \varphi \quad \Rightarrow \neg\varphi}{\Rightarrow \psi}$$

where '*mMP*' stands for 'meta Modus Ponens' and '*mExp*' stands for 'meta Explosion'. Lastly, **TS** is non-reflexive; in a language without the means to express semantic values, it has no validities at all, though it validates many classically valid metainferences, such as *mMP*, *mExp* and

[6] This claim holds under the so-called local definition of metavalidity (see Section 3.2). Under the alternative global definition, in a language without the means to express semantic values, **CL** and **ST** have the same valid metainferences.

$$mId\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

where '*mId*' stands for 'meta Identity'.

Enough preambles. We can now tackle our proposal.

3 Inferential Logics

We first exhibit our novel language. Then, we provide a semantic definition of validity. Lastly, we develop a proof theory for each of the systems concerned.

3.1 The Language

As we anticipated, we shall define what we call an *inferential language*, that is, a language where atomic formulas are inferences. So, let $\mathcal{L}_1$ be a language with connectives $\wedge_1$, $\vee_1$ and $\neg_1$, of the same arities and intended meanings as the corresponding connectives of $\mathcal{L}_0$. We define the set of $\mathcal{L}_1$-formulas, denoted by $FOR(\mathcal{L}_1)$, as the absolutely free $\mathcal{L}_1$-algebra generated by $\mathrm{INF}(\mathcal{L}_0)$—where $\mathrm{INF}(\mathcal{L}_0)$ is, remember, the set of all $\mathcal{L}_0$-inferences. To illustrate,

$$\neg_1 \langle p \Rightarrow q \rangle$$
$$\langle p \Rightarrow q \rangle \wedge_1 \langle q \Rightarrow p \rangle$$
$$\langle p \Rightarrow q \rangle \vee_1 \langle q \Rightarrow p \rangle$$

are $\mathcal{L}_1$-formulas. We call the members of $\mathrm{INF}(\mathcal{L}_0)$ the *atomic formulas* of $\mathcal{L}_1$. When no confusion threatens, we shall omit the subscripts '1' and '0' from $\mathcal{L}_1$ and $\mathcal{L}_0$ connectives, respectively.

Now, we define inferences in our new language. As in the sentential case, we assume a multiple-conclusion setting.

Definition 5. An inference on $\mathcal{L}_1$ (or *metainference*, for short) is any element of $\mathcal{P}(FOR(\mathcal{L}_1)) \times \mathcal{P}(FOR(\mathcal{L}_1))$.

So, metainferences are just pairs $\langle \Gamma, \Delta \rangle$, where $\Gamma, \Delta \subseteq FOR(\mathcal{L}_1)$. We let $\mathrm{INF}(\mathcal{L}_1)$ denote the set of all metainferences. To improve readability, we will often replace the central comma of a metainference with the arrow '$\Rrightarrow$', drop the curly brackets '{' and '}' and enlarge the outermost corner brackets '⟨' and '⟩'. Thus, the metainference $\langle \{\langle p \Rightarrow q \rangle, \langle q \Rightarrow r \rangle\}, \{\langle p \Rightarrow r \rangle\} \rangle$ will be more easily displayed as $\big\langle \langle p \Rightarrow q \rangle, \langle q \Rightarrow r \rangle \Rrightarrow \langle p \Rightarrow r \rangle \big\rangle$. Also, we shall sometimes display metainferences in a rule-like manner. Thus, the above metainference can be written:

$$\frac{\langle p \Rightarrow q \rangle \qquad \langle q \Rightarrow r \rangle}{\langle p \Rightarrow r \rangle}$$

Notice that, with our language $\mathcal{L}_1$, we can easily express metainferences such as

$$\frac{\langle p \Rightarrow p \rangle \quad \langle q \Rightarrow q \rangle}{\langle p \Rightarrow p \rangle \wedge \langle q \Rightarrow q \rangle} \qquad \frac{\langle p \Rightarrow p \rangle \vee \langle q \Rightarrow q \rangle}{\langle p \Rightarrow p \rangle \quad \langle q \Rightarrow q \rangle} \qquad \frac{}{\neg \langle p \vee \neg p \Rightarrow p \wedge \neg p \rangle}$$

In the Introduction, we argued that these are, at the very least, the kinds of metainferences that the current literature cannot express. Lastly, let say that a metainference $\big\langle \Gamma \Rrightarrow \Delta \big\rangle$ is *structural* if $\Gamma, \Delta \subseteq \text{INF}(\mathcal{L}_0)$, that is, it is a metainference in the (now) traditional sense. If a metainference is not structural, it is *operational.* Validity of a structural metainference does not depend on the meanings of the inferential constants. Validity of an operational metainference may or may not depend on these meanings.

3.2 Semantics

The aim of this section is to provide a definition of validity for metainferences, relative to any of our logics for the base language $\mathcal{L}_0$. We start by defining satisfaction of formulas:

Definition 6. Let x and y be standards, $v \in V \subseteq Val$, and $\varphi \in FOR(\mathcal{L}_1)$. We write $v \Vdash^V_{xy} \varphi$ to abbreviate that v *satisfies* φ *relative to* $\vDash^V_{xy}$, and assume the following conditions:

(a) If φ is atomic, then $v \Vdash^V_{xy} \varphi$ just in case v is not a counterexample to $\vDash^V_{xy} \varphi$.
(b) If $\varphi \equiv \neg_1 \psi$, then $v \Vdash^V_{xy} \varphi$ just in case $v \nVdash^V_{xy} \psi$.
(c) If $\varphi \equiv \psi \vee_1 \delta$, then $v \Vdash^V_{xy} \varphi$ just in case $v \Vdash^V_{xy} \psi$ or $v \Vdash^V_{xy} \delta$.
(d) If $\varphi \equiv \psi \wedge_1 \delta$, then $v \Vdash^V_{xy} \varphi$ just in case $v \Vdash^V_{xy} \psi$ and $v \Vdash^V_{xy} \delta$.

If $\Gamma \subseteq FOR(\mathcal{L}_1)$, by $v \Vdash^V_{xy} \Gamma$ we mean that $v \Vdash^V_{xy} \gamma$ for each γ in Γ.

The intuitive justification of the above semantics is straightforward. The atomic formulas of $\mathcal{L}_1$ are all inferences on $\mathcal{L}_0$, so satisfying them *relative* to a consequence relation $\vDash^V_{xy}$ is nothing more than not being a counterexample to the claim that they are $\vDash^V_{xy}$-valid. Moreover, satisfaction of an inference is a bivalent property in all the systems we address. Hence, satisfaction of complex $\mathcal{L}_1$ formulas can be dealt with classically: a negated formula is satisfied just in case the formula is not, a disjunctive formula is satisfied just in case some disjunct is, and a conjunctive formula is satisfied just in case both conjuncts are.

Now, we shall give a 'recipe' to define metainferential consequence relations. The matter is not trivial, though, for there are at least two competing notions of metainferential validity in the literature, called local and global (see, e.g. [20, 48]).[7] They are usually addressed within a single-conclusion framework lacking any explicit means to express compounds of inferences (negations, conjunctions, and the like). Thus, consider any structural metainference $\langle \Gamma \Rightarrow \delta \rangle$. It is said to be *globally* valid relative to a consequence relation $\vDash^V_{xy}$ just in case, if all the γs in Γ are $\vDash^V_{xy}$-valid, then δ is $\vDash^V_{xy}$-valid. In contrast, it is said to be *locally* valid relative to $\vDash^V_{xy}$ just in case, for every interpretation $v \in V$, if v xy-satisfies all the γs in Γ, then it also xy-satisfies δ. Global validity is meant to capture preservation of validity between inferences. Local validity is meant to capture preservation of satisfaction.

We want to extend these usual notions to a multiple-conclusion framework and to our novel inferential language, that is, to metainferences of the form $\langle \Gamma \Rightarrow \Delta \rangle$, where $\Gamma, \Delta \subseteq FOR(\mathcal{L}_1)$. The local notion is quite straightforward:

Definition 7. A metainference $\langle \Gamma \Rightarrow \Delta \rangle$ is locally valid relative to a consequence relation $\vDash^V_{xy}$, in symbols $\vDash^V_{xy} \langle \Gamma \Rightarrow^{\ell} \Delta \rangle$, just in case, for every $v \in V$, if $v \Vdash^V_{xy} \Gamma$, then $v \Vdash^V_{xy} \delta$ for some $\delta \in \Delta$.

[7] A third interesting though less explored notion is called *absolute global validity*; it can be found in, e.g. [17] and [28].

Thus, a metainference is locally valid, relative to a given sentential logic, just in case, for all relevant interpretations, if all the premises of the metainference are satisfied, some of the conclusions is satisfied as well.

The global notion, however, is more cumbersome. At first glance, one might feel tempted to define it as follows:

Definition 8. A metainference $\langle \Gamma \Rrightarrow \Delta \rangle$ is globally$_1$ valid relative to a consequence relation $\vDash^V_{xy}$, in symbols $\vDash^V_{xy} \langle \Gamma \Rrightarrow^{g_1} \Delta \rangle$, just in case, if $v \Vdash^V_{xy} \Gamma$ for every $v \in V$, then there is a $\delta \in \Delta$ such that $v \Vdash^V_{xy} \delta$ for every $v \in V$.

Thus, a metainference is globally$_1$ valid, relative to a given sentential logic, just in case, if all its premises are valid, some conclusion is valid. As the authors in [16] note, however, this definition might be not entirely felicitous. In the single-conclusion framework, locally valid metainferences constitute a proper subset of globally valid ones. But according to the above definition, that is no longer the case in a multiple conclusion setting. For example, the metainference

$$\frac{\varnothing}{p \Rightarrow \quad \Rightarrow p}$$

would be locally valid but globally invalid relative to **CL**, for each classical interpretation v satisfies all of the premises and at least one conclusion, but no conclusion is satisfied by all classical interpretations. Thus, the authors recommend a different formulation:

Definition 9. A metainference $\langle \Gamma \Rrightarrow \Delta \rangle$ is globally$_2$ valid relative to a consequence relation $\vDash^V_{xy}$, in symbols $\vDash^V_{xy} \langle \Gamma \Rrightarrow^{g_2} \Delta \rangle$, just in case, if $v \Vdash^V_{xy} \Gamma$ for every $v \in V$, then, for every $v \in V$ there is a $\delta \in \Delta$ such that $v \Vdash^V_{xy} \delta$.

Thus, a metainference is globally$_2$ valid, relative to a given sentential logic, just in case, if all its premises are valid, then all relevant interpretations satisfy at least one conclusion. However, we do not think that this definition is entirely satisfactory either. The standard informal reading of global validity is as preservation of validity between inferences, and the above definition is clearly at odds with such a reading.

While we think that both candidates for a definition of global validity are to some extent plausible, we found no conclusive evidence that tips the scales in favor of one of them. Moreover, there are some good reasons for sticking to the local notion anyway. First, as noted in [5], the global notion seems pretty useless to study validity as a unified phenomenon. In the authors' words,

> [W]ere we to adopt [the] global reading for the study of all kinds of inferences, we would need to do this for ordinary inferences too. This would require of valid inferences that, instead of preserving truth (...), they preserve the property of being a tautology. But this is seriously non-standard, as it would deem that the inference $p \Rightarrow q$ is valid—just because p is not a tautology.

Second, local and global validity collapse under fairly weak conditions. In fact, Teijeiro showed in [48] that, for metainferences with a single conclusion, the notions are coextensive as long as the base language has constants for each of the possible values in the semantics and we consider schemes rather that tokens. In what follows, then, we shall not take a stance on the issue of how to define global validity, and focus on local validity instead.

In the next section, we provide recipe for building calculi for the locally valid metainferences of the systems we address.

3.3 Proof Theory

Let us use $\mathbb{L}$ to denote any of the logics we address (that is, **CL**, **LP**, **K3**, **ST** or **TS**). Moreover, assume that there is sequent calculus $\mathcal{S}^0_{\mathbb{L}}$ that is sound and complete with respect to the valid inferences of $\mathbb{L}$, and the rules of $\mathcal{S}^0_{\mathbb{L}}$ have multiple conclusions, in the sense that they allow multiple sequents below the horizontal line. In what follows, we provide a recipe for defining a sequent calculus $\mathcal{S}^1_{\mathbb{L}}$ that is sound and (under certain conditions) complete with respect to the metainferences locally valid in $\mathbb{L}$. Intuitively, rules of $\mathcal{S}^1_{\mathbb{L}}$ go from metainferences to metainferences. Thus, they are objects of the form

$$\frac{\dfrac{\Gamma_1}{\Delta_1} \quad \cdots \quad \dfrac{\Gamma_n}{\Delta_n}}{\dfrac{\Sigma}{\Pi}}$$

where Greek letters stand for sets of $\mathcal{L}_1$-formulas.[8] The bold line separates the *premises* of the rule from the *conclusion*.

Definition 10. The sequent calculus $\mathcal{S}^1_{\mathbb{L}}$ consists of the following rules. First, for each axiom or rule

$$(*)\ \frac{\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_j \Rightarrow \Delta_j}{\Sigma_1 \Rightarrow \Pi_1, \ldots, \Sigma_k \Rightarrow \Pi_k}$$

of $\mathcal{S}^0_{\mathbb{L}}$, $\mathcal{S}^1_{\mathbb{L}}$ contains the axiom

$$(**)\ \frac{\varnothing}{\dfrac{\langle \Gamma_1 \Rightarrow \Delta_1\rangle, \ldots, \langle \Gamma_j \Rightarrow \Delta_j\rangle}{\langle \Sigma_1 \Rightarrow \Pi_1\rangle, \ldots, \langle \Sigma_j \Rightarrow \Pi_j\rangle}}$$

Second, $\mathcal{S}^1_{\mathbb{L}}$ includes the following *operational rules*:

$$\neg C\ \frac{\dfrac{\Gamma, \varphi}{\Delta}}{\dfrac{\Gamma}{\Delta, \neg\varphi}} \qquad\qquad \neg P\ \frac{\dfrac{\Gamma}{\Delta, \varphi}}{\dfrac{\Gamma, \neg\varphi}{\Delta}}$$

$$\wedge P\ \frac{\dfrac{\Gamma, \varphi, \psi}{\Delta}}{\dfrac{\Gamma, \varphi \wedge \psi}{\Delta}} \qquad \wedge C\ \frac{\dfrac{\Gamma}{\Delta, \varphi} \quad \dfrac{\Gamma}{\Delta, \psi}}{\dfrac{\Gamma}{\Delta, \varphi \wedge \psi}} \qquad \vee C\ \frac{\dfrac{\Gamma}{\Delta, \varphi, \psi}}{\dfrac{\Gamma}{\Delta, \varphi \vee \psi}} \qquad \vee P\ \frac{\dfrac{\Gamma, \varphi}{\Delta} \quad \dfrac{\Gamma, \psi}{\Delta}}{\dfrac{\Gamma, \varphi \vee \psi}{\Delta}}$$

Lastly, $\mathcal{S}^1_{\mathbb{L}}$ contains the following *structural rules*:

[8] In more precise terms, the rules go from *schematic* metainferences to *schematic* metainferences. Accordingly, Greek letters stand for sets of schematic $\mathcal{L}_1$-formulas.

$$MId\ \frac{\varnothing}{\frac{\varphi}{\varphi}} \qquad\qquad PW\ \frac{\frac{\Gamma}{\Delta}}{\frac{\Gamma,\varphi}{\Delta}} \qquad\qquad CW\ \frac{\frac{\Gamma}{\Delta}}{\frac{\Gamma}{\Delta,\varphi}} \qquad\qquad MCut\ \frac{\frac{\Gamma}{\varphi,\Delta} \qquad \frac{\varphi,\Sigma}{\Pi}}{\frac{\Gamma,\Sigma}{\Delta,\Pi}}$$

We give a couple of examples of our calculi in action, so the reader can have a better grasp of how they work. First, we show how they recover what we called structural metainferences, that is, metainferences lacking $\mathcal{L}_1$-connectives. Consider the following one:

$$\frac{\varnothing}{A \wedge B \Rightarrow A \wedge B}$$

It is locally valid in classical logic, **CL**, and it has no premises. Thus, any calculus $\mathcal{S}^0_{\mathbf{CL}}$ for classical logic contains a derivation of it. Suppose that the derivation runs as follows:

$$\frac{\dfrac{\overline{A \Rightarrow A}}{A \wedge B \Rightarrow A} \qquad \dfrac{\overline{B \Rightarrow B}}{A \wedge B \Rightarrow B}}{A \wedge B \Rightarrow A \wedge B}$$

Then, our calculus $\mathcal{S}^1_{\mathbf{CL}}$ contains the derivation

$$MCut\ \dfrac{MCut\ \dfrac{MCut\ \dfrac{(**)\ \dfrac{}{\frac{\varnothing}{A \Rightarrow A}} \qquad (**)\ \dfrac{}{\frac{A \Rightarrow A}{A \wedge B \Rightarrow A}}}{\frac{\varnothing}{A \wedge B \Rightarrow A}} \qquad (**)\ \dfrac{}{\frac{A \wedge B \Rightarrow A \quad A \wedge B \Rightarrow B}{A \wedge B \Rightarrow A \wedge B}}}{\frac{A \wedge B \Rightarrow B}{A \wedge B \Rightarrow A \wedge B}} \qquad MCut\ \dfrac{(**)\ \dfrac{}{\frac{\varnothing}{B \Rightarrow B}} \qquad (**)\ \dfrac{}{\frac{B \Rightarrow B}{A \wedge B \Rightarrow B}}}{\frac{\varnothing}{A \wedge B \Rightarrow B}}}{\frac{\varnothing}{A \wedge B \Rightarrow A \wedge B}}$$

As is easy to see, only axioms and *MCut* have been applied. In general, any structural metainference locally valid in $\mathbb{L}$ is provable in $\mathcal{S}^1_{\mathbb{L}}$ using just axioms and *MCut*, provided it is derivable in $\mathcal{S}^0_{\mathbb{L}}$ [9]. Secondly, we give an example of what we called an operational metainference, that is, one that features some $\mathcal{L}_1$-connectives:

$$\frac{\varnothing}{\neg(\langle\Gamma \Rightarrow \Delta\rangle \wedge \neg\langle\Gamma \Rightarrow \Delta\rangle)}$$

The metainference is locally valid in $\mathbb{L}$, for any choice of $\mathbb{L}$. Moreover, it is provable in $\mathcal{S}^1_{\mathbb{L}}$, for any choice of $\mathcal{S}^0_{\mathbb{L}}$:

$$\neg C\ \dfrac{\wedge P\ \dfrac{\neg P\ \dfrac{MId\ \dfrac{}{\frac{\langle\Gamma \Rightarrow \Delta\rangle}{\langle\Gamma \Rightarrow \Delta\rangle}}}{\frac{\langle\Gamma \Rightarrow \Delta\rangle \quad \neg\langle\Gamma \Rightarrow \Delta\rangle}{\varnothing}}}{\frac{\langle\Gamma \Rightarrow \Delta\rangle \wedge \neg\langle\Gamma \Rightarrow \Delta\rangle}{\varnothing}}}{\frac{\varnothing}{\neg(\langle\Gamma \Rightarrow \Delta\rangle \wedge \neg\langle\Gamma \Rightarrow \Delta\rangle)}}$$

[9] We say a metainference $\Gamma \Rightarrow \Delta$ (with $\Gamma, \Delta \subseteq \mathrm{INF}(\mathcal{L}_0)$) is *derivable* in $\mathcal{S}^0_{\mathbb{L}}$ if there is a proof of Δ when each $\gamma \in \Gamma$ is added as an axiom to $\mathcal{S}^0_{\mathbb{L}}$

Now, we are ready to exhibit the soundness and (conditional) completeness results.

Fact 1. $\mathcal{S}^1_{\mathbb{L}}$ is sound with respect to the metainferences locally valid in $\mathbb{L}$.

Proof. It is straightforward to check that every rule or axiom of $\mathcal{S}^1_{\mathbb{L}}$ preserves $\mathbb{L}$-satisfaction from premises to conclusion. □

Fact 2. Suppose that a structural metainference is derivable in $\mathcal{S}^0_{\mathbb{L}}$ just in case it is locally valid in $\mathbb{L}$. Then, $\mathcal{S}^1_{\mathbb{L}}$ is complete with respect to metainferences locally valid in $\mathbb{L}$.

Proof. (Sketch) We proceed by induction on the number of inferential connectives in the metainference.

Base step: $n = 0$. If a metainference without inferential connectives is locally valid in $\mathbb{L}$, it is derivable in $\mathcal{S}^0_{\mathbb{L}}$. So, consider any $\mathcal{S}^0_{\mathbb{L}}$-derivation of it. Since each step in the derivation is an instance of an axiom or rule (*) of $\mathcal{S}^0_{\mathbb{L}}$, it corresponds to an instance of an axiom (**) of $\mathcal{S}^1_{\mathbb{L}}$. We just need to apply the $\mathcal{S}^1_{\mathbb{L}}$-rule of *MCut* a finite amount of times on the corresponding instances of the (**) $\mathcal{S}^1_{\mathbb{L}}$-axioms (as per the example on p. 10) to get a $\mathcal{S}^1_{\mathbb{L}}$-proof of the metainference we address.

Inductive step: $1 \leq n$. There are six cases. We will consider just two of them.

- The metainference is of the form

$$\frac{\Gamma, \neg_1 \phi}{\Delta}$$

If it is locally valid in $\mathbb{L}$, then the metainference

$$\frac{\Gamma}{\Delta, \phi}$$

is locally valid in $\mathbb{L}$ as well. But the latter has a number $m < n$ of $\mathcal{L}_1$-connectives. Thus, by inductive hypothesis, it has a proof in $\mathcal{S}^1_{\mathbb{L}}$. Take the proof in question. Apply the rule $\neg P$ at the last step. We obtain a proof the metainference we are aiming at.

- The metainference is of the form

$$\frac{\Gamma}{\Delta, \phi \wedge_1 \psi}$$

If it is locally valid in $\mathbb{L}$, then the metainferences

$$\frac{\Gamma}{\Delta, \phi} \qquad\qquad \frac{\Gamma}{\Delta, \psi}$$

are both locally valid in $\mathbb{L}$. But they have numbers $l, j < n$ of $\mathcal{L}_1$-connectives. Thus, by inductive hypothesis, they have proofs in $\mathcal{S}^1_{\mathbb{L}}$. Take the proofs in question. Apply the rule $\wedge C$ at their last steps. We obtain a proof the metainference we are aiming at.

The remaining cases are similar.

□

To sum up. Let $\mathbb{L}$ be any of the logics we address. We have made explicit a recipe to build a sequent calculus $\mathcal{S}^1_{\mathbb{L}}$ for the locally valid metainferences of $\mathbb{L}$, provided that there is a base calculus $\mathcal{S}^0_{\mathbb{L}}$ that is sound and complete with respect to the valid inferences of $\mathbb{L}$, and the rules of $\mathcal{S}^0_{\mathbb{L}}$ have multiple conclusions—viz. they allow multiple sequents below the horizontal line. Our calculus $\mathcal{S}^1_{\mathbb{L}}$ will be complete if, additionally, derivability and local validity coincide in $\mathcal{S}^0_{\mathbb{L}}$. As far as we know, there are no examples of such systems in the literature. For instance, Dicher and Paoli, in [20], have proven that $\mathcal{LK}^-_{\mathcal{INV}}$ is not only sound and complete with respect to **ST**, but it is also such that every locally valid metainference of **ST** is derivable and the other way around. Nevertheless, $\mathcal{LK}^-_{\mathcal{INV}}$ does not admit metainferences with multiple conclusions. We hope to develop such sequent-calculi with multiple conclusions, where derivability and local validity coincide, for **ST**, **TS** and the remaining logics addressed in future work.

That was our technical proposal. In the next section, we provide several reasons to think that it has fruitful philosophical applications.

4 Philosophical Motivation

We argue that our inferential language has interesting philosophical applications. First, we show that it allows us to elegantly talk about various properties of inferences that are being studied nowadays. Then, we argue that it allows to better grasp the relationship between some of the logics we address, as for example **ST** and **LP**. Thirdly, we claim that it sheds further light upon an idea that appears from time to time in the literature, namely, that most non-classical systems assume a classical metatheory. Lastly, we show that not every non-classical logics is such that their inferential logic turns out to be classical.

4.1 The many properties of inferences

Arguably, validity is not the only property of inferences that is worth studying. Works [14, 2, 32, 47] define and analyze various other properties in depth. We show that these properties can all be easily expressed with the resources of our inferential language.

First, we introduce the properties at stake. There is no terminological consistency in the literature; for instance, what Scambler [47] calls *antivalidity* is quite a different thing from what Cobreros, La Rosa and Tranchini [14] (from now on, CLT) address under the same label. We propose a terminology of our own, with the hope that it results intuitive and contributes to unification. In each case, we include a footnote disclosing how our labeling relates to that of our sources. We will follow CLT and focus on single-premised and single-conclusioned arguments, though not much hinges on this.

Definition 11. Let v be an arbitrary interpretation in Val

(a) The interpretation v *xy-unsatisfies* an inference $\langle \varphi \Rightarrow \psi \rangle$ just in case $v \nVdash^V_{xy} \langle \varphi \Rightarrow \psi \rangle$, viz. $v(\varphi) \in x$ and $v(\psi) \notin y$. An inference $\langle \varphi \Rightarrow \psi \rangle$ is *xy-antivalid* over the valuation space V if and only if every $v \in V$ xy-unsatisfies it.[10]

(b) The interpretation v *xy-unsoundifies* an inference $\langle \varphi \Rightarrow \psi \rangle$ just in case, if $v(\varphi) \in x$, then $v(\psi) \notin y$. An inference $\langle \varphi \Rightarrow \psi \rangle$ is *xy-antisound* over the valuation space V if and only if every $v \in V$ xy-unsoundifies it.[11]

[10] This is what [2, 47, 32] call antivalidity. We think that it could also be reasonably called *unsatisfiability*. Note that a valuation unsatisfies an inference just in case it is what we define as a counterexample to it.

[11] This is what CLT call *unsatisfiability*. We think the authors' label may be not entirely felicitous, since inferences can have this property and at the same be satisfied by many valuations. (For instance, $\langle p \wedge \neg p \Rightarrow q \rangle$

(c) The interpretation v *xy-supersatisfies* an inference $\langle \varphi \Rightarrow \psi \rangle$ just in case, if $v(\varphi) \notin x$, then $v(\psi) \in y$. An inference $\langle \varphi \Rightarrow \psi \rangle$ is *xy-supervalid* over the valuation space V if and only if every $v \in V$ xy-supersatisfies it.[12]

(d) The interpretation v *xy-inversesatisfies* an inference $\langle \varphi \Rightarrow \psi \rangle$ just in case, if $v(\varphi) \notin x$, then $v(\psi) \notin y$. An inference $\langle \varphi \Rightarrow \psi \rangle$ is *xy-inversevalid* over the valuation space V if and only if every $v \in V$ xy-inversesatisfies it.[13]

The ideas behind the labels intend to be pretty intuitive. To illustrate informally, take the case of **CL**. An inference $\langle \varphi \Rightarrow \psi \rangle$ is antivalid if it cannot be satisfied, so φ is always true and ψ always false; for one such case, take e.g. $\langle p \vee \neg p \Rightarrow q \wedge \neg q \rangle$. An inference $\langle \varphi \Rightarrow \psi \rangle$ is antisound if it cannot be sound, so whenever φ is true, ψ is false; take e.g. $\langle p \Rightarrow \neg p \rangle$. An inference $\langle \varphi \Rightarrow \psi \rangle$ is supervalid if whenever φ is false, ψ is true; take e.g. $\langle p \Rightarrow q \vee \neg q \rangle$. Lastly, an inference $\langle \varphi \Rightarrow \psi \rangle$ is inversevalid if whenever φ is false, ψ is false as well—so, if validity is a strict conditional, inversevalidity is its inversion; take e.g. $\langle p \vee q \Rightarrow q \rangle$.

Next, we show that, when an inference has one of the above properties according to standards x and y in a valuation space V, this fact is expressed by the claim that certain metainference is locally valid according to the same standards, in the same space:

Fact 3. For any standards x, y and $V \subseteq Val$:

(a) An inference $\langle \varphi \Rightarrow \psi \rangle$ is xy-antivalid in the space of valuations V if and only if $\vDash^V_{xy} \big\langle \varnothing \Rightarrow \neg \langle \varphi \Rightarrow \psi \rangle \big\rangle$

(b) An inference $\langle \varphi \Rightarrow \psi \rangle$ is xy-antisound in the space of valuations V if and only if $\vDash^V_{xy} \big\langle \varnothing \Rightarrow \langle \varphi \Rightarrow \varnothing \rangle \vee \neg \langle \varnothing \Rightarrow \psi \rangle \big\rangle$

(c) An inference $\langle \varphi \Rightarrow \psi \rangle$ is xy-supervalid in the space of valuations V if and only if $\vDash^V_{xy} \big\langle \varnothing \Rightarrow \neg \langle \varphi \Rightarrow \varnothing \rangle \vee \langle \varnothing \Rightarrow \psi \rangle \big\rangle$

(d) An inference $\langle \varphi \Rightarrow \psi \rangle$ is xy-inversevalid in the space of valuations V if and only if $\vDash^V_{xy} \big\langle \varnothing \Rightarrow \neg \langle \varphi \Rightarrow \varnothing \rangle \vee \neg \langle \varnothing \Rightarrow \psi \rangle \big\rangle$

(The proof is by inspection of the definitions; we leave the details to the reader.) In our opinion, Fact 3 shows that our inferential language provides a quite elegant and simple way of talking about different properties that inferences may have. Indeed, we suggest that it improves some characterizations of these properties given in the literature.

CLT provide one such characterization. The authors show that an inference has one of the properties (b), (c) or (d) in the valuation space Val, according to some combination of standards S and T, just in case certain other inference is valid in the same valuation space, according to some *other* combination of S and T standards. For instance, an inference $\langle \varphi \Rightarrow \psi \rangle$ is (what we call) antisound in **ST** just in case it is valid in **TS**. This result certainly provides a way of talking about the properties (b), (c) and (d) in terms of simple *validity*. It requires, however, having more than one notion of validity in play. To talk about all these properties in logic **ST**, for example, we would have to resort to **K3**, **LP** and **TS**. In contrast, our inferential language allows us to reduce, so to say, the talk about the properties

is antisound and valid in **CL** as well as in many other logics.)

[12] This is what CLT call supervalidity; so, we coincide with the authors in this respect.

[13] This is what CLT call *antivalidity*. We choose a different label for two reasons. First, the notion of antivalidity is more strongly associated in the literature with property (a). Second, we reckon that our label "inversevalidity" is more descriptive of how an inference with this property behaves across valuations.

(a) to (d) in a given logic to the talk about metainferential validity in *this very logic*. That is, we take it, expressively desirable.

Maybe, CLT could retort by appealing to *meta-metainferences*, which are, intuitively speaking, inferences between metainferences. For instance, in **ST**, an inference $\varphi \Rightarrow \psi$ is antisound just in case the meta-metainference

$$\dfrac{\dfrac{\varphi \Rightarrow \varnothing}{\varnothing}}{\dfrac{\varnothing \Rightarrow \psi}{\varnothing}}$$

is locally valid. Likewise, in each of the logics addressed, properties (a) to (d) of inferences can all be expressed using certain meta-metainferences. So, when we analyze any one of these logics, there is no need to resort to the others, after all.

We grant that the answer is correct. However, we did not want to suggest that our inferential language provides the *only* way of talking about properties (a) to (d) within the very logic where these properties are assessed. Our observation is just that it provides a quite simple and flexible way of doing so. To make an analogy, one could also capture the idea that p is false by appealing to the inference $\langle p \Rightarrow \varnothing \rangle$, and then express the classical validity $\langle \neg p \Rightarrow \neg(p \wedge q) \rangle$ with the metainference

$$\frac{\langle p \Rightarrow \varnothing \rangle}{\langle p \wedge q \Rightarrow \varnothing \rangle}$$

however, the existence of such 'reductions' is not usually taken to show that we can dispense with negation in the sentential language. In a similar vein, the fact that what we can say with our connectives at the metainferential level can also be said without them at the meta-metainferential level should not be taken to establish that our connectives are dispensable.

4.2 Failures of Translation

Lately, there has been an interesting discussion about when two logics can be taken to be similar or even identical. In a number of papers, Cobreros, Egré, Ripley, and van Rooij [11, 13, 10, 9] and Ripley [38, 39, 40, 41] advocated system **ST** as a suitable logic to deal with paradoxes of vagueness and self-reference. One of the main arguments in favor of **ST** is that, as mentioned in Section 2, it has the same valid inferences as classical logic.[14] The authors have always been explicit about the discrepancies between **ST** and classical logic at the metainferential level; however, sometimes they seemed to rely on the inferential agreement to speak as if **ST** were just classical logic:

> This paper has presented and explored a logical framework (...) for adding transparent truth to classical logic [13, 863]

The discussion arose in light of a number of technical results to be found in [37], [3] and [20]. In a nutshell, the results show that there is a strong connection between the metainferences

[14] Moreover, naive theories of vagueness and truth based on **ST** conservatively extend classically valid inferences to the non-logical fragments of their respective languages.

of **ST** and the inferences of **LP**. Under some plausible procedures for translating metainferences into inferences, a metainference is locally valid in **ST** just in case its translation is a valid inference in **LP**.[15] Barrio, Rosenblatt and Tajer [3] have used this fact to argue that, far from being a mere presentation of classical logic, **ST** is in a relevant sense akin to **LP**, and thus, problems usually attributed to theories based on the latter will tend to affect theories based on the former (p. 19).[16] Setting aside an assessment of the authors' argument, the translation result is often taken to illuminate an important kinship between the two systems. Moreover, a similar translation result can be shown to relate systems **TS** and **K3**—that is, a metainference is locally valid in **TS** just in case its translation is a valid inference in **K3**.[17]

Our inferential language allows us to better understand the nature of the kinship involved, for it shows that it is rather limited. In fact, the translation results break once we consider not only structural but also operational metainferences. To see this, consider the following plausible procedure to translate formulas from our inferential to our sentential language:

Definition 12. Let $\tau : FOR(\mathcal{L}_1) \to FOR(\mathcal{L}_0)$ be a function defined as follows, where $\Gamma, \Delta \subseteq FOR(\mathcal{L}_0)$:

$$\tau(\langle \Gamma \Rightarrow \Delta \rangle) = \bot \vee \bigvee\{\neg\gamma : \gamma \in \Gamma) \vee \bigvee\{\delta : \delta \in \Delta\}$$
$$\tau(\neg_1 \varphi) = \neg_0 \tau(\varphi)$$
$$\tau(\varphi \wedge_1 \psi) = \tau(\varphi) \wedge_0 \tau(\psi)$$
$$\tau(\varphi \vee_1 \psi) = \tau(\varphi) \vee_0 \tau(\psi)$$

The atomic case of τ is similar in all relevant respects to translation procedures typically used in the literature, with the only proviso that, following [18], we allow premises and conclusions of inferences to be empty. As for the non-atomic cases, they are self-explanatory. If $\langle \Gamma \Rightarrow \Delta \rangle$ is a metainference, by $\tau(\langle \Gamma \Rightarrow \Delta \rangle)$ we denote the inference $\langle \{\tau(\gamma) : \gamma \in \Gamma\} \Rightarrow \{\tau(\delta) : \delta \in \Delta\} \rangle$. Then, it is straightforward to deploy counterexamples to the the correspondence between **ST** (**TS**) metainferences and **LP** (**K3**) inferences. For the case of **ST** and **LP**, we have

$$\frac{\langle \varnothing \Rightarrow p \rangle \quad \neg\langle \varnothing \Rightarrow p \rangle}{\langle \varnothing \Rightarrow q \rangle} \qquad\qquad p, \neg p \Rightarrow q$$

For **TS** and **K3**, in turn, we have

$$\frac{}{\langle \varnothing \Rightarrow q \rangle \qquad \neg\langle \varnothing \Rightarrow q \rangle} \qquad\qquad \Rightarrow q \vee \neg q$$

The metainference on the left is locally valid in **ST** (**TS**), while its translation on the right is (a paradigmatic case of) an invalid inference in **LP** (**K3**). The intuitive explanation of why the translation results fail in our extended language is simple. As we already mentioned, satisfaction conditions for inferences in *any* of the logics we consider, and thus in particular in **ST** and **TS**, are bivalent; as a consequence, inferential constants in **ST** and **TS** satisfy classical principles such as Explosion and the Law of Excluded Middle. In contrast, truth conditions for sentences in **LP** and **K3** obey a three-valued schema that invalidates the

[15] There are some subtleties concerning equivalent statements of this result, but they need not detain us.
[16] As Cobreros et. al. [12] put it, the charge is that **ST** is nothing more than "**LP** in sheep's clothing".
[17] Even though, as far as we know, the proof is not explicitly present in the literature, it can be done by a simple induction.

principles mentioned. Hence, it comes as no surprise that the metainferential rendering of the principles mentioned is valid in **ST** (**TS**) but its translation is invalid in **LP** (**K3**).

At this point, an objection might arise that, if successful, could undermine the semantics we gave in Section 3.2. The objection goes more or less like this. There are other possible accounts of what it means to *negate* an inference. In particular, authors in [16] and [18] propose to understand the negation of a sequent $\Gamma \Rightarrow \Delta$ as a *negative sequent* $\Gamma \Rightarrow^{-} \Delta$, whose satisfaction conditions are as follows:

Definition 13. An interpretation v is said to *satisfy* $\Gamma \Rightarrow^{-} \Delta$ (with respect to a given pair of standards, in a given valuation space) just in case v satisfies each sequent in the set $\{\Rightarrow \gamma : \gamma \in \Gamma\} \cup \{\delta \Rightarrow : \delta \in \Delta\}$ (with respect to the same standards, in the same space).

The account has some independent appeal, for as shown in [16], it allows to make sense of the usual claim that logics **TS** and **ST** are dual at the metainferential level. Moreover, it is routine to check that the addition of negative sequents to the traditional framework preserves the usual translation results. For concreteness, we exhibit the case of **ST** and **LP**. Let $\Gamma^{(-)}$ denote a set Γ each of whose members is either an ordinary or a negative sequent. For the time being, we loosen our official Definition 5, and call a pair of the form $\langle \Gamma^{(-)} \Rrightarrow \Delta^{(-)} \rangle$ a metainference. Also, we assume that local validity is defined for metainferences of this sort in the standard way, that is, as preservation of satisfaction at every valuation.

Definition 14. Let τ^* be a function from ordinary or negative sequents into $\mathcal{L}_0$, identical to τ for ordinary sequents, and defined for negative sequents as follows:

$$\tau^*(\Gamma \Rightarrow^{-} \Delta) = \top \wedge \bigwedge\{\gamma : \gamma \in \Gamma\} \wedge \bigwedge\{\neg\delta : \delta \in \Delta\}$$

As above, if $\langle \Gamma^{(-)} \Rrightarrow \Delta^{(-)} \rangle$ is a metainference, by $\tau^*(\langle \Gamma^{(-)} \Rrightarrow \Delta^{(-)} \rangle)$ we denote the inference $\langle \{\tau^*(\gamma) : \gamma \in \Gamma^{(-)}\} \Rightarrow \{\tau^*(\delta) : \delta \in \Delta^{(-)}\} \rangle$.

Fact 4. A metainference $\langle \Gamma^{(-)} \Rrightarrow \Delta^{(-)} \rangle$ is locally valid in **ST** just in case the inference $\tau^*(\langle \Gamma^{(-)} \Rrightarrow \Delta^{(-)} \rangle)$ is valid in **LP**

(A proof can be found in [18].) So, to recap, the account of negation in [16] and [18] vindicates the usual translation results between **ST** and **LP** (**TS** and **K3**). Besides—the objection goes—these results are a key piece in the growing consensus that **ST** is in a relevant sense similar to **LP** (and likewise for **TS** to **K3**). Therefore, the account mentioned should be preferred over the given in this paper.

To begin with, we think that the objection is legitimate, in the sense that determining which is the correct account of, e.g., inferential negation is essential for the project of enriching the traditional framework for metainferences. However, in this paper we claim neither that our account is the *only* reasonable, nor that it is the *best*—these bold theses have to wait for the time being. We are content to put our account on the table, and argue that it provides and interesting and hopefully useful extension of the traditional framework. Hence, giving a decisive answer to the objection is beyond the scope of this paper.

That being said, we do not think that the account of inferential negation in [16] and [18] fares better than ours. The reason is that, if there is some consensus around the notion of *negation*, it is that it expresses some kind of semantic opposition (see, e.g., [27]). Opposition may take many forms, but one conspicuous way to understand it assumes that negation is what is called a *contradiction-forming operator* ([36], [19]). The idea is that the negation of a proposition α is contradictory with α; in turn, two propositions are contradictory just in case they can neither be jointly true nor jointly false. However, negative sequents from [16] and [18] are not negations in this sense. This is easy to see, considering any valuation v such that $v(p) = 1/2$. At such a valuation, expressions

$$\varnothing \Rightarrow p \qquad\qquad \varnothing \Rightarrow^{-} p$$

are both satisfied in **ST**, and both dissatisfied in **TS**. Thus, if by saying that a (negative or ordinary) sequent is true at a valuation we mean that it is satisfied at that valuation, it is not in general the case that $\Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow^{-} \Delta$ cannot be jointly true, or that they cannot be jointly false at a valuation.[18]

Maybe, it could be argued that negative sequents constitute some kind of *weak* negation. That might indeed be the case. However, what would be the grounds for embracing a weak negation at the metainferential level? In the object language of our theories we can encounter various phenomena (e.g., vagueness, paradoxes, contingents futures) that are often taken to motivate the abandonment of bivalence and the adoption of some subclassical logic where sentential negation is weak. In the inferential language, however, no such phenomena seem to arise. In all systems we consider, and at any given valuation, an inference is either satisfied or not; at least one of the alternatives holds, and never both. Moreover, the conditions under which any of the alternatives occurs are well-defined and decidable. Hence, we find no need to retreat to any kind of weak metainferential negation.

The moral of this section is the following. The kinship between **ST** and **LP** (**TS** and **K3**) is restricted to those metainferences whose validity does not depend on the meanings of the inferential constants; these comprise, but are not limited to, what we called the structural metainferences. In contrast, whenever the validity of a metainference depends on those meanings, it will be determined by classical standards—the goal of the next section is to give a precise statement of this idea.

4.3 Schematic Classicality

As we have just shown, when we add our inferential connectives to **ST** and **TS**, both systems start to validate certain metainferences whose translations turn out to be invalid in **LP** and **K3**, respectively. The fact that motivates this section is that the metainferences in question are valid not just in **ST** or **TS**, but in each of the logics we address. Indeed, each of these logics validates every instance of the following schemas:

$$MExp\ \frac{\phi, \neg\phi}{\psi} \qquad\qquad MLEM\ \frac{}{\phi \vee \neg\phi}$$

where ϕ and ψ range over formulas of $\mathcal{L}_1$. In what follows, we show that *MExp* and *MLEM* are just examples of a more general phenomenon. Informally, we prove that, in each of the

[18] If the language has the means to express the semantic value $1/2$, there are cases in which both $\langle \Gamma \Rightarrow \Delta \rangle$ and $\langle \Gamma \Rightarrow^{-} \Delta \rangle$ are both valid (invalid) in **ST** (**TS**). Thus, if by saying that a (negative or ordinary) sequent is true *simpliciter* we mean that it is valid, it is also not the case that $\Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow^{-} \Delta$ cannot be jointly true, or that they cannot be jointly false *simpliciter*.

logics we address, a metainference is valid under substitution just in case it is translatable to an inference that is valid in classical logic. To do so, we introduce some bits of technical terminology.

First, let a *substitution* be any function $\sigma : \mathrm{INF}(\mathcal{L}_0) \to FOR(\mathcal{L}_1)$, that is, any function that takes atomic $\mathcal{L}_1$ formulas and delivers (perhaps complex) $\mathcal{L}_1$ formulas. We write $\sigma[\Gamma]$ to denote the set $\{\sigma(\gamma) \mid \gamma \in \Gamma\}$, and $\sigma[\langle\Gamma \Rrightarrow \Delta\rangle]$ to denote the metainference $\langle\sigma[\Gamma] \Rrightarrow \sigma[\Delta]\rangle$. Second, we stipulate that a metainference $\langle\Gamma \Rrightarrow \Delta\rangle$ is *valid under substitution* in $\mathbb{L}$ just in case, for every substitution σ, $\sigma[\langle\Gamma \Rrightarrow \Delta\rangle]$ is locally valid in $\mathbb{L}$ (remember we use $\mathbb{L}$ to denote any of the logics we address). Third and last, we need a procedure to translate metainferences into inferences. This procedure will be different from τ above, for it will be blind to the internal structure of inferences:

Definition 15. Let $\rho : FOR(\mathcal{L}_1) \to FOR(\mathcal{L}_0)$ be an injective function such that:[19]

$$\rho(\phi) \in \mathrm{Var} \text{ for } \phi \in \mathrm{INF}(\mathcal{L}_0)$$
$$\rho(\neg_1 \phi) = \neg_0\, \rho(\phi)$$
$$\rho(\phi \vee_1 \psi) = \rho(\phi) \vee_0 \rho(\psi)$$
$$\rho(\phi \wedge_1 \psi) = \rho(\phi) \vee_0 \rho(\psi)$$

By $\rho[\Gamma]$ we denote the set $\{\rho(\gamma) \mid \gamma \in \Gamma\}$, and by $\rho[\langle\Gamma \Rrightarrow \Delta\rangle]$ the inference $\langle\rho[\Gamma] \Rightarrow \rho[\Delta]\rangle$.

The difference between ρ and τ, then, is that ρ assigns propositional variables to inferences. We say that $\rho[\langle\Gamma \Rrightarrow \Delta\rangle]$ is the *inference translation* of the metainference $\langle\Gamma \Rrightarrow \Delta\rangle$.

Now, we are ready to state the result:

Fact 5. $\langle\Gamma \Rrightarrow \Delta\rangle$ is valid under substitution in $\mathbb{L}$ if and only if its inference translation is valid in **CL**.

Proof. First, the right to left side. We shall appeal to the following Lemma:

Lemma 6. For any $\langle\Gamma \Rrightarrow \Delta\rangle$, σ and $v \in V$, there is a $v^* \in V_{\mathbf{CL}}$ such that, for any $\phi \in \Gamma \cup \Delta$, $v \Vdash_{\mathbb{L}} \sigma(\phi)$ iff $v^*(\rho(\phi)) = 1$.

Proof. We proceed by induction on the complexity of ϕ.

Base step. ϕ is atomic. Since ρ is injective, $\rho(\phi)$ is logically independent of $\rho(\psi)$ for any other atomic ψ in $\Gamma \cup \Delta$. Moreover, given any $v \in V$, either $v \Vdash_{\mathbb{L}} \sigma(\phi)$ or $v \nVdash_{\mathbb{L}} \sigma(\phi)$. Thus, there is a $v^* \in V_{\mathbf{CL}}$ such that $v^*(\rho(\phi)) = 1$ if and only if $v \Vdash_{\mathbb{L}} \phi$.

Inductive step. There are three cases. We consider just two of them.

- ϕ is $\neg_1\psi$. Suppose $v \Vdash_{\mathbb{L}} \sigma(\neg_1\psi)$. Then, by Def. 6, $v \nVdash_{\mathbb{L}} \sigma(\psi)$. Thus, by inductive hypothesis, $v^*(\rho(\psi)) = 0$. Hence, by the classicality of v^*, $v^*(\neg_0\rho(\psi)) = 1$. Therefore, by the definition of ρ, $v^*(\rho(\neg_1\psi)) = 1$. The converse is obtained by similar reasoning.
- ϕ is $\psi \wedge_1 \chi$. Suppose $v \Vdash_{\mathbb{L}} \sigma(\psi \wedge_1 \chi)$. Then, by Def. 6, $v \Vdash_{\mathbb{L}} \sigma(\psi)$ and $v \Vdash_{\mathbb{L}} \sigma(\chi)$. Thus, by inductive hypothesis, $v^*(\rho(\psi)) = v^*(\rho(\chi)) = 1$. Hence, by the classicality of v^*, $v^*(\rho(\psi) \wedge_0 \rho(\chi)) = 1$. Therefore, by the definition of ρ, $v^*(\rho(\psi \wedge_1 \chi)) = 1$. The converse is obtained by similar reasoning.

□

[19] Var is, remember, the set of propositional variables generating the formulas of our base language $\mathcal{L}_0$.

Now, suppose that $\langle \Gamma \Rightarrow \Delta \rangle$ is not valid under substitution in $\mathbb{L}$. Thus, there is a substitution σ such that $\sigma[\langle \Gamma \Rightarrow \Delta \rangle]$ is locally invalid in $\mathbb{L}$. Let $v \in V$ be a witness of this latter fact. By Lemma 6, there is a classical valuation v^* that is a counterexample to $\rho[\langle \Gamma \Rightarrow \Delta \rangle]$.

Secondly, the left to right side. We shall appeal to the following Lemma:

Lemma 7. For any $\rho[\langle \Gamma \Rightarrow \Delta \rangle]$ and any $v \in V_{\mathbf{CL}}$, there is a σ and a $v^* \in V$ such that, for any $\rho(\phi)$ in $\rho[\Gamma] \cup \rho[\Delta]$, $v(\rho(\phi)) = 1$ if and only if $v^* \Vdash_{\mathbb{L}} \sigma(\phi)$

Proof. We proceed by induction on the complexity of ϕ.

Base step. ϕ is atomic.

(i) If $v(\rho(\phi)) = 0$, then if $\mathbb{L}$ has antivalid inferences, let $\sigma(\phi)$ be an antivalid inference of $\mathbb{L}$. If $\mathbb{L}$ has no antivalid inferences, let $\sigma(\phi)$ be $\neg_1 \psi$, for some inference ψ valid in $\mathbb{L}$.

(ii) If $v(\rho(\phi)) = 1$, then if $\mathbb{L}$ has valid inferences, let $\sigma(\phi)$ be a valid inference of $\mathbb{L}$. If $\mathbb{L}$ has no valid inferences, let $\sigma(\phi)$ be $\neg_1 \psi$, for some inference ψ antivalid in $\mathbb{L}$.

Inductive step. There are three cases, each one analogous to the corresponding case of Lemma 6. We leave the details to the reader. □

Now, suppose that $\rho[\langle \Gamma \Rightarrow \Delta \rangle]$ is invalid in $\mathbb{L}$. Let the classical valuation $v \in V_{\mathbf{CL}}$ be a counterexample. By Lemma 7, there is a substitution σ and a valuation $v \in V$ that constitute a counterexample to the validity under substitution of $\langle \Gamma \Rightarrow \Delta \rangle$ in $\mathbb{L}$. □

In the literature on non-classical logics, we often find the claim that most non-classical logicians appeal to classical logic when proving facts about their own favorite logical systems—in other words, they embrace a classical metatheory. The claim often takes the form of a complaint. For example, we have Burgess [7] asking

> How far can a logician who professes to hold that [her favored logic provides] the correct criterion of a valid argument, but who freely accepts and offers standard mathematical proofs, in particular for theorems about [this] logic itself, be regarded as sincere or serious in objecting to classical logic?

The complaint, then, is that non-classical logicians who use a classical metatheory are somehow hypocrite or incoherent, and thus, following Rosenblatt's words [45], should be 'embarrassed'. There have been various proposals to avoid the embarrassment. Some authors take seriously the project of developing a non-classical metatheory for their favored logical systems (see [21] for the case of intuitionistic, and [1] for the case of paraconsistent logic). Others, adopt what has come to be known as the 'recapture strategy'; in few words, they claim that, while classical logic does not provide the correct standard of valid argument in general, it can be justifiably applied to certain specific domains of discourse that are 'safe'—in a sense to be made precise; crucially, among the safe domains we find mathematics (see [26, 30, 49] for arguments against, and [22, 44, 46] for arguments in favor of this strategy).

We think that Fact 5 sheds further light upon the debate and, hopefully, sparks some intuitions among the contenders. The result can be taken to show, in a formal fashion, that there is a sense in which the 'logic of inferences' of any of the systems we address is classical. The systems diverge in some metainferences whose validity depends on the *internal structure* of the inferences involved. Thus, for instance,

$$(Cut)\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

is (as mentioned is Section 2) locally valid in **CL** but invalid in **ST**, for if a valuation assigns $1/2$ to a formula ϕ, then it ST-satisfies $\langle \varnothing \Rightarrow \phi \rangle$ and $\langle \phi \Rightarrow \varnothing \rangle$ but not $\langle \varnothing \Rightarrow \varnothing \rangle$. Now, suppose that we want to, so to speak, 'abstract away' the internal structure of inferences—in other words, treat inferences as non-logical expressions. This seems reasonable when we study the 'logic of inferences', as much as it is seems reasonable to treat atomic sentences as non-logical expressions when we study sentential logic, or the 'logic of sentences'. The obvious way to do so (indeed, the only reasonable way that comes to our mind) is to consider metainferences whose validity is schematic, in the sense that they are valid under substitution. Then, Fact 5 tells us that the systems we address are all coextensive with one another; indeed, they are all, *modulo* translation, coextensive with classical logic.

We do not take a definite stance on whether non-classical logicians who assume a classical metatheory are guilty of incoherence. However, we suggest that, as long as the semantics we provided in Section 3.2 are taken to be correct, it provides evidence in favor of the idea that, at least to some extent, non-classical logicians *are* justified in reasoning classically about their favored systems. This claim has concrete philosophical applications. For example, many metatheoretical proofs appeal to the metarule of contraposition,

$$MCon\,\frac{\phi \rightarrow \psi}{\neg\psi \rightarrow \neg\phi}$$

Sentential contraposition (viz. $\langle \phi \rightarrow \psi \Rightarrow \neg\psi \rightarrow \neg\phi \rangle$) is invalid in **K3**. By Fact 5, however, *MCon* is valid in this system. Thus, the metarule should be acceptable by the lights of paracomplete logicians who endorse **K3** and accept our framework for inferential connectives. They would not be hypocrite by using this metarule, not anymore than a classical logician who performs a Modus Ponens.

4.4 Deviant Metalogics

The non-classical logics that we study in this paper are such that, in a sense, their 'logic of inferences' turns out to be classical. This could perhaps rise some doubts about the relevance of the present project. Virtually all logical systems that we find in the literature are similar to the systems we analyze here in that validity is a bivalent property: an inference is either valid or invalid, and never both. Then, one could conjecture that all those systems satisfy a result analogous to Fact 5, that is, their 'logic of inferences' is classical as well. But if this is the case—the thought goes—then our inferential constants may be of not much use after all. They would not serve to show differences between logical systems. On the contrary, they could be regarded as little more than a regimentation of our classical reasoning about various non-classical logics.[20]

We admit that it would be interesting to develop systems where validity is *not* a bivalent property. This would surely induce a non-classical behavior in the inferential constants. However, such systems are not necessary to vindicate the theoretical interest of the inferential constants. The reason is that, even if we restrict ourselves to systems where validity is

[20]We thank an anonymous reviewer for pressing us on this point.

bivalent, Fact 5 does not hold in general. That is, there are systems whose metainferences valid under substitution do not coincide with those of classical logic, once the inferential constants are added. The aim of this subsection is to show this fact.

Chemla et al. [8] introduce the notion of a *mixed* consequence relation. Intuitively, a consequence relation is mixed if and only if it can be characterized through two possibly different sets of designated values, one for the premises, the other for the conclusions. A logic, in turn, is mixed just in case its consequence relation is mixed. (Notice, then, that these notions are quite encompassing: for instance, the five logics we have analyzed so far are all mixed.) Pailos [31] applies these notions to the analysis of metainferences, thus defining what he calls *mixed metainferential logics*. One of those systems is **ST/TS**:

Definition 16. A metainference $\langle \Gamma \Rightarrow \Delta \rangle$ is locally valid in **ST/TS** just in case, for every $v \in Val$, if $v \Vdash_{\mathbf{ST}} \Gamma$ then $v \Vdash_{\mathbf{TS}} \delta$ for some $\delta \in \Delta$.

Pailos argues that **ST/TS** is a *metainferentially empty* logic, and in fact he proves that all single-conclusion structural metainferences (that is, metainferences of the form $\langle \Gamma \Rightarrow \delta \rangle$, where δ and each of the things in Γ are inferences) are invalid in the system. Intuitively, the reason why this happens is that while in **ST** (the standard for premises) no inference is unsatisfiable, in **TS** (the standard for conclusions) no inference is valid. The emptiness result is robust under the extension to a multiple-conclusion framework. Remarkably, however, it fails when we consider operational metainferences. For instance, *MExp* and *MLEM*, which for the readers comfort we exhibit again

$$MExp\ \frac{\phi, \neg\phi}{\psi} \qquad\qquad MLEM\ \frac{}{\phi \vee \neg\phi}$$

both become locally valid in **ST/TS**. At the same time, the system does not satisfy a result analogous to Fact 5, because, for example, the metainference

$$\frac{p \Rightarrow p}{p \Rightarrow p}$$

will have an inference translation that is valid in **CL**, but is not locally valid in **ST/TS** (to wit, consider the valuation that assigns $1/2$ to p), and so *a fortiori* it is not valid under substitution. In short, one can argue, first, that the apparent emptiness of **ST/TS** was just a byproduct of a too poor language and that, once that language is properly enriched, the system is not empty at all. Second, the 'logic of inferences' of **ST/TS** is *not* classical logic; indeed, it is not even reflexive.[21]

Let us use $\overline{\varnothing}$ as a nickname for the set $\{1, 1/2, 0\}$. Another mixed metainferential logic that is worth mentioning in this context is $\overline{\varnothing}\varnothing/\varnothing\overline{\varnothing}$:

Definition 17. A metainference $\langle \Gamma \Rightarrow \Delta \rangle$ is locally valid in $\overline{\varnothing}\varnothing/\varnothing\overline{\varnothing}$ just in case, for every $v \in Val$, if $v \Vdash_{\overline{\varnothing}\varnothing} \Gamma$ then $v \Vdash_{\varnothing\overline{\varnothing}} \delta$ for some $\delta \in \Delta$.

[21] For the record, we do not know (and it would be interesting to investigate) what is the sentential logic that corresponds, *modulo translation*, with the schematic metainferential validities of **ST/TS**. A similar remark applies to system $\overline{\varnothing}\varnothing/\varnothing\overline{\varnothing}$ to be found below.

Pailos argues that $\overline{\varnothing}\varnothing/\varnothing\overline{\varnothing}$ is a *metainferentially trivial* logic, and again, he proves this contention for single-conclusion structural metainferences. Intuitively, the reason why all metainferences of this kind are valid in this system is that while $\overline{\varnothing}\varnothing$ (the standard for premises) is empty, $\varnothing\overline{\varnothing}$ (the premise standard for conclusions) is trivial. The triviality result is robust under extension to a multiple-conclusion framework. But, as the reader may expect already, it fails for operational metainferences. For instance, the schemas

$$\frac{\neg\phi}{\neg\phi} \qquad\qquad\qquad \frac{}{\neg\phi}$$

become both invalid in $\overline{\varnothing}\varnothing/\varnothing\overline{\varnothing}$. Of course, this also implies that the system does not satisfy a result analogous to Fact 5, since the leftmost of the above schemas has invalid instances whose inference translations are valid in **CL**. Once more, we can draw two conclusions: first, that the triviality of $\overline{\varnothing}\varnothing/\varnothing\overline{\varnothing}$ is relative to the language, and that the 'logic of inferences' of this system is not classical.

We think that the above considerations suffice to show that our inferential connectives *do* contribute to a fruitful comparison between different logical systems; indeed, they even help to reveal certain attributes of these systems that tend to remain hidden when only structural metainferences are under consideration.

That's it for the philosophical motivations of our work. We have defined logical constants that help us expressing some metainferences that are not usually studied in the traditional framework. One might wonder whether our framework can be extended further, to so-called arbitrary metainferential levels. In the next section we give a positive answer and briefly explain how to do so.

5 Generalizing to Higher Levels

Perhaps unsurprisingly, the literature on metainferences is not limited to the study of metainferences proper. Authors in [5, 33, 34, 47] define and work with so-called *hierarchies* of metainferential languages. Roughly, ordinary inferences are taken to be metainferences of level 0, or meta$_0$inferences. In turn, metainferences of level n, or meta$_n$inferences, are defined as pairs of sets of metainferences of level $n-1$. Thus, proper metainferences are meta$_1$inferences, meta-metainferences are meta$_2$inferences, and so on. In this section, we show how to extend our framework to arbitrary metainferential levels.

We start by presenting our languages. Recall that $\mathcal{L}_0$ is a standard sentential language. For any $n > 0$, we shall define a *meta$_n$inferential language*, that is, a language where the atomic formulas are metainferences of level n. Notice that $\mathcal{L}_1$ is a meta$_0$inferential language, for its atomic formulas are inferences. Similarly, each of our languages $\mathcal{L}_i$ will be a meta$_{i-1}$inferential language.

That being said, for any $n > 0$, let $\mathcal{L}_n$ be a language with connectives $\wedge_n$, $\vee_n$ and $\neg_n$, of the same arities and intended meanings as the corresponding connectives of $\mathcal{L}_{n-1}$. We define the set of $\mathcal{L}_n$-formulas, denoted by $FOR(\mathcal{L}_n)$, as the absolutely free $\mathcal{L}_n$-algebra generated by $\text{INF}(\mathcal{L}_{n-1})$, where $\text{INF}(\mathcal{L}_{n-1})$ is the set $\mathcal{P}(FOR(\mathcal{L}_{n-1})) \times \mathcal{P}(FOR(\mathcal{L}_{n-1}))$. Thus, for instance,

$$\neg_2\big\langle \neg_1\neg_1\langle\Gamma\Rightarrow\Delta\rangle \Rightarrow \langle\Gamma\Rightarrow\Delta\rangle\big\rangle \tag{1}$$

is a $\mathcal{L}_2$-formula. Meta$_n$inferences are defined in the expected way:

Definition 18. For $n \geq 0$, a *meta$_n$inference* is any element of $\mathcal{P}(FOR(\mathcal{L}_n)) \times \mathcal{P}(FOR(\mathcal{L}_n))$

Thus, for any $n \geq 0$, INF$(\mathcal{L}_n)$ is the set of all meta$_n$inferences. To improve readability, we shall use a labeled arrow $\Rightarrow^n$ to indicate a metainference of level n. Thus, for example,

$$\varnothing \Rightarrow^2 \neg_2\big\langle \neg_1\neg_1\langle\Gamma\Rightarrow^0\Delta\rangle \Rightarrow^1 \langle\Gamma\Rightarrow^0\Delta\rangle\big\rangle$$

is the meta$_2$inference without premises that has the $\mathcal{L}_2$-formula (1) as only conclusion.

Next, we extend the semantics provided in Section 3.2.

Definition 19. Let x and y be standards, $v \in V \subseteq Val$, and $\varphi \in FOR(\mathcal{L}_n)$, with $n > 0$. We write $v \Vdash^V_{xy} \varphi$ to abbreviate that v *satisfies* φ *relative to* $\vDash^V_{xy}$, and assume the conditions:

(a) If φ is atomic, viz. of the form $\big\langle\Gamma \Rightarrow^{n-1} \Delta\big\rangle$, then $v \Vdash^V_{xy} \varphi$ just in case $v \nVdash^V_{xy} \Gamma$ or $v \Vdash^V_{xy} \delta$ for some $\delta \in \Delta$

(b) If $\varphi \equiv \neg_n \psi$, then $v \Vdash^V_{xy} \varphi$ just in case $v \nVdash^V_{xy} \psi$.

(c) If $\varphi \equiv \psi \vee_n \delta$, then $v \Vdash^V_{xy} \varphi$ just in case $v \Vdash^V_{xy} \psi$ or $v \Vdash^V_{xy} \delta$.

(d) If $\varphi \equiv \psi \wedge_n \delta$, then $v \Vdash^V_{xy} \varphi$ just in case $v \Vdash^V_{xy} \psi$ and $v \Vdash^V_{xy} \delta$.

If $\Gamma \subseteq FOR(\mathcal{L}_n)$, by $v \Vdash^V_{xy} \Gamma$ we mean that $v \Vdash^V_{xy} \gamma$ for each γ in Γ.

Lastly, we define local validity at every metainferential level n:

Definition 20. A metainference $\big\langle\Gamma \Rightarrow^n \Delta\big\rangle$ is locally valid relative to a consequence relation $\vDash^V_{xy}$, in symbols $\vDash^V_{xy} \big\langle\Gamma \Rightarrow^{n^\ell} \Delta\big\rangle$, just in case, for every $v \in V$, if $v \Vdash^V_{xy} \Gamma$, then $v \Vdash^V_{xy} \delta$ for some $\delta \in \Delta$.

As the reader can see, we have just generalized our Definition 7 of local validity at the meta$_1$inferential level.

The remaining definitions and results of Sections 3 and 4 can be extended to arbitrary metainferential levels without major difficulties. For one example, let $n > 0$ and suppose we want a calculus for the meta$_n$inferences locally valid in $\mathbb{L}$. Assume that there is a calculus $\mathcal{S}^{n-1}_{\mathbb{L}}$ that has rules with multiple conclusions and is sound and complete with respect to the meta$_{n-1}$inferences locally valid in $\mathbb{L}$. Then, our calculus $\mathcal{S}^n_{\mathbb{L}}$ can be characterized just as per Definition 10, with the only difference that all schematic letters will range over $\mathcal{L}^{n-1}$-formulas (and thus, in particular, axioms and rules (*) will be part of $\mathcal{S}^{n-1}_{\mathbb{L}}$). For another example, Fact 5 can be easily extended to arbitrary metainferential levels: for any $n > 0$, a schematic meta$_n$inference is valid in $\mathbb{L}$ just in case its translation into an inference is valid in classical logic. We leave it to the reader to fill in the details.

6 Final remarks

In this paper we added inferential connectives to some well known logical systems (namely, **CL**, **LP**, **K3**, **ST** and **TS**), in order to expand the expressive resources of the usual framework for studying what we call the logic of inferences. We provided a semantic and a proof-theoretic analysis of validity for our new inferential language. Also, we gave various reasons to think that the language in question is of philosophical interest. Lastly, we briefly explained how to generalize our framework to arbitrary metainferential levels.

To finish, we hint at some prospects for future work in this line of research. First, it would be interesting to expand the new language with modal and/or first order expressive resources, and study the various phenomena that this expansion might bring into play (e.g. an object-language representation of the distinction between global and local validity, the arousal of semantic paradoxes, among others). Secondly, it would be exciting to study non-classical notions of satisfaction, both from a philosophical and a technical perspective; in particular, gappy and/or glutty notions of satisfaction could be developed, and inferential constants that accord with these notions could be defined. Lastly (and as advanced in fn. 21), we would like to know what are the sentential logics that, *modulo* translation, correspond with metainferences valid under substitution in **ST/TS** and $\overline{\emptyset}\emptyset/\emptyset\overline{\emptyset}$. Also, we would like to investigate how to define logical systems that are 'truly' empty and trivial in their valid metainferences (that is, they are empty and trivial even when our inferential connectives are considered). Though interesting, all these issues will have to wait for another day.

References

[1] G. Badia, P. Girard, and Z. Weber. What is an inconsistent truth table? *Australasian Journal of Philosophy*, 94(3):533–548, 2016.

[2] E. Barrio and F. Pailos. Validities, antivalidities and contingencies: A multi-standard approach. *Journal of Philosophical Logic*, pages 1–24, 2021.

[3] E. Barrio, L. Rosenblatt, and D. Tajer. The logics of strict-tolerant logic. *Journal of Philosophical Logic*, 44(5):551–571, 2015.

[4] E. A. Barrio, F. Pailos, and J. T. Calderón. Anti-exceptionalism, truth and the ba-plan. *Synthese*, pages 1–26, 2021.

[5] E. A. Barrio, F. Pailos, and D. Szmuc. A hierarchy of classical and paraconsistent logics. *Journal of Philosophical Logic*, 49(1), 2020.

[6] E. A. Barrio, F. Pailos, and D. Szmuc. Substructural logics, pluralism and collapse. *Synthese*, 198(20):4991–5007, 2021.

[7] J. P. Burgess. No requirement of relevance. In S. Shapiro, editor, *The Oxford Handbook of Philosophy of Mathematics and Logic*. Oxford University Press, 2014.

[8] E. Chemla, P. Égré, and B. Spector. Characterizing logical consequence in many-valued logic. *Journal of Logic and Computation*, 27(7):2193–2226, 2017.

[9] P. Cobreros, P. Egré, D. Ripley, and R. van Rooij. Tolerance and mixed consequence in the s'valuationist setting. *Studia logica*, 100(4):855–877, 2012.

[10] P. Cobreros, P. Egré, D. Ripley, and R. van Rooij. Tolerant, classical, strict. *Journal of Philosophical Logic*, 41(2):347–385, 2012.

[11] P. Cobreros, P. Egré, D. Ripley, and R. van Rooij. Vagueness, truth and permissive consequence. In *Unifying the philosophy of truth*, pages 409–430. Springer, 2015.

[12] P. Cobreros, P. Egré, D. Ripley, and R. van Rooij. Inferences and metainferences in st. *Journal of Philosophical Logic*, 49(6):1057–1077, 2020.

[13] P. Cobreros, P. Egré, D. Ripley, and R. V. Rooij. Reaching transparent truth. *Mind*, 122(488):841–866, 2013.

[14] P. Cobreros, E. La Rosa, and L. Tranchini. (i can't get no) antisatisfaction. *Synthese*, pages 1–15, 2020.

[15] B. Da Ré and F. Pailos. Sequent-calculi for metainferential logics. *Studia Logica*, 110(2):319–353, 2022.

[16] B. Da Re, F. Pailos, D. Szmuc, and P. Teijeiro. Metainferential duality. *Journal of Applied Non-Classical Logics*, 0(0):1–23, 2020.

[17] B. Da Ré, D. Szmuc, and P. Teijeiro. Derivability and metainferential validity. *Journal of Philosophical Logic*, pages 1–27, 2021.

[18] B. Da Re, P. Teijeiro, and M. Rubin. What really is a paraconsistent logic. Forthcoming in *Logic and Logical Philosophy*.

[19] M. De and H. Omori. There is more to negation than modality. *Journal of Philosophical logic*, 47(2):281–299, 2018.

[20] B. Dicher and F. Paoli. St, lp and tolerant metainferences. In *Graham Priest on dialetheism and paraconsistency*, pages 383–407. Springer, 2019.

[21] M. Dummett. *Elements of Intuitionism*. Clarendon Press, Oxford, 1977 (2nd edition, 2000).

[22] H. Field. The power of naïve truth. *The Review of Symbolic Logic*, pages 1–34, 2020.

[23] R. French. Structural reflexivity and the paradoxes of self-reference. *Ergo, an Open Access Journal of Philosophy*, 3, 2016.

[24] G. Gentzen. Untersuchungen iiber das logische schliessen. *Mathematische Zeitschrift*, 39:176–210 and 405–431, 1934–35.

[25] R. Golan. Metainferences from a proof-theoretic perspective, and a hierarchy of validity predicates. *Journal of Philosophical Logic*, pages 1–31, 2021.

[26] V. Halbach and C. Nicolai. On the Costs of Nonclassical Logic. *Journal of Philosophical Logic*, 47:227–257, 2018.

[27] L. R. Horn and H. Wansing. Negation. In E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Spring 2020 edition, 2020.

[28] L. Humberstone. *The connectives*. MIT Press, 2011.

[29] S. C. Kleene. *Introduction to metamathematics*. North-Holland, Amsterdam, 1952.

[30] J. Murzi and L. Rossi. Generalized revenge. *Australasian Journal of Philosophy*, 98(1):153–177, 2020.

[31] F. Pailos. A family of metainferential logics. *Journal of Applied Non-Classical Logics*, 29(1):97–120, 2019.

[32] F. Pailos. Empty logics. *Journal of Philosophical Logic*, 2021. DOI: https://doi.org/10.1007/s10992-021-09622-8.

[33] F. M. Pailos. A fully classical truth theory characterized by substructural means. *The Review of Symbolic Logic*, 13(2):249–268, 2020.

[34] B. Porter. Supervaluations and the strict-tolerant hierarchy. *Journal of Philosophical Logic*, pages 1–20, 2021.

[35] G. Priest. The logic of paradox. *Journal of Philosophical logic*, 8(1):219–241, 1979.

[36] G. Priest, R. Routley, and J. Norman. *Paraconsistent Logic: Essays on the Inconsistent.* Philosophia Verlag, 1989.

[37] A. Pynko. Gentzen's cut-free calculus versus the logic of paradox. *Bulletin of the Section of Logic*, 39(1/2):35–42, 2010.

[38] D. Ripley. Conservatively extending classical logic with transparent truth. *Review of Symbolic Logic*, 5(02):354–378, 2012.

[39] D. Ripley. Paradoxes and failures of cut. *Australasian Journal of Philosophy*, 91(1):139–164, 2013.

[40] D. Ripley. Revising up: Strengthening classical logic in the face of paradox. *Philosophers*, 13, 2013.

[41] D. Ripley. Comparing substructural theories of truth. *Ergo, an Open Access Journal of Philosophy*, 2, 2015.

[42] D. Ripley. One step is enough. *Journal of Philosophical Logic*, pages 1–27, 2021.

[43] A. J. Roffé and F. Pailos. Translating metainferences into formulae: satisfaction operators and sequent calculi. *Australasian Journal of Logic*, 3, 2021.

[44] L. Rosenblatt. Paradoxicality without paradox. *Erkenntnis*, pages 1–20, 2021.

[45] L. Rosenblatt. Should the non-classical logician be embarrassed? *Philosophy and Phenomenological Research*, 2021.

[46] L. Rosenblatt and C. Fiore. Recapture results and classical logic. *Unpublished*.

[47] C. Scambler. Classical logic and the strict tolerant hierarchy. *Journal of Philosophical Logic*, 49(2):351–370, 2020.

[48] P. Teijeiro. Strength and stability. *Análisis Filosófico*, 41(2):337–349, 2021.

[49] J. Woods. Logical Partisanhood. *Philosophical Studies*, 176:1203–1224, 2019.