CHAPTER 9

Numbers, Ontologically Speaking: Plato on Numerosity

Florin George Calian

The conceptualisation of numbers is culturally bound. This may seem like a counterintuitive claim, but one illustration thereof is the limitations of the resemblance of the ancient Greek concept of number to that in modern mathematics.[1] We may take Greek mathematics as familiar and transferable, and are accustomed to perceive Euclid's *Elements* as universal and transcultural. Natural numbers, the most common numbers, are currently conceived in a Euclidean key as quantitative units taken together, distributed into sequences initiated by a unit.[2] As Aristotle (*Metaph.* 1080ª22–23) put it, each unit of a number has the same value. This would imply an understanding of ἀριθμός and its units as devoid of any qualitative features relative to each other.

But this apparent continuity of understanding with the Greeks is a projection of our own mathematical understanding. Even if Thomas Kuhn did not explicitly discuss a mathematical paradigm shift, his theory leads us to question the equivalences drawn between modern and ancient mathematics. As Bruce Pourciau highlighted, 'Kuhnian revolutions in mathematics are logically possible, in the sense of not being inconsistent with the nature of

1 Most strikingly, some peoples do not count higher than three or six, or do not count at all, since they do not have words in their natural language to designate these discrete quantitative concepts. This contradicts the idea that every culture has developed a concept of number which is translatable and understandable by any human being, and could confirm the classical Whorfian thesis that language can determine thought, and, in this case, arithmetical skills. An example is given by the Pirahã tribe, from Amazon, who use a 'one, two, many' system of counting, for which see P. Gordon, 'Numerical Cognition Without Words', *Science* 15 (2004), 496–499. For a critique of Gordon see S. Laurence and E. Margolis, 'Linguistic Determinism and the Innate Basis of Number', in P. Carruthers, P. Laurence and S. Stich (eds), *The Innate Mind* (Oxford: Oxford University Press, 2008), 139–169. Further, C. Everett, *Numbers and the Making of Us: Counting and the Course of Human Cultures* (Cambridge, MA: Harvard University Press, 2017), 60–100.

2 Euclid (VII, Def. 2) defines it thus: 'a number is a multitude composed of units'. See T. Heath, *A History of Greek Mathematics*, ii: *From Aristarchus to Diophantus* ([Oxford: Clarendon Press, 1921] New York: Dover, 1981), 69.

 | DOI:10.1163/9789004467224_011

mathematics'.[3] And more pointedly: 'the one science where Kuhn apparently believed his ideas on incommensurability did not apply [mathematics], is the science that reveals the deepest incommensurability of all'.[4] Ancient Greek mathematicians and philosophers understood numbers as integers, and beyond the integer there was nothing that could be called number. However, the fifth-century discovery of incommensurable magnitudes,[5] a real achievement of Greek mathematics, could be taken as a case of paradigm shift because it challenged the reflex assumption of number as integer. This conceptual shift, which took place within the world of mathematics itself, is instructional for modern mathematics in its partial inheritance of ancient mathematical impasses.[6]

The difference between the modern and the ancient understanding of numbers is not only marked by the initial ancient Greek inability to work with irrational numbers, or by post-Renaissance algebraisation of mathematics,[7] but by the iconoclastic modern understanding of numbers as devoid of ontolo-

3 B. Pourciau, 'Intuitionism as a (Failed) Kuhnian Revolution in Mathematics', *SHPS*, 31 (2000), 297–329, at 297.

4 Pourciau, 'Intuitionism', 328. For an up-to-date discussion see M. Sialaros (ed.) *Revolutions and Continuity in Greek Mathematics* (Berlin: De Gruyter, 2018), especially the introduction by Sialaros, 'Introduction: Revolutions in Greek Mathematics', 1–15.

5 For example, there is no unit that makes it possible for a square to be commensurable with both its side and its diagonal. Plato (*Resp*. 5.546c4–5) calls the diagonal of the square ἄρρητον (irrational). Euclid calls it ἀσύμμετρος (incommensurable). For a detailed discussion, see Heath, *Greek Mathematics*, ii, 90–91. For a contextualisation of the testimony, see L. Zhmud, *Pythagoras and the Early Pythagoreans*, trans. K. Windle and R. Ireland (Oxford: Oxford University Press, 2012), 263–265.

6 As seen, for example, in the question allegedly posed by Leopold Kronecker to Ferdinand von Lindemann: 'Why study such problems [the proof that Pi is transcendental], when irrational numbers do not exist?' See R.S. Wolf, *A Tour Through Mathematical Logic* (Washington DC: Mathematical Association of America, 2005), 323.

7 J. Klein, *Greek Mathematical Thought and the Origin of Algebra*, trans. Eva Brann (New York: Dover, 1992 [1968]) stresses that our concept of number should be traced to the Renaissance, when algebra found its place within mathematics. This revolutionary idea is taken further by several scholars, including P. Pritchard, who underscores the Greek understanding of unity and number (ἀριθμός) as distinct from 'post-Renaissance number notions', and Gottlob Frege's 'Platonic' understanding of numbers, see P. Pritchard, *Plato's Philosophy of Mathematics* (Sankt Augustin: Academia Verlag, 1995), esp. chapter 4. Pritchard's thesis of discontinuity between Greek and post-Renaissance mathematics is followed by M. Burnyeat, 'Plato on Why Mathematics is Good for the Soul', in T. Smiley (ed.), *Mathematics and Necessity: Essays in the History of Philosophy* (Oxford: Oxford University Press, 2000), 1–81, and 'Platonism and Mathematics: A Prelude to Discussion', in A. Graeser (ed.), *Mathematics and Metaphysics in Aristotle* (Bern: Paul Hapt, 1987), 213–240. For a critical review of Pritchard's analysis of Plato see V. Harte's review in *JHS*, 118 (1998), 227.

gical or symbolical value.[8] For Greek philosophers, mathematical questions are ontological questions, and ontological questions have their corresponding mathematical expressions.

Some Greek philosopher-mathematicians, especially the Pythagoreans and the early Platonists, understood numbers as 'generated' (Arist. *Metaph.* 1081^{a}14, 1098^{b}7). For the Pythagoreans and early Academics, the world is likewise 'generated' and numbers have a pivotal role in the generation process: fundamental ingredients (unity, unity and duality, odd and even, etc.) generate and inform numbers which, in their turn, generate geometrical entities, which then generate physical bodies.[9] Plato himself dealt dialectically with the problem of the generation of the world, of the soul, and, as I argue, of numbers. Several scholars and mathematicians take for granted that Plato's philosophy of mathematics was devoted only to the elaboration of the famous number-form theory, in which number is eternal, unchanged, and ungenerated.[10] Nevertheless, an argument from the *Parmenides* (142b1–144a4), which is seldom mentioned in Plato's philosophy of numbers, explains numbers in terms of generation. Talking about numbers in terms of generation, and not in terms of forms, implies a scaling down of numbers to basic principal ingredients, instead of the assertion of numbers as simple and uncomposed abstract and eternal entities.[11]

8 The Greek concept of number of course had a pragmatic function (number calculations), but the Pythagoreans also endowed it with religious and ontological importance. Modern number theory leaves little room for this more speculative aspect; though some, such us Kepler and Newton, had a 'superstitious' understanding of numbers that complemented their scientific endeavours: see J. Henry, *Religion, Magic, and the Origins of Science in Early Modern England* (London: Routledge, 2018).

9 Some pieces of evidence in this regard are provided by Aristotle (*Metaph.* 1084^{a}10, 25, 1090^{b}20–24, *De an.* 404^{b}20–24, *Ph.* 206^{b}33). Even if these testimonies come to us filtered through the Aristotelian lens, they nevertheless provide a substantial starting point for understanding at least the controversies around number theories and the concept of generation.

10 See, for example, *Phaedo* (101c5): 'no other reason for their coming to be two, save participation in twoness: things that are going to be two must participate in that, and whatever is going to be one must participate in oneness'. The place of these number-forms at the level of forms or as intermediary objects (between intelligible forms and participated things) is an ongoing debate which has meaning only if one accepts that Plato conceived for each number a correspondent number-form. Even if this number-form theory is the source for modern 'Platonism' in mathematics, it is far from clear whether Plato's conception on number was 'Platonist' all throughout his dialogues. The 'Platonism' of Plato is only one facet of his explorations into mathematical philosophy: Aristotle attributed at least seven partly contradictory views to Plato, see F.G. Calian, 'One, Two, Three … A Discussion on the Generation of Numbers in Plato's *Parmenides*', *New Europe College*, (2015), 50–51.

11 The subject has received little attention from scholarship, but the now classic treatments are those of R.E. Allen, 'The Generation of Numbers in Plato's *Parmenides*', *CPh*, 65, (1970),

Being generated does not automatically, however, imply that numbers are less abstract and eternal entities, but that they are composed (i.e. not simple) and reducible to more basic entities, and therefore they cannot be plain forms.[12]

Even if isolated, the argument from the *Parmenides* draws on a type of onto-mathematics which resembles Pythagorean mathematical philosophy, foreshadows elements from later Platonic dialogues, such as the *Sophist* (254b–264b) and the *Timaeus* (34c–35b), and anticipates some of the ideas of the early Academy.[13] The exegesis of this argument for the generation of numbers provides us with another facet of Plato's ontology and philosophy of numbers. If one takes the argument *ad litteram*, Plato has here an understanding of numbers that displays Pythagorean elements (as I show below), and, since the argument contains ideas present in later dialogues, this trait should not be ignored as a peculiarity of the second part of the *Parmenides* alone. Moreover, the hidden premises of the argument are also relevant for the historiography of Ancient Greek numerical thinking.

1 The Generation of Numbers

The *Parmenides* is conventionally divided into two parts: the first part (126a–137c) stands as an outstanding critique of the theory of forms, while the second part (137c–166c) is an elaborate debate on the concept of ἕν – 'one'.[14] The

30–34, followed by 'Unity and Infinity: Parmenides 142b–145a', *RMeta*, 27 (1974), 697–725. For a recent revision of the argument in the context of an alternative understanding of Platonic philosophy and mathematics, see S. Negrepontis, 'The Anthyphairetic Revolutions of the Platonic Ideas', in M. Sialaros (ed.), *Revolutions and Continuity in Greek Mathematics* (Berlin: De Gruyter, 2018), 335–381. For a detailed analytic presentation of the argument see F.G. Calian, 'One, Two, Three', 52–53.

12 One of the reasons for attempting to articulate a formula for the generation of numbers could be that it is difficult to conceive an infinite number of forms, corresponding to the infinite string of numbers.

13 According to Aristotle (*Metaphysics*, Books M and N), early Academics, such as Speusippus and Xenocrates, considered not only numbers, but also forms and beings, as generated from two principles ('One' and its opposite principle, sometimes called 'the indefinite dyad').

14 Alternately, 'the one', 'the Parmenidean one', 'unity', 'number one'. G. Ryle, 'Plato's *Parmenides*', *Mind*, 48/190 (1939), 129–151, translates as 'unity', as does R.E. Allen, *Plato's* Parmenides (rev. New Haven: Yale University Press, 1997). G.E.L. Owen, 'Notes On Ryle's Plato', in *Logic, Science and Dialectic* (London: Duckworth, 1986), 85–103 prefers to translate as 'one' and 'the one' and uses them interchangeably. S. Scolnicov, *Plato's* Parmenides (Berkeley: University of California Press, 2003) translates as 'one', and I do the same since there is a numerical context in which ἕν is conceived in its numerical value.

two parts seem to have been written at different moments and for different purposes, and later brought together to form one text. The dialogue's second part consists of a series of eight inferences, which are usually referred to as 'arguments' (Scolnicov), 'deductions' (Kahn, Ryle, Owen, Allen, Rickless), or 'hypotheses' (Cornford).[15] I focus on the beginning of the second argument, specifically the first (ontological) argument (142b1–143a2) therein, and, correspondingly, the second (mathematical) argument (143a4–144a4). For reasons that are apparent below, I refer to the first argument as the ontological argument, and to the second argument as the mathematical argument. I refer to both these arguments, when considered together, as the general argument for the generation of numbers.

The general argument is exceptional in several ways. I quote Kahn for a good illustration of its richness: 'Although all the deductions make some positive contribution, Deduction 2 presents philosophical thought on an entirely different scale, as an outline theory of the conceptual properties required for spatial-temporal being and becoming.'[16] The main lines of the general argument, along which Plato differentiates 'one' and 'being' and institutes numbers, are as follows:

– The first argument (the ontological argument)
  Although 'one' participates in 'being', 'one' is not 'being', since 'is' signifies something other than 'one'; 'one' is a whole, 'being' and 'one' are its parts (μόρια), each of the two parts possesses oneness and being, and by necessity, it always comes to be 'two', it is never 'one' (ἀνάγκη δύ' ἀεὶ γιγνόμενον μηδέποτε ἓν εἶναι) (142e7–143a1).[17] The 'one' is infinitely many (πολλά), unlimited (ἄπειρον) and multitude (πλῆθος) (143a1).
– The second argument (the mathematical argument)
  'One' is not different from 'being' because of its oneness, and 'being' is not different from 'one' by virtue of being itself but because of 'difference'; therefore, there is 'difference' and it is distinct from 'one' and 'being'. Since there are three distinct entities the argument (143c3) goes further by picking out τινε (pairs).[18] The pairs (143c4) are called ἀμφοτέρω (both/couple),[19]

15 For a rejection of the term 'hypothesis' see Scolnicov, *Plato's* Parmenides, 3.
16 C. Kahn, *Plato and the Post-Socratic Dialogue: The Return to the Philosophy of Nature* (Cambridge: Cambridge University Press, 2013), 21.
17 All translations from the *Parmenides* are my own.
18 It is not clear if there is an exact correspondent in Greek for 'pair'. The Greek dual τινε means literally 'two somethings'. A more neutral conceptualisation of τινε should be understood in this argument as, for example, (a,b), (b,c), (a,c).
19 ἀμφοτέρω is understood as: 'x', 'y' = 'both (x, y)'.

and what is called both is δύο (two).[20] From here, the argument switches back to 'one': each of the 'two' is 'one' (δύο ἦτον > ἓν εἶναι) (143d2–3, 4–5), and, further, one added to any sort of pair is three (τρία γίγνεται) (143d7).[21]

And from here, if there is two and three, then there are all the numbers.[22] The transition from ontology towards numbers may be represented graphically as follows:

The argument for the generation of numbers: *if one is*

The first argument (the ontological argument, 142b1–143a2)

→ I. the first demonstration (142b–c): *one partakes in being*.

→ II. the second demonstration (142d–143a1): *one is many*.

→ The second argument (the mathematical argument, 143a–144a4): *if one is, there is number*.

Both Plato's condensed arguments are not easily intelligible, and, as I have already emphasised, they contradict Plato's view on numbers in terms of the theory of forms in other dialogues (e.g. *Phaedo*) and are unique among philosophies of mathematics. Plato's intention—namely to show that 'one' is not only 'one', but that 'one' is also 'many'—appears to be logically incongruent. There is an ambiguity in how 'being' is understood, but also how 'part' is used.[23] Deciphering the texture of the arguments is important for getting to grips with Plato's understanding of the ontology of numbers, for Plato builds upon the conclusions of these arguments (especially the ontological one) in later dialogues where there is no recourse to a theory of forms.[24]

The ontological argument states that, if 'one' is, then 'one' has 'being', and thus 'one' and 'being' are separate and distinct entities (142b5–143a2), and thus

20 δύο is identified as a set with two members corresponding to the cardinal number two.

21 A set of three members corresponding to the cardinal number three.

22 See also Calian, 'One, Two, Three', 52–54.

23 Thus, Bertrand Russell, *Introduction to Mathematical Philosophy* (London: Allen & Unwin, 1919; rev. repr. London: Routledge, 1993) believed that 'This argument is fallacious, partly because 'being' is not a term having any definite meaning, and still more because, if a definite meaning were invented for it, it would be found that numbers do not have being—they are, in fact, what are called logical fictions' (138). The same idea is reinforced by F.M. Cornford, *Plato and Parmenides: Parmenides'* Way of Truth *and Plato's* Parmenides (London: Kegan Paul, 1939), 139; M. Schofield, 'A Neglected Regress Argument in the Parmenides', *CQ*, 23 (1973), 44; W. Kelsey, *Troubling Play: Meaning and Entity in Plato's* Parmenides (New York: SUNY Press, 2012), 94.

24 For example, two of the very basic elements for generating numbers, such as 'difference' and 'being', are found among the greatest kinds of the *Sophist* (254b–264b), and in the generation of the soul in the *Timaeus* (35a1–b3).

the 'is'-ness of 'one' could be conceived independently from 'one'. It is not clear here whether 'one' gets multiplied or divided by two.[25] The text might favour the second option, rather than multiplication as repeated addition.[26] It is relevant that the Greek word used by Plato for 'part' is τὸ μόριον (also 'piece' or 'member'), but it can also mean, in arithmetic, 'fraction'. Diophantus (*Arith.* 1.23, 3.19, 5.20) later uses τὸ μόριον as 'fraction with one for numerator', 'fraction in general' or 'denominator of a fraction'. Another usage that would incline the balance towards division rather than multiplication are the expressions μορίου or ἐν μορίῳ 'divided by'. As a sub-unitary process of building pairs from 'one' and 'being', the line between division and multiplication is actually blurred,[27] as each item of the 'one-being' pair (the 'one' or 'being') becomes even more divided. It must be acknowledged that, ultimately, the question over multiplication versus division must yield to the overarching ontological conception of one as many. This ontological argument (142b5–143a2) would thus be enough to justify the generation of 'two' by division or multiplication, since 'one' turning into 'one' and 'being' creates only 'twos' (142e7–143a1). Yet the obtained duos from the ontological argument are not yet the placeholders of numerical twos, but an attempt to conceptualise how 'one' implies multiplicity in its being.

The second, mathematical, argument (143a4–144a4) reaffirms the division of 'one', announcing, in conjunction with 'one' and 'being', the logical operator of 'difference' that makes possible the identification of 'one' and 'being'. We get the feeling as we read through the argument that the emphasis on 'difference' as an equal player in the argument—just like 'one' and 'being'—makes it not only a logical operator, but an ontological entity as well. 'One' is not different from 'being' by virtue of its oneness (of being 'one'), nor is 'being' different from 'one' because of its 'is-ness' (of being 'being'), but because of 'difference' or otherness.

25 In the *Phaedo* (101b8–10), Plato keeps this ambiguity regarding the origin of two, that is, whether it comes about by the operation of addition or division: 'Then would you not avoid saying that when one is added to one it is the addition and when it is divided it is the division that is the cause of two?' (trans. G.M.A. Grube).

26 Even if the verb γίγνεσθαι is commonly used for mathematical products (cf. Pl. *Tht.* 148a; Euc. VII, Def. 18).

27 Cornford reads the argument as using division. Later, where numbers are brought into discussion, he interprets addition and multiplication as an alternative to division, i.e. 'The sort of division here intended can only be the mental act of distinguishing the two elements in 'One Entity'' (*Plato and Parmenides*, 138–140, at 139). Examining the natural cause of things (Pl. *Phd.* 96e6–97b3), Socrates shows his perplexity by giving the example of the becoming (generation) of two, asserting that two is formed by the addition of one to another one or by the division of one thing, and thus getting to two things. Socrates'

It is this triadic differentiation—'one', 'being', and 'difference'—that imperceptibly brings about numerical generation, but its presence in the discussion does not draw our attention to numbers in themselves. Nevertheless, this shift does happen. With no explanation, the argument strangely continues by picking up pairs. Plato asserts that since there are three discrete entities, we can form three sorts of pairs (τινε) (143c3): 'being' and 'difference', or 'being' and 'one', or 'one' and 'difference'. The name of such pairs is 'both' (ἀμφοτέρω) (143c4); and both, subsequently, is 'two' (δύο) (143d2). Thus, we witness the derivation of cardinality δύο, which refers to groups of two elements and to number two, from the collective dual ἄμφω.[28] Drawing on the numerical value of the pairs created, Plato moves from an ontological discussion towards a mathematical one, thereby opening the argument on the generation of numbers.[29] Additionally, focusing on the semantic layers of these assemblages, the mathematical argument reviews dualities in their collective, cardinal, and ordinal meanings.

The collective (linguistic) duality of ἄμφω seems independent from the ordinality or the cardinality of number two, and stays as a precondition for them: duality stays as an ontological and linguistic token that is *prior* to any counting operation. The cardinality of δύο, adds Plato, is subsequent and a natural consequence of the pair condition of ἀμφοτέρω. Here Plato seems to be 'deceived' by the nature of the Greek language which uses dual as a distinct grammatical number to refer to objects that come in pairs. The argument suggests that this 'linguistic priority' is also ontological, prior to effective cardinality as two. The reverse might be expected: to stipulate cardinality first, and then to advance the possibility of building pairs.[30] On the contrary, Plato's argument does not endorse the idea that duality is deduced from the cardinality and counting of two instances. Should the argument advance cardinality first, its

perplexity arises from the fact that one cannot have two opposite causes—addition and division—for reaching the same result, which is two.

28 See also Calian, 'One, Two, Three', 58.

29 D. Blyth, 'Platonic Number in the *Parmenides* and *Metaphysics XIII*', *IJPS*, 8 (2000), 23–45, bases all numerical argumentation on the ability to count. He distinguishes between form-numbers, originally ordinal and so differentiated by position, and cardinals as mathematical numbers. But Blyth's interpretation falls short in justifying why Plato uses 'one', 'being' and 'difference' as the *primordia* of counting.

30 Aristotle criticises Plato for not pointing to cardinality first, but the dyad: 'for it follows that not the dyad but number is first, i.e., that the relative is prior to the absolute' (*Metaph.* 990^{b}18–20). Aristotle probably has this specific argument in mind, see Calian, 'One, Two, Three', 58.

whole structure would be meaningless. To summarise, first there is 'pair' (τινε) (143c3), and since the pair is called 'both' (ἀμφοτέρω) (143c4), we then get to 'two' (δύο) (143d2), with each of the 'two' (δύο) as 'one' (ἓν εἶναι) (143d4–5).

Since Plato's argument does not start from 'one' in order to build the number series but determines the number series starting with number two, the argument assumes number two as the first actual number.[31] This method of obtaining number δύο—positing a set with two elements, from a pair relation (ἀμφοτέρω)—is indeed unusual but it is not inconsistent with Greek mathematics, which understood the first number of the number series as being number two. For Greek mathematicians, the unit lacks a proper definition, since it is not a number but the condition of numbers, while for Plato the unit is not the condition, but the 'numerical' derivation from two (δύο). Thus 'one' is not the fundamental unit for 'two', but rather on the contrary, 'two' is the condition for 'one'—the 'unit' for calculation. Both Plato and Greek mathematicians, by different logical routes, seem to agree that the number series starts with 'two'.

Although one would expect to proceed from δύο to number three, we go instead back (or forth) to number one. Plato does not consider ἕν, from which the ontological and mathematical argument starts, as having any numerical value since he does not consider it countable. The ontological argument stated the foundation of the ontological multiplicity and stressed that the initial ἕν must be understood not as a unitary being, but as a part of the 'one-being' pair—'since [one] always proves to be two, it must never be one' (142e7). The subsequent mathematical argument mirrors the division of 'one' into 'twos', concluding that 'each of the two is one'. The direction is from ἕν towards δύο, and from δύο towards another type of ἕν. In Plato's words, from ἂν δύο ἦτον (if there are two), to ἑκάτερον αὐτοῖν ἓν εἶναι (each of the two to be one) (143d3), the argument enforces the idea that by way of duality we are given an account of 'one'. Thus, the initial ἕν is the ontological basis for the numerical ἕν.

Having generated the numerical one, Plato goes further towards obtaining number three (143d7). The new ἕν, in its role as numerical unit, would have populated, for the modern mind, the whole of the numeric axis by mere successive addition, and should have easily led to number three from plain self-addition (i.e. 1+1+1). Moreover, if the pairs 'one-being', 'being-difference',

31 See also A. Wedberg, *Plato's Philosophy of Mathematics* (Stockholm: Almqvist & Wiksel, 1955), 23; D. Ross (ed.), *Aristotle's* Physics (Oxford: Clarendon Press, 1936), 604.

and 'difference-one' lead to the revelation of duality and thereof, to numerical two, would not 'one', 'being', and 'difference' suggests an analogous trinity, and thereof, a numerical three? If we do not follow Plato's line of argumentation thoroughly, we can definitely be misled, as David Ross was when he took 'one', 'being' and 'difference' as the three first countable entities.[32] However, the argument is again surprising and offers solutions for the generation of numbers that go against what the reader would expect. Any initial structure of three concepts (or any other instantiations) would not be enough, since the general argument seems to develop arbitrarily from 'any two' towards 'one', and from 'any two' and 'one' towards three.

A question that one could ask is whether one could reach number two and number three from *any* two or three given concepts. One could perhaps think that any triadic structure could be the starting point for further insights into the generation of numbers. Could this initial triad be any triad or is it bound to be a conceptual triad, made up out of specific ontological concepts? Should the constituents of such a triad be necessarily and precisely the three concepts of 'one', 'being' and 'difference', or any other three? The construction of the whole argument, namely from an ontological argument towards a mathematical argument, and the interplay between one-multiple (ontologically speaking) and one-multiple (numerically understood) might testify that numbers could not have been articulated unless we proceed from an ontological one to a numerical one.[33]

The initial three entities which are different from each other—'difference is not the same as oneness or being' (143b6–7)—are not straightforwardly counted to obtain number three, but, on the contrary, the stress is on 'one' added to a pair. I venture to say that a possible reason for getting to three through such an elaborate and unexpected operation might be the need to highlight oddness—that one is what is *added* to any *pair*. Reaching number three just from plain counting of 'one', 'being' and 'difference' would not stress the oddness of number three, a matter in which Plato is very interested. After reaching three (143d9–e2) from 2 + 1, the next step is to emphasise that three is

32 D. Ross, *Plato's Theory of Ideas* (Oxford: Clarendon Press, 1951), 187: 'The difference […] is different, so that we already have three things. And three is odd'.

33 G.E.M. Anscombe, *From Parmenides to Wittgenstein*, i: *Collected Philosophical Papers* (Oxford: Basil Blackwell, 1991) commenting on this argument, notices that 'one itself is infinitely divided, each of the numbers being one' (25). If numbers are unitary because of 'one', then it follows that their 'existence' is given by 'being' and their identity is made possible by 'difference'.

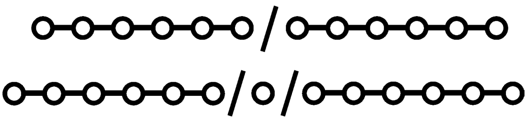

FIGURE 9.1 Knorr's depiction of the formula for 2k+1

odd (τρία … περιττά), and, of course, that two is even (δύο ἄρτια). Through the operation of 2+1 the argument brings into focus the idea of oddness, rather than the actual numerical value of three. This becomes more evident if we consider that in the *Phaedo* (105c) Plato states that oneness makes an odd number odd: 'if asked the presence of what in a number makes it odd (περιττός), I will not say oddness (περιττότης) but oneness (μονάς)'.[34] Knorr's visual representation of even numbers (Figure 9.1) highlights how one is essential to the definition of odd numbers.[35] Plato's understanding of odd numbers, at least according to this argument, would be to identify odd numbers as being of type 2k+1, while the 2k expression defines even numbers.

After the classification of oddness and evenness is established, the next step is to use another arithmetical operation.[36] After addition, used to obtain three, multiplication is introduced (143e5–e7): there will be even times even (ἄρτια ἀρτιάκις), odd times odd (περιττὰ περιττάκις), odd times even (ἄρτια περιττάκις), and even times odd (περιττὰ ἀρτιάκις). Since multiplication is immediately added, and since one (the condition for three and odd) is derived from two (which is even), we need only number two, and the generation of the rest of the numbers is assured. Hence the argument seems to stress that if there is one, there are numbers one, two, three, but not in standard 'chronological' order. If one is, numbers are subsequently 2, 1, and 2+1 (3), and by multiplication,

34 The Greek περιττός has the basic meaning of something that is beyond the average, something in excess, more in quantity, and it could have also referred to the *additional* one.

35 W.R. Knorr, *The Evolution of the Euclidean Elements: A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry* (Dordrecht: D. Reidel, 1975), 140.

36 The very use of a mathematical operation here actually goes against the frame of Plato's mathematical discussions elsewhere in his dialogues. As J.M. Moravcsik, 'Forms and Dialectic in the Second Half of the *Parmenides*', in M. Schofield and M.C. Nussbaum (eds), *Language and Logos: Studies in Ancient Greek Philosophy Presented to G.E.L. Owen* (Cambridge: Cambridge University Press, 1982) puts it: 'There is nothing in Plato's ontology that corresponds to mathematical operations; the ontology reflects only mathematical truths' (144).

the rest of the numbers.[37] Two, one, and three—in this order of generation—stand thus as the basic numbers for Plato's generation of numbers.[38] This could be understood as an innovation from the 'one' and the model of the 'first four numbers' (1, 2, 3, 4—the *tetractys*) that were the elementary ingredients for the generation of numbers for the Pythagoreans.[39] In addition to having all the numbers generated through two and three by multiplying them, and by bringing into discussion evenness and oddness, the argument traces the operations that lead to the identification of some kind of *primordia*, or generating conditions, for numbers.

2 Odd, Even and Prime Numbers

It may seem unnecessary for the argument to jump to the discussion of evenness and oddness, since two and three would do the multiplication process without the necessity of classification. However, by avoiding the direct resol-

37 J.M. Moravcsik, 'Forms and Dialectic', 144 draws attention to this feature of number three: 'Plato might add the number 3 as basic if 1 is not acknowledged as a number'.

38 This reminds us of intuitionism in mathematics. For example, for L.E.J. Brouwer, number generation starts from the intuition of pure twoness: 'This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely' ('Intuitionism and Formalism' [1912], in P. Benacerraf and H. Putnam (eds), *Philosophy of Mathematics: Selected Readings* (Cambridge: Cambridge University Press, 1984), 77–89, at 80). See also M. Panza and A. Sereni, *Plato's Problem: An Introduction to Mathematical Platonism* (New York: Palgrave Macmillan, 2013), 88. However, Brouwer's intuitionism with regards to duality is not based on Plato's argument, but on Kant's. J. Annas, *Aristotle's Metaphysics: Books M and N* (Oxford: Clarendon Press, 1976), 43, also draws a parallel with Brouwer, when referring to the Aristotelian critique of Plato, namely that according to the so-called 'unwritten doctrine' numbers are generated from one and 'indefinite two'.

39 In Pythagorean philosophy there is an intermingling between cosmology and generation of numbers. I agree on this point with Cornford's classical study, which argued that Pythagoras 'could not yet distinguish between a purely logical 'process' such as the 'generation' of the series of numbers, and an actual process in time such as the generation of the visible Heaven [...]. The cosmological process was thus confused with the generation of numbers from One', see F.M. Cornford, 'Mystery Religions and Pre-Socratic Philosophy', in J.B. Bury, S.A. Cook and F.E. Adcock (eds), *The Cambridge Ancient History*, iv: *The Persian Empire and the West* (Cambridge: Cambridge University Press, 1926), 522–678, at 550–551. Aristotle mentions more than once that Plato followed the Pythagoreans (e.g. *Metaph.* 987^{a}29). For a reassessment of the Pythagoreanism of Plato, see P.S. Horky, *Plato and Pythagoreanism* (Oxford: Oxford University Press, 2013).

ution of the generation process that could be achieved by multiplying only two and three, and by instead insisting on multiplying oddness and evenness, Plato makes a leap into the field of number properties. This is not only a dialogic device, but a logical development of the whole argument. The abrupt emphasis on evenness and oddness is a progression from a particular—there must be 'twice two' (δύο δίς, 143e3), and there must be 'thrice three' (τρία τρίς, 143e3), thus there must be twice three (τρία δίς) and thrice two (δύο τρίς, 143e5)—to a universal rule (143e7): there will be even times even (ἄρτια ἀρτιάκις), odd times odd (περιττὰ περιττάκις), odd times even (ἄρτια περιττάκις), and even times odd (περιττὰ ἀρτιάκις). It appears that one of the most important aims within the argument is to obtain the first odd and the first even; these are not features of numbers; rather numbers are features and derivations of odd and even.[40] Plato is thus consistent with his views in other dialogues, at least at the epistemological level, where the knowledge of numbers is the knowledge of the odd and even.[41]

The specific classification of numbers as odd and even shows also the use of a specific *differentia* of numbers in order to classify them. Hence, what the mathematical argument offers us is not a linear progression of numbers, but a generation and classification of numbers according to odd an even. If we emphasise this distribution, odd and even work for the classification of numbers from two elementary categories (odd and even) towards four composite categories (odd-even, even-odd, even-even, and odd-odd).[42] It is probable that the ancient Greeks understood even, odd, and odd-even as species.[43] Similar ways of classifying numbers, by odd and even, are described by Philolaus: 'Number, indeed, has two proper kinds, odd and even, and a third mixed together from both, the even-odd (αρτιοπέριττον)'.[44] Whether there is a link between

40 As it is pointed out in *Euthyphro* (12d–e): 'Shame is a part of fear just as odd is a part of number, with the result that it is not true that *where there is number there is also oddness, but that where there is oddness there is also number*' (emphasis added).

41 *Resp*. 7.524d, *Tht*. 198a, *Grg*. 453e, *Chrm*. 166a. See further, for example, L. Zhmud, *The Origin of the History of Science in Classical Antiquity*, trans. Alexander Chernoglazov (Berlin: De Gruyter, 2006), 223.

42 From the four composite categories only three are distinct, because odd-even and even-odd share the same multitude of elements.

43 The mixture of odd and even is thus a derivate of the two main species. See in this regard J. Klein, 'The Concept of Number in Greek Mathematics and Philosophy', in R.B. Williamson and E. Zuckerman (eds), *Lectures and Essays* (Annapolis, MD: St John's College Press, 1985), 43–52, at 47.

44 Philolaus fr. 5 DK, see C.A. Huffman, *Philolaus of Croton: Pythagorean and Presocratic* (Cambridge: Cambridge University Press, 1993), 178.

Philolaus here and Plato's understanding of numbers in terms of odd and even, is still an issue to be clarified,[45] but it is significant that Euclid continues the same classification.[46]

As stated earlier, we would expect to derive numbers with the help of addition rather than multiplication. Apart from setting out oddness and evenness, another possible motivation for the use of multiplication may be that it offers a simpler and more schematic pattern for numbers. They are to be factorially generated from the fundamentals two and three, rather than through an increase of one (e.g., operations with units: $6 = 1+1+1+1+1+1$ versus, *planius*, $6 = 3\times2$ or 2×3).[47] Factorial deduction of numbers would contradict Aristotle's view, according to which 'each number is said to be many because it consists of ones and because each number is measurable by one' (*Metaph.* 1056^{b}23).

Even if the argument ends with the conclusion that there is no number left that does not necessarily exist (144a3), we cannot get the entire number series by reducing numbers to odd and even and their multiplication. The argument fails to include prime numbers—Aristotle notes Plato's failure to address the subject (*Metaph.* 987^{b}29–988^{a}1)—since these numbers have no divisors other than themselves and one; after two and three the rest of the prime numbers cannot be generated. Conceivable solutions to the generation of primes would eventually be limited to the subsequent operations. We could combine addition and multiplication, and therefore get to a prime number like 5 as the result of $2\times2+1$. Primes could also be the result of odd times odd, and numbers like 5 or 7 would be 5×1 or 7×1, which would be possible if 1 is considered odd, or

45 C. Meinwald, 'Plato's Pythagoreanism', *AncPhil*, 22 (2002), 87–101, at 87, rightly points out that 'Pythagorean scholarship is too diverse and contentious to be a starting-point for reading Plato'; still, she identifies Philolaic remnants in Plato. Another resemblance with Philolaus could be found in fr. 8, where 'one' is understood as the 'principle of all things'. But, as Huffman suggests (*Philolaus*, 346), this last fragment may be spurious. Hence, according to both Plato and Philolaus, numbers gravitate around one (or the 'one that is', in the case of Plato), which gives every number its unitary identity (see also footnote 33 in this article). When Plato pictures numbers as originating in one (that is) he could be reiterating a Philolaic idea, but by bringing 'being' and 'difference' into discussion he nevertheless develops the idea further. Plato's incursion into the problem of the generation of numbers may thus echo a Pythagorean discussion on the generation of the world and of numbers.

46 As T. Heath, *Greek Mathematics*, ii, 72, noticed: 'Euclid's classification does not go much beyond this [Plato's classification]'.

47 There are similarities here with the series of numbers used by some cultures: some Indigenous Australians limited their number systems to one and two (i.e., a binary system), and out of them composed numbers up to six, e.g. three is made by two and one, while six is made by two and two and two. See T. Dantzig, *Number: The Language of Science* (1930; 4th edn., New York: Macmillan, 1954), 14.

primes could be a subcategory of odd numbers. The question of how exactly one gets to primes remains, however, unanswered. Given that any solution would eventually be controversial, for the moment I propose that Plato tries to achieve a governing law of the number series encompassing their *primordia* and their related operations: oddness, evenness, and multiplication. In stressing a general law for the generation of numbers, Plato might have oversimplified the whole discussion and, as a result, failed to give a satisfying and explicit account of primes. Primes could have been intentionally left out since there is no law for their generation (only ways to validate them), and ancient mathematicians knew that.[48] But beyond these geometrical attempts to 'work' with primes, no arithmetic operation clarified how non-composite (i.e. prime) numbers came to be.[49] For Plato, one could go as far as speculating that each prime number would have a corresponding prime number-form, while the rest—reachable through factorial operations—would have corresponding combinations of one or two forms (e.g., seven would participate in the prime number form, while six would participate in even-odd).

Conclusion

The line on which Plato develops his thoughts seems to move imperceptibly from ontology towards arithmetic, as if there is a continuum from the ontological differentiation between 'one' and 'being' towards number differentiation and thus their generation and classification. Once the ontological differentiation is made (by the atypical instrumentalisation of 'difference' that introduces *differentia*—perceived as an ontological-logical device), arithmetical inferences follow. The arguments are consequential: there are numbers (the second argument) only because 'one' is 'one' and 'being' (the first argu-

48 Greek mathematicians were aware that each integer number could undergo prime factorisation, and they were the first to study prime numbers in themselves (πρώτοι ἀριθμοί). According to Iamblichus, the Pythagorean mathematician Thymaridas of Paros (400–350 BCE) called prime numbers 'rectilinear' since they can be represented only as one-dimensional segments, while non-prime numbers can be represented in two-dimensional planes. Euclid, in Books VII and IX of the *Elements*, which deals with number theory, discusses thoroughly the problem of prime numbers. One of the biggest achievements of Euclid, in number theory, was to show that any number is either a prime, or divisible by a prime number (VII, 32), and that there are infinitely many prime numbers (IX, 20).

49 Only recent research into the question of primes advances algorithms as key for identifying prime numbers, e.g., R. Crandall and C. Pomerance, *Prime Numbers: A Computational Perspective* (New York: Springer, 2006).

ment). The 'one' and its related concepts of 'being' and 'difference' are distinguished as separate entities for an account of the question of generation. These distinct concepts do not lead us to discover twoness or threeness: their very distinctiveness endorses the idea of multiplicity. The first duality observed (i.e., 'one' and 'being') is the pair 'one-multiple'. In itself, it has no mathematical meaning, but it does establish a basis for understanding ontological multiplicity, and thus the numeric value of multiplicity. Without recognising this shift from ontology to mathematics as purposeful, and not fallacious, one can only agree with Sabetai Unguru, that 'it is impossible for a modern man to think like an ancient Greek.'[50] Plato did not just understand the role of mathematics differently, he also explored the ontology of numbers differently. Though there may be little value for modern mathematics in Plato's argument, what the *Parmenides* brings forth is a philosophical discourse on the ontological fundamentals of mathematics.

Bibliography

Allen, R.E., 'The Generation of Numbers in Plato's *Parmenides*', *CPh*, 65 (1970), 30–34.

Allen, R.E., 'Unity and Infinity: Parmenides 142b–145a', *RMeta*, 27 (1974), 697–725.

Allen, R.E., *Plato's* Parmenides (rev. New Haven: Yale University Press, 1997).

Annas, J., *Aristotle's Metaphysics: Books M and N* (Oxford: Clarendon Press, 1976).

Anscombe, G.E.M., *From Parmenides to Wittgenstein*, i: *Collected Philosophical Papers* (Oxford: Basil Blackwell, 1991).

Blyth, D., 'Platonic Number in the *Parmenides* and *Metaphysics XIII*', *IJPS*, 8 (2000), 23–45.

Brouwer, L.E.J., 'Intuitionism and Formalism' [1912], in P. Benacerraf and H. Putnam (eds), *Philosophy of Mathematics: Selected Readings* (Cambridge: Cambridge University Press, 1983), 77–89.

Burnyeat, M.F., 'Platonism and Mathematics: A Prelude to Discussion', in A. Graeser (ed.), *Mathematics and Metaphysics in Aristotle* (Bern: Paul Hapt, 1987), 213–240.

Burnyeat, M.F., 'Plato on Why Mathematics is Good for the Soul', in T. Smiley (ed.), *Mathematics and Necessity: Essays in the History of Philosophy* (Oxford: Oxford University Press, 2000), 1–81.

Calian, F.G., 'One, Two, Three… A Discussion on the Generation of Numbers in Plato's *Parmenides*', *New Europe College*, (2015), 49–78.

50 S. Unguru, 'Counter-Revolutions in Mathematics', in Michalis Sialaros (ed.), *Revolutions and Continuity in Greek Mathematics* (Berlin: De Gruyter, 2018), 19–34, at 29.

Cornford, F.M., 'Mystery Religions and Pre-Socratic Philosophy', in J.B. Bury, S.A. Cook and F.E. Adcock (eds), *The Cambridge Ancient History*, iv: *The Persian Empire and the West* (Cambridge: Cambridge University Press, 1926), 522–678.

Cornford, F.M., *Plato and Parmenides: Parmenides'* Way of Truth *and Plato's* Parmenides (London: Kegan Paul, 1939).

Crandall, R. and C. Pomerance, *Prime Numbers: A Computational Perspective* (New York: Springer, 2006).

Dantzig, T., *Number: The Language of Science* (1930; 4th edn., New York: Macmillan, 1954).

Everett, C., *Numbers and the Making of Us: Counting and the Course of Human Cultures* (Cambridge, MA: Harvard University Press, 2017).

Gordon, P., 'Numerical Cognition Without Words', *Science*, 15 (2004), 496–499.

Harte, V., review of P. Pritchard, *Plato's Philosophy of Mathematics* (Sankt Augustin: Academia Verlag, 1995), in *JHS*, 118 (1998), 227.

Heath, T., *A History of Greek Mathematics*, ii: *From Aristarchus to Diophantus* ([Oxford: Clarendon, 1921] New York: Dover, 1981).

Henry, J., *Religion, Magic, and the Origins of Science in Early Modern England* (London: Routledge, 2018).

Horky, P.S., *Plato and Pythagoreanism* (Oxford: Oxford University Press, 2013).

Huffman, C.A., *Philolaus of Croton: Pythagorean and Presocratic* (Cambridge: Cambridge University Press, 1993).

Kahn, C., *Plato and the Post-Socratic Dialogue: The Return to the Philosophy of Nature* (Cambridge: Cambridge University Press, 2013).

Kelsey, W., *Troubling Play: Meaning and Entity in Plato's* Parmenides (New York: SUNY Press, 2012).

Klein, J., 'The Concept of Number in Greek Mathematics and Philosophy', in R.B. Williamson and E. Zuckerman (eds), *Lectures and Essays* (Annapolis, MD: St John's College Press, 1985), 43–52.

Klein, J., *Greek Mathematical Thought and the Origin of Algebra*, trans. Eva Brann (Cambridge, MA: MIT Press, 1968; repr. New York: Dover, 1992).

Knorr, W.R., *The Evolution of the Euclidean Elements: A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry* (Dordrecht: D. Reidel, 1975).

Laurence, S. and E. Margolis, 'Linguistic Determinism and the Innate Basis of Number', in P. Carruthers, P. Laurence and S. Stich (eds), *The Innate Mind* (Oxford: Oxford University Press, 2008), 139–169.

Meinwald, C., 'Plato's Pythagoreanism', *AncPhil*, 22 (2002), 87–101.

Moravcsik, J.M., 'Forms and Dialectic in the Second Half of the *Parmenides*', in M. Schofield and M.C. Nussbaum (eds), *Language and Logos: Studies in Ancient Greek Philosophy Presented to G.E.L. Owen* (Cambridge: Cambridge University Press, 1982), 135–154.

Negrepontis, S., 'The Anthyphairetic Revolutions of the Platonic Ideas', in M. Sialaros (ed.), *Revolutions and Continuity in Greek Mathematics* (Berlin: De Gruyter, 2018), 335–381.

Owen, G.E.L., 'Notes On Ryle's Plato', in *Logic, Science and Dialectic* (London: Duckworth, 1986), 85–103.

Panza, M. and A. Sereni, *Plato's Problem: An Introduction to Mathematical Platonism* (New York: Palgrave Macmillan, 2013).

Pourciau, B., 'Intuitionism as a (Failed) Kuhnian Revolution in Mathematics', *SHPS*, 31 (2000), 297–329.

Pritchard, P., *Plato's Philosophy of Mathematics* (Sankt Augustin: Academia Verlag, 1995).

Rickless, S.C., *Plato's Forms in Transition: A Reading of the Parmenides* (Cambridge: Cambridge University Press, 2006).

Ross, D., (ed.), *Aristotle's* Physics (Oxford: Clarendon Press, 1936).

Ross, D., *Plato's Theory of Ideas* (Oxford: Clarendon Press, 1951).

Russell, B., *Introduction to Mathematical Philosophy* (London: Allen & Unwin, 1919; rev. repr. London: Routledge, 1993).

Ryle, G., 'Plato's *Parmenides*', *Mind*, 48/190 (1939), 129–151.

Schofield, M., 'A Neglected Regress Argument in the *Parmenides*', *CQ*, 23/1 (1973), 29–44.

Scolnicov, S., *Plato's* Parmenides (Berkeley: University of California Press, 2003).

Sialaros, M., 'Introduction: Revolutions in Greek Mathematics', in M. Sialaros (ed.), *Revolutions and Continuity in Greek Mathematics* (Berlin: De Gruyter, 2018), 1–15.

Unguru, S., 'Counter-Revolutions in Mathematics', in M. Sialaros (ed.), *Revolutions and Continuity in Greek Mathematics* (Berlin: De Gruyter, 2018), 19–34.

Wedberg, A., *Plato's Philosophy of Mathematics* (Stockholm: Almqvist & Wiksel, 1955).

Wolf, R.S., *A Tour Through Mathematical Logic* (Washington DC: Mathematical Association of America, 2005).

Zhmud, L., *The Origin of the History of Science in Classical Antiquity*, trans. Alexander Chernoglazov (Berlin: De Gruyter, 2006).

Zhmud, L., *Pythagoras and the Early Pythagoreans*, trans. K. Windle and R. Ireland (Oxford: Oxford University Press, 2012).