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Strong Large Deviations Principles of Non-Freidlin-Wentzell Type -Optimal Control Problem with Imperfect Information -Jumps Phenomena in Financial Markets

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Abstract. The paper presents, a new large deviations principles (SLDP) of non-Freidlin-Wentzell type, corresponding to the solutions Colombeau-Ito's SDE. Using SLDP we present a new approach to construct the Bellman function $v(t, \boldsymbol{x})$ and optimal control $\boldsymbol{u}(t, \boldsymbol{x})$ directly by way of using strong large deviations principle for the solutions Colombeau-Ito's SDE. As important application such SLDP, the generic imperfect dynamic models of air-to-surface missiles are given in addition to the related simple guidance law. A four, examples have been illustrated proposed approach and corresponding numerical simulations have been illustrated and analyzed. Using SLDP approach, Jumps phenomena, in financial markets, also is considered. Jumps phenomena, in financial markets is explained from the first principles, without any reference to Poisson jump process. In contrast with a phenomenological approach we explain such jumps phenomena from the first principles, without any reference to Poisson jump process.

Keywords: Optimal control, Bellman equation, Colombeau-Ito's SDE,Large deviations principles, Algebra of Colombeau generalized functions, Poisson jump process, Jumps phenomena, in financial markets.

1. Introduction

What new scalable mathematics is needed to replace the traditional Partial Differential Equations (PDE) approach to differential games?

Let$\mathfrak{C} = (\Omega, \Sigma. P)$be a probability space. Any stochastic process on$\mathbb{R}^n$ is aΣ-measurable mapping$X: \Omega \times [0,T] \to \mathbb{R}^n$. Many stochastic optimal control problems essentially come down to constructing a function$u(t,x)$ that has the properties:

(1) $u(t,x) = \inf_\alpha \left[\bar{\mathbf{J}}\left(\left\{X_{s,D}^x(\omega)\right\}_{a\in[0,t]}; \{\alpha(s)\}_{a\in[0,t]}\right)\right]$ and

(2) $u(t,x) = \inf_\alpha \left[\bar{\mathbf{J}}\left(\left\{X_{s,D}^x(\omega)\right\}_{a\in[0,t]}; \{\alpha(s)\}_{a\in[0,t]}\right) + u\left(t, X_{t,D}^x(\omega)\right)\right]$, where $\alpha(t) \in U \subsetneq R^n$.

Here $\bar{\mathbf{J}} = E_\Omega\left[\int_0^t \left(g\left(X_{s,D}^x(\omega), s\right)\right) ds\right]$ is the termination payoff: functional,$\alpha(t)$is a control and $X_{t,D}^x(\omega)$ is some Markov process governed by some stochastic Ito's equation driven by a Brownian motion of the form

$$X_{t,D}^x(\omega) = x + \int_0^t f\left(X_{s,D}^x(\omega), \alpha(s)\right) ds + \sqrt{D} W(t,\omega). \quad (3)$$

Here$W(t,\omega)$ is the Brownian motion. Traditionally the function $u(t,x)$ has been computed by way of solving the associated *Bellman equation*, for which various numerical techniques mostly variations of the finite difference scheme have been developed. Another approach, which takes advantage of the recent developments in computing technology and allows one to construct the function$u(x,t)$ by way of backward induction governed by Bellman's principle such that described in [1]. In paper [1] Equation (3) is *approximated* by an equation with affine coefficients which admits an explicit solution in terms of integrals of the exponential Brownian motion. Using Colombeau approach proposed in paper [2], [3],[4] we have replaced Equation (3) by Colombeau-Ito's equation [4-6]:

$$\left(X_{t,D,\varepsilon'}^{x,\varepsilon}(\omega,\varpi)\right)_{\varepsilon'} = x + \left(\int_0^t f_{\varepsilon'}\left(X_{t,D,\varepsilon'}^{x,\varepsilon}(\omega,\varpi), \alpha(s)\right) ds\right)_{\varepsilon'}$$

$$+\sqrt{D}\left(\int_0^t w_{\varepsilon'}(s,\varpi) ds\right)_{\varepsilon'} + \sqrt{\varepsilon}\left(W(t,\omega)\right)_{\varepsilon'}.$$

Here $\varepsilon, \varepsilon' \in (0,1]$,$\omega \in \Omega_1, \varpi \in \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$, where $w(t,\omega)$ is the white noise on $\mathbb{R}^n$ i.e., $w(t,\omega) = d/dt\, W(t,\omega)$ almost surelyin D' and $w_{\varepsilon'}(t,\varpi)$ is the smoothed white noiseon $\mathbb{R}^n$i.e., $w_{\varepsilon'}(t,\varpi) = \langle w(t,\omega), \phi_{\varepsilon'}(s-t)\rangle$, and $\phi_{\varepsilon'}$ is a model delta net [2], [4]. Fortunately in contrast with Equation (3) one can solve Equation (4) without any approximation using strong large deviations principleof Non-Freidlin-Wentzelltype [5],[6],[7].

Statement of the novelty and uniqueness of the proposed idea: A new approach, which is proposed in this paper allows one to construct the Bellman function$v(t,x)$ and optimal control $\alpha(t,x)$ directly, i.e., without any reference to the Bellman equation, by way of using strong large deviations principle for the solutionsColombeau-Ito's SDE (CISDE).

2. Proposed Approach

Let$\mathfrak{C}_i = (\Omega_i, \boldsymbol{\Sigma}_i, \mathbf{P}_i), i = 1,2$ be a probability spaces such that: $\Omega_1 \cap \Omega_2 = \emptyset$. Let us consider m-persons Colombeau-Ito differential game$CIDG_{m;T}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{y}, G^n(\mathbb{R}^n), \mathfrak{C}_1, \mathfrak{C}_2)$, withthe termination payoff functional for the i-th player is:

$$\left(\bar{\mathbf{J}}^{\varepsilon}_{\varepsilon',j}\right)_{\varepsilon'} = \mathbf{E}_{\Omega_1}\mathbf{E}_{\Omega_2}\left[\left(\int_0^T g_{\varepsilon',i}\left(x^{x,\varepsilon}_{t,D,\varepsilon'}(\omega,\varpi), \boldsymbol{\alpha}(t), t, \varepsilon\right) dt\right)_{\varepsilon'}\right] +$$
$$+\mathbf{E}_{\Omega_1}\mathbf{E}_{\Omega_2}\left[\left(\sum_{i=1}^{n}\left[x^{x,\varepsilon}_{T,D,\varepsilon';i}(\omega,\varpi) - y_i\right]^2\right)_{\varepsilon'}\right] \quad (1)$$

and with stochastic nonlinear dynamics:

$$\left(\dot{\boldsymbol{x}}^{x_0,\varepsilon}_{t,D,\varepsilon'}(\omega,\varpi)\right)_{\varepsilon'} = \left(\boldsymbol{f}_{\varepsilon'}\left(\boldsymbol{x}^{x,\varepsilon}_{t,D,\varepsilon'}(\omega,\varpi), \sqrt{D}w_{\varepsilon'}(t,\varpi), \boldsymbol{\alpha}(t), t, \varepsilon\right)\right)_{\varepsilon'} + \sqrt{\varepsilon}\left(\boldsymbol{w}(t,\omega)\right)_{\varepsilon'} \quad (2)$$

$\varepsilon, \varepsilon' \in (0,1], \omega \in \Omega_1, \varpi \in \Omega_2$.

Here $\forall t \in [0,T]$: $\left(\boldsymbol{x}_{\varepsilon'}(t)\right)_{\varepsilon'} \in \widehat{\mathbb{R}}^n$; $\boldsymbol{x}^{x_0,\varepsilon}_{0,D,\varepsilon'}(\omega,\varpi) = \boldsymbol{x}_0 \in \mathbb{R}^n, \forall \boldsymbol{\varepsilon} \in (0,1]$:$\boldsymbol{f} = [(\boldsymbol{f}_{\varepsilon'})_{\varepsilon'}], \boldsymbol{g} = [(\boldsymbol{g}_{\varepsilon'})_{\varepsilon'}]$; $\boldsymbol{f}(\boldsymbol{x},\circ,\circ,\circ,\circ), \boldsymbol{g}(\boldsymbol{x},\circ,\circ,\circ) \in G^n(\mathbb{R}^n), \boldsymbol{\alpha}(t) = \{\alpha_1(t), \dots, \alpha_m(t)\}; \alpha_i(t) \in U_i \subsetneq \mathbb{R}^{k_i}, i = 1, \dots, m$,

And m–persons Colombeau-Ito differential game $CIDG_{m;T}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{y}, G^n(\mathbb{R}^n), \boldsymbol{\beta}(t), \boldsymbol{\varphi}(t), \mathfrak{C}_1, \mathfrak{C}_2)$ with imperfect measurements and with imperfect information about the system [5], [6]. The corresponding stochastic nonlinear dynamics is:

$$\left(\dot{\boldsymbol{x}}^{x_0,\varepsilon}_{t,D,\varepsilon'}(\omega,\varpi)\right)_{\varepsilon'} = \left(\boldsymbol{f}_{\varepsilon'}\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,D,\varepsilon'}(\omega,\varpi), \sqrt{D}w_{\varepsilon'}(t,\varpi), \boldsymbol{\varphi}(t), \alpha\left(t, x^{x_0,\varepsilon}_{t,D,\varepsilon'} + \boldsymbol{\beta}(t)\right), t, \varepsilon\right)\right)_{\varepsilon'} +$$

$+\sqrt{\varepsilon}\big(w(t,\omega)\big)_{\varepsilon'}; \varepsilon,\varepsilon' \in (0,1], \omega \in \Omega_1, \varpi \in \Omega_2$ and the playoff for the i-th player is:

$$\left(\bar{\mathbf{J}}^{\varepsilon}_{\varepsilon',j}\right)_{\varepsilon'} = \mathbf{E}_{\Omega_1}\mathbf{E}_{\Omega_2}\left[\left(\int_0^T g_{\varepsilon',i}\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,D,\varepsilon'}(\omega,\varpi), \alpha\big(t,\boldsymbol{\beta}(t)\big), t, \varepsilon\right) dt\right)_{\varepsilon'}\right] +$$

$$+\mathbf{E}_{\Omega_1}\mathbf{E}_{\Omega_2}\left[\left(\sum_{i=1}^{n}\left[x^{x_0,\varepsilon}_{T,D,\varepsilon';i}(\omega,\varpi) - y_i\right]^2\right)_{\varepsilon'}\right]. \quad (3)$$

Here $\boldsymbol{\beta}(t) = (\beta_1(t),\dots,\beta_n(t))$, $\boldsymbol{\varphi}(t) = \big(\varphi_1(t),\dots,\varphi_n(t)\big)$ and $\forall t \in [0,T]: \big(x_{\varepsilon'}(t)\big)_{\varepsilon'} \in \widehat{\mathbb{R}}^n; x^{x_0,\varepsilon}_{0,D,\varepsilon'}(\omega,\varpi) = x_0, \forall \boldsymbol{\varepsilon} \in (0,1]$:

$\boldsymbol{f} = [(\boldsymbol{f}_{\varepsilon'})_{\varepsilon'}], \boldsymbol{g} = [(\boldsymbol{g}_{\varepsilon'})_{\varepsilon'}]; \boldsymbol{f}(x,\circ,\circ,\circ,\circ,\circ), \boldsymbol{g}(x,\circ,\circ,\circ) \in G^n(\mathbb{R}^n)$ or

$\boldsymbol{f}(x,\circ,\circ,\circ,\circ), \boldsymbol{g}(x,\circ,\circ) \in G^n_{P,r}(E), \alpha(t) = \{\alpha_1(t),\dots,\alpha_m(t)\}; \alpha_i(t) \in U \subsetneq \mathbb{R}^{k_i}, i = 1,\dots,m, \boldsymbol{\beta}(t) = \{\beta_1(t),\dots,\beta_n(t)\}, \boldsymbol{\varphi}(t) = \{\varphi_1(t),\dots,\varphi_n(t)\}$.

Here $\mathbb{R}$ is a field of the real numbers, $G(\mathbb{R}^n)$ is the algebra of Colombeau generalized functions [8],[9],[12] , $G^n(\mathbb{R}^n) = G(\mathbb{R}^n) \times \dots \times G(\mathbb{R}^n)$, $G_{P,r}(E) = \frac{\mathcal{F}_{P,r}(E)}{\mathrm{K}_{P,r}(E)}$ is the Colombeau type algebra[13],[14], E is an appropriate algebra of functions, which is a locally convex vector space over field ₵, $G^n_{P,r}(E) = G_{P,r}(E) \times \dots \times G_{P,r}(E)$, $\widehat{\mathbb{R}}$ is the ring of Colombeau generalized numbers [11], $\widehat{\mathbb{R}}^{\boldsymbol{n}} = \widehat{\mathbb{R}} \times \dots \times \widehat{\mathbb{R}}$, $t \to \alpha_i(t)$ is the control chosen by the i-th player, within a set of admissible control values U_i.

Here $t \mapsto \left\{\left(x^{x_0,\varepsilon}_{t,D,\varepsilon';1}(\omega,\varpi)\right)_{\varepsilon'},\dots,\left(x^{x_0,\varepsilon}_{t,D,\varepsilon';n}(\omega,\varpi)\right)_{\varepsilon'}\right\}$ isthe trajectoryof the Equation (2). Optimal control problem for the i-th player is:

$$\left(\bar{\bar{\mathbf{J}}}^{\varepsilon}_{\varepsilon',i}\right)_{\varepsilon'} = \left(\min_{\alpha_i(t)\in U_i}\left(\max_{\alpha_j(t)\in U_j}\bar{\mathbf{J}}^{\varepsilon}_{\varepsilon',j\neq i}\right)\right)_{\varepsilon'}. \quad (4)$$

We remind now some classical definitions.

Let us consider now Ito's SDE:

$$d\boldsymbol{x}_t = \boldsymbol{b}(\boldsymbol{x}_t,t)dt + \sum_{r=1}^{k}\sigma_r(\boldsymbol{x}_t,t)dW_r(t,\omega), \quad (5)$$

$$\boldsymbol{x}_0 = \boldsymbol{x}_0(\omega), \boldsymbol{x} \in \mathbb{R}^n.$$

Theorem1.[15]-[16]. Let the vectors $\boldsymbol{b}(\boldsymbol{x},t), \boldsymbol{\sigma}(\boldsymbol{x},t)$ be continuous functions of $(\boldsymbol{x},t)$ such that for some constants D and C the following conditionshold:

$$\|\boldsymbol{b}(\boldsymbol{x},t)-\boldsymbol{b}(\boldsymbol{y},t)\|+\sum_{r=1}^{k}|\sigma_r(\boldsymbol{x},t)-\sigma_r(\boldsymbol{y},t)|\leq D\|\boldsymbol{x}-\boldsymbol{y}\|, \qquad (6)$$

$$\|\boldsymbol{b}(\boldsymbol{x},t)\|+\sum_{r=1}^{k}|\sigma_r(\boldsymbol{x},t)|\leq C(1+\|\boldsymbol{x}\|). \qquad (7)$$

Then: (1) For every random variable $\boldsymbol{x}(\omega)$ independent of the processes $W_r(t,\omega), r = 1,2,\dots,k$ there exists a solution $\boldsymbol{x}_t$ of the Ito's SDE(5) which is an almost surely continuous Markov process and
(2) Two solutions $\boldsymbol{x}_{t,1}$ and $\boldsymbol{x}_{t,2}(\omega)$ is unique up to equivalence: $\mathbf{P}[\boldsymbol{x}_{t,1}(\omega)=\boldsymbol{x}_{t,2}(\omega)]=1$, for all $t\in[0,\infty)=I_\infty$.

Remark1.[15],[17]. It well known, that the boundedness assumption on $\boldsymbol{b}(\boldsymbol{x},t)$ and $\boldsymbol{\sigma}(\boldsymbol{x},t)$ can be weakened, but somekind of restriction on the $\boldsymbol{b}(\boldsymbol{x},t)$ and $\boldsymbol{\sigma}(\boldsymbol{x},t)$ is necessary in order to guarantee the existence of a global solution i.e., a solution defined for all $t\in[0,\infty)$. If we remove this condition of boundedness, then a solution of Ito's SDE (5) does exist locally but, in general, *blows up* (or *explodes)* in finite time.

Definition1. Let $\grave{\mathbb{R}}^n=\mathbb{R}^n\cup\{\boldsymbol{\Delta}\}$ be the one-point compactification of $\mathbb{R}^n$ and $\grave{\boldsymbol{W}}^n=\{\boldsymbol{w}|[0,\infty)\ni t\mapsto\boldsymbol{w}(t)\in\grave{\mathbb{R}}^n$ is continuous and such that if $\boldsymbol{w}(\boldsymbol{t})=\boldsymbol{\Delta}$, then $\boldsymbol{w}(t')=\boldsymbol{\Delta}$ for all $t'\geq t\}$. Let $\mathfrak{H}(\grave{\mathbb{R}}^n)$ be the σ-field generated by Borel cylinder sets. For $\boldsymbol{w}\in\grave{\boldsymbol{W}}^n$ we set

$$e(\boldsymbol{w})=\inf\{t|\boldsymbol{w}(t)=\Delta\} \qquad (8)$$

and call the *explosion time* of the trajectory $\boldsymbol{w}(t), t\in[0,\infty)$.

Definition2.[15]. By a solution $\boldsymbol{x}_t(\omega)$ of the equation(5) we mean a $\left(\grave{\boldsymbol{W}}^n,\mathfrak{H}(\grave{\mathbb{R}}^n)\right)$- valued random variable defined on a probability space $\mathfrak{C}=(\Omega,\boldsymbol{\Sigma},\mathbf{P})$ with a reference family $(\boldsymbol{\Sigma}_t)_{t\geq0}$ such that:

(i) there exists an n-dimensional $(\boldsymbol{\Sigma}_t)$-Brownian motion $\boldsymbol{W}(t,\omega)=\left(W_1(t,\omega),\dots,W_n(t,\omega)\right)$ with $\boldsymbol{W}(0,\omega)=0$,

(ii) for each $(t,\omega)\mapsto\boldsymbol{x}_t(\omega)\in\grave{\mathbb{R}}^n$ is $\boldsymbol{\Sigma}_t$ -measurable and

(iii) if $e(\omega)=e(\boldsymbol{x}_t(\omega))$ is the explosion time of $\boldsymbol{x}_t(\omega)$then for almost all ω,

$$\boldsymbol{x}_t(\omega)-\boldsymbol{x}_0(\omega)=$$

$$=\int_0^t \boldsymbol{b}(\boldsymbol{x}_s(\omega),t)\,ds+\sum_{r=1}^{k}\int_0^t \sigma_r(\boldsymbol{x}_s(\omega),t)\,dW_r(s,\omega),\quad(9)$$

for all $t\in[0,e(\omega))$.

Theorem2.[15],[16].(1)Given $\grave{\mathbb{R}}^n$ −continuous$\boldsymbol{b}(\boldsymbol{x},t)$and $\boldsymbol{\sigma}(\boldsymbol{x},t)$consider the equation (5).Then for any probability μ on$\left(\grave{\boldsymbol{W}}^n,\mathfrak{H}(\grave{\mathbb{R}}^n)\right)$with compact support, there exists a solution
of (5) such thatthe law of $\boldsymbol{x}_0(\omega)$ coincides with μ.

(2) Suppose$\boldsymbol{b}(\boldsymbol{x},t)and\boldsymbol{\sigma}(\boldsymbol{x},t)$are locally Lipschitz continuous,
i.e., for every $N>0$ there exists a constant$D_N>0$ such that

$$\|\boldsymbol{b}_n(\boldsymbol{x},t)-\boldsymbol{b}_n(\boldsymbol{y},t)\|+\sum_{r=1}^{k}\left|\sigma_{r,n}(\boldsymbol{x},t)-\sigma_{r,n}(\boldsymbol{y},t)\right|\leq D_N\|\boldsymbol{x}-\boldsymbol{y}\| \quad (10)$$

for every$\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{B}_N$,$\boldsymbol{B}_N=\{\boldsymbol{z}|\|\boldsymbol{z}\|\leq N\}$.Then for any probability μ on$\left(\grave{\boldsymbol{W}}^n,\mathfrak{H}(\grave{\mathbb{R}}^n)\right)$with compact support, there exists a solution
of (5) such thatthe law of $\boldsymbol{x}_0(\omega)$ coincides with μ.

Theorem3.[16].Let$\boldsymbol{x}_{t,n}(t), n=1,2,\ldots$ be the solutions
of the Ito's SDE's

$$d\boldsymbol{x}_{t,n}=\boldsymbol{b}_n(\boldsymbol{x}_{t,n},t)dt+\sum_{r=1}^{k}\sigma_{r,n}(\boldsymbol{x}_{t,n},t)dW_r(t,\omega), \quad (11)$$

$$\boldsymbol{x}_{0,n}=\boldsymbol{x}(\omega)x\in\mathbb{R}^n.$$

Assume that: (i) let the vectors $\boldsymbol{b}_n(\boldsymbol{x},t),\boldsymbol{\sigma}_n(\boldsymbol{x},t)$becontinuous functions of $(\boldsymbol{x},t)$such that for some constants$D$and C the following conditions hold

$$\|\boldsymbol{b}_n(\boldsymbol{x},t)-\boldsymbol{b}_n(\boldsymbol{y},t)\|+\sum_{r=1}^{k}\left|\sigma_{r,n}(\boldsymbol{x},t)-\sigma_{r,n}(\boldsymbol{y},t)\right|\leq D\|\boldsymbol{x}-\boldsymbol{y}\|, \quad (12)$$

$$\|\boldsymbol{b}_n(\boldsymbol{x},t)\|+\sum_{r=1}^{k}\left|\sigma_{r,n}(\boldsymbol{x},t)\right|\leq C(1+\|\boldsymbol{x}\|), \quad (13)$$

(ii)$\mathbf{E}[\boldsymbol{x}^2(\omega)] < \infty,$ (14)

(iii)$\forall N > 0$:

$$\lim_{n\to\infty}\sup_{\|x\|\leq N}\left[\|\boldsymbol{b}_n(\boldsymbol{x},t) - \boldsymbol{b}_0(\boldsymbol{y},t)\| + \sum_{r=1}^{k}\left|\sigma_{r,n}(\boldsymbol{x},t) - \sigma_{r,0}(\boldsymbol{y},t)\right|\right] = 0. \quad (15)$$

Then

$$\lim_{n\to\infty}\sup_{0\leq t\leq T}\mathbf{E}\left[\boldsymbol{x}_{t,n}(\omega) - \boldsymbol{x}_{t,0}(\omega)\right]^2 = 0. \quad (16)$$

Corollary 1. Let $\boldsymbol{x}_{t,n}(t), n = 1,2,\ldots$ be the solutionsof the Ito's SDE's

$d\boldsymbol{x}_{t,n} = \boldsymbol{b}_n(\boldsymbol{x}_{t,n}, t)dt + \sum_{r=1}^{k}\sigma_{r,n}dW_r(t,\omega)\,\boldsymbol{x}_{0,n} = \boldsymbol{x}(\omega)x \in \mathbb{R}^n$.(17)

Assume that: (i) Let the vectors $\boldsymbol{b}_n(\boldsymbol{x},t)$,becontinuous functions of $(\boldsymbol{x},t)$ and $\sigma_n = const$ such that for some constants $D$and C the following conditions hold

$$\|\boldsymbol{b}_n(\boldsymbol{x},t) - \boldsymbol{b}_n(\boldsymbol{y},t)\| \leq D\|\boldsymbol{x} - \boldsymbol{y}\| \quad (18)$$

$$\|\boldsymbol{b}_n(\boldsymbol{x},t)\| + \sum_{r=1}^{k}\left|\sigma_{r,n}\right| \leq C(1 + \|\boldsymbol{x}\|), \quad (19)$$

(ii) $\mathbf{E}[\boldsymbol{x}^2(\omega)] < \infty,$ (20)

(iii)$\forall N > 0$:

$$\lim_{n\to\infty}\sup_{\|x\|\leq N}\left[\|\boldsymbol{b}_n(\boldsymbol{x},t) - \boldsymbol{b}_0(\boldsymbol{y},t)\| + \sum_{r=1}^{k}\left|\sigma_{r,n}\right|\right] = 0. \quad (21)$$

Then

$$\lim_{n\to\infty}\sup_{0\leq t\leq T}\mathbf{E}\left[\boldsymbol{x}_{t,n}(\omega) - \boldsymbol{x}_{t,0}(\omega)\right]^2 = 0. \quad (22)$$

Here $\boldsymbol{x}_{t,0}(\omega)$ is the solution of the ODE:

$$d\boldsymbol{x}_{t,0} = \boldsymbol{b}_0(\boldsymbol{x}_{t,0}, t)dt, \boldsymbol{x}_{0,0} = \boldsymbol{x}(\omega), x \in \mathbb{R}^n. \quad (23)$$

Remark 2. Note that Theorem 3 in fact asserts that under conditions (12)-(15) any solution $\boldsymbol{x}_t(\omega)$ of the Ito's SDE (5) is continuously depend on functions $\boldsymbol{b}(\boldsymbol{x},t)$ and $\boldsymbol{\sigma}(\boldsymbol{x},t)$. Note that the assumptions of the Lipschitz continuously (12) and boundedness (13) on $\boldsymbol{b}(\boldsymbol{x},t)$ and $\boldsymbol{\sigma}(\boldsymbol{x},t)$ in the Theorem 3 cannot be weakened.

Theorem. Assume that: (1) Let $\boldsymbol{x}_{t,n}(t), n = 1,2,\dots$ be the solutions of the Ito's SDE's

$$\boldsymbol{x}_{t,n} = \boldsymbol{b}_n(\boldsymbol{x}_{t,n}, t)dt + \boldsymbol{\sigma}_n(\boldsymbol{x}_{t,n}, t)d\boldsymbol{W}(t,\omega),$$
$$\boldsymbol{x}_{0,n} = \boldsymbol{x}(\omega), x \in \mathbb{R}^n.$$

and let $\widetilde{\boldsymbol{x}}_{t,n}(t), n = 1,2,\dots$ be the solutions of the Ito's SDE's

$$\widetilde{\boldsymbol{x}}_{t,n} = \widetilde{\boldsymbol{b}}_n(\widetilde{\boldsymbol{x}}_{t,n}, t)dt + \widetilde{\boldsymbol{\sigma}}_n(\widetilde{\boldsymbol{x}}_{t,n}, t)d\boldsymbol{W}(t,\omega),$$

$$\widetilde{\boldsymbol{x}}_{0,n} = \boldsymbol{x}(\omega), x \in \mathbb{R}^n.$$

Here

$$\boldsymbol{\sigma}_n(\boldsymbol{x}_{t,n}, t)d\boldsymbol{W}(t,\omega) = \sum_{r=1}^{k} \sigma_{r,n}(\boldsymbol{x}_{t,n}, t)dW_r(t,\omega),$$

$$\widetilde{\boldsymbol{\sigma}}_n(\widetilde{\boldsymbol{x}}_{t,n}, t)d\boldsymbol{W}(t,\omega) = \sum_{r=1}^{k} \tilde{\sigma}_{r,n}(\boldsymbol{x}_{t,n}, t)dW_r(t,\omega).$$

(2) The inequalities

$$\|\boldsymbol{b}_n(\boldsymbol{x},t)\| + \|\boldsymbol{\sigma}_n(\boldsymbol{x},t)\| \le K_n(1 + \|\boldsymbol{x}\|),$$

$$\|\boldsymbol{b}_n(\boldsymbol{x},t) - \boldsymbol{b}_n(\boldsymbol{y},t)\| + \|\boldsymbol{\sigma}_n(\boldsymbol{x},t) - \boldsymbol{\sigma}_n(\boldsymbol{x},t)\| \le K_n\|\boldsymbol{x} - \boldsymbol{y}\|,$$

$$\left\|\widetilde{\boldsymbol{b}}_n(\boldsymbol{x},t)\right\| + \|\widetilde{\boldsymbol{\sigma}}_n(\boldsymbol{x},t)\| \le K_n(1 + \|\boldsymbol{x}\|),$$

$$\left\|\widetilde{\boldsymbol{b}}_n(\boldsymbol{x},t)-\widetilde{\boldsymbol{b}}_n(\boldsymbol{y},t)\right\|+\|\widetilde{\boldsymbol{\sigma}}_n(\boldsymbol{x},t)-\widetilde{\boldsymbol{\sigma}}_n(\boldsymbol{x},t)\|\leq K_n\|\boldsymbol{x}-\boldsymbol{y}\|,$$

$$\left\|\boldsymbol{b}_n(\boldsymbol{x},t)-\widetilde{\boldsymbol{b}}_n(\boldsymbol{x},t)\right\|\leq\delta_{1,n}\|\boldsymbol{x}\|,$$

$$\|\boldsymbol{\sigma}_n(\boldsymbol{x},t)-\boldsymbol{\sigma}_n(\boldsymbol{x},t)\|\leq\delta_{2,n}\|\boldsymbol{x}\|$$

where $0\leq t\leq T$, is satisfied. Then the inequality

$$\sup_{0\leq t\leq T}\mathbf{E}\left[\left\|\boldsymbol{x}_{t,n}-\widetilde{\boldsymbol{x}}_{t,n}\right\|^2\right]\leq e^{L_n}\left(\delta_{1,n}^2+\delta_{2,n}^2\right)E\left[\int_0^T\left\|\widetilde{\boldsymbol{x}}_{t,n}\right\|^2dt\right]$$

is satisfied.

Proof. See Appendix.

Remark 3.[17].If conditions(6)-(7)are valid only in every cylinder $U_R\times I_\infty$, with $C=C(R),D=D(R)$, one can construct a sequenceof functions $\boldsymbol{b}_n(\boldsymbol{x},t)and\boldsymbol{\sigma}_n(\boldsymbol{x},t)$such that for$\|\boldsymbol{x}\|<n$

$$\boldsymbol{b}_n(\boldsymbol{x},t)=\boldsymbol{b}(\boldsymbol{x},t),\boldsymbol{\sigma}_n(\boldsymbol{x},t)=\boldsymbol{\sigma}(\boldsymbol{x},t),\tag{24}$$

and therefore for each$\boldsymbol{b}_n(\boldsymbol{x},t),\boldsymbol{\sigma}_{\boldsymbol{n}}(\boldsymbol{x},t)$satisfy conditions(6)-(7)everywhere in $\mathbb{R}^n$. By Theorem 1,there exists a sequence of Markov processes $\boldsymbol{x}_{t,n}(\omega)$corresponding to the functions$\boldsymbol{b}_n(\boldsymbol{x},t)$and $\boldsymbol{\sigma}_{\boldsymbol{n}}(\boldsymbol{x},t)$.

Assumption1. Suppose now that the distribution of$\boldsymbol{x}_0(\omega)$hascompact support in$\mathbb{R}^n$.Then as, well known, that the first exit random times$\boldsymbol{\tau}_m(\omega)$ of the processes $\boldsymbol{x}_{t,,m}(\omega)$from the set $\|\boldsymbol{x}\|<n$ are identical for $m\geq n$[15] [18],[19]. Let this common value be$\boldsymbol{\tau}_n(\omega)$. Itis also clear that the processes themselves coincide up to time$\boldsymbol{\tau}_n(\omega)$, i.e.

$$\mathbf{P}\left[\sup_{0\leq\tau(\omega)\leq\tau_n(\omega)}\left\|\boldsymbol{x}_{\boldsymbol{\tau}(\omega),n}(\omega)-\boldsymbol{x}_{\boldsymbol{\tau}(\omega),m}(\omega)\right\|>0\right]=0,m>n.\tag{25}$$

Orin the equivalent form

$$\mathbf{P}\left[\sup_{0\le t\le\tau_n(\omega)}\left\|\boldsymbol{x}_{t,n}(\omega)-\boldsymbol{x}_{t,m}(\omega)\right\|>0\right]=0, m>n. \qquad (27)$$

Definition1. (i) Let $\boldsymbol{\tau}_\infty(\omega)$denote the (finite or infinite) limit of the monotone increasing sequence $\tau_n(\omega)$as $n\to\infty$. We call the random variable $\boldsymbol{\tau}_\infty(\omega)$the first exit time from every bounded domain, or briefly the *explosion* time.
(ii)We now define a new stochastic process $\boldsymbol{x}_t(\omega)$by setting[17]:

$$\boldsymbol{x}_t(\omega)=\boldsymbol{x}_{t,n}(\omega)\text{for } t<\boldsymbol{\tau}_n(\omega). \qquad (28)$$

It well known, that this is always a Markov process for $t<\boldsymbol{\tau}_n(\omega)$[18],[19].
We also can to define a new stochastic process $\boldsymbol{x}_t(\omega)$by setting

$$\boldsymbol{x}_t(\omega)=\mathbf{P}-\lim_{n\to\infty}\boldsymbol{x}_{t,n}(\omega) \qquad (29)$$

If finite or infinite limit in RHS of Eq.(29) exist.

(iii) In general case we set

$$\left(\boldsymbol{x}_{t,\varepsilon'}(\omega)\right)_{\boldsymbol{\varepsilon}'}=\left(\boldsymbol{x}_{t,n}(\omega)\right)_n, n=\frac{1}{\varepsilon'}. \qquad (30)$$

We note that the Colombeau-Ito's equation

$$\left(\boldsymbol{x}_{t,\varepsilon'}(\omega)\right)_{\varepsilon'}-\left(\boldsymbol{x}_{0,\varepsilon}(\omega)\right)_{\varepsilon'}=$$
$$\left(\int_{\mathbf{0}}^{\boldsymbol{t}}\boldsymbol{b}\left(\boldsymbol{x}_{s,\varepsilon'}(\omega),s\right)ds\right)_{\varepsilon'}+\sum_{r=1}^{k}\left(\int_0^t\sigma_r\left(\boldsymbol{x}_{s,\varepsilon'}(\omega),t\right)dW_r(s,\omega)\right)_{\varepsilon'} \qquad (31)$$

is satisfied for all $t\in\left[0,\boldsymbol{\tau}_\infty(\omega)\right)$.

(iv) Markov process $\boldsymbol{x}_t(\omega)$is *regular* if for all $s<\infty,\boldsymbol{x}\in\mathbb{R}^n$

$$\mathbf{P}^{s,x}\{\boldsymbol{\tau}_\infty(\omega)=\infty\}=1. \quad (32)$$

Assumption2. We assume now that:(1)$\forall\epsilon\in(0,1]:\left(\boldsymbol{b}_{\boldsymbol{\varepsilon}'}(\boldsymbol{x},t,\epsilon)\right)_{\boldsymbol{\varepsilon}'}\triangleq\left(b_{1,\boldsymbol{\varepsilon}'}(\boldsymbol{x},t,\epsilon),\dots,b_{n,\boldsymbol{\varepsilon}'}(\boldsymbol{x},t,\epsilon)\right)_{\boldsymbol{\varepsilon}'}\in G^n(\mathbb{R}^n)\left(\text{or } G^n_{P,r}(E)\right)$,$\epsilon=(\epsilon_1,\dots,\epsilon_n),\epsilon\in(0,1]:^n$for all $t\in[0,\infty)$and

(2) $\forall\epsilon\in(0,1]^n$there exist infinite Colombeau constants $\left(C^{\varepsilon}_{\varepsilon'}\right)_{\varepsilon'}and\left(D^{\varepsilon}_{\varepsilon'}\right)_{\varepsilon'}$,such that$\forall\epsilon\in(0,1]$:

$(i)(\|\boldsymbol{b}_{\boldsymbol{\varepsilon}'}(\boldsymbol{x},t,\epsilon)\|^2)_{\boldsymbol{\varepsilon}'}\leq\left(\left(C^{\varepsilon}_{\varepsilon'}\right)_{\boldsymbol{\varepsilon}'}\right)(1+\|\boldsymbol{x}\|^2),\varepsilon'\in(0,1]$,(33)

(ii)$\left(\left\|\boldsymbol{b}_{\boldsymbol{i},\boldsymbol{\varepsilon}'}(\boldsymbol{x},t,\epsilon)-\boldsymbol{b}_{i,\varepsilon'}(\boldsymbol{y},t,\epsilon)\right\|\right)_{\boldsymbol{\varepsilon}'_i}\leq\left(\left(D^{\varepsilon}_{\varepsilon'}\right)_{\boldsymbol{\varepsilon}'_i}\right)\|\boldsymbol{x}-\boldsymbol{y}\|$(34)

for all $t\in[0,\infty)$and for all $\boldsymbol{x}\in\mathbb{R}^{\boldsymbol{n}}$and for all$\boldsymbol{y}\in\mathbb{R}^n$.

Definition 2. [4] 1.Let $\mathfrak{C}=(\Omega,\Sigma.P)$be a probability space. Let $\mathcal{E}R$ be the space of nets $\left(X_\varepsilon(\omega)\right)_\varepsilon$of measurable functions on Ω.

Let $\mathcal{E}R_M$ be the space of nets $(X_\varepsilon)_\varepsilon\in\mathcal{E}R,\varepsilon\in(0,1]$,with the property that for almost all $\omega\in\Omega$ there exist constants $r,C>0$ and$\varepsilon_0\in(0,1]$ such that $|(X_\varepsilon)_\varepsilon|\leq C\varepsilon^{-r},\varepsilon\leq\varepsilon_0$.

2.Let NR is the space of nets$(X_\varepsilon)_\varepsilon\in\mathcal{E}R,\varepsilon\in(0,1]$,with the property that foralmost all $\omega\in\Omega$and all $b\in\mathbb{R}_+$there exist constants$C>0$and $\varepsilon_0\in(0,1]$ such that $|(X_\varepsilon)_\varepsilon|\leq C\varepsilon^b,\varepsilon\leq\varepsilon_0$.Thedifferential algebra $GR$of Colombeau generalized random variables is thefactor algebra $GR=\mathcal{E}R/NR$.

Let us consider now a family$\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,\epsilon,\varepsilon'}\right)_{\varepsilon'}$ of the solutions Colombeau-Ito's SDE:

$$\left(d\boldsymbol{x}^{x_0,\varepsilon}_{t,,\epsilon,\varepsilon'}(\omega)\right)_{\varepsilon'}=\left(\boldsymbol{b}_{\varepsilon'}\left(x^{x_0,\varepsilon}_{t,,\epsilon,\varepsilon'}(\omega),t,\epsilon\right)\right)_{\varepsilon'}+\sqrt{\varepsilon}\left(d\boldsymbol{W}(t,\omega)\right)_{\varepsilon'}, \quad (35)$$

$$\left(\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{0,\epsilon,,\varepsilon'}\right)_{\varepsilon'}=\left(\boldsymbol{x}^{x_0}_{\boldsymbol{\varepsilon}'}(\omega)\right)_{\boldsymbol{\varepsilon}'},\in GR,\left(\mathbf{E}\left[\boldsymbol{x}^{x_0,\varepsilon}_{0,\epsilon,\varepsilon'}\right]\right)_{\varepsilon'}=\boldsymbol{x}_0\in\widehat{R}^{\,n}, \quad (36)$$

$$t\in[0,T],\varepsilon,\varepsilon'\in(0,1].$$

Here (i)$\boldsymbol{W}(t,\omega)=\left(W_1(t,\omega),\dots,W_n(t,\omega)\right)$ is n-dimensional Brownian motion,(ii)$\forall t\in[0,T]:\left(\boldsymbol{b}_{\varepsilon'}(x,t,\epsilon)\right)_{\varepsilon'}\in G^n(\mathbb{R}^n)$, $\left(\text{or } G^n_{P,r}(E)\right)$, $\boldsymbol{b}_0(\boldsymbol{x},t,\epsilon)\equiv\boldsymbol{b}_{\varepsilon'=0}(\boldsymbol{x},t,,\epsilon):\mathbb{R}^n\to\mathbb{R}^n$is a polynomial on variable $\boldsymbol{x} = (x_1, \dots, x_n)$,i.e.

$$b_{0,i}(\boldsymbol{x}, t, \epsilon) = \sum_{\alpha,|\alpha|\leq r} b_{0,i}^{\alpha}(t, \epsilon)x^{\alpha}, \quad (37)$$

$\alpha = (i_1, \dots, i_n), |\alpha| = \sum_{j=1}^{n} i_j, 0 \leq i_j \leq p$, or

(iii)) $\forall t \in [0, T]: \left(\boldsymbol{b}_{\varepsilon'}(x, t, \varepsilon)\right)_{\varepsilon'} \in G_{P,r}^{n}(E), \boldsymbol{b}_0(\boldsymbol{x}, t, \epsilon) \equiv \boldsymbol{b}_{\varepsilon'=0}(\boldsymbol{x}, t, \epsilon): \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbb{R}$-analytic function on variable $\boldsymbol{x} = (x_1, \dots, x_n)$,i.e.

$$b_{0,i}(\boldsymbol{x}, t, \epsilon) = \sum_{r=1}^{\infty} \sum_{\alpha,|\alpha|\leq r} b_{0,i}^{\alpha}(t, \epsilon)x^{\alpha}, \quad (38)$$

$\alpha = (i_1, \dots, i_n), |\alpha| = \sum_{j=1}^{n} i_j, 0 \leq i_j \leq p$ and

(iv) $\lim_{\|\boldsymbol{x}\|\to\infty} \|\boldsymbol{b}_0(\boldsymbol{x}, t, \epsilon)\| / \|\boldsymbol{x}\| = \infty$,

(v) $b_{i,\varepsilon'}(\boldsymbol{x}(t), t, \epsilon) = b_{i,0}(\boldsymbol{x}_{\boldsymbol{\varepsilon}'}(t), t, \epsilon)$.(39)

Here $\boldsymbol{x}_{\boldsymbol{\varepsilon}'}(t) = \left(x_{1,\varepsilon'}(t), \dots, x_{n,\varepsilon'}(t)\right)$ and

$$x_{i,\varepsilon'}(t) = \begin{cases} x_{i,\varepsilon'}(t) = \frac{x_i(t)}{1+(\boldsymbol{\varepsilon}')^{2l}\boldsymbol{x}_i^{2l}(t)}, l \geq 1 \\ \text{or} \\ x_{i,\varepsilon'}(t) = x_i(t)\theta_{\epsilon_i}[x_i(t)]. \end{cases} \quad (40)$$

$$i = 1, \dots, n.$$

Here $\theta_{\epsilon_i}[z] \in C^{\infty}(\mathbb{R}), supp\left(\theta_{\epsilon_i}[z]\right) \subseteq [-\nu(\epsilon_i), \nu(\epsilon_i)]$

$$\begin{cases} \theta_{\epsilon_i}[z] = 1 \leftrightarrow z \in [-\nu_1(\epsilon_i), \nu_1(\epsilon_i)] \subsetneq [-\nu(\epsilon_i), \nu(\epsilon_i)], \\ \theta_{\epsilon_i}[z] = 0 \leftrightarrow z \in \mathbb{R} \backslash [-\nu(\epsilon_i), \nu(\epsilon_i)], \\ 0 \leq \theta_{\epsilon_i}[z] \leq 1 \leftrightarrow z \in [-\nu(\epsilon_i), \nu(\epsilon_i)] \backslash [-\nu_1(\epsilon_i), \nu_1(\epsilon_i)]. \end{cases}$$

Remark 5. By Theorem 1 for every Colombeau generalized random variable$\left(\boldsymbol{x}_{\boldsymbol{\varepsilon}'}^{x_0}(\omega)\right)_{\boldsymbol{\varepsilon}'}, \in GR$ such that $\left(\mathbf{E}\left[\boldsymbol{x}_{0,\varepsilon'}^{x_0,\varepsilon}\right]\right)_{\varepsilon'} = \boldsymbol{x}_0 \in \widehat{R}^{\,n}$,and independent of the processes $W_1(t,\omega), \ldots, W_n(t,\omega)$ there exist Colombeau generalized stochastic process $\left(\boldsymbol{x}_{t,\epsilon,\varepsilon'}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}(\omega)\right)_{\boldsymbol{\varepsilon}'}, \varepsilon' \in (0,1]$, such that$\left(\boldsymbol{x}_{0,\epsilon,\varepsilon'}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}(\omega)\right)_{\boldsymbol{\varepsilon}'} = \left(\boldsymbol{x}_{\boldsymbol{\varepsilon}'}^{x_0}(\omega)\right)_{\boldsymbol{\varepsilon}'}$,and $\left(\boldsymbol{x}_{t,,\epsilon,\varepsilon'}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}(\omega)\right)_{\boldsymbol{\varepsilon}'}$ is the solution of the Colombeau-Ito's SDE (35)-(36),which is an almost surely continuous Colombeau generalized stochastic process and is unique up to equivalence
$\left(\mathbf{P}\left[\left\|\boldsymbol{x}_{\boldsymbol{t},,\boldsymbol{\epsilon},\boldsymbol{\varepsilon}',1}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}(\omega) - \boldsymbol{x}_{\boldsymbol{t},,\boldsymbol{\epsilon},\boldsymbol{\varepsilon}',2}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}(\omega)\right\| > 0\right]\right)_{\boldsymbol{\varepsilon}'} = 0$, for all $t \in [0,\infty)$.

Remark 6. One can to construct a sequence of Colombeau generalized functions $\left(\boldsymbol{b}_{\varepsilon',n}(x,t,\epsilon)\right)_{\varepsilon'}$ such that for $\|x\| < n$:

$$\boldsymbol{b}_{\varepsilon',n}(x,t,\epsilon) = \boldsymbol{b}_{\boldsymbol{\varepsilon}'}(x,t,\epsilon), \varepsilon' \in (0,1], \epsilon \in (0,1]^n,$$

and therefore for each $\boldsymbol{b}_{\varepsilon',n}(x,t,\epsilon)$, satisfy conditions (18)-(19) everywhere in $\mathbb{R}^n$. By Theorem 1, there exists a sequence of Colombeau generalized stochastic processes $\left(\boldsymbol{x}_{t,,\varepsilon',n}^{x_0,\varepsilon}(\omega)\right)_{\varepsilon'}$corresponding to Colombeau generalized functions$\left(\boldsymbol{b}_{\varepsilon',n}(x,t,\epsilon)\right)_{\varepsilon'}$. Suppose now that for each $\varepsilon' \in (0,1], \epsilon \in (0,1]^n$ the distribution of $\boldsymbol{x}_{0,\varepsilon'}^{x_0}(\omega)$ has compact support in $\mathbb{R}^n$. Then there exit times of the processes $\boldsymbol{x}_{t,,\epsilon,\varepsilon',m}^{x_0,\varepsilon}(\omega), \varepsilon', \varepsilon \in (0,1]$, from the set $\|x\| < n$ are identical for $m \geq n$. Let this common value be$\boldsymbol{\tau}_{\varepsilon',n}(\omega,\epsilon)$. It is also clear that the processes$\left(\boldsymbol{x}_{t,,\epsilon,\varepsilon',n}^{x_0,\varepsilon}(\omega)\right)_{\varepsilon'}$ and $\left(\boldsymbol{x}_{t,,\epsilon,\varepsilon',m}^{x_0,\varepsilon}(\omega)\right)_{\varepsilon'}$themselves coincide up to time$\left(\boldsymbol{\tau}_{\varepsilon',n}(\omega,\epsilon)\right)_{\varepsilon'}$ i.e.,

$$\left(\mathbf{P}\left[\sup_{0\le t\le \boldsymbol{\tau}_{\varepsilon',n}(\omega,\epsilon)}\left\|\boldsymbol{x}^{x_0,\varepsilon}_{t,,,\epsilon,\varepsilon',m}(\omega)-\boldsymbol{x}^{x_0,\varepsilon}_{t,,,\epsilon,\varepsilon',n}(\omega)\right\|>0\right]\right)_{\varepsilon'}=0, \quad (41)$$

for all $m \geq n$.

Definition3. (i) Let $\boldsymbol{\tau}_{\varepsilon'}(\omega,\varepsilon,\epsilon),\varepsilon',\epsilon\in(0,1]^n$denote the (finite or infinite) limit of the monotone increasing sequence $\boldsymbol{\tau}_{\varepsilon',n}(\omega,\varepsilon,\epsilon)$as $n\to\infty$. We call the generalized random variable $\left(\boldsymbol{\tau}_{\varepsilon'}(\omega,\varepsilon,\epsilon)\right)_{\varepsilon'},\varepsilon'\in(0,1]$the first exit time of the sample function from every bounded domain, or briefly thegeneralized*explosion* time.

(ii) We now define Colombeau generalized stochastic process $\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,,\in,\varepsilon'}(\omega)\right)_{\varepsilon'}$ by setting

$$\boldsymbol{x}^{x_0,\varepsilon}_{t,,\epsilon,\varepsilon'}(\omega)=\boldsymbol{x}^{x_0,\varepsilon}_{t,,\epsilon,\varepsilon',n}(\omega)\text{for } t=t(\omega)<\boldsymbol{\tau}_{\varepsilon',n}(\omega,\varepsilon,\epsilon). \quad (42)$$

(iii)That this is always a Markov process for $t=t(\omega)<\left(\boldsymbol{\tau}_{\varepsilon',n}(\omega,\varepsilon,\epsilon)\right)_{\varepsilon'}$.

(iv) Colombeau generalized stochastic process $\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,,\epsilon,\varepsilon'}(\omega)\right)_{\varepsilon'}$defined by setting (42) on the random generalized interval $\left[0,\left(\boldsymbol{\tau}_{\varepsilon',n}(\omega,\varepsilon,\epsilon)\right)_{\varepsilon'}\right]$is *regular* ,if for any $s<\infty,\boldsymbol{x}\in\mathbb{R}^n,\boldsymbol{\epsilon}\in(0,1]^n$:

$$(\mathbf{P}^{s,x}\{\boldsymbol{\tau}_{\varepsilon'}(\omega,\varepsilon,\epsilon)=\infty\})_{\varepsilon'}=1,\varepsilon'\in(0,1](43)$$

(vi) Colombeau generalized stochastic process $\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,\epsilon,,\varepsilon'}(\omega)\right)_{\varepsilon'}$, defined by setting (42) is a *strongly regular* if for any $s<\infty,\boldsymbol{x}\in\mathbb{R}^n$,$\varepsilon'\in[0,1],\boldsymbol{\epsilon}\in(0,1]^n$:

$$(\mathbf{P}^{s,,x}\{\boldsymbol{\tau}_{\varepsilon'}(\omega,\varepsilon,\epsilon)=\infty\})_{\varepsilon'}=1. \quad (44)$$

Remark7. Wenote that: (iii) does not imply (iv).

Proposition1. Assume that Colombeau generalized stochastic process$\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon'}(\omega)\right)_{\varepsilon'}$

defined by setting (42)is a strongly regular.

Then (1)$\forall\boldsymbol{\epsilon},\boldsymbol{\epsilon}\in(0,1]^n,\forall\delta,\delta>0$:

$$\underline{\lim}_{\varepsilon'\to 0}\mathbf{E}\left[\left\|\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{t,,\epsilon,\varepsilon'}(\omega)-\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{t,\epsilon,\varepsilon'=0}(\omega)\right\|^{\mathbf{2}}\right]=0.\quad (45.a)$$

$$\underline{\lim}_{\varepsilon'\to 0}\mathbf{P}\left\{\left\|\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{t,,\epsilon,\varepsilon'}(\omega)-\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{t,\epsilon,\varepsilon'=0}(\omega)\right\|>\delta\right\}=0.\quad (45.b)$$

(2)$\forall\delta,\delta>0$:

$$\underline{\lim}_{\varepsilon'\to 0,\epsilon\to 0}\mathbf{E}\left[\left\|\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{t,,\epsilon,\varepsilon'}(\omega)-\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{t,\epsilon=0,\varepsilon'=0}(\omega)\right\|^{2}\right]=0.\quad (45.c)$$

$$\underline{\lim}_{\varepsilon'\to 0,\epsilon\to 0}\mathbf{P}\left\{\left\|\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{t,,\epsilon,\varepsilon'}(\omega)-\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{t,\epsilon=0,\varepsilon'=0}(\omega)\right\|>\delta\right\}=0.\quad (45.d)$$

Proof. Immediately follows fromTheoremA1.(I) (see appendix A)anddefinitions1,3.

Let us consider now a family $\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,\epsilon,\varepsilon'}(\omega)\right)_{\varepsilon'}$ of the solutions of the Colombeau SDE:

$$\left(d\boldsymbol{x}^{x_0,\varepsilon}_{t,\epsilon,,\varepsilon'}(\omega)\right)_{\varepsilon',\epsilon}=\left(\boldsymbol{b}_{\boldsymbol{\varepsilon}',\boldsymbol{\epsilon}}\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,,\epsilon,\varepsilon'}(\omega),t,\omega\right)\right)_{\varepsilon'}+\sqrt{\varepsilon}d\boldsymbol{W}(t,\omega),\ (46)$$

$$\left(\boldsymbol{x}^{x_0,\varepsilon}_{0,\varepsilon'}\right)_{\varepsilon'}=\boldsymbol{x}_0\in\widehat{R}^{\,n},t\in[0,T],\varepsilon,\varepsilon',\in(0,1],\epsilon\in(0,1].^{n}$$

Here $\boldsymbol{W}(t)$ is n-dimensional Brownian motion,

and$\forall\epsilon\in(0,1]^n,\forall t\in[0,T]$ andfor almost al $\omega\in\Omega$:$\left(\boldsymbol{b}_{\varepsilon',\epsilon}(x,t,\omega)\right)_{\varepsilon'}\in G^n(\mathbb{R}^n)$, $\boldsymbol{b}_{0,0}(\cdot,t)\equiv$ $\boldsymbol{b}_{\varepsilon'=0,\epsilon=0}(\cdot,t,\omega):\mathbb{R}^n\to\mathbb{R}^n$is a polynomialvector-function on a variable $\boldsymbol{x}=(x_1,\dots,x_n)$ i.e., $b_{i,0,0}(\boldsymbol{x},t)=\sum_{\alpha,|\alpha|\leq r}b^{\alpha}_{i,0,0}(t)x^{\alpha},\alpha=(i_1,\dots,i_n),|\alpha|=\sum_{j=1}^{n}i_j\,,0\leq i_j\leq p,$ and

$$b_{i,\varepsilon',\epsilon}(\boldsymbol{x}(t),t,\omega)=b_{i,0,0}\left(\boldsymbol{x}_{\varepsilon',\epsilon}(t,\omega),t\right).\qquad (47)$$

Here $\boldsymbol{x}_{\varepsilon',\epsilon}(t,\omega)=\left(x_{1,\varepsilon',\varepsilon,\epsilon}(t,\omega),\dots,x_{n,\varepsilon',\varepsilon,\epsilon}(t,\omega)\right)$,

$$x_{i,\varepsilon',\epsilon}(t,\omega)=\frac{x_i(t)}{1+\varepsilon' x_i^{2l}(t)+\varepsilon'\left[\epsilon_i\int_0^t\theta_{\epsilon_i}[x_i(\tau)]x_i^{2l}(\tau)d\tau+\sqrt{\delta}W_i(t)\right]^2},\quad(48)$$

$i=1,..,n.$ Now we let

$$u_i(t)=\epsilon_i\int_0^t\theta_{\epsilon_i}[x_i(\tau)]x_i^{2l}(\tau)d\tau+\sqrt{\delta}W_i(t)(49)$$

and rewrite Eq.(46) of the canonical Colombeau-Ito form:

$$\left(d\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega)\right)_{\varepsilon',\epsilon}=\left(\boldsymbol{b}_{\varepsilon',\epsilon}\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega),\boldsymbol{u}^{\delta}_{t,,\varepsilon',,\epsilon}(\omega),t\right)\right)_{\varepsilon'}+\sqrt{\varepsilon}d\boldsymbol{W}(t,\omega),\boldsymbol{u}^{\delta}_{t,,\varepsilon',\varepsilon,\epsilon}(\omega)=$$

$$\left(u^{\delta}_{1,t,,\varepsilon',\epsilon}(\omega),\dots,u^{\delta}_{n,t,,\varepsilon',\epsilon}(\omega)\right),\quad(50)$$

$$\left(du^{\delta}_{i,t,,\varepsilon',\epsilon}(\omega)\right)_{\varepsilon'}=$$

$$\epsilon_i\left(\theta_{\epsilon_i}\left[x^{x_0,\delta}_{i,t,\varepsilon',\epsilon}(\omega)\right]\left[x^{x_0,\delta}_{i,t,,\varepsilon',\epsilon}(\omega)\right]^{2l}\right)_{\varepsilon'}+$$

$$\sqrt{\delta}dW_i(t),\quad(51)$$

$$i=1,\dots,n,\left(x^{x_0,\varepsilon}_{0,\varepsilon'}\right)_{\varepsilon'}=\boldsymbol{x}_0\in\widehat{R}^{\,n},t\in[0,T],\varepsilon,\varepsilon',\epsilon,\delta\in(0,1].$$

Theorem3. Let us consider a pair of the Colombeau-Ito's SDE:

$$\left(d\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon}',\mu}(\omega)\right)_{\boldsymbol{\varepsilon}'}=\left(\boldsymbol{g}^{\mu}_{\boldsymbol{\varepsilon}'}\left(\boldsymbol{x}^{\boldsymbol{x_0},\varepsilon}_{t,,\varepsilon'}(\omega),t\right)\right)_{\boldsymbol{\varepsilon}'}+\sqrt{\varepsilon}\left(d\boldsymbol{W}(t,\omega)\right)_{\boldsymbol{\varepsilon}'},\quad(52)$$

$$\left(\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{0,\boldsymbol{\varepsilon}',\mu}\right)_{\boldsymbol{\varepsilon}'}=\boldsymbol{x}_0\in\widehat{R}^{\,m},t\in[0,T],\varepsilon,\varepsilon'\in(0,1],\mu=1,2.\quad(53)$$

Assume now that:(1) Conditions (33) and (34)is satisfied.

(2)For a given $N>0,\forall\boldsymbol{x}\in\mathbb{R}^n$such that $\|\boldsymbol{x}\|\leq N$: $\boldsymbol{g}^1_{\boldsymbol{\varepsilon}'}(\boldsymbol{x},t)=\boldsymbol{g}^2_{\boldsymbol{\varepsilon}'}(\boldsymbol{x},t)$.

Let$\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon}',\mu}(\omega),\mu=1,2$ be a pair of the solutions of the Colombeau- Ito's SDE (52)-(53) and let$\mathcal{F}^{N,t}_{\varepsilon',\mu}(\omega),\mu=1,2$be a set$\mathcal{F}^{N}_{\varepsilon',\mu}(\omega)=\left\{t|\sup_{0\leq s\leq t}\left\|\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{s},,\boldsymbol{\varepsilon}',\mu}(\omega)\right\|\leq N\right\}$.We let now$\boldsymbol{\tau}^{N}_{\varepsilon',\mu}(\omega)=\sup\left\{t|t\in\mathcal{F}^{N}_{\varepsilon,\varepsilon',\mu}(\omega)\right\}$.

Then$\forall\varepsilon'\in(0,1]$:

(i) $\mathbf{P}\{\boldsymbol{\tau}^{N}_{\varepsilon,\varepsilon',1}(\omega) = \boldsymbol{\tau}^{N}_{\varepsilon,\varepsilon',2}(\omega)\} = 1$ and

(ii) $\mathbf{P}\left\{\sup_{0\leq s\leq\tau_1}\left\|\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon}',1}(\omega) - \boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon}',2}(\omega)\right\| = 0\right\} = 1.$

Proof. A proof of this statement, complete similarly, to a classical case. For example see[15],chapt.2, subsect.6,theorem2.

Let us rewrite now Eq.(50)-Eq.(51) in the next form (with $\theta_{\epsilon_i} \equiv 1$)

$$\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\delta)\right)_{\varepsilon',\epsilon} =$$

$$\boldsymbol{x}_0 + \left(\int_0^t \boldsymbol{b}_{\varepsilon',\epsilon}\left(\boldsymbol{x}^{x_0,\varepsilon}_{\tau,,\varepsilon',\epsilon}(\omega,\delta), \boldsymbol{u}^{\delta}_{\tau,,\varepsilon',\epsilon}(\omega),\tau\right)d\,\tau\right)_{\varepsilon'} +$$

$$\sqrt{\varepsilon}\boldsymbol{W}(t,\omega), \tag{54}$$

$$\boldsymbol{u}^{\delta}_{t,,\varepsilon',\epsilon}(\omega) = \left(u^{\delta}_{1,t,,\varepsilon',\epsilon}(\omega),\dots,u^{\delta}_{n,t,,\varepsilon',\epsilon}(\omega)\right),$$

$$\left(u^{\delta}_{i,t,,\varepsilon',\epsilon}(\omega)\right)_{\varepsilon'} = \epsilon_i\left(\int_0^t\left[x^{x_0,\varepsilon}_{i,\tau,,\varepsilon',,\epsilon}(\omega,\delta)\right]^{2l}d\tau\right)_{\varepsilon'} + \sqrt{\delta}W_i(t), \tag{55}$$

$i = 1,\dots,n,$

Let $G_N(\boldsymbol{y}), \boldsymbol{y} \in \mathbb{R}^n$bea function: (i) $G_N(\boldsymbol{y}) = \boldsymbol{y}if\|\boldsymbol{y}\| \leq N(ii)G_N(\boldsymbol{y}) = 0if\|\boldsymbol{y}\| > N$.

We set now $\boldsymbol{b}^{N}_{\boldsymbol{\varepsilon}',\epsilon}(\boldsymbol{x},\boldsymbol{u},t) = \boldsymbol{b}_{\boldsymbol{\varepsilon}',\epsilon}(G_N(\boldsymbol{x}),G_N(\boldsymbol{u}),t)$**.**

Let$\left(\boldsymbol{y}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\delta,N)\right)_{\varepsilon'} = \left\{\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\delta,N)\right)_{\varepsilon'}, \boldsymbol{u}^{\delta}_{t,,\varepsilon',,\epsilon}(\omega,N)\right\}$

be a family of the solution of the Colombeau-Ito's SDE:

$$\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\delta,N)\right)_{\varepsilon'} = \boldsymbol{x}_0 + \left(\int_0^t \boldsymbol{b}^{N}_{\varepsilon',\epsilon}\left(\boldsymbol{x}^{x_0,\varepsilon}_{\tau,,\varepsilon',\epsilon}(\omega,\delta,N), \boldsymbol{u}^{\delta}_{\tau,,\varepsilon',\epsilon}(\omega),\tau,N\right)d\,\tau\right)_{\varepsilon'} + \tag{56}$$

$$+\sqrt{\varepsilon}\boldsymbol{W}(t,\omega),$$

$$\boldsymbol{u}^{\delta}_{t,,\varepsilon',\epsilon}(\omega,N) = \left(u^{\delta}_{1,t,,\varepsilon',\epsilon}(\omega,N),\dots,u^{\delta}_{n,t,,\varepsilon',\epsilon}(\omega,N)\right),$$

$$\left(u^{\delta}_{i,t,,\varepsilon',\epsilon}(\omega,N)\right)_{\varepsilon'} =$$

$$\epsilon_i\left(\int_0^t\left[G_N\left(x^{x_0,\varepsilon}_{i,\tau,,\varepsilon',,\epsilon}(\omega,\delta,N)\right)\right]^{2l}d\tau\right)_{\varepsilon'} +$$

$$+\sqrt{\delta}W_i(t), \tag{57}$$

$i = 1, \dots, n,$

Definition4.(1) Let $\left\{\left(\boldsymbol{y}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\delta,N)\right)_{\varepsilon',\epsilon}\right\}_{N=1}^{\infty}$ be a sequence of the solution of the Colombeau-Ito's SDE(54)- (55). Let$\mathcal{F}^{N}_{\varepsilon,\varepsilon',\epsilon}(\omega,\delta)$ be a set

$$\mathcal{F}^{N}_{\varepsilon,\varepsilon',\epsilon}(\omega,\delta) = \left\{t|\sup_{0\leq s\leq t}\left\|\boldsymbol{y}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon}',\boldsymbol{\epsilon}}(\omega,\delta,N)\right\| \leq N\right\}. \quad (58)$$

(2) We let now

$$\boldsymbol{\tau}^{N}_{\varepsilon,\varepsilon',\epsilon}(\omega,\delta) = \sup\{t|t \in \mathcal{F}^{N}_{\varepsilon,\varepsilon',\epsilon}(\omega)\}, \quad (59)$$

$$\boldsymbol{\tau}^{\infty}_{\varepsilon,\varepsilon',\epsilon}(\omega,\delta) = \lim_{N\to\infty}\boldsymbol{\tau}^{N}_{\varepsilon,\varepsilon',\epsilon}(\omega,\delta).(60)$$

(3) Let$\widetilde{\boldsymbol{y}}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon}',\epsilon}(\omega,\delta)$be a net of the stochastic processes defined by setting

$$\widetilde{\boldsymbol{y}}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon}',\boldsymbol{\epsilon}}(\omega,\delta) = \boldsymbol{y}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t},,\varepsilon',\epsilon}(\omega,\delta,N)\text{iff } t < \boldsymbol{\tau}^{N}_{\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',\epsilon}(\omega,\delta)(61)$$

(4) Let$\left[\left(\boldsymbol{y}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon}',\boldsymbol{\epsilon}}(\omega,\delta)\right)_{\boldsymbol{\varepsilon}'}\right]$be Colombeaugeneralized stochastic process defined by setting

$$\left[\left(\boldsymbol{y}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon}',\boldsymbol{\epsilon}}(\omega,\delta)\right)_{\boldsymbol{\varepsilon}'}\right] = \left[\left(\widetilde{\boldsymbol{y}}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{t,,\varepsilon',\epsilon}(\omega,\delta)\right)_{\boldsymbol{\varepsilon}'}\right]. \quad (62)$$

Remark5. We note that according to the Theorem3 $\forall M(M \geq N)$one obtain

$$\mathbf{P}\left\{\sup_{0\leq t\leq \boldsymbol{\tau}^{N}_{\varepsilon,,\varepsilon',\epsilon}(\omega)}\left\|\boldsymbol{y}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon}',\boldsymbol{\epsilon}}(\omega,\delta,N) - \boldsymbol{y}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon}',\boldsymbol{\epsilon}}(\omega,\delta,M)\right\| > 0\right\} = 0,$$

Therefore definitions (61)-(62) is correct.

Definition5.Let$\left(\boldsymbol{y}^{x_0,\varepsilon}_{t,\varepsilon',\epsilon}\right)_{\boldsymbol{\varepsilon}'}$be a family of the solutions Colombeau-Ito's SDE (56)-(57).

(1) A family$\left(\boldsymbol{y}^{x_0,\varepsilon}_{t,\varepsilon',\epsilon}\right)_{\boldsymbol{\varepsilon}'}$, $\varepsilon,\varepsilon' \in (0,1]$,$\epsilon \in (0,1]^{n}$is regular if

$$\left(\lim_{c\to\infty}\mathbf{P}\left\{\left\|\boldsymbol{y}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega,\delta)\right\| > c\right\}\right)_{\varepsilon'} = 0.(63)$$

Or in the next equivalent form

$$\left(\mathbf{P}\{\boldsymbol{\tau}_{\varepsilon,\varepsilon',\epsilon}^{\infty}(\omega,\delta) = \infty\}\right)_{\varepsilon'} = 1. \qquad (64)$$

(2) A family$\left(\boldsymbol{y}_{t,\varepsilon',\epsilon}^{x_0,\varepsilon}\right)_{\varepsilon'}$is a *strongly* regular if $\forall\varepsilon',\varepsilon' \in [0,1]$, $\forall\epsilon,\epsilon \in (0,1]^{\,n}$:

$$\left(\lim_{c\to\infty}\mathbf{P}\left\{\left\|\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega,\delta)\right\| > c\right\}\right)_{\varepsilon'} = 0.(65)$$

or in the next equivalent form: $\forall\varepsilon',\varepsilon' \in [0,1], \forall\epsilon,\epsilon \in (0,1]^{\,n}$:

$$\left(\mathbf{P}\{\boldsymbol{\tau}_{\varepsilon,\varepsilon',\epsilon}^{\infty}(\omega,\delta) = \infty\}\right)_{\varepsilon'} = 1.(66)$$

Definition6. Let$\left(\boldsymbol{y}_{t,\varepsilon',\epsilon}^{x_0,\varepsilon}\right)_{\varepsilon'}$bea family of the solutions Colombeau-Ito's SDE (56)-(57). A family $\left(\boldsymbol{y}_{t,\varepsilon',\epsilon}^{x_0,\varepsilon}\right)_{\varepsilon'}$, $\varepsilon,\varepsilon' \in (0,1], \epsilon \in (0,1]^{\,n}$isanon-regular if

$$\exists t'\forall t \geq t': \left(\lim_{c\to\infty}\mathbf{P}\left\{\left\|\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega,\delta)\right\| > c\right\}\right)_{\varepsilon'} \neq 0. \qquad (67)$$

Or in the next equivalent form

$$\left(\mathbf{P}\{\boldsymbol{\tau}_{\varepsilon,\varepsilon',\epsilon}^{\infty}(\omega,\delta) < \infty\}\right)_{\varepsilon'} = 1. \qquad (68)$$

Proposition2. Assume that Colombeau generalized stochastic process $\left(\boldsymbol{y}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega,\delta)\right)_{\varepsilon'}$

defined by setting (62) is a strongly regular. Then

(1) $\underline{\lim}_{\substack{\varepsilon'\to 0\\ \epsilon\to 0,}}\underline{\lim}_{\delta\to 0}\mathbf{E}\left[\left\|\boldsymbol{y}^{\boldsymbol{x_0,\varepsilon}}_{\boldsymbol{t},,\varepsilon',\epsilon}(\omega,\delta)-\boldsymbol{y}^{\boldsymbol{x_0,\varepsilon}}_{\boldsymbol{t}}(\omega)\right\|^{\mathbf{2}}\right]=0$ (69.a)

(2) $\forall\sigma>0$:

$$\underline{\lim}_{\substack{\varepsilon'\to 0\\ \epsilon\to 0,}}\underline{\lim}_{\delta\to 0}\mathbf{P}\left\{\left\|\boldsymbol{y}^{\boldsymbol{x_0,\varepsilon}}_{t,,\varepsilon',\epsilon}(\omega,\delta)-\boldsymbol{y}^{\boldsymbol{x_0,\varepsilon}}_{t}(\omega)\right\|>\sigma\right\}=0. \qquad (69.b)$$

Here:$\boldsymbol{y}^{\boldsymbol{x_0,\varepsilon}}_{t}(\omega)=\boldsymbol{y}^{\boldsymbol{x_0,\varepsilon}}_{t,,\varepsilon'=0,\epsilon=0}(\omega,\delta=0)$**.**

Proof. Immediately follows fromTheorem3 andTheoremA.1 (see appendix A).

Proposition3. Let$\left(\boldsymbol{y}^{\boldsymbol{x_0,\varepsilon}}_{\boldsymbol{t},,\varepsilon',\epsilon}(\omega,\delta)\right)_{\boldsymbol{\varepsilon'}}=\left\{\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,\varepsilon',\epsilon}\right)_{\varepsilon'},\left(\boldsymbol{u}^{\delta}_{t,,\varepsilon',\epsilon}(\omega)\right)_{\varepsilon'}\right\}$be a family of the solutions Colombeau-Ito's SDE (56)-(57)with$\theta_\epsilon[z]\equiv 1$.A family $\left(\boldsymbol{y}^{x_0,\varepsilon}_{t,\varepsilon',\epsilon}\right)_{\boldsymbol{\varepsilon'}}$, $\varepsilon,\varepsilon'\in(0,1]$,$\boldsymbol{\epsilon}\in(0,1]^{n}$is regular.

Proof. Assume that: process $\left(\boldsymbol{y}^{\boldsymbol{x_0,\varepsilon}}_{\boldsymbol{t},,\varepsilon',\epsilon}(\omega,\delta)\right)_{\boldsymbol{\varepsilon'}}$ is a non-regular. Therefore$(\mathbf{P}^{s,,x}\{\boldsymbol{\tau}_{\boldsymbol{\varepsilon'}}(\omega,\varepsilon)<\infty\})_{\boldsymbol{\varepsilon'}}=1$ and consequently

$$\left(\mathbf{P}^{s,,x}\left\{\boldsymbol{y}^{\boldsymbol{x_0,\varepsilon}}_{\boldsymbol{\tau}_{\varepsilon'}(\omega,\varepsilon),,\varepsilon',\epsilon}(\omega,\delta)=\infty\right\}\right)_{\boldsymbol{\varepsilon'}}>0. \qquad (70)$$

But the other hand from Eq.(56)-Eq. (57) we obtain

$$\begin{aligned}&\left(\boldsymbol{x}^{x_0,\varepsilon}_{\boldsymbol{\tau}_{\varepsilon'}(\omega,\varepsilon),,\varepsilon',\epsilon}(\omega,\delta)\right)_{\varepsilon'}=\\&\boldsymbol{x}_0+\left(\int_0^{\boldsymbol{\tau}_{\varepsilon'}(\omega,\varepsilon)}\boldsymbol{b}_{\varepsilon',\epsilon}\left(\boldsymbol{x}^{x_0,\varepsilon}_{\boldsymbol{v},,\varepsilon',\epsilon}(\omega,\delta),\boldsymbol{u}^{\delta}_{v,,\varepsilon',\epsilon}(\omega),v,\varepsilon\right)dv\right)_{\varepsilon'}+\\&+\left(\sqrt{\varepsilon}\boldsymbol{W}(\boldsymbol{\tau}_{\boldsymbol{\varepsilon'}}(\omega,\varepsilon),\omega)\right)_{\varepsilon'},\end{aligned} \qquad (71)$$

$$\boldsymbol{u}^{\delta}_{v,,\varepsilon',\epsilon}(\omega)=\left(u^{\delta}_{1,v,,\varepsilon',\epsilon}(\omega),\dots,u^{\delta}_{n,v,,\varepsilon',\epsilon}(\omega)\right),$$

$$\left(u^{\delta}_{i,\boldsymbol{\tau}_{\varepsilon'}(\omega,\varepsilon),,\varepsilon',\epsilon}(\omega)\right)_{\varepsilon'}=\epsilon\left(\int_0^{\boldsymbol{\tau}_{\varepsilon'}(\omega,\varepsilon)}\left[x^{x_0,\varepsilon}_{i,v,,\varepsilon',,\epsilon}(\omega,\delta)\right]^{2l}dv\right)_{\varepsilon'}+\left(\sqrt{\delta}W_i\left(\boldsymbol{\tau}_{\boldsymbol{\varepsilon'}}(\omega,\varepsilon)\right)\right)_{\varepsilon'},\ i=1,\dots,n, \qquad (72)$$

From (70) and Eq. (71)-Eq. (72) we obtain

$$\left(\mathbf{P}^{0,,x_0}\left\{\int_0^{\tau_{\varepsilon'}(\omega,\varepsilon)} \boldsymbol{b}_{\varepsilon',\epsilon}\left(\boldsymbol{x}^{x_0,\varepsilon}_{v,,\varepsilon',\epsilon}(\omega,\delta),\boldsymbol{u}^{\delta}_{v,,\varepsilon',\epsilon}(\omega),v,\varepsilon\right)d\,v=\infty\right\}\right)_{\boldsymbol{\varepsilon'}}=0,$$

and therefore

$$\left(\mathbf{P}^{0,,x_0}\left\{\boldsymbol{x}^{x_0,\varepsilon}_{\boldsymbol{\tau}_{\boldsymbol{\varepsilon'}}(\omega,\varepsilon),,\varepsilon',\epsilon}(\omega,\delta)=\infty\right\}\right)_{\varepsilon'}=$$

$$\left(\mathbf{P}^{0,,x_0}\left\{\boldsymbol{x}_0+\int_0^{\tau_{\varepsilon'}(\omega,\varepsilon)} \boldsymbol{b}_{\varepsilon',\epsilon}\left(\boldsymbol{x}^{x_0,\varepsilon}_{v,,\varepsilon',\epsilon}(\omega,\delta),\boldsymbol{u}^{\delta}_{v,,\varepsilon',\epsilon}(\omega),v,\varepsilon\right)d\,v+\sqrt{\varepsilon}\boldsymbol{W}(\boldsymbol{\tau}_{\boldsymbol{\varepsilon'}}(\omega,\varepsilon),\omega)=\infty\right\}\right)_{\boldsymbol{\varepsilon'}}$$
$$=0,$$

$$\left(\mathbf{P}^{0,,0}\left\{u^{\delta}_{i,\boldsymbol{\tau}_{\boldsymbol{\varepsilon'}}(\omega,\varepsilon),,\varepsilon',\epsilon}(\omega)=\infty\right\}\right)_{\varepsilon'}=$$

$$\left(\mathbf{P}^{s,,\boldsymbol{x}}\left\{\epsilon\int_0^{\boldsymbol{\tau}_{\boldsymbol{\varepsilon'}}(\boldsymbol{\omega},\boldsymbol{\varepsilon})}\left[x^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{i},\boldsymbol{v},,\boldsymbol{\varepsilon'},,\epsilon}(\omega,\delta)\right]^{\boldsymbol{2l}}dv+\sqrt{\delta}W_i\left(\boldsymbol{\tau}_{\boldsymbol{\varepsilon'}}(\omega,\varepsilon)\right)=\infty\right\}\right)_{\boldsymbol{\varepsilon'}}=0.$$

Thus

$$\left(\mathbf{P}^{s,,\boldsymbol{x}}\left\{\boldsymbol{y}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{\tau}_{\boldsymbol{\varepsilon'}}(\omega,\varepsilon),,\varepsilon',\epsilon}(\omega,\delta)=\infty\right\}\right)_{\boldsymbol{\varepsilon'}}=0.$$

But this is the contradiction. This contradiction completed the proof.

Definition7.CISDE(35)-(36) is $\widehat{\mathbb{R}}$-dissipative if there exist

Lyapunov candidate function$\left(V_{\varepsilon'}(\boldsymbol{x},t)\right)_{\varepsilon'}:\widehat{\mathbb{R}}^n\times[0,T]\to\widehat{\mathbb{R}}$and positive infinite

Colombeau constants $\widetilde{C}=[(C_{\varepsilon'})_{\varepsilon'}]\in\widehat{\mathbb{R}}_+$,

$\tilde{r}=[(r_{\varepsilon'})_{\varepsilon'}]\in\widehat{\mathbb{R}}_+$,such that:

(1)$\forall\varepsilon'\in(0,1]:V_{*,\varepsilon'}=\lim_{R\to\infty}\left(\inf_{\|\mathbf{x}\|>R}V_{\varepsilon'}(\boldsymbol{x},t)\right)=\infty$, and

(2)$\forall[(x_{\varepsilon'})_{\varepsilon'}]([(\|x_{\varepsilon'}\|)_{\varepsilon'}]\geq\tilde{r})$theinequality

$$\left[\left(\dot{V}_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t;\boldsymbol{b}_{\varepsilon'})\right)_{\varepsilon'}\right] \leq \tilde{C}\left[\left(V_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'}\right] \quad (73)$$

is satisfied.Here

$$\left[\left(\dot{V}_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t;\boldsymbol{b}_{\varepsilon'})\right)_{\varepsilon'}\right] \equiv$$

$$\left[\left(\frac{\partial V_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t)}{\partial t}\right)_{\varepsilon'}\right] + \sum_{i=1}^{n}\left[\left(\frac{\partial V_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t)}{\partial x_{i,\varepsilon'}} b_{i,\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'}\right]. \quad (74)$$

Or in the next equivalent form:

CISDE (35)-(36) is $\widehat{\mathbb{R}}$-dissipative if there existLyapunovcandidate function$\left(V_{\varepsilon'}(\boldsymbol{x},t)\right)_{\varepsilon'}: \widehat{\mathbb{R}}^n \times [0,T] \to \widehat{\mathbb{R}}$and positive infinite Colombeau constants$\widetilde{C} = [(C_{\varepsilon'})_{\varepsilon'}] \in \widehat{\mathbb{R}}_+$,

$\tilde{r} = [(r_{\varepsilon'})_{\varepsilon'}] \in \widehat{\mathbb{R}}_+$, such that:

(1)$\forall\varepsilon' \in (0,1] : V_{*,\varepsilon'} = \lim_{R\to\infty}\left(\inf_{\|\mathrm{x}\|>R} V_{\varepsilon'}(\boldsymbol{x},t)\right) = \infty$, and

(2′)$\forall\varepsilon' \in (0,1]\forall\boldsymbol{x}_{\varepsilon'}[(\boldsymbol{x}_{\varepsilon'} \in \mathbb{R}^n) \wedge (\|\boldsymbol{x}_{\varepsilon'}\| \geq r_{\varepsilon'})]$the inequality

$$\left(\dot{V}_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t;\boldsymbol{b}_{\varepsilon'})\right)_{\varepsilon'} \leq ((C_{\varepsilon'})_{\varepsilon'})\left(V_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'} \quad (75)$$

is satisfied. Here

$$\left(\dot{V}_{\varepsilon'}(x_{\varepsilon'},t;b_{\varepsilon'})\right)_{\varepsilon'} \equiv \left(\frac{\partial V_{\varepsilon'}(x_{\varepsilon'},t)}{\partial t}\right)_{\varepsilon'} + \left(\sum_{i=1}^{n}\frac{\partial V_{\varepsilon'}(x_{\varepsilon'},t)}{\partial x_{\varepsilon'}} b_{i,\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'}. \quad (76)$$

Definition 8. CISDE (35)-(36) isa strongly$\widehat{\mathbb{R}}$-dissipative if Lyapunov candidate function$\left(V_{\varepsilon'}(\boldsymbol{x},t)\right)_{\varepsilon'}: \widehat{\mathbb{R}}^n \times [0,T] \to \widehat{\mathbb{R}}$, $\varepsilon' \in [0,1]$andpositive finite Colombeau constants $\widetilde{C} = [(C_{\varepsilon'})_{\varepsilon'}] \in \widehat{\mathbb{R}}_+, \tilde{r} = [(r_{\varepsilon'})_{\varepsilon'}] \in \widehat{\mathbb{R}}_+$,such that:

(1)$\forall\varepsilon' \in (0,1] : V_{*,\varepsilon'} = \lim_{r\to\infty}\left(\inf_{\|\mathrm{x}\|>r} V_{\varepsilon'}(\boldsymbol{x},t)\right) = \infty$,and(2)$\forall[(\boldsymbol{x}_{\varepsilon'})_{\varepsilon'}]([(\|\boldsymbol{x}_{\varepsilon'}\|)_{\varepsilon'}] \geq \tilde{r})$ the inequality

$$\left[\left(\dot{V}_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t;\boldsymbol{b}_{\varepsilon'})\right)_{\varepsilon'}\right] \leq \tilde{C}\left[\left(V_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'}\right] (77)$$

is satisfied.

Here

$$\left[\left(\dot{V}_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t;\boldsymbol{b}_{\varepsilon'})\right)_{\varepsilon'}\right] \equiv$$

$$\left[\left(\frac{\partial V_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t)}{\partial t}\right)_{\varepsilon'}\right] + \sum_{i=1}^{n}\left[\left(\frac{\partial V_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t)}{\partial x_{i,\varepsilon'}} b_{i,\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'}\right]. (78)$$

Proposition4. Let$\left(\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega)\right)_{\varepsilon'}$) be generalized stochastic process satisfying Colombeau-Ito's SDE(35)-(36) on the time interval$[s,T]$and $\left(\boldsymbol{\tau}_{\varepsilon',U}(\omega,\epsilon)\right)_{\varepsilon'}$- is a generalizedrandomvariableequaltothetimeatwhichthesamplefunctionofthegeneralized process $\left(\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega)\right)_{\varepsilon'}$) first leaves the bounded neighborhood U, and let$\left(\boldsymbol{\tau}_{\varepsilon',U}(\omega,t,\epsilon)\right)_{\varepsilon'} = \left(\min\{\boldsymbol{\tau}_{\varepsilon',U}(\omega,\epsilon),t\}\right)_{\varepsilon'}$.Supposemoreover that$\forall\varepsilon' \in (0,1]$:$\mathbf{P}\left\{\boldsymbol{x}_{s,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega) \in U\right\} = 1$.Then

$$\left(\mathbf{E}\left[V_{\varepsilon'}\left(\boldsymbol{x}_{\boldsymbol{\tau}_{\varepsilon',U}(\omega,t,\epsilon),,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega),\boldsymbol{\tau}_{\varepsilon',U}(\omega,t,\epsilon)\right) - V_{\varepsilon'}\left(\boldsymbol{x}_{s,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega),s\right)\right]\right)_{\varepsilon'} =$$

$$= \left(\mathbf{E}\left[\int_s^{\boldsymbol{\tau}_{\varepsilon',U}(\omega,t,\epsilon)} \dot{V}_{\varepsilon'}\left(\boldsymbol{x}_{u,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega),u\right)du\right]\right)_{\varepsilon'}.$$

Proof. Similarly as the proof of the corresponding classical result, see [17]Lemma 3.2.

Theorem4. (1) Assume that: (i)for CISDE (35)-(36**)** the inequalities(33)and(34) is satisfied and (ii)CISDE (35)-(36) is$\widehat{\mathbb{R}}$-dissipative.

Then (1) Colombeau generalized stochastic process $\left(\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega)\right)_{\varepsilon'}$, $\varepsilon' \in (0,1]$, $\epsilon \in (0,1]^n$defined by setting (42) is regular, and (2) the inequality

$$\left(\mathbf{E}\left[V_{\varepsilon'}\left(\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega),t\right)\right]\right)_{\varepsilon'} \leq \left(\mathbf{E}\left[V_{\varepsilon'}\left(\boldsymbol{x}_{t_0,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega),t_0\right)\right]\right)_{\varepsilon'} \exp[(C_{\varepsilon'})_{\varepsilon'}(t-t_0)](79)$$

is satisfied.

Proof.**(1)** From (76) it follows that the Colombeau generalized function$\left(W_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'} = \left(V_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'}\exp[-(C_{\varepsilon'})_{\varepsilon'}(t-t_0)]$ is satisfies the inequality:$\left(\dot{W}_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'} \leq 0$.Hence, by

Proposition4,for$\left(\boldsymbol{\tau}_{\boldsymbol{\varepsilon}',n}(\omega,t,\epsilon)\right)_{\boldsymbol{\varepsilon}'} = \left(\min\{\boldsymbol{\tau}_{\boldsymbol{\varepsilon}',n}(\omega,\epsilon),t\}\right)_{\boldsymbol{\varepsilon}'}$wehave

$\left(\mathbf{E}\left[V_{\varepsilon'}\left(\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{\tau}_{\varepsilon',,\epsilon,n}(\omega,t,\epsilon),,\varepsilon'}(\omega),\boldsymbol{\tau}_{\boldsymbol{\varepsilon}',n}(\omega,t,\epsilon)\right)\exp[-(C_{\varepsilon'})_{\varepsilon'}(\boldsymbol{\tau}_{\boldsymbol{\varepsilon}',n}(\omega,t,\epsilon)-t_0)]\right]\right)_{\varepsilon'} -$

$\left(\mathbf{E}\left[V_{\varepsilon'}\left(\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t_0},,\boldsymbol{\varepsilon}',\boldsymbol{\epsilon}}(\omega),t_0\right)\right]\right)_{\varepsilon'} = \left(\mathbf{E}\left[\int_{\boldsymbol{t_0}}^{\tau_{\varepsilon',n}(\omega,t,\epsilon)}\dot{W}_{\varepsilon'}\left(\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{u,,\boldsymbol{\varepsilon}',\boldsymbol{\epsilon}}(\omega),u\right)du\right]\right)_{\varepsilon'} \leq$

$0.$ (80)

This, together with the inequalities$\left(\boldsymbol{\tau}_{\boldsymbol{\varepsilon}',n}(\omega,t,\epsilon)\right)_{\boldsymbol{\varepsilon}'} \leq t,$

$\left(V_{\varepsilon'}(\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'} \geq 0$, implies

$\left(\mathbf{E}\left[V_{\varepsilon'}\left(\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{\tau}_{\varepsilon',\epsilon,n}(\omega,t),,\varepsilon'}(\omega),\boldsymbol{\tau}_{\boldsymbol{\varepsilon}',U}(\omega,t,\epsilon)\right)\right]\right)_{\varepsilon'} \leq \left(\mathbf{E}\left[V_{\varepsilon'}\left(\widetilde{\boldsymbol{x}}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t_0},\boldsymbol{\epsilon},\boldsymbol{\varepsilon}'}(\omega),t_0\right)\right]\right)_{\varepsilon'}\exp[(C_{\varepsilon'})_{\varepsilon'}(t-t_0)]$

(81)

From (81) one derive the estimate

$$\left(\mathbf{P}\{\boldsymbol{\tau}_{\boldsymbol{\varepsilon}',n}(\omega,\epsilon)<t\}\right)_{\varepsilon'} \leq$$

$$\leq \frac{\exp[(C_{\varepsilon'})_{\varepsilon'}(t-t_0)]\left(\mathbf{E}\left[V_{\varepsilon'}\left(\widetilde{\boldsymbol{x}}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t_0},,\boldsymbol{\varepsilon}'}(\omega),t_0\right)\right]\right)_{\varepsilon'}}{\left(\inf_{\|\mathbf{x}\|\geq \mathrm{n},\mathrm{u}>t_0}V_{\varepsilon'}(\boldsymbol{x},u)\right)_{\boldsymbol{\varepsilon}'}}$$

Letting $n\to\infty$ and making use of theDefinition7we now get(64).

(2)Assume that CISDE (35)-(36) is a strongly

$\widehat{\mathbb{R}}$-dissipative. Then(1) Colombeau generalized stochastic process $=\left\{\left(\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon}'}(\omega)\right)_{\boldsymbol{\varepsilon}'}\right\}\varepsilon' \in$

$[0,1],\epsilon\in[0,1],^n$defined by setting (42) is a strongly regular and (2) the inequality

$$\left(\mathbf{E}\left[V_{\varepsilon'}\left(\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon}'}(\omega),t\right)\right]\right)_{\varepsilon'} \leq$$

$\leq \left(\mathbf{E}\left[V_{\varepsilon'}\left(\boldsymbol{x}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}_{\boldsymbol{t_0},,\boldsymbol{\varepsilon}'}(\omega),t_0\right)\right]\right)_{\varepsilon'}\exp[(C_{\varepsilon'})_{\varepsilon'}(t-t_0)]$(82)

is satisfied.

Proof.(2) Similarly as the proof of the Theorem4.

Theorem5. Weset now$\theta_{\epsilon_i}[\boldsymbol{z}] \equiv 1, i = 1, \dots, n$. For any solution

$$\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\delta)\right)_{\varepsilon'} = \left(x^{x_0,\varepsilon}_{1,t,,\varepsilon',\epsilon}, \dots, x^{x_0,\varepsilon}_{n,t,,\varepsilon',\epsilon}\right)_{\varepsilon'}$$

ofa strongly$\widehat{\mathbb{R}}$–dissipative CISDE(46)-(48) and any$\mathbb{R}$-valued parameters$\lambda_1, \dots, \lambda_n$, there exist finite Colombeau constant $\widetilde{C}' = \left[\left(C'_{\varepsilon'}\right)_{\varepsilon'}\right] > 0$, such that $\forall \boldsymbol{\lambda}[\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)]$, the inequality

$$\underline{\lim}_{\substack{\varepsilon\to 0\\ \varepsilon'\to 0\\ \epsilon\to 0\\ \left(\frac{\varepsilon'}{\varepsilon}\right)\to 0}} \underline{\lim}_{\delta\to 0}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}^{\boldsymbol{x_0,\varepsilon}}_{\boldsymbol{t,,\varepsilon',\epsilon}}(\omega,\delta) - \boldsymbol{\lambda}\right\|^{\boldsymbol{2}}\right] \leq \tilde{C}'\|\boldsymbol{U}(t,\boldsymbol{\lambda})\|^{\boldsymbol{2}} \quad (83)$$

is satisfied. Or in the next equivalent form: for a sufficiently small$\epsilon \approx 0$and for a sufficiently small $\varepsilon \approx 0, \varepsilon' \approx 0$such that
$\frac{\varepsilon'}{\varepsilon} \approx 0$,the inequality

$$\left[\left(\underline{\lim}_{\delta\to 0}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}^{\boldsymbol{x_0,\varepsilon}}_{\boldsymbol{t,,\varepsilon',\epsilon}}(\omega,\delta) - \boldsymbol{\lambda}\right\|^{2}\right]\right)_{\varepsilon'}\right] \leq \tilde{C}'\|\boldsymbol{U}(t,\boldsymbol{\lambda})\|^2 \quad (84)$$

is satisfied.
Here thevector-function $\boldsymbol{U}(t,\boldsymbol{\lambda}) = (U_1(\mathrm{t},\boldsymbol{\lambda}),\dots,U_n(\mathrm{t},\boldsymbol{\lambda}))$is the solutionof the differential master equation:

$$\dot{\boldsymbol{U}}(t,\boldsymbol{\lambda}) = \mathbf{J}[\boldsymbol{b_0}(\boldsymbol{\lambda},t)]\boldsymbol{U}(t,\boldsymbol{\lambda}) + \boldsymbol{b_0}(\boldsymbol{\lambda},t), \boldsymbol{U}(0,\boldsymbol{\lambda}) = \boldsymbol{x_0} - \boldsymbol{\lambda}, \qquad (85)$$

Here$\mathbf{J} = \mathbf{J}[\boldsymbol{b_0}(\boldsymbol{\lambda},t)]$ is a Jacobian i.e.,**J**is $n \times n$-matrix:

$$\mathbf{J}[\boldsymbol{b_0}(\boldsymbol{\lambda},t)] = \mathbf{J}\left[\partial \boldsymbol{b_{0,i}}(\boldsymbol{x},t)/\partial x_{\boldsymbol{j}}\right]_{\boldsymbol{x}=\boldsymbol{\lambda}}. \qquad (86)$$

Proof. We let now

$$\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon} - \boldsymbol{\lambda} = \boldsymbol{y}^{\boldsymbol{x_0,\varepsilon}}_{\boldsymbol{t,,\varepsilon',\epsilon}}. \qquad (86)$$

Replacement $\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon} = \boldsymbol{y}^{\boldsymbol{x_0,\varepsilon}}_{\boldsymbol{t,,\varepsilon',\epsilon}} + \boldsymbol{\lambda}$ into Eq.(50)-Eq.(51)gives

$$\left(d\boldsymbol{y}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\delta)\right)_{\varepsilon'}=\left(\boldsymbol{b}_{\varepsilon',\epsilon}\left(\boldsymbol{y}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\delta)+\boldsymbol{\lambda},\boldsymbol{u}^{\delta}_{t,,\varepsilon',\varepsilon,\epsilon}(\omega),t,\varepsilon\right)\right)_{\varepsilon'}++\sqrt{\varepsilon}d\boldsymbol{W}(t,\omega),\boldsymbol{u}^{\delta}_{t,,\varepsilon',\varepsilon,\epsilon}(\omega)=$$

$$\left(u^{\delta}_{1,t,,\varepsilon',\varepsilon,\epsilon}(\omega),\dots,u^{\delta}_{n,t,,\varepsilon',\varepsilon,\epsilon}(\omega)\right),\quad(87)$$

$$\left(du^{\delta}_{i,t,,\varepsilon',\varepsilon,\epsilon}(\omega)\right)_{\varepsilon'}=\epsilon\left(\left[x^{x_0,\varepsilon}_{i,t,,\varepsilon',,\varepsilon,\epsilon}(\omega,\delta)\right]^{2l}\right)_{\varepsilon'}+\sqrt{\delta}dW_i(t),$$

$$i=1,\dots,n,\left(x^{x_0,\varepsilon}_{0,\varepsilon',\epsilon}\right)_{\varepsilon'}=\boldsymbol{x}_0\in\widehat{R}^{\,n},t\in[0,T],\varepsilon,\varepsilon',\epsilon,\delta\in(0,1].$$

Thus we need to estimate the quantity

$$\underline{\lim}_{\substack{\varepsilon\to0\\ \varepsilon'\to0\\ \epsilon\to0\\ \left(\frac{\varepsilon'}{\varepsilon}\right)\to0}}\underline{\lim}_{\delta\to0}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{y}^{\boldsymbol{x_0,\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon'},\boldsymbol{\epsilon}}(\omega,\delta)\right\|^{\mathbf{2}}\right].$$

Application of the Theorem B.4 (see Appendix B) to Eq.(87) gives the inequality (83) directly.

Theorem 6.(Strong large deviations principle) [5],[7].

Assume that CISDE (35)-(36) is a strongly$\widehat{\mathbb{R}}$-dissipative. Then:

(1) For any solution

$$\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega)\right)_{\varepsilon'}=\left(x^{x_0,\varepsilon}_{1,t,,\varepsilon',\epsilon},\dots,x^{x_0,\varepsilon}_{n,t,,\varepsilon',\epsilon}\right)_{\varepsilon'}$$

of a strongly$\widehat{\mathbb{R}}$–dissipative CISDE(35)-(40) and any$\mathbb{R}$-valued parameters$\lambda_1,\dots,\lambda_n$, there exist finite Colombeau constant $\widetilde{C}'=\left[\left(C'_{\varepsilon'}\right)_{\varepsilon'}\right]>0$, such that $\forall\boldsymbol{\lambda}[\boldsymbol{\lambda}=(\lambda_1,\dots,\lambda_n)]$the inequality

$$\underline{\lim}_{\substack{\varepsilon\to0\\ \varepsilon'\to0\\ \epsilon\to0\\ \left(\frac{\varepsilon'}{\varepsilon}\right)\to0}}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}^{\boldsymbol{x_0,\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon'},\boldsymbol{\epsilon}}(\omega)-\boldsymbol{\lambda}\right\|^{\mathbf{2}}\right]\le\tilde{C}'\|\boldsymbol{U}(t,\boldsymbol{\lambda})\|^2(8\,8)$$

is satisfied. Or in the next equivalent form: for a sufficiently small $\epsilon\approx0$ and for a sufficiently small $\varepsilon\approx0,\varepsilon'\approx0$ such that

$\varepsilon'/\varepsilon\approx0$,the inequality

$$\left[\left(\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}^{\boldsymbol{x_0,\varepsilon}}_{\boldsymbol{t},,\boldsymbol{\varepsilon'},\boldsymbol{\epsilon}}(\omega)-\boldsymbol{\lambda}\right\|^{\mathbf{2}}\right]\right)_{\varepsilon'}\right]\le\tilde{C}'\|\boldsymbol{U}(t,\boldsymbol{\lambda})\|^2.$$

is satisfied.

(2) For any solution

$$\left(\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega)\right)_{\varepsilon'} = \left(x_{1,t,,\varepsilon',\epsilon'}^{x_0,\varepsilon},\ldots,x_{n,t,,\varepsilon',\epsilon}^{x_0,\varepsilon}\right)_{\varepsilon'}$$

of a strongly$\widehat{\mathbb{R}}$–dissipative CISDE(35)-(40) andany$\mathbb{R}$-valued parameters$\lambda_1,\ldots,\lambda_n$, there exist finite Colombeauconstant $\widetilde{C}' = \left[\left(C'_{\varepsilon'}\right)_{\varepsilon'}\right] > 0$, such that $\forall\boldsymbol{\lambda}[\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_n)]$the inequality

$$\underline{\lim}_{\varepsilon\to 0}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}_{t,,\boldsymbol{\varepsilon}'=0,\boldsymbol{\epsilon}=0}^{\boldsymbol{x}_0,\boldsymbol{\varepsilon}}(\omega)-\boldsymbol{\lambda}\right\|^2\right] \leq \tilde{C}'\|\boldsymbol{U}(t,\boldsymbol{\lambda})\|^2 \qquad (89)$$

issatisfied.Here the vector-function $\boldsymbol{U}(t,\boldsymbol{\lambda}) = (U_1(\mathrm{t},\boldsymbol{\lambda}),\ldots,U_n(\mathrm{t},\boldsymbol{\lambda}))$is the solutionof the differential master equation:

$$\dot{\boldsymbol{U}}(t,\boldsymbol{\lambda}) = \mathbf{J}[\boldsymbol{b}_0(\boldsymbol{\lambda},t)]\boldsymbol{U}(t,\boldsymbol{\lambda}) + \boldsymbol{b}_0(\boldsymbol{\lambda},t),\ \boldsymbol{U}(0,\boldsymbol{\lambda}) = \boldsymbol{x}_0 - \boldsymbol{\lambda}, \qquad (90)$$

where$\mathbf{J} = \mathbf{J}[\boldsymbol{b}_0(\boldsymbol{\lambda},t)]$is aJacobian i.e.,$\mathbf{J}$is $n\times n$-matrix:

$$\mathbf{J}[\boldsymbol{b}_0(\boldsymbol{\lambda},t)] = \mathbf{J}\left[\partial\boldsymbol{b}_{0,i}(\boldsymbol{x},t)/\partial x_j\right]_{x=\lambda}.$$

Proof**1.**Fromtheequality

$$\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega)-\boldsymbol{\lambda}\right\|^2\right] = \mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\left[\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega)-\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega,\delta)\right]++\left[\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega,\delta)-\boldsymbol{\lambda}\right]\right\|^2\right],$$

by using the triangle inequality, one obtain

$$\sqrt{\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega)-\boldsymbol{\lambda}\right\|^2\right]} \leq \sqrt{\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega)-\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega,\delta)\right\|^2\right]} +$$

$+\sqrt{\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega,\delta)-\boldsymbol{\lambda}\right\|^2\right]}$.Therefore statement (1) immediately follows fromTheoremA1 (see appendix A), Proposition 2andTheorem5.

2.From the equality

$$\mathbf{E}_{\Omega}\left[\left\|\boldsymbol{x}_{t,,\varepsilon'=0,\epsilon=0}^{x_0,\varepsilon}(\omega)-\boldsymbol{\lambda}\right\|^2\right]=\mathbf{E}_{\Omega}\left[\left\|\left[\boldsymbol{x}_{t,,\varepsilon'=0,\epsilon=0}^{x_0,\varepsilon}(\omega)-\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega)\right]+\left[\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega)-\boldsymbol{\lambda}\right]\right\|^2\right],$$

byusingthetriangle inequality, one obtain

$$\sqrt{\mathbf{E}_{\Omega}\left[\left\|\boldsymbol{x}_{t,,\varepsilon'=0,\epsilon=0}^{x_0,\varepsilon}(\omega)-\boldsymbol{\lambda}\right\|^2\right]}\leq\sqrt{\mathbf{E}_{\Omega}\left[\left\|\boldsymbol{x}_{t,,\varepsilon'=0,\epsilon=0}^{x_0,\varepsilon}(\omega)-\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega)\right\|^2\right]}$$

$+\sqrt{\mathbf{E}_{\Omega}\left[\left\|\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega)-\boldsymbol{\lambda}\right\|^2\right]}$.Therefore statement (2) immediately follows fromTheoremA1 (see appendix A), Proposition 1andstatement (1).

Remark.5. We note that in general case the inequality

$$\left[\left(\delta_{\varepsilon'}(t)\right)_{\varepsilon'}\right]\equiv\left[\left(\underline{\lim}_{\varepsilon\to0}\mathbf{E}_{\Omega}\left[\left\|\boldsymbol{x}_{t,,\varepsilon'}^{x_0,\varepsilon}(\omega)-\boldsymbol{x}_{t,,\varepsilon'}^{x_0,\varepsilon=0}\right\|^2\right]\right)_{\varepsilon'}\right]\neq0$$

is satisfied, see Example 1.

Example 1.Figures 1-2.

$$\dot{x}_t^{x_0,\varepsilon}=-a\cdot\left(x_t^{x_0,\varepsilon}\right)^3-b\cdot\left(x_t^{x_0,\varepsilon}\right)^2-c\cdot x_t^{x_0,\varepsilon}-\sigma\cdot t^n-\qquad(91)$$
$$-\chi\cdot t^m\cdot\sin(\Omega\cdot t^k)+\sqrt{\varepsilon}w(t)\,,x_0^{x_0,\varepsilon}=x_{0.}$$

From Eq.(91) and general differential master equation (90) one obtain the next linear differential master equation:

$$\dot{u}(t)=-(3a\lambda^2+2b\lambda+c)u(t)-(a\cdot\lambda^3+b\cdot\lambda^2+c\cdot\lambda)-\qquad(92)$$
$$-\sigma\cdot t^n-\chi\cdot t^m\cdot\sin(\Omega\cdot t^k)\,,u(0)=x_{0.}-\lambda.$$

From the differential master equation (92) one obtain the transcendental master equation:

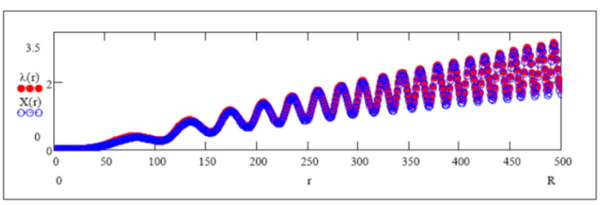

Figure 1.The solution of the Equation (8) in a comparison with a corresponding solution $x(t)$ of the ODE (10).

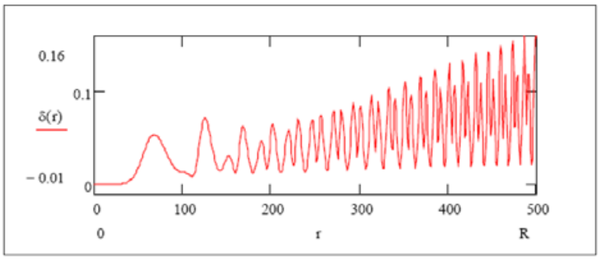

Figure 2. δ(r)versus**R.**

$$\left(x_0-\lambda(t)\right)\exp\left[-\left(3a\cdot\lambda^2(t)+2b\cdot\lambda(t)\right)\cdot t\right]-$$
$$-\int_0^t[\sigma\cdot\tau^n+\chi\cdot\tau^m\cdot\sin(\Omega\cdot\tau^k)+a\cdot\lambda^3(t)+b\cdot\lambda^2(t)]\times \quad (93)$$
$$\times\exp\left[-\left(3a\cdot\lambda^2(t)+2b\cdot\lambda(t)\right)\cdot(t-\tau)\right]d\tau=0.$$

Example 1.Numerical simulation: Figures 1 and **2**.

$$a=1,b=5,c=1,\sigma=\chi=-2,m=n=k=2,\Omega=5,$$
$$x_{0.}=0,T=5,R=T/0.001.$$

$$\delta(r)=\underline{\lim}_{\varepsilon\to 0}\mathbf{E}\left[\left\|x_t^{x_0,\varepsilon}-x_t^{x_0,\varepsilon=0}\right\|^2\right]. \quad (94)$$

$$\dot{x}_t^0=-a(x_t^0)^3-b(x_t^0)^2-cx_t^0-\sigma\cdot t^n-\chi\cdot t^m\cdot\sin(\Omega\cdot t^k).(95)$$

Let $\mathfrak{C}=(\Omega,\boldsymbol{\Sigma},\mathbf{P})$be a probability space. Let us consider now m–persons Colombeau-Ito's stochastic differential game$\mathrm{CIDG}_{m;T}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{y},G^n(\mathbb{R}^n),\mathfrak{C})$with nonlinear dynamics:

$$\left(\dot{\boldsymbol{x}}_{t,\varepsilon'}^{x_0,\varepsilon}(\omega)\right)_{\varepsilon'}=\left(\boldsymbol{f}_{\varepsilon'}\left(\boldsymbol{x}_{t,\varepsilon'}^{x_0,\varepsilon}(\omega),\boldsymbol{\alpha}(t),t\right)\right)_{\varepsilon'}+\sqrt{\varepsilon}\left(\boldsymbol{w}(t,\omega)\right)_{\varepsilon'}(96)$$

Here$\varepsilon,\varepsilon'\in(0,1],\varepsilon\ll 1;\forall t\in[0,T]$: $\left(\boldsymbol{x}_{\varepsilon'}(t)\right)_{\varepsilon'}\in\hat{R}^n;\boldsymbol{x}_{0,\varepsilon'}^{x_0,\varepsilon}(\omega)=\boldsymbol{x}_0\in\mathbb{R}^n,\boldsymbol{f}_{\varepsilon'}=$ $\left(f_{\varepsilon',\mathbf{1}},\dots,\boldsymbol{f}_{\varepsilon',n}\right),\boldsymbol{f}=[(\boldsymbol{f}_{\varepsilon'})_{\varepsilon'}],\boldsymbol{g}=[(\boldsymbol{g}_{\varepsilon'})_{\varepsilon'}];$

$$\boldsymbol{f}(x,\circ,\circ),\boldsymbol{g}(x,\circ,\circ)\in G^n(\mathbb{R}^n),\boldsymbol{\alpha}(t)=\{\alpha_1(t),\dots,\alpha_m(t)\};\alpha_i(t)\in U_i\subsetneq\mathbb{R}^{k_i},i=1,\dots,m.$$

Here$t\mapsto\alpha_i(t)$, is the control chosen by the i-th player, within a set of admissible control values$U_i\subsetneq\mathbb{R}^{k_i}$ and the playoff of the

i-th player is

$$\left(\bar{\mathbf{J}}^{\varepsilon}_{\varepsilon',i}\right)_{\varepsilon'}=\mathbf{E}\left[\left(\int_0^T g^2_{\varepsilon',i}\left(x^{x_0,\varepsilon}_{t,\varepsilon'}(\omega),\boldsymbol{\alpha}(t),t\right)dt\right)_{\varepsilon'}\right]+ \quad (97)$$

$$+\mathbf{E}\left[\left(\sum_{i=1}^{n}\left[x^{x,\varepsilon}_{T,\varepsilon';i}(\omega)-y_i\right]^2\right)_{\varepsilon'}\right].$$

Definition 9. $\mathrm{CIDG}_{m;T}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{y},G^n(\mathbb{R}^n),\mathfrak{C})$ (96)-(97) is a strongly $\widehat{\mathbb{R}}$–dissipative if CISDE(96) is a strongly $\widehat{\mathbb{R}}$–dissipative.

Theorem.7. Suppose that:

(1) $\mathrm{CIDG}_{m;T}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{y},G^n(\mathbb{R}^n),\mathfrak{C})$(96)-(97) isa strongly $\widehat{\mathbb{R}}$–dissipative,

(2) $\boldsymbol{f}_0(\circ,\boldsymbol{\alpha},t)\equiv\boldsymbol{f}_{\boldsymbol{\varepsilon}'=0}(\circ,\boldsymbol{\alpha},t):\mathbb{R}^{\boldsymbol{n}}\to\mathbb{R}^{\boldsymbol{n}}$ is a polynomial on a variable $\boldsymbol{x}=(x_1,\dots,x_n)$ and a linear function on a variable $\boldsymbol{\alpha}(t)=\{\alpha_1(t),\dots,\alpha_m(t)\}$ i.e., $f_{\boldsymbol{0},\boldsymbol{i}}(\boldsymbol{x},\boldsymbol{\alpha},t)=\sum_{\boldsymbol{\mu},|\boldsymbol{\mu}|\le r}f^{\boldsymbol{\mu}}_{\boldsymbol{0},\boldsymbol{i}}(t)\boldsymbol{x}^{\boldsymbol{\mu}}+\sum_{\boldsymbol{l=1}}^{\boldsymbol{m}}c_{l.i}(t)\alpha_l(t)\,,\boldsymbol{\mu}=(\boldsymbol{i_1},\dots,\boldsymbol{i_n}),|\boldsymbol{\mu}|=\sum_{\boldsymbol{j=1}}^{\boldsymbol{n}}i_j\,,0\le\boldsymbol{i_j}\le p,$

(3) $\boldsymbol{g}_0(\circ,\boldsymbol{\alpha},t)\equiv\boldsymbol{g}_{\boldsymbol{\varepsilon}'=0}(\circ,\boldsymbol{\alpha},t):\mathbb{R}^{\boldsymbol{n}}\to\mathbb{R}^{\boldsymbol{n}}$ is a polynomial on a variable $\boldsymbol{x}=(x_1,\dots,x_n)$ and a linear function on a variable $\boldsymbol{\alpha}(t)=\{\alpha_1(t),\dots,\alpha_m(t)\}$ i.e., $g_{\boldsymbol{0},\boldsymbol{i}}(\boldsymbol{x},\boldsymbol{\alpha},t)=\sum_{\boldsymbol{\mu},|\boldsymbol{\mu}|\le r}g^{\boldsymbol{\mu}}_{\boldsymbol{0},\boldsymbol{i}}(t)\boldsymbol{x}^{\boldsymbol{\mu}}+\sum_{\boldsymbol{l=1}}^{\boldsymbol{m}}d_{l,i}(t)\alpha_l(t)\,,\boldsymbol{\mu}=(\boldsymbol{i_1},\dots,\boldsymbol{i_n}),|\boldsymbol{\mu}|=\sum_{\boldsymbol{j=1}}^{\boldsymbol{n}}i_j\,,0\le\boldsymbol{i_j}\le p.$

Then For any solution

$$\left\{\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon'};\breve{\boldsymbol{\alpha}}(t)\right\}=\left(\left\{x^{x_0,\varepsilon}_{1,t,,\varepsilon'},\dots,x^{x_0,\varepsilon}_{n,t,,\varepsilon'}\right\};\{\alpha_1(t),\dots,\alpha_m(t)\}\right)(98)$$

of the $\mathrm{CIDG}_{m;T}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{y},G^n(\mathbb{R}^n),\mathfrak{C})$ (96) -(97) and any $\mathbb{R}$-valued parameters $\{\boldsymbol{\lambda}^{(1)},\boldsymbol{\lambda}^{(2)}\}=\left\{\left(\lambda^{(1)}_1,\dots,\lambda^{(1)}_n\right),\left(\lambda^{(2)}_1,\dots,\lambda^{(2)}_m\right)\right\}$ there exist finite Colombeau constant $\tilde{C}'=\left[\left(C'_{\varepsilon'}\right)_{\varepsilon'}\right]>0$, such that $\forall\boldsymbol{\lambda}\left[\boldsymbol{\lambda}=\{\boldsymbol{\lambda}^{(\mathbf{1})},\boldsymbol{\lambda}^{(\mathbf{2})}\}\right]$ the inequalities

$$(1)\underline{\lim}_{\substack{\varepsilon\to0\\ \varepsilon'\to0\\ \left(\frac{\varepsilon'}{\varepsilon}\right)\to0}}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}^{\boldsymbol{x_0,\varepsilon}}_{\boldsymbol{t,,\varepsilon'}}(\omega)-\boldsymbol{\lambda}^{(\mathbf{1})}\right\|^{\mathbf{2}}\right]\le\tilde{C}'\left\|\boldsymbol{U}\left(t,\boldsymbol{\lambda}^{(\mathbf{1})}\right)\right\|^{\mathbf{2}},$$

$$(2)\ \underline{\lim}_{\substack{\varepsilon\to0\\ \varepsilon'\to0\\ \left(\frac{\varepsilon'}{\varepsilon}\right)\to0}}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}^{\boldsymbol{x_0,\varepsilon}}_{T,,\boldsymbol{\varepsilon'}}(\boldsymbol{\omega})-\boldsymbol{\lambda}^{(2)}\right\|^{\mathbf{2}}\right]\le\tilde{C}'\left\|\boldsymbol{V}\left(T,\boldsymbol{\lambda}^{(\mathbf{1})}\right)\right\|^{\mathbf{2}},(99.a)$$

(3) $\underline{\lim}_{\substack{\varepsilon\to 0\\ \varepsilon'\to 0\\ \left(\frac{\varepsilon'}{\varepsilon}\right)\to 0}} \mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}^{x_0,\varepsilon}_{T,,\varepsilon'}(\omega)-\boldsymbol{y}\right\|^2\right]\leq \tilde{C}'\|\boldsymbol{V}(T,\boldsymbol{y})\|^2,$

(4) $\underline{\lim}_{\substack{\varepsilon\to 0\\ \varepsilon'\to 0\\ \left(\frac{\varepsilon'}{\varepsilon}\right)\to 0}} \left(\bar{\mathbf{J}}^{\varepsilon}_{\varepsilon',j}\right)\leq |V_i(T,\boldsymbol{y},0)|+\|\boldsymbol{U}(T,\boldsymbol{y})\|^2, i=1,\dots,m,$

(5) $\underline{\lim}_{\varepsilon\to 0}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon'=0}(\omega)-\boldsymbol{\lambda}^{(1)}\right\|^2\right]\leq \tilde{C}'\left\|\boldsymbol{U}\left(t,\boldsymbol{\lambda}^{(1)}\right)\right\|^2,$

(6) $\underline{\lim}_{\varepsilon\to 0}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}^{x_0,\varepsilon}_{T,,\varepsilon'=0}(\omega)-\boldsymbol{\lambda}^{(2)}\right\|^2\right]\leq \tilde{\boldsymbol{C}}'\left\|\boldsymbol{V}\left(T,\boldsymbol{\lambda}^{(1)}\right)\right\|^2,$

(7) $\underline{\lim}_{\varepsilon\to 0}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}^{x_0,\varepsilon}_{T,,\varepsilon'=0}(\omega)-\boldsymbol{y}\right\|^2\right]\leq \tilde{C}'\|\boldsymbol{V}(T,\boldsymbol{y})\|^2,$

(8) $\underline{\lim}_{\varepsilon\to 0}\left(\bar{\mathbf{J}}^{\varepsilon}_{\varepsilon'=0,j}\right)\leq |V_i(T,\boldsymbol{y},0)|+\|\boldsymbol{U}(T,\boldsymbol{y})\|^2, i=1,\dots,m$

issatisfied.Or in the next equivalent form: for a sufficiently small $\varepsilon\approx 0,\varepsilon'\approx 0$such that $\frac{\varepsilon'}{\varepsilon}\approx 0$,the inequalities

(1) $\left[\left(\liminf_{\varepsilon\to 0}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon'}(\omega)-\boldsymbol{\lambda}^{(1)}\right\|^2\right]\right)_{\varepsilon'}\right]\leq \tilde{C}'\left\|\boldsymbol{U}\left(t,\boldsymbol{\lambda}^{(1)}\right)\right\|^2,$

(2) $\left[\left(\liminf_{\varepsilon\to 0}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{z}^{z_0,\varepsilon}_{T,,\varepsilon'}(\omega)-\boldsymbol{\lambda}^{(2)}\right\|^2\right]\right)_{\varepsilon'}\right]\leq \tilde{C}'\|\boldsymbol{V}(T,\boldsymbol{\lambda})\|^2,$ (99.b)

(3) $\left[\left(\liminf_{\varepsilon\to 0}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}^{x,\varepsilon}_{T,\varepsilon'}(\omega)-\boldsymbol{y}\right\|^2\right]\right)_{\varepsilon'}\right]\leq \tilde{C}'\|\boldsymbol{U}(T,\boldsymbol{y})\|^2.$

(4) $\left(\bar{\mathbf{J}}^{\varepsilon}_{\varepsilon',j}\right)_{\varepsilon'}\leq |V_i(T,\boldsymbol{y},0)|+\|\boldsymbol{U}(T,\boldsymbol{y})\|^2, i=1,\dots,m$

issatisfied.Here $\boldsymbol{z}^{z_0,\varepsilon}_{T,,\varepsilon'}(\omega)=\int_0^T \boldsymbol{g}_{\varepsilon'}\left(\boldsymbol{x}^{x_0,\varepsilon}_{T,\varepsilon'}(\omega),\boldsymbol{\alpha}(t),t\right)dt.$

Here a function$\boldsymbol{W}(t,\boldsymbol{\lambda})=\{\boldsymbol{U}(t,\boldsymbol{\lambda}),\boldsymbol{V}(t,\boldsymbol{\lambda})\}^{\mathbf{t}}=$

$\left\{\left(U_1(t,\boldsymbol{\lambda}),\dots,U_n(t,\boldsymbol{\lambda})\right);\left(V_1(t,\boldsymbol{\lambda}),\dots,V_m(t,\boldsymbol{\lambda})\right)\right\}^{\mathbf{t}}$is the solution

of the differential master game with linear dynamics:

$\dot{\boldsymbol{W}}(t,\boldsymbol{\lambda})$= $\mathbf{J}\left[\widehat{\boldsymbol{b}}_0(\boldsymbol{\lambda},t)\right]\boldsymbol{W}(t,\boldsymbol{\lambda})+\widehat{\boldsymbol{b}}_0(\boldsymbol{\lambda},t)+\langle\boldsymbol{d}(t),\breve{\boldsymbol{\alpha}}_{\boldsymbol{l}}(t)\rangle$,(100)

$$\boldsymbol{U}(0,\boldsymbol{\lambda})=\boldsymbol{x_0}-\boldsymbol{\lambda},\boldsymbol{V}(0,\boldsymbol{\lambda})=0.$$

And with the playoff of the$i$-th player is:

$\breve{\mathbf{J}}_{\mathbf{i}}=|V_i(T,\boldsymbol{y},\mathbf{0})|+\|\boldsymbol{U}(T,\boldsymbol{y})\|^2$.(101)

Here

$\widehat{\boldsymbol{b}}_0(\boldsymbol{\lambda},t)=\{\boldsymbol{f}(\boldsymbol{\lambda},0,t);\boldsymbol{g}(\boldsymbol{\lambda},0,t)\}^{\mathrm{t}}$(102)

and

$\mathbf{J}=\mathbf{J}\left[\widehat{\boldsymbol{b}}_{\mathbf{0}}(\boldsymbol{\lambda},t)\right]$(103)

isJacobian i.e., **J** is $(n+m)\times(n+m)$-matrix:

$$\mathbf{J}\left[\widehat{\boldsymbol{b}}_{\mathbf{0}}(\boldsymbol{\lambda},t)\right]=\mathbf{J}\left[\partial\widehat{\boldsymbol{b}}_{\mathbf{0},\boldsymbol{i}}(\boldsymbol{x},t)/\partial x_{\boldsymbol{j}}\right]_{\boldsymbol{x}=\boldsymbol{\lambda}}.(104)$$

Proof. Let us rewrite Eqs.(96)-(97) of the next equivalent form

(i) $$\left(\dot{\boldsymbol{x}}_{t,\varepsilon'}^{x_0,\varepsilon}(\omega)\right)_{\varepsilon'}=\left(\boldsymbol{f}_{\varepsilon'}\left(\boldsymbol{x}_{t,\varepsilon'}^{x_0,\varepsilon}(\omega),\boldsymbol{\alpha}(t),t\right)\right)_{\varepsilon'}++\sqrt{\varepsilon}\big(w(t,\omega)\big)_{\varepsilon'} \quad (105)$$

(ii) (ii) $\dot{\boldsymbol{z}}_{t,,\boldsymbol{\varepsilon}'}^{\boldsymbol{z}_0,\boldsymbol{\varepsilon}}(\omega)=\boldsymbol{g}_{\boldsymbol{\varepsilon}'}^{\mathbf{2}}\left(\boldsymbol{x}_{\boldsymbol{T},\boldsymbol{\varepsilon}'}^{x_0,\varepsilon}(\omega),\boldsymbol{\alpha}(t),t\right)++\sqrt{\boldsymbol{\varepsilon}}\big(w(t,\omega)\big)_{\boldsymbol{\varepsilon}'}$.(106)

Then the playoff of the i-th player is

$$\left(\bar{\mathbf{J}}_{\varepsilon',i}^{\varepsilon}\right)_{\varepsilon'}=\mathbf{E}\left[\left(z_{T,,\boldsymbol{\varepsilon}',\boldsymbol{i}}^{\boldsymbol{z}_0,\boldsymbol{\varepsilon}}(\omega)\right)_{\varepsilon'}\right]+\mathbf{E}\left[\left(\sum_{i=1}^n\left[x_{T,\varepsilon';i}^{x,\varepsilon}(\omega)-y_i\right]^2\right)_{\varepsilon'}\right].$$

Here $z_{t,,\boldsymbol{\varepsilon}',\boldsymbol{i}}^{\boldsymbol{z}_0,\boldsymbol{\varepsilon}}(\omega)=\int_0^t g_{\boldsymbol{\varepsilon}',\boldsymbol{i}}\left(\boldsymbol{x}_{\boldsymbol{T},\boldsymbol{\varepsilon}'}^{\boldsymbol{x}_0,\boldsymbol{\varepsilon}}(\omega),\boldsymbol{\alpha}(t),t\right)dt\,,\boldsymbol{z}_0=0.$

The inequalities (99) immediately follow from Eq.(105), Eq.(106),Theorem 6and definitions.

Example.2.*2*-Persons Ito's stochastic differential game, with a small white noise.

$$(1)\dot{x}_1 = x_2, \dot{x}_2 = -kx_2^3 + \alpha_1(t) + \alpha_2(t) + \sqrt{\varepsilon}w(t,\omega); k > 0, (107)$$
$$t \in [0,T], x_1(0) = x_{10}, x_2(0) = x_{20}\ ;\ \varepsilon \ll 1;$$
$$(2)\ \alpha_1(t) \in [-\rho_1, \rho_1], \alpha_2(t) \in [-\rho_2, \rho_2];$$
$$(3) J_i = x_1^2(T), i = 1,2.$$

Optimal control problem for the first player is:

$$\min_{\alpha_1(t) \in [-\rho_1,\rho_1]} \left(\max_{\alpha_2(t) \in [-\rho_2,\rho_2]} [x_1^2(T)] \right) (108)$$

and optimal control problem for the second player is:

$$\max_{\alpha_2(t) \in [-\rho_2,\rho_2]} \left(\min_{\alpha_1(t) \in [-\rho_1,\rho_1]} [x_1^2(T)] \right). \qquad (109)$$

Using Equation (100) one obtained the corresponding linear master game:

$$(1)\dot{u}_1 = u_2, \dot{u}_2 = -3k\lambda_2^2 u_2 - k\lambda_2^3 + \breve{\alpha}_1(t) + \breve{\alpha}_2(t), \qquad (110)$$
$$u_1(0) = x_{10} - \lambda_1, x_2(0) = x_{20} - \lambda_2\ ;$$
$$(2)\breve{\alpha}_1(t) \in [-\rho_1, \rho_1], \breve{\alpha}_2(t) \in [-\rho_2, \rho_2];$$
$$(3)\bar{\boldsymbol{J}}_i = u_1^2(T), i = 1,2.$$

Optimal control problem for the first player is:

$$\min_{\breve{\alpha}_1(t) \in [-\rho_1,\rho_1]} \left(\max_{\breve{\alpha}_2(t) \in [-\rho_2,\rho_2]} [u_1^2(T)] \right), \qquad (111)$$

and optimal control problem for the second player is:

$$\max_{\breve{\alpha}_2(t)\in[-\rho_2,\rho_2]}\left(\min_{\breve{\alpha}_1(t)\in[-\rho_1,\rho_1]}[u_1^2(T)]\right)(112)$$

Having solved by standard way [20]-[22] linear master game (110)-(112),one obtain optimal feedback control for
the first player[5]-[6]:

$$\alpha_1(t)=\breve{\alpha}_1\big(t,x_1(t),x_2(t)\big)= \qquad (113)$$

$$=-\rho_1\mathrm{sign}[x_1(t)+[\boldsymbol{\Theta}_{\boldsymbol{\tau}}(t)]\cdot x_2(t)]$$

and optimal feedback control for the second player:

$$\alpha_2(t)=\breve{\alpha}_2\big(t,x_1(t),x_2(t)\big)= \qquad (114)$$

$$=\rho_2\mathrm{sign}[x_1(t)+[\boldsymbol{\Theta}_{\boldsymbol{\tau}}(t)]\cdot x_2(t)].$$

Here

$$\boldsymbol{\Theta}_{\boldsymbol{\tau}}(t)=\theta_{\boldsymbol{\tau}}\big(\eta_{\boldsymbol{\tau}}(t)\big),\boldsymbol{\theta}_{\boldsymbol{\tau}}=\boldsymbol{\tau}-t,\eta_{\boldsymbol{\tau}}(t)=t-\left(\mathrm{ceil}\left(\frac{\boldsymbol{t}}{\boldsymbol{\tau}}\right)-1\right)\cdot\boldsymbol{\tau}, \quad (115)$$

And where $\mathrm{ceil}(x)$ is a part-whole of a number $x\in\mathbb{R}$. Thus, for numerical simulation we obtain nonlinear ODE:

$$\dot{x}_1=x_2,\dot{x}_2=-kx_2^3+\alpha_2(t)-\rho_1\mathrm{sign}[x_1(t)+[\boldsymbol{\Theta}_{\boldsymbol{\tau}}(t)]\cdot x_2(t)].\,(116)$$

Numerical simulation: Figures 3-6.

$$k=1,|\rho_1|\leq 400,x_1(0)=300m,x_2(0)=30\frac{m}{sec},T=80sec,$$

$$\alpha_2(t)=A\sin^2(\omega\cdot t),A=100,\omega=5.$$

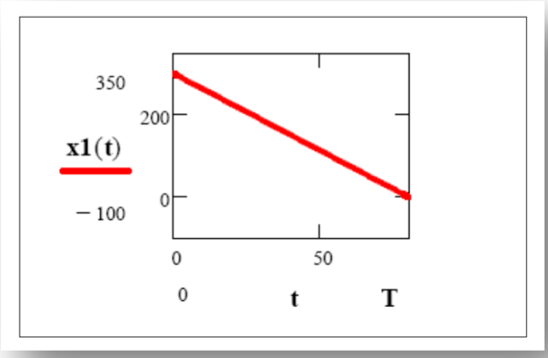

Figure.3. Optimal trajectory:$x_1(t).x_1(T) = 0.4$m

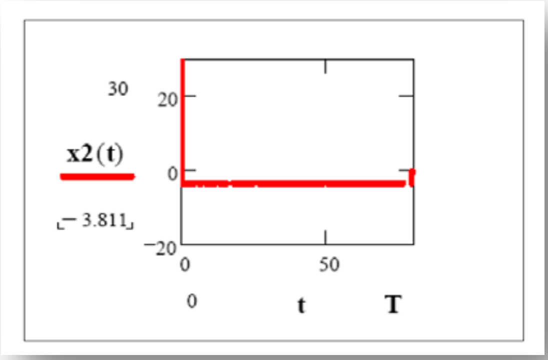

Figure.4. Optimal velocity: $x_2(t).x_2(T) = 0.4$ m/sec

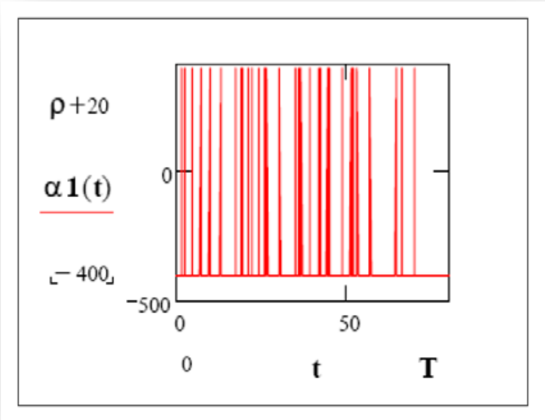

Figure.5. Optimal control of the first player:$\alpha_2(t) = \rho_2 \text{sign}[x_1(t) + [\boldsymbol{\Theta}_{\boldsymbol{\tau}}(t)] \cdot x_2(t)]$.

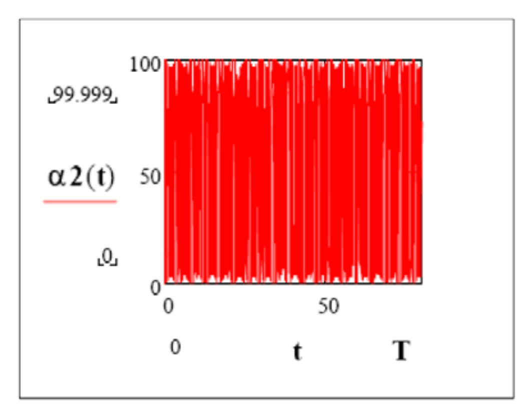

Figure.6. Control of the second player: $\alpha_2(t)$.

Let $\mathfrak{C} = (\Omega, \boldsymbol{\Sigma}, \mathbf{P})$be a probability space. Let us consider now m -persons Colombeau-Ito's differential game $\mathrm{CIDG}_{m;T}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{y}, G^n(\mathbb{R}^n), \boldsymbol{\beta}(t), \boldsymbol{\varphi}(t), \mathfrak{C})$with imperfect measurements and with imperfect information about the system [5],[6]. The corresponding stochastic nonlinear dynamics is:

$$\left(\dot{\boldsymbol{x}}_{t,,\varepsilon'}^{x_0,\varepsilon}(\omega)\right)_{\varepsilon'} = \left(\boldsymbol{f}_{\varepsilon'}\left(\boldsymbol{x}_{t,\varepsilon'}^{x_0,\varepsilon}(\omega), \boldsymbol{\varphi}(t), \alpha\left(t, \boldsymbol{x}_{t,\varepsilon'}^{x_0,\varepsilon} + \boldsymbol{\beta}(t)\right), t\right)\right)_{\varepsilon'} \quad (117)$$

$$+\sqrt{\varepsilon}\big(w(t,\omega)\big)_{\varepsilon'}; \varepsilon, \varepsilon' \in (0,1], \omega \in \Omega,$$

and the playoff of the i-th player is:

$$\left(\bar{\mathbf{J}}_{\varepsilon',j}^{\varepsilon}\right)_{\varepsilon'} = \mathbf{E}_{\Omega}\left[\left(\int_{\mathbf{0}}^{\boldsymbol{T}} \boldsymbol{g}_{\varepsilon',i}\left(\boldsymbol{x}_{t,\varepsilon'}^{x_0,\varepsilon}(\omega), \boldsymbol{\alpha}\big(t, \boldsymbol{\beta}(t)\big), t\right) dt\right)_{\varepsilon'}\right] + \quad (118)$$

$$+\mathrm{E}_{\Omega}\left[\left(\sum_{i=1}^{n}\left[\boldsymbol{x}_{T,D,\varepsilon';i}^{x_0,\varepsilon}(\omega) - y_i\right]^2\right)_{\varepsilon'}\right].$$

Here$\boldsymbol{\beta}(t) = (\beta_1(t), \dots, \beta_n(t))$, $\boldsymbol{\varphi}(t) = \big(\varphi_1(t), \dots, \varphi_n(t)\big)$ and $\forall t \in [0,T]$: $\big(\boldsymbol{x}_{\varepsilon'}(t)\big)_{\varepsilon'} \in \widehat{\mathbb{R}}$; $\boldsymbol{x}_{\mathbf{0},\varepsilon'}^{x_0,\varepsilon}(\omega) = x_0$,$\boldsymbol{f} = [(\boldsymbol{f}_{\varepsilon'})_{\varepsilon'}]$, $\boldsymbol{g} = [(\boldsymbol{g}_{\varepsilon'})_{\varepsilon'}]$; $\boldsymbol{f}(\boldsymbol{x},\circ,\circ,\circ), \boldsymbol{g}(\boldsymbol{x},\circ,\circ) \in \boldsymbol{G}^{\boldsymbol{n}}(\mathbb{R}^{\boldsymbol{n}})$, $\boldsymbol{\alpha}(t) = \{\alpha_1(t), \dots, \alpha_m(t)\}$; $\boldsymbol{\alpha_i}(t) \in U_i \subsetneq \mathbb{R}^{k_i}$, $i = 1, \dots, m$, $\boldsymbol{\beta}(t) = \{\beta_1(t), \dots, \beta_n(t)\}$, $\boldsymbol{\varphi}(t) = \{\varphi_1(t), \dots, \varphi_n(t)\}$.

Definition10.$CIDG_{m;T}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{y},G^{\boldsymbol{n}}(\mathbb{R}^{\boldsymbol{n}}),\mathfrak{C})$isa strongly$\widehat{\mathbb{R}}$–dissipative if corresponding CISDE (117) is a strongly $\widehat{\mathbb{R}}$–dissipative.

Theorem.8. Suppose that:

(1)$\mathrm{CIDG}_{m;T}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{y},G^{n}(\mathbb{R}^{n}),\boldsymbol{\beta}(t),\boldsymbol{\varphi}(t),\mathfrak{C})$ (117)-(118) is a strongly $\widehat{\mathbb{R}}$–dissipative,

(2)$\boldsymbol{f}_0(\circ,\boldsymbol{\varphi}(\boldsymbol{t}),\boldsymbol{\alpha},t)\equiv \boldsymbol{f}_{\boldsymbol{\varepsilon}'=0}(\circ,\boldsymbol{\varphi}(t),\boldsymbol{\alpha},t):\mathbb{R}^{\boldsymbol{n}}\to\mathbb{R}^{\boldsymbol{n}}$is a polynomial on a variable$\boldsymbol{x}=(x_1,\dots,x_n)$and a linear function on a variable$\boldsymbol{\alpha}(t)=\{\alpha_1(t),\dots,\alpha_m(t)\}$i.e.,$f_{\boldsymbol{0},\boldsymbol{i}}(\boldsymbol{x},\boldsymbol{\varphi}(t),\boldsymbol{\alpha},t)=\sum_{\boldsymbol{\mu},|\boldsymbol{\mu}|\leq \boldsymbol{r}} f_{\boldsymbol{0},\boldsymbol{i}}^{\boldsymbol{\mu}}(t)\boldsymbol{x}^{\boldsymbol{\mu}}+\sum_{\boldsymbol{l}=\boldsymbol{1}}^{\boldsymbol{m}} c_{l.i}(t)\alpha_l(t)\,,\boldsymbol{\mu}=(i_1,\dots,i_n),|\boldsymbol{\mu}|=\sum_{\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}} i_{\boldsymbol{j}}\,,0\leq \boldsymbol{i}_{\boldsymbol{j}}\leq p,$

(3)$\boldsymbol{g}_0(\circ,\boldsymbol{\alpha},t)\equiv \boldsymbol{g}_{\boldsymbol{\varepsilon}'=0}(\circ,\boldsymbol{\alpha},t):\mathbb{R}^{\boldsymbol{n}}\to\mathbb{R}^{\boldsymbol{n}}$is a polynomial on a variable$\boldsymbol{x}=(x_1,\dots,x_n)$and a linear function on a variable$\boldsymbol{\alpha}(t)=\{\alpha_1(t),\dots,\alpha_m(t)\}$i.e.,$g_{\boldsymbol{0},\boldsymbol{i}}(\boldsymbol{x},\boldsymbol{\alpha},t)=\sum_{\boldsymbol{\mu},|\boldsymbol{\mu}|\leq \boldsymbol{r}} g_{\boldsymbol{0},\boldsymbol{i}}^{\boldsymbol{\mu}}(t)\boldsymbol{x}^{\boldsymbol{\mu}}+\sum_{\boldsymbol{l}=\boldsymbol{1}}^{\boldsymbol{m}} d_{l,i}(t)\alpha_l(t)\,,\boldsymbol{\mu}=(i_1,\dots,i_n),|\boldsymbol{\mu}|=\sum_{\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}} i_{\boldsymbol{j}}\,,0\leq \boldsymbol{i}_{\boldsymbol{j}}\leq p.$

Then for any solution $\left\{\boldsymbol{x}_{\boldsymbol{t},,\boldsymbol{\varepsilon}'}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}};\breve{\boldsymbol{\alpha}}(\boldsymbol{t})\right\}=\left(\left\{\boldsymbol{x}_{\boldsymbol{1},\boldsymbol{t},,\boldsymbol{\varepsilon}'}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}},\dots,\boldsymbol{x}_{\boldsymbol{n},\boldsymbol{t},,\boldsymbol{\varepsilon}'}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}\right\};\{\boldsymbol{\alpha_1}(\boldsymbol{t}),\dots,\boldsymbol{\alpha_m}(\boldsymbol{t})\}\right)$ ofthe $\mathrm{CIDG}_{\boldsymbol{m};\boldsymbol{T}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{y},\boldsymbol{G}^{\boldsymbol{n}}(\mathbb{R}^{\boldsymbol{n}}),\boldsymbol{\beta}(t),\boldsymbol{\varphi}(t),\mathfrak{C})$ (117)-(118) and for any $\mathbb{R}$-valued parameters $\{\boldsymbol{\lambda}^{(\boldsymbol{1})},\boldsymbol{\lambda}^{(\boldsymbol{2})}\}=\left\{\left(\boldsymbol{\lambda}_{\boldsymbol{1}}^{(\boldsymbol{1})},\dots,\boldsymbol{\lambda}_{\boldsymbol{n}}^{(\boldsymbol{1})}\right),\left(\boldsymbol{\lambda}_{\boldsymbol{1}}^{(\boldsymbol{2})},\dots,\boldsymbol{\lambda}_{\boldsymbol{m}}^{(\boldsymbol{2})}\right)\right\}$ there exist finite Colombeau constant $\widetilde{\boldsymbol{C}}'=\left[\left(\boldsymbol{C}'_{\boldsymbol{\varepsilon}'}\right)_{\boldsymbol{\varepsilon}'}\right]>0$, such that$\forall\boldsymbol{\lambda}[\boldsymbol{\lambda}=\{\boldsymbol{\lambda}^{(\boldsymbol{1})},\boldsymbol{\lambda}^{(\boldsymbol{2})}\}]$the inequalities

(1)$\left[\left(\liminf_{\varepsilon\to0}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}_{\boldsymbol{t},,\boldsymbol{\varepsilon}'=0}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}(\omega)-\boldsymbol{\lambda}^{(\boldsymbol{1})}\right\|^{\boldsymbol{2}}\right]\right)_{\boldsymbol{\varepsilon}'}\right]\leq\widetilde{\boldsymbol{C}}'\left\|\boldsymbol{U}\left(t,\boldsymbol{\lambda}^{(\boldsymbol{1})}\right)\right\|^{\boldsymbol{2}},$

(2)$\left[\left(\liminf_{\varepsilon\to0}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{z}_{\boldsymbol{T},,\boldsymbol{\varepsilon}'=0}^{\boldsymbol{z_0},\boldsymbol{\varepsilon}}(\omega)-\boldsymbol{\lambda}^{(\boldsymbol{2})}\right\|^{\boldsymbol{2}}\right]\right)_{\boldsymbol{\varepsilon}'}\right]\leq\widetilde{\boldsymbol{C}}'\|\boldsymbol{V}(T,\boldsymbol{\lambda})\|^{\boldsymbol{2}},$(119)

(3)$\left[\left(\liminf_{\varepsilon\to0}\mathbf{E}_{\boldsymbol{\Omega}}\left[\left\|\boldsymbol{x}_{\boldsymbol{T},\boldsymbol{\varepsilon}'=0}^{\boldsymbol{x},\boldsymbol{\varepsilon}}(\omega)-\boldsymbol{y}\right\|^{\boldsymbol{2}}\right]\right)_{\boldsymbol{\varepsilon}'}\right]\leq\widetilde{\boldsymbol{C}}'\|\boldsymbol{U}(T,\boldsymbol{y})\|^{\boldsymbol{2}}.$

(4)$\left(\bar{\mathbf{J}}_{\boldsymbol{\varepsilon}',\boldsymbol{j}}^{\boldsymbol{\varepsilon}}\right)_{\boldsymbol{\varepsilon}'}\leq|\boldsymbol{V}_{\boldsymbol{i}}(T,\boldsymbol{y},0)|+\|\boldsymbol{U}(T,\boldsymbol{y})\|^{\boldsymbol{2}},\ i=1,\dots,m,$

is satisfied. Here$\boldsymbol{z}_{\boldsymbol{T},,\boldsymbol{\varepsilon}'}^{\boldsymbol{z_0},\boldsymbol{\varepsilon}}(\omega) = \int_{\mathbf{0}}^{\boldsymbol{T}} \boldsymbol{g}_{\boldsymbol{\varepsilon}'}\left(\boldsymbol{x}_{\boldsymbol{T},\boldsymbol{\varepsilon}'}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}(\omega), \boldsymbol{\alpha}(t), t\right) dt$.

Here a function$\boldsymbol{W}(t,\boldsymbol{\lambda}) = \{\boldsymbol{U}(t,\boldsymbol{\lambda}), \boldsymbol{V}(t,\boldsymbol{\lambda})\}^{\mathbf{t}} =$

$\left\{\left(\boldsymbol{U_1}(t,\boldsymbol{\lambda}), \dots, \boldsymbol{U_n}(t,\boldsymbol{\lambda})\right); \left(\boldsymbol{V_1}(t,\boldsymbol{\lambda}), \dots, \boldsymbol{V_m}(t,\boldsymbol{\lambda})\right)\right\}^{\mathbf{t}}$is thesolution of the differential master game with linear imperfect dynamics and with imperfect measurements:

$$\dot{\boldsymbol{W}}(t,\boldsymbol{\lambda}) = \mathbf{J}\left[\widehat{\boldsymbol{b}}_{\mathbf{0}}(\boldsymbol{\lambda}, \boldsymbol{\varphi}(t), t)\right]\boldsymbol{W}(t,\boldsymbol{\lambda}) + \widehat{\boldsymbol{b}}_{\mathbf{0}}(\boldsymbol{\lambda}, \boldsymbol{\varphi}(t), t) + \tag{120}$$

$$+\langle \boldsymbol{d}(t), \breve{\boldsymbol{\alpha}}_{\boldsymbol{l}}(t, \boldsymbol{\beta}(t))\rangle, \boldsymbol{U}(0,\boldsymbol{\lambda}) = \boldsymbol{x_0} - \boldsymbol{\lambda}, \boldsymbol{V}(0,\boldsymbol{\lambda}) = 0$$

and the playoff of the i-th player is:

$$\breve{\mathbf{J}}_{\mathbf{i}} = |V_i(T, \boldsymbol{y}, \mathbf{0})| + \|\boldsymbol{U}(T, \boldsymbol{y})\|^2. \tag{121}$$

Here

$$\widehat{\boldsymbol{b}}_0(\boldsymbol{\lambda}, \boldsymbol{\varphi}(t), t) = \{\boldsymbol{f}(\boldsymbol{\lambda}, \boldsymbol{\varphi}(t), 0, t); \boldsymbol{g}(\boldsymbol{\lambda}, 0, t)\}^{\mathbf{t}}, \tag{122}$$

And here $\mathbf{J} = \mathbf{J}\left[\widehat{\boldsymbol{b}}_0(\boldsymbol{\lambda}, t)\right]$is Jacobian i.e., $\mathbf{J}$ is $(n+m) \times (n+m)$-matrix:

$$\mathbf{J}\left[\widehat{\boldsymbol{b}}_{\mathbf{0}}(\boldsymbol{\lambda}, \boldsymbol{\varphi}(\boldsymbol{t}), t)\right] = \mathbf{J}\left[\partial \widehat{\boldsymbol{b}}_{\mathbf{0},\boldsymbol{i}}(\boldsymbol{x}, \boldsymbol{\varphi}(t), t)/\partial x_j\right]_{\boldsymbol{x}=\boldsymbol{\lambda}}. \tag{123}$$

Proof. The proof is completely to similarly a proof of the Theorem 7.

Example.3.$2m$-Persons Ito's stochastic differential game with a small white noise, and with imperfect measurements.

(1)$\dot{x}_1 = x_2, \dot{x}_2 = -k_1 x_2^3 + k_2 x_2^2 + \alpha_1[t, x_1(t), x_2(t) + \beta(t)] + \alpha_2(t) + \sqrt{\varepsilon} w(t,\omega); k_1 > 0,$ (124)

$$t \in [0,T], x_1(0) = x_{10}, x_2(0) = x_{20}\,;\ \varepsilon \ll 1;$$

$$(2)\ \alpha_1(t) \in [-\rho_1, \rho_1], \alpha_2(t) \in [-\rho_2, \rho_2];$$

$$(3) J_i = x_1^2(T), i = 1,2.$$

$$(4) \beta(t) = A \cdot \sin^2(\omega \cdot t).$$

From Equation (120) one obtained corresponding linear master game:

$(1)\dot{u}_1 = u_2, \dot{u}_2 =$
$-(3k_1\lambda_2^2 - 2k_2\lambda_2)u_2 - k\lambda_2^3 +$
$\breve{\alpha}_1[t, u_1(t), u_2(t) + \beta(t)] + \breve{\alpha}_2(t),$ (125)

$$u_1(0) = x_{10} - \lambda_1, x_2(0) = x_{20} - \lambda_2;$$
$$(2)\breve{\alpha}_1(t) \in [-\rho_1, \rho_1], \breve{\alpha}_2(t) \in [-\rho_2, \rho_2];$$
$$(3)\bar{\boldsymbol{J}}_i = u_1^2(T), i = 1,2.$$

Having solved by standard way linear master game (124) one obtain local optimal feedback control of the first player [5]:

$$\alpha_1(t_{n+1}) = -\rho_1 \text{sign}[x_1(t_n) + (t_{n+1} - t_n)(x_2(t_n) + \beta(t_n))], \quad (126)$$

and local optimal feedback control of the second player:

$$\alpha_2(t_{n+1}) = \rho_2 \text{sign}[x_1(t_n) + (t_{n+1} - t_n)(x_2(t_n) + \beta(t_n))]. \quad (127)$$

Thus, finally we obtain global optimal control of the next form [5]:
$\alpha_1(t) = -\rho_1 \text{sign}[x_1(t) + [\boldsymbol{\Theta}_{\boldsymbol{\tau}}(t)] \cdot (x_2(t) + \beta(t))],$ (128)
$\alpha_2(t) = \rho_2 \text{sign}[x_1(t) + [\boldsymbol{\Theta}_{\boldsymbol{\tau}}(t)] \cdot (x_2(t) + \beta(t))].$ (129)
Here $\boldsymbol{\Theta}_{\boldsymbol{\tau}}(t) = \theta_{\boldsymbol{\tau}}(\eta_{\boldsymbol{\tau}}(t)), \boldsymbol{\theta}_{\boldsymbol{\tau}} = \boldsymbol{\tau} - t, \eta_{\boldsymbol{\tau}}(t) = t - \left(\text{ceil}\left(\frac{t}{\tau}\right) - 1\right) \cdot \boldsymbol{\tau},$ whereceil$(x)$is a part-whole of a number $x \in \mathbb{R}$. Thus, for numerical simulation we obtain ODE:

$\dot{x}_1 = x_2, \dot{x}_2 = -k_1 x_2^3 + k_2 x_2^2 - \rho_1 \text{sign}[x_1(t)[\boldsymbol{\Theta}_{\boldsymbol{\tau}}(t)](x_2(t) + \beta(t))] + +\rho_2 \text{sign}[x_1(t) + [\boldsymbol{\Theta}_{\boldsymbol{\tau}}(t)] \cdot (x_2(t) + \beta(t))].$ (130)

Numerical simulation: Figures 7-12. Game with imperfect measurements(red curves $x_1(t)$and$x_2(t)$) in comparison with a classical game with perfect measurements: blue curves $y_1(t)$and$y_2(t)$,$\beta(t) = A \cdot \sin^2(\omega \cdot t), A = 100, \omega = 5.$

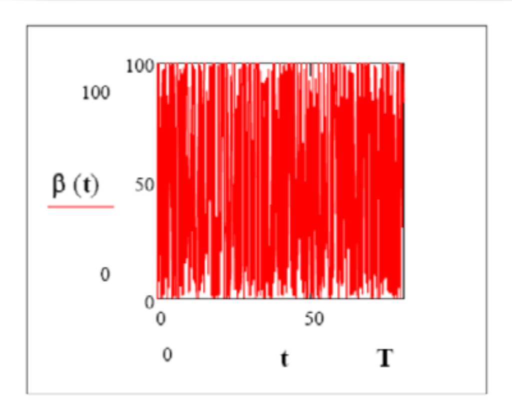

Figure7. Uncertainty of speed measurements $\beta(t)$

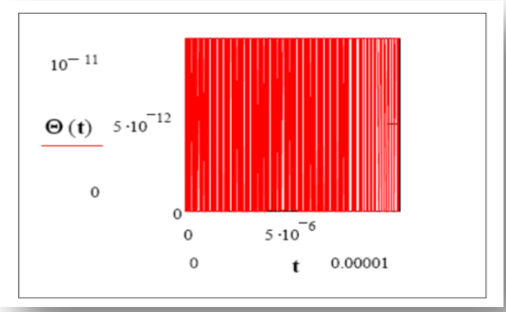

Figure8.Cutting function. $\boldsymbol{\Theta_{\tau}(t), \tau = 10^{-11}}$.

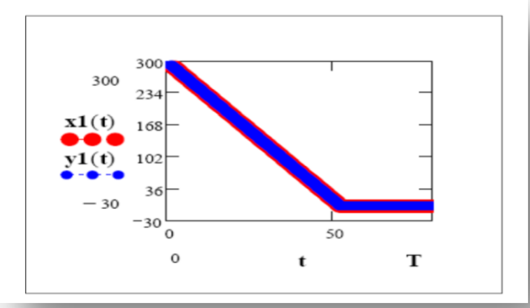

Figure9. Optimal trajecto

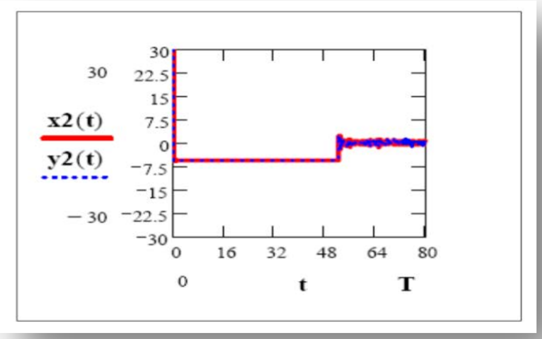

Figure 10.Optimal velocity.

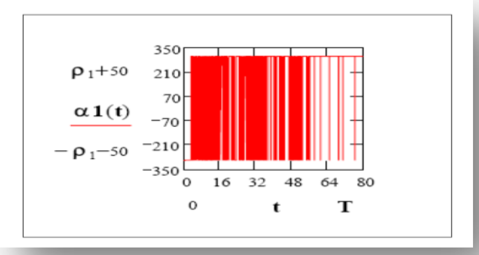

Figure 11.Optimal control of the first player.

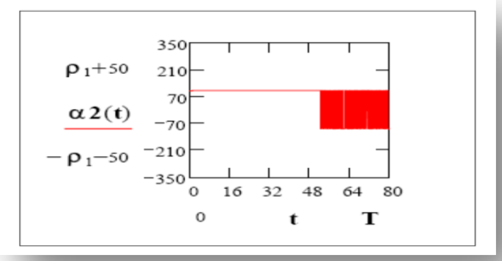

Figure 12.Optimal control of the second player.

3. Homing Missile Guidance with Imperfect Measurements Capable to Defeat in Conditions of Hostile Active Radio-electronic Jamming

Homing missile guidance strategies (guidance laws) dictate the manner in which the missile will guide to intercept, or rendezvous with, the target [20]- [21]. The feedback nature of homing guidance allows the guided missile (or, more generally, the pursuer) to tolerate some level of (sensor) measurement uncertainties, errors in the assumptions used to model the engagement (e. g., unanticipated target maneuver), and errors in modeling missile capability (e.g., deviation of actual missile speed of response to guidance commands from the design assumptions). Nevertheless, the selection of a guidance strategy and its subsequent mechanization are crucial design factors that can have substantial impact on guided missile performance. Key drivers to guidance law design include the type of targeting sensor to be used (passive IR, active or semi-active RF, etc.), accuracy of the targeting and inertial measurement unit (IMU) sensors, missile maneuverability, and, finally yet important, the types of targets to be engaged and their associated maneuverability levels.
Figure 13 shows the intercept geometry of a missile in planar pursuit of a target. Taking the origin of the reference frame to be the instantaneous position of the missile, the equation of motion in polar form are [22]:

$$\ddot{R} = R\dot{\sigma}^2 + a_M^r\left[t, \tilde{R}(t), \tilde{\dot{R}}(t)\right] + a_T^r(t), \qquad (131)$$

$$R\ddot{\sigma} + 2\dot{R}\dot{\sigma} = a_M^n\left[t, \tilde{\sigma}(t), \tilde{\dot{\sigma}}(t)\right] + a_T^n(t), \qquad (132)$$

$$a_M^r(t) \in [-\bar{a}_M^r, \bar{a}_M^r], a_T^r(t) \in [-\bar{a}_T^r, \bar{a}_T^r],$$

$$a_M^n(t) \in [-\bar{a}_M^n, \bar{a}_M^n], a_T^n(t) \in [-\bar{a}_T^n, \bar{a}_T^n].$$

1. The variable$R = R(t)$ denotes a true target-to-missile range $R_{TM}(t)$.

2. The variable $\tilde{R} = \tilde{R}(t)$ denotes it is *measured* target-to-missile range: $R_{TM}(t)$

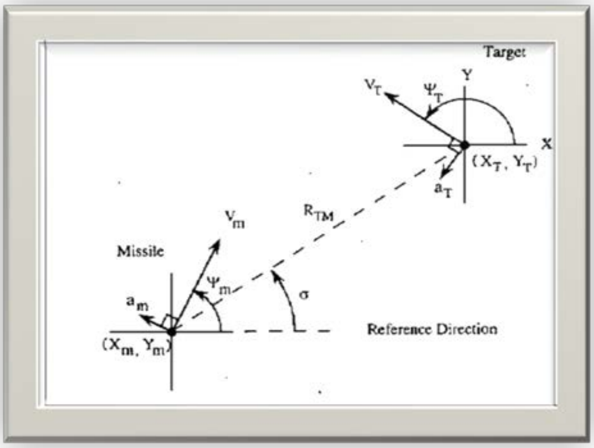

Figure13. Planar intercept geometry.

3. The variable$\sigma = \sigma(t)$ denotes a true line-of-sight angle (***LOST***) i. e., the it is *true angle* between the constant reference direction and target-to-missile direction.

4. The variable$\tilde{\sigma} = \tilde{\sigma}(t)$denotes it's *really measured* line-of-sight angle (***LOSM***) i.e., it is *measured angle* between the constant reference direction and target-to-missile direction.

5. The variable$a_M^n(t) = a_M^n[t, \sigma(t), \dot{\sigma}(t)]$denotes the missiles acceleration along direction which perpendicularly to line-of-sight direction.

6. The variable$a_M^r(t) = a_M^r\left[t, R(t), \dot{R}(t)\right]$denotes the missile acceleration along target-to-missile direction.

7. The variable$a_T^n(t)$ denotes the target acceleration along direction which perpendicularly to line-of-sight direction.

8. The variable$a_T^r(t)$ denotes the target acceleration along target-to-missile direction.

Using replacement$\dot{z} = R\dot{\sigma}$ into Equation (131)-(132) one
obtain:

$$\ddot{R} = \frac{\dot{z}^2}{R} + a_M^r\left[t, \tilde{R}(t), \tilde{\dot{R}}(t)\right] + a_T^r(t), \quad (133)$$

$$\ddot{z} = -\frac{\dot{R}\dot{z}}{R} + a_M^n\left[t, \tilde{z}(t), \tilde{\dot{z}}(t)\right] + a_T^n(t), (134)$$

$$\tilde{z}(t) = \tilde{R}(t)\tilde{\sigma}(t), \quad (135)$$

$$\tilde{\dot{z}}(t) = \tilde{\dot{R}}(t)\tilde{\sigma}(t) + \tilde{R}(t)\tilde{\dot{\sigma}}(t). \quad (136)$$

Here we denoted:

$\tilde{\dot{R}}(t) \triangleq \dot{\tilde{R}}, \tilde{\dot{\sigma}}(t) \triangleq \dot{\tilde{\sigma}}, \tilde{\dot{z}}(t) \triangleq \dot{\tilde{z}}(t),$ (137)

$\tilde{\ddot{\sigma}}(t) \triangleq \ddot{\tilde{\sigma}}(t), \tilde{\ddot{z}}(t) \triangleq \ddot{\tilde{z}}(t).$ (138)

Suppose that:

$\tilde{R}(t) = R(t) + \beta_1(t), \tilde{\sigma}(t) = \sigma(t) + \beta_2(t).$ (139)

Therefore:

$\tilde{\dot{R}}(t) = \dot{R}(t) + \dot{\beta}_1(t) = \dot{R}(t) + \bar{\beta}_1(t), \bar{\beta}_1(t) = \dot{\beta}_1(t).$ (140)

$\tilde{\dot{\sigma}}(t) = \dot{\sigma}(t) + \dot{\beta}_2(t), \tilde{\ddot{\sigma}}(t) = \ddot{\sigma}(t) + \ddot{\beta}_2(t).$(141)

$\tilde{\dot{z}}(t) = \tilde{R}(t)\tilde{\dot{\sigma}}(t) = [R(t) + \beta_1(t)]\left[\dot{\sigma}(t) + \dot{\beta}_2(t)\right] =$(142)

$R(t)\dot{\sigma}(t) + \beta_1(t)\left[\dot{\sigma}(t) + \dot{\beta}_2(t)\right] + R(t)\dot{\beta}_2(t) \approx$

$\approx \dot{z}(t) + \beta_1(t)\left[\tilde{\dot{\sigma}}(t) + \dot{\beta}_2(t)\right] + \tilde{R}(t)\dot{\beta}_2(t) = \dot{z}(t) + \bar{\beta}_3(t),$ (143)

$\bar{\beta}_3(t) = \beta_1(t)\left[\tilde{\dot{\sigma}}(t) + \dot{\beta}_2(t)\right] + \tilde{R}(t)\dot{\beta}_2(t),$(144)

$\tilde{z}(t) = z(t) + \int_0^t \bar{\beta}_3(\tau)d\tau = z(t) + \bar{\beta}_4(t).$(145)

Let us consider antagonistic Colombeau differential game $CIDG_{2;T}(f, g, y, G^n(\mathbb{R}^n), \boldsymbol{\beta}(t), \mathfrak{C})$, $\boldsymbol{\beta}(t) = (\beta_1(t), \beta_2(t))$ with non-linear dynamics given by Eq.(133)-Eq.(134), and imperfect measurements [5]-[6]. Using replacement$\dot{R}(t) = V_r(t)$,$\dot{z}(t) = \eta(t)$, from Eq.(133)-Eq.(145) one obtain:

$\dot{R} = V_r,$ (146)

$\dot{V}_r = \frac{\dot{z}^2}{R} + a_M^r(t) + a_T^r(t),$ (147)

$a_M^r(t) = \breve{a}_M^r[t, \tilde{R}(t), \tilde{V}_r(t)] - k_1\tilde{V}_r^3(t),$ (148)

$\tilde{R}(t) = R(t) + \beta_1(t), \tilde{V}_r(t) = V_r + \bar{\beta}_1(t),$ (149)

$\dot{z} = \eta,$ (150)

$\dot{\eta} = -\frac{V_r\eta}{R} + a_M^n(t) + a_T^n(t),$ (151)

$a_M^n(t) = \breve{a}_M^n[t, \tilde{z}(t), \tilde{\eta}(t)] - k_2\tilde{\eta}^3(t),$(152)

$\tilde{\eta}(t) = \eta(t) + \bar{\beta}_3(t), \tilde{\dot{\eta}}(t) = \dot{\eta}(t) + \bar{\beta}_4(t),$ (153)

$\breve{a}_M^r(t) \in [-\bar{a}_M^r\ ,\ \bar{a}_M^r], \breve{a}_T^r(t) \in [-\bar{a}_T^r\ ,\ \bar{a}_T^r],$ (154)

$$\breve{a}_M^n(t) \in [-\bar{a}_M^n, \bar{a}_M^n], \breve{a}_T^n(t) \in [-\bar{a}_T^n, \bar{a}_T^n], \quad (155)$$

$$\mathbf{J}_i = R^2(t_1), i = 1,2. \quad (156)$$

Optimal control problem of the first player is:

$$\bar{J}_1 = \left\{ \begin{matrix} \min[\Upsilon_1(t_1)] \\ \breve{a}_M^r(t) \in [-\bar{a}_M^r, \bar{a}_M^r], \breve{a}_M^n(t) \in [-\bar{a}_M^n, \bar{a}_M^n] \end{matrix} \right\}, \quad (157)$$

where

$$\Upsilon_1(t_1) = \left\{ \begin{matrix} \max[R^2(t_1)] \\ \breve{a}_T^r(t) \in [-\bar{a}_T^r, \bar{a}_T^r], \breve{a}_T^n(t) \in [-\bar{a}_T^n, \bar{a}_M^n] \end{matrix} \right\}. \quad (158)$$

Optimal control problem of the second player is:

$$\bar{J}_2 = \left\{ \begin{matrix} \max[\Upsilon_2(t_1)] \\ \breve{a}_T^r(t) \in [-\bar{a}_T^r, \bar{a}_M^r], \breve{a}_T^n(t) \in [-\bar{a}_T^n, \bar{a}_M^n] \end{matrix} \right\}, \quad (159)$$

where

$$\Upsilon_2(t_1) = \left\{ \begin{matrix} \min[R^2(t_1)] \\ \breve{a}_M^r(t) \in [-\bar{a}_M^r, \bar{a}_M^r], \breve{a}_M^n(t) \in [-\bar{a}_M^n, \bar{a}_M^n] \end{matrix} \right\}, \quad (160)$$

From Equations (146)-(160) one obtain corresponding linear master game:

$$\dot{r}(t) = v_r(t) + \lambda_2, \quad (161)$$

$$\dot{v}_r(t) = -3k_1\lambda_2^2\tilde{v}_r(t) - k_1\lambda_2^3 + \breve{a}_M^r(t) + \breve{a}_T^r(t), \quad (162)$$

$$\breve{a}_M^r(t) = \breve{a}_M^r[t, \tilde{r}(t), \tilde{v}_r(t)], \quad (163)$$

$$\tilde{r}(t) = \lambda_1 + r(t) + \beta_1(t), \quad (164)$$

$$\tilde{v}_r(t) = \lambda_2 + v_r(t) + \bar{\beta}_1(t), \quad (165)$$

$$\dot{z}_1(t) = \eta_1(t) + \lambda_3, \quad (166)$$

$$\dot{\eta}_1(t) = -3k_2\lambda_3^2\tilde{\eta}_1(t) - k_2\lambda_3^3 + \breve{a}_M^n(t) + \breve{a}_T^n(t), \quad (167)$$

$\tilde{\eta}_1(t) = \lambda_3 + \eta_1(t) + \bar{\beta}_3(t),$ (168)

$\breve{a}_M^r(t) \in [-\bar{a}_M^r, \bar{a}_M^r], \breve{a}_T^r(t) \in [-\bar{a}_T^r, \bar{a}_T^r],$ (169)

$\breve{a}_M^n(t) \in [-\bar{a}_M^n, \bar{a}_M^n], \breve{a}_T^n(t) \in [-\bar{a}_T^n, \bar{a}_T^n],$ (170)

$\mathbf{J}_i = r^2(t_1), i = 1,2.$ (171)

Optimal control problem of the first player is:

$$\bar{\mathbf{J}}_1 = \left\{ \begin{matrix} \min[\Upsilon_1(t_1)] \\ \breve{a}_M^r(t) \in [-\bar{a}_M^r, \bar{a}_M^r], \breve{a}_M^n(t) \in [-\bar{a}_M^n, \bar{a}_M^n] \end{matrix} \right\}, \quad (172)$$

Where

$$\Upsilon_1(t_1) = \left\{ \begin{matrix} \max[J_1(t_1)] \\ \breve{a}_T^r(t) \in [-\bar{a}_T^r, \bar{a}_T^r], \breve{a}_T^n(t) \in [-\bar{a}_T^n, \bar{a}_M^n] \end{matrix} \right\}, \quad (173)$$

Optimal control problem of the second player is:

$$\bar{\mathbf{J}}_2 = \left\{ \begin{matrix} \max[\Upsilon_2(t_1)] \\ \breve{a}_T^r(t) \in [-\bar{a}_T^r, \bar{a}_M^r], \breve{a}_T^n(t) \in [-\bar{a}_T^n, \bar{a}_M^n] \end{matrix} \right\}, \quad (174)$$

where

$$\Upsilon_2(t_1) = \left\{ \begin{matrix} \min[J_2(t_1)] \\ \breve{a}_M^r(t) \in [-\bar{a}_M^r, \bar{a}_M^r], \breve{a}_M^n(t) \in [-\bar{a}_M^n, \bar{a}_M^n] \end{matrix} \right\}. \quad (175)$$

From Equations (24) we obtain quasi optimal solution for the antagonistic differential game $CIDG_{2;T}(f, g, y, G^n(\mathbb{R}^n), \boldsymbol{\beta}(t), \mathfrak{C})$ given by Equations (21-23). Optimal control $\{a_M^r(t), a_M^n(t)\}$of
the first player are [5]-[6]:

$\breve{a}_M^r[t, \tilde{R}(t), \tilde{V}_r(t)] =$
$-\bar{a}_M^r \text{sign}\{[R(t) + \beta_1(t)] +$
$\Theta_\tau(t)[V_r(t) + +\bar{\beta}_1(t)]\},$ (176)

$$\breve{a}_M^n\left[t, \tilde{z}(t), \tilde{\dot{z}}(t)\right] = \quad (177)$$

$$= -\bar{a}_M^n \text{sign}\left\{\left[z(t) + \bar{\beta}_4(t)\right] + \Theta_\tau(t)\left[\dot{z}(t) + \bar{\beta}_3(t)\right]\right\}$$

Thus, for numerical simulation we obtain ODE's:

$$\dot{R} = V_r, \quad (178)$$

$$\dot{V}_r = \frac{\dot{z}^2}{R} - \bar{a}_M^r \text{sign}\left\{\left[R(t) + \beta_1(t)\right] + \Theta_\tau(t)\left[V_r(t) + \bar{\beta}_1(t)\right]\right\} - -k_1\left[V_r(t) + \bar{\beta}_1(t)\right]^3 + a_T^r(t), \quad (179)$$

$$\dot{z} = \eta, \quad (180)$$

$$\dot{\eta} = -\frac{V_r\eta}{R} - \bar{a}_M^n \text{sign}\left\{\left[z(t) + \bar{\beta}_4(t)\right] + \Theta_\tau(t)\left[\dot{z}(t) + \bar{\beta}_3(t)\right]\right\} -$$

$$-k_2\left[z(t) + \bar{\beta}_4(t)\right]^3. \quad (181)$$

Example.4. Figures 14-24. $\tau = 10^{-3}, k_1 = k_2 = 10^{-3}, \bar{a}_T^r = 20m/sec^2, \bar{a}_T^\tau = 20m/sec^2, R(0) = 200m, V_r(0) = 10m/sec,$
$z(0) = 60, \dot{z}(0) = 40,\ a_T^r(t) = \bar{a}_T^r\left(sin(\omega \cdot t)\right)^p, a_T^\tau(t) = \bar{a}_T^\tau\left(sin(\omega \cdot t)\right)^q, \beta_i(t) = \bar{\beta}\left(sin(\omega \cdot t)\right)^p, i = 1,2,3,4, \omega = 50, \bar{\beta} = 20, p = 2, q = 1.$

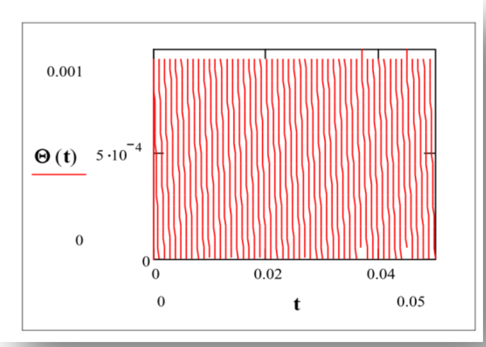

Figure 14. Cutting function:$\Theta_\tau(t)$.

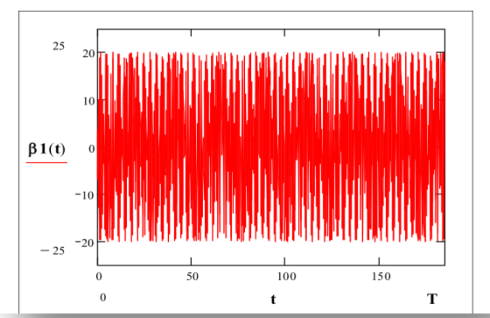

Figure 15. Uncertainty$\beta_1(t)$ of measurements of a variable $R(t)$.

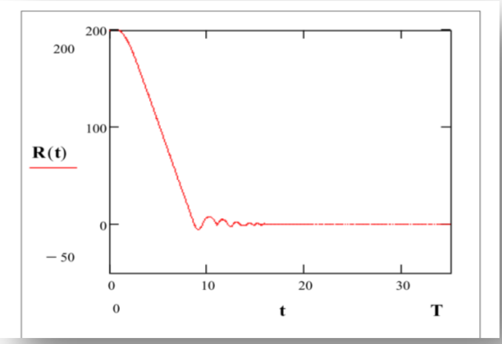

Figure 16.Target-to-missile range$\mathbf{R}(t)$.$\mathbf{R}(30) = 7.2 \times 10^{-3}m$.

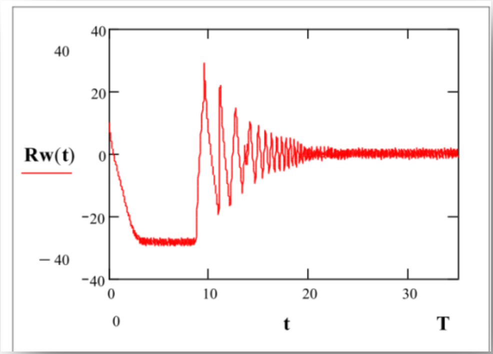

Figure 17. Speed of rapprochement missile-to-target:$\dot{R}(t)$.

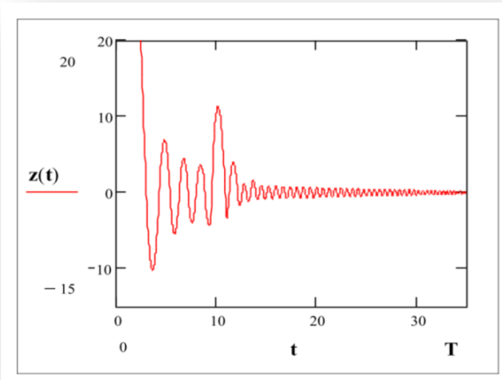

Figure18.Variable$\dot{z}(t) = R(t)\dot{\sigma}(t)$**.**

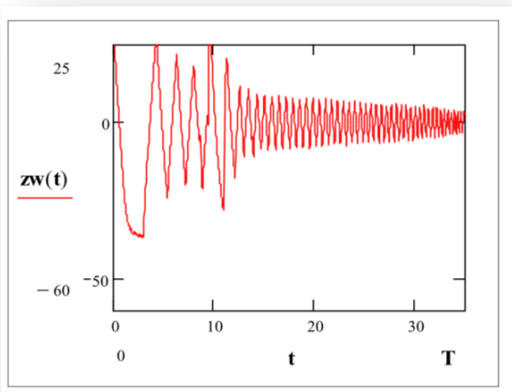

Figure 19.Variable$\ddot{z}(t)$. $\ddot{z}(T) = 2.172$.

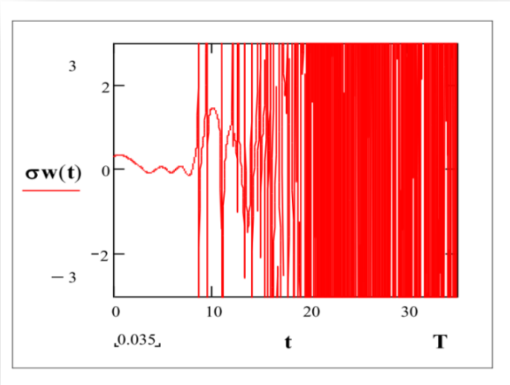

Figure 20. Variable$\dot{\sigma}(t)$. $\dot{\sigma}(0) = 0.3$.

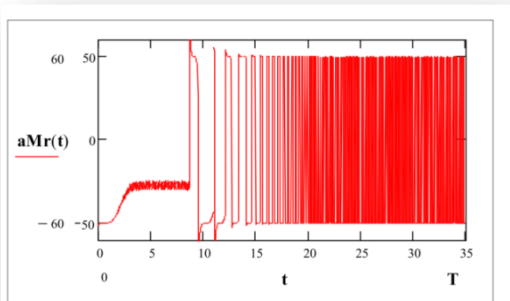

Figure 21. Missile acceleration along target-to-missile direction: $a_M^r(t)$**.**

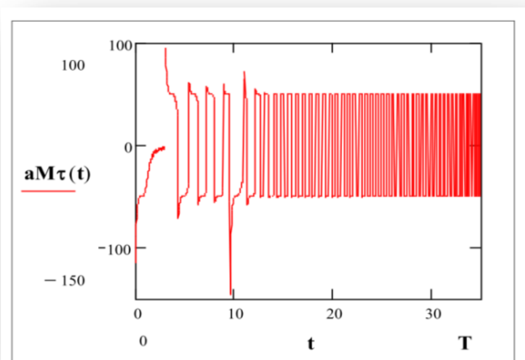

Figure22. Missile acceleration along direction which perpendicularly to

line-of-sight direction: $a_M^n(t)$.

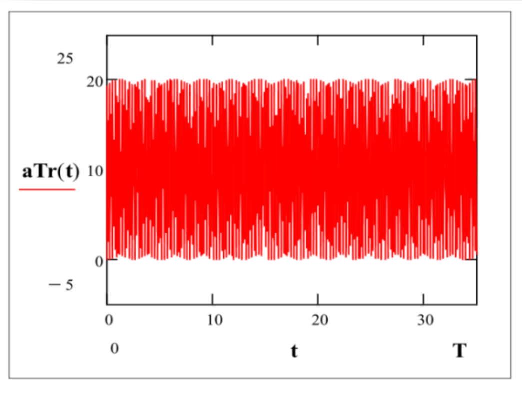

Figure 23. Target acceleration along target-to-missile direction: $a_T^r(t)$.

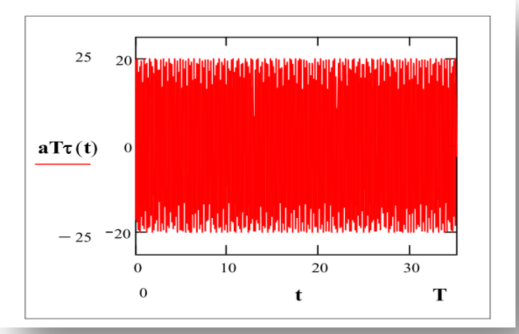

Figure 24. Target acceleration along direction which perpendicularly to line-of-sight direction: $a_T^n(t)$.

4.Jumps in Financial Markets

A classical model of financial market return process, such as the Black-Scholes [23], is the lognormal diffusion process, such that the log-return process has a normal distribution. However, real markets exhibit several deviations from this ideal, although useful, model. The market distribution, say for stocks, should have several realistic properties not found in the ideal log-normal model: (1) the model must permit

large random fluctuations such as crashes or sudden upsurges, (2) the log-return distribution should be skew since large downward outliers are larger than upward outliers, and (3)the distribution should be leptokurtic since the mode is usually higher and the tails thicker than for a normal distribution. For modeling these extra properties
phenomenologically, a jump-diffusion process with log-uniform jump-amplitude Poisson process it is usually applied. Let $S(t)$ be the price of a stock or stock fund satisfies a Markov, continuous-time, jump-diffusion stochastic differential equation

$$dS(t) = S(t)[\mu dt + \sigma dZ(t) + J(Q)dP(t)], S(0) = S_0, \quad (182)$$

Where $\mu$is the mean return rate, $\sigma$is the diffusive volatility, $Z(t)$
is a one-dimensional stochastic diffusion process, $J(Q)$ is a log-return mean $\mu_j$and varianceσ_j^2random jump-amplitude and $P(t)$is a simple Poisson jump process with jump rate λ. The processes $Z(t)$ and $P(t)$ are pairwise independent, while $J(Q)$ is also independent except that it is conditioned on the existence of a jump in $dP(t)$ it is conditioned on the existence of a jump in $dP(t)$.The numerical simulation a jumps in financial markets based on , jump-diffusion stochastic differential equation(182), was considered in many papers, see for example [23]-[25],
In contrast with a phenomenological approach we explain jumps phenomena in financial markets from the first principles, without any reference to Poisson jump process.
We claim that jumps phenomena in financial markets completely induced by nonlinearity and additive "small" white noise in corresponding Colombeau-Ito's stochastic equations.
Let$\mathfrak{C}_i = (\Omega_i, \mathbf{\Sigma}_i, \mathbf{P}_i), i = 1,2$ be a probability spaces such that: $\Omega_1 \cap \Omega_2 = \emptyset$.Let $\boldsymbol{W}(t,\omega)$be a Wiener process on $\mathfrak{C}_1$and let
$\boldsymbol{W}(t,\varpi)$be a Wiener process on $\mathfrak{C}_2$.Let us consider now a family $\left(\dot{\boldsymbol{x}}_{t,D,\varepsilon'}^{x_0,\varepsilon}(\omega,\varpi)\right)_{\varepsilon'}$ of the Colombeau generalized stochastic processes which is a solution of the Colombeau-Ito's

stochastic equation with stochastic coefficients:

$$\left(\dot{\boldsymbol{x}}^{x_0,\varepsilon}_{t,D,\varepsilon'}(\omega,\varpi)\right)_{\varepsilon'} = \left(\boldsymbol{b}_{\varepsilon'}\left(\varpi,\boldsymbol{x}^{x,\varepsilon}_{t,D,\varepsilon'}(\omega,\varpi),t\right)\right)_{\varepsilon'} +$$

$$+\sqrt{D}\boldsymbol{w}_{\varepsilon'}(t,\varpi) + \sqrt{\varepsilon}\boldsymbol{w}(t,\omega); \varepsilon,\varepsilon' \in (0,1], \omega \in \Omega_1, \varpi \in \Omega_2. \qquad (183)$$

Here $\boldsymbol{w}(t,\varpi)$ and $\boldsymbol{w}(t,\omega)$ is a white noise on $\mathbb{R}^n$ i.e.,

$$\boldsymbol{w}(t,\varpi) = \frac{d}{dt}\boldsymbol{W}(t,\varpi), \boldsymbol{w}(t,\omega) = \frac{d}{dt}\boldsymbol{W}(t,\omega) \qquad (184)$$

is almost surely in D', and $\boldsymbol{w}_{\varepsilon'}(t,\varpi)$ is the smoothed white noise on $\mathbb{R}^n$ i.e.,

$$\boldsymbol{w}_{\varepsilon'}(t,\varpi) = \langle \boldsymbol{w}(t,\varpi), \varphi_{\varepsilon'}(s-t)\rangle, \qquad (185)$$

where $\varphi_{\varepsilon'}$ is a model delta net [2], [4].

Definition 11. (I) CISDE (183) is $\widehat{\mathbb{R}}$-dissipative if there exist Lyapunov candidate function $\left(V_{\varepsilon'}(\varpi,\boldsymbol{x},t)\right)_{\varepsilon'}: \widehat{\mathbb{R}}^n \times [0,T] \to \widehat{\mathbb{R}}$ andpositiveinfiniteColombeau constants $\widetilde{C} = [(C_{\varepsilon'})_{\varepsilon'}] \in \widehat{\mathbb{R}}_+$, $\tilde{r} = [(r_{\varepsilon'})_{\varepsilon'}] \in \widehat{\mathbb{R}}_+$, such that:

$$(1) \forall \varepsilon' \in (0,1] : \mathbf{P}_2\left\{V_{*,\varepsilon'}(\varpi) = \infty\right\} = 1,$$

where $V_{*,\varepsilon'}(\varpi) = \lim_{R\to\infty}\left(\inf_{\|\mathbf{x}\|>R} V_{\varepsilon'}(\varpi,\boldsymbol{x},t)\right)$, and

(2) $\forall [(x_{\varepsilon'})_{\varepsilon'}]([(\|x_{\varepsilon'}\|)_{\varepsilon'}] \geq \tilde{r})$ theinequality

$$\left[\left(\dot{V}_{\varepsilon'}(\varpi,\boldsymbol{x}_{\varepsilon'},t;\boldsymbol{b}_{\varepsilon'})\right)_{\varepsilon'}\right] \leq \tilde{C}\left[\left(V_{\varepsilon'}(\varpi,\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'}\right]\mathbf{P}_2 - \text{o.s.} \qquad (186)$$

is satisfied. Here

$$\left[\left(\dot{V}_{\varepsilon'}(\varpi,\boldsymbol{x}_{\varepsilon'},t;\boldsymbol{b}_{\varepsilon'})\right)_{\varepsilon'}\right] \equiv$$

$$\left[\left(\frac{\partial V_{\varepsilon'}(\varpi,x_{\varepsilon'},t)}{\partial t}\right)_{\varepsilon'}\right] + \sum_{i=1}^{n}\left[\left(\frac{\partial V_{\varepsilon'}(\varpi,x_{\varepsilon'},t)}{\partial x_{i,\varepsilon'}} b_{i,\varepsilon'}(\varpi,\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'}\right]. \quad (187)$$

Or in the next equivalent form:

CISDE (183) is $\widehat{\mathbb{R}}$-dissipative if there existLyapunovcandidate function$\left(V_{\varepsilon'}(\varpi,\boldsymbol{x},t)\right)_{\varepsilon'}: \widehat{\mathbb{R}}^n \times [0,T] \to \widehat{\mathbb{R}}$and positive infinite Colombeau constants$\widetilde{C} = [(C_{\varepsilon'})_{\varepsilon'}] \in \widehat{\mathbb{R}}_+$,

$\tilde{r} = [(r_{\varepsilon'})_{\varepsilon'}] \in \widehat{\mathbb{R}}_+$, such that:

$$(1)\forall\varepsilon' \in (0,1] : \mathbf{P}_2\{V_{*,\varepsilon'}(\varpi) = \infty\} = 1,$$

where $V_{*,\varepsilon'}(\varpi) = \lim_{R\to\infty}\left(\inf_{\|\mathbf{x}\|>R} V_{\varepsilon'}(\varpi,\boldsymbol{x},t)\right)$, and

$(2')\forall\varepsilon' \in (0,1]\forall\boldsymbol{x}_{\varepsilon'}[(\boldsymbol{x}_{\varepsilon'} \in \mathbb{R}^n) \wedge (\|\boldsymbol{x}_{\varepsilon'}\| \geq r_{\varepsilon'})]$ the inequality

$$\left(\dot{V}_{\varepsilon'}(\varpi,\boldsymbol{x}_{\varepsilon'},t;\boldsymbol{b}_{\varepsilon'})\right)_{\varepsilon'} \leq ((C_{\varepsilon'})_{\varepsilon'})\left(V_{\varepsilon'}(\varpi,\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'}\mathbf{P}_2\text{- a.s.} \quad (188)$$

is satisfied. Here

$$\left(\dot{V}_{\varepsilon'}(\varpi,\boldsymbol{x}_{\varepsilon'},t;b_{\varepsilon'})\right)_{\varepsilon'} \equiv$$

$$\left(\frac{\partial V_{\varepsilon'}(\varpi,x_{\varepsilon'},t)}{\partial t}\right)_{\varepsilon'} + \left(\sum_{i=1}^{n}\frac{\partial V_{\varepsilon'}(\varpi,x_{\varepsilon'},t)}{\partial x_{i,\varepsilon'}} b_{i,\varepsilon'}(\varpi,\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'}. \quad (189)$$

(II) CISDE (183) is a strongly$\widehat{\mathbb{R}}$-dissipative if Lyapunov candidate function$\left(V_{\varepsilon'}(\varpi,\boldsymbol{x},t)\right)_{\varepsilon'}: \widehat{\mathbb{R}}^n \times [0,T] \to \widehat{\mathbb{R}}$, $\varepsilon' \in [0,1]$andpositive finite Colombeau constants $\widetilde{C} = [(C_{\varepsilon'})_{\varepsilon'}] \in \widehat{\mathbb{R}}_+, \tilde{r} = [(r_{\varepsilon'})_{\varepsilon'}] \in \widehat{\mathbb{R}}_+$,such that:

$(1)\forall\varepsilon' \in (0,1] : V_{*,\varepsilon'} = \lim_{r\to\infty}\left(\inf_{\|\mathbf{x}\|>r} V_{\varepsilon'}(\boldsymbol{x},t)\right) = \infty$,and$(2)\forall[(\boldsymbol{x}_{\varepsilon'})_{\varepsilon'}]([(\|\boldsymbol{x}_{\varepsilon'}\|)_{\varepsilon'}] \geq \tilde{r})$ the inequality

$\left[\left(\dot{V}_{\varepsilon'}(\varpi,\boldsymbol{x}_{\varepsilon'},t;\boldsymbol{b}_{\varepsilon'})\right)_{\varepsilon'}\right] \leq \tilde{C}\left[\left(V_{\varepsilon'}(\varpi,\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'}\right]\mathbf{P}_2$- a.s.

is satisfied. Here

$$\left[\left(\dot{V}_{\varepsilon'}(\varpi,\boldsymbol{x}_{\varepsilon'},t;\boldsymbol{b}_{\varepsilon'})\right)_{\varepsilon'}\right] \equiv$$

$$\left[\left(\frac{\partial V_{\varepsilon'}(\varpi,\boldsymbol{x}_{\varepsilon'},t)}{\partial t}\right)_{\varepsilon'}\right] + \sum_{i=1}^{n}\left[\left(\frac{\partial V_{\varepsilon'}(\varpi,\boldsymbol{x}_{\varepsilon'},t)}{\partial x_{i,\varepsilon'}} b_{i,\varepsilon'}(\varpi,\boldsymbol{x}_{\varepsilon'},t)\right)_{\varepsilon'}\right].$$

Let us consider now a family $\left(\boldsymbol{x}_{t,\epsilon,\varepsilon'}^{x_0,\varepsilon}(\varpi)\right)_{\varepsilon'}$ of the solutions of the Colombeau SDE:

$$\left(d\boldsymbol{x}_{t,\epsilon,,\varepsilon'}^{x_0,\varepsilon}(\omega,\varpi,\delta)\right)_{\varepsilon'} = \left(\boldsymbol{b}_{\boldsymbol{\varepsilon}',\boldsymbol{\epsilon}}\left(\boldsymbol{x}_{t,,\epsilon,\varepsilon'}^{x_0,\varepsilon}(\omega,\varpi,\delta),t,\varpi\right)\right)_{\varepsilon'} + \sqrt{D}d\boldsymbol{W}(t,\varpi) + \sqrt{\varepsilon}\boldsymbol{w}(t,\omega),$$

(190)

$$\left(\boldsymbol{x}_{0,\varepsilon'}^{x_0,\varepsilon}\right)_{\varepsilon'} = \boldsymbol{x}_0 \in \widehat{R}^{\,n}, t \in [0,T], \varepsilon, \varepsilon', \in (0,1], \epsilon \in (0,1].^{n}$$

Here $\boldsymbol{W}(t)$ is n-dimensional Brownian motion,

and$\forall\epsilon \in (0,1]^n, \forall t \in [0,T]$ and for almost all $\varpi \in \Omega_2$:$\left(\boldsymbol{b}_{\varepsilon',\epsilon}(x,t,\varpi)\right)_{\varepsilon'} \in G^n(\mathbb{R}^n)$, $\boldsymbol{b}_{0,0}(\cdot,t,\varpi) \equiv \boldsymbol{b}_{\varepsilon'=0,\epsilon=0}(\cdot,t,\varpi)\colon \mathbb{R}^n \to \mathbb{R}^n$is a polynomialvector-function on a variable $\boldsymbol{x} = (x_1,\ldots,x_n)$ i.e., $b_{i,0,0}(\boldsymbol{x},t,\varpi) = \sum_{\alpha,|\alpha|\leq r} b_{i,0,0}^{\alpha}(t,\varpi)x^{\alpha}, \alpha = (i_1,\ldots,i_n), |\alpha| = \sum_{j=1}^{n} i_j, 0 \leq i_j \leq p$, and

$$b_{i,\varepsilon',\epsilon}(\boldsymbol{x}(t),t,\varpi) = b_{i,0,0}\left(\boldsymbol{x}_{\varepsilon',\epsilon}(t,\varpi),t,\varpi\right).$$

Here$\boldsymbol{x}_{\varepsilon',\epsilon}(t,\omega) = \left(x_{1,\varepsilon',\varepsilon,\epsilon}(t,\varpi),\ldots,x_{n,\varepsilon',\varepsilon,\epsilon}(t,\varpi)\right),$

$$x_{\boldsymbol{i},\varepsilon',\epsilon}(t,\varpi) = \frac{x_i(t)}{1+\varepsilon' x_i^{2l}(t)+\varepsilon'\left[\epsilon_i\int_0^t \theta_{\epsilon_i}[x_i(\tau)]x_i^{2l}(\tau)d\tau+\sqrt{\delta}W_i(t,\varpi)\right]^2},$$

$i = 1,..,n.$ Now we let

$$u_i(t) = \epsilon_i \int_0^t \theta_{\epsilon_i}[x_i(\tau)]x_i^{2l}(\tau)d\tau + \sqrt{\delta}W_i(t,\varpi),$$

and rewrite Eq.(190) in the canonical Colombeau-Ito form:

$$\left(d\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega,\varpi,\delta)\right)_{\varepsilon'}$$

$$= \left(\boldsymbol{b}_{\varepsilon',\epsilon}\left(\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega,\varpi,\delta),\boldsymbol{u}_{t,,\varepsilon',,\epsilon}^{\delta}(\omega,\varpi),t\right)\right)_{\varepsilon'} + +\sqrt{D}\boldsymbol{w}_{\varepsilon'}(t,\varpi)$$

$$+\sqrt{\varepsilon}\boldsymbol{w}(t,\omega), \boldsymbol{u}_{t,,\varepsilon',\varepsilon,\epsilon}^{\delta}(\omega,\varpi) = \left(u_{1,t,,\varepsilon',\epsilon}^{\delta}(\omega,\varpi),\dots,u_{n,t,,\varepsilon',\epsilon}^{\delta}(\omega,\varpi)\right),$$

$$\left(du_{i,t,,\varepsilon',\epsilon}^{\delta}(\omega,\varpi)\right)_{\varepsilon'} =$$

$$\epsilon_i\left(\theta_{\epsilon_i}\left[x_{i,t,\varepsilon',\epsilon}^{x_0,\delta}(\omega,\varpi)\right]\left[x_{i,t,,\varepsilon',\epsilon}^{x_0,\delta}(\omega,\varpi)\right]^{2l}\right)_{\varepsilon'} +$$

$$\sqrt{\delta}dW_i(t,\varpi), \quad (191)$$

$$i = 1,\dots,n, \left(x_{0,\varepsilon'}^{x_0,\varepsilon}\right)_{\varepsilon'} = \boldsymbol{x}_0 \in \widehat{R}^n, t \in [0,T], \varepsilon,\varepsilon',\epsilon,\delta \in (0,1].$$

Theorem 9.(Strong large deviations principle SLDP)

We set now$\theta_{\boldsymbol{\epsilon_i}}[\boldsymbol{z}] \equiv 1, i = 1,\dots,n.$

(**I**)Assume that CISDE (191) is a strongly$\widehat{\mathbb{R}}$-dissipative. Thenfor any solution

$$\left(\boldsymbol{x}_{t,,\varepsilon',\boldsymbol{\epsilon}}^{x_0,\varepsilon}(\omega,\varpi,\delta)\right)_{\varepsilon'} = \left(x_{1,t,,\varepsilon',\boldsymbol{\epsilon}}^{x_0,\varepsilon}(\omega,\varpi,\delta),\dots,x_{n,t,,\varepsilon',\boldsymbol{\epsilon}}^{x_0,\varepsilon}(\omega,\varpi,\delta)\right)_{\varepsilon'}$$

of a strongly$\widehat{\mathbb{R}}$–dissipative CISDE(191) and any$\mathbb{R}$-valued parameters$\lambda_1,\dots,\lambda_n$, there exist finite Colombeau constant $\widetilde{C}' = \left[\left(C'_{\varepsilon'}\right)_{\varepsilon'}\right] > 0$, such that $\forall\boldsymbol{\lambda}[\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n)]$, the inequality

$$\underline{\lim}_{\substack{\varepsilon\to 0\\ \varepsilon'\to 0\\ \epsilon\to 0\\ \left(\frac{\varepsilon'}{\varepsilon}\right)\to 0}} \underline{\lim}_{\delta\to 0}\mathbf{E}_{\boldsymbol{\Omega_1}}\left[\left\|\boldsymbol{x}_{\boldsymbol{t},,\boldsymbol{\varepsilon}',\epsilon}^{\boldsymbol{x_0},\boldsymbol{\varepsilon}}(\omega,\varpi,\delta)-\boldsymbol{\lambda}\right\|^{\boldsymbol{2}}\right] \leq \tilde{C}'\|\boldsymbol{U}(\varpi,t,\boldsymbol{\lambda})\|^{\boldsymbol{2}}(192)$$

$\mathbf{P}_2$- a.s.is satisfied. Or in the next equivalent form: for a sufficiently small $\epsilon \approx 0$ and for a sufficiently small $\varepsilon \approx 0, \varepsilon' \approx 0$ such that $\frac{\varepsilon'}{\varepsilon} \approx 0$, the inequality

$$\left[\left(\underline{\lim}_{\delta\to 0}\mathbf{E}_{\mathbf{\Omega_1}}\left[\left\|\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\varpi,\delta)-\boldsymbol{\lambda}\right\|^2\right]\right)_{\varepsilon'}\right]\leq\tilde{C}'\|\boldsymbol{U}(\varpi,t,\boldsymbol{\lambda})\|^2$$

$\mathbf{P}_2 -$ a.s. is satisfied. $\mathbf{P}_2$.

Here the vector-functionis the solution of the differential master equation:

$\dot{\boldsymbol{U}}(\varpi,t,\boldsymbol{\lambda})=\mathbf{J}[\boldsymbol{b_0}(\boldsymbol{\lambda},t,\varpi)]\boldsymbol{U}(\varpi,t,\boldsymbol{\lambda})+\boldsymbol{b}_0(\boldsymbol{\lambda},t,\varpi)$, $\boldsymbol{U}(0,\boldsymbol{\lambda},\varpi)=\boldsymbol{x_0}-\boldsymbol{\lambda}$.

Here$\mathbf{J}=\mathbf{J}[\boldsymbol{b}_0(\boldsymbol{\lambda},t,\varpi)]$is a Jacobian i.e., $\mathbf{J}$ is $n\times n$-matrix:

$\mathbf{J}[\boldsymbol{b}_0(\boldsymbol{\lambda},t,\varpi)]=\mathbf{J}\left[\partial\boldsymbol{b}_{0,\boldsymbol{i}}(\boldsymbol{x},t,\varpi)/\partial x_{\boldsymbol{j}}\right]_{\boldsymbol{x}=\boldsymbol{\lambda}}$.

(**II**) Assume that CISDE (183) is a strongly$\widehat{\mathbb{R}}$-dissipative. Then for any solution$\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\varpi)\right)_{\varepsilon'}=\left(x^{x_0,\varepsilon}_{1,t,,\varepsilon',\epsilon}(\omega,\varpi),\ldots,x^{x_0,\varepsilon}_{n,t,,\varepsilon',\epsilon}(\omega,\varpi)\right)_{\varepsilon'}$ of a strongly$\widehat{\mathbb{R}}$–dissipative CISDE (185) and any$\mathbb{R}$-valued parameters$\lambda_1,\ldots,\lambda_n$, there exist finite Colombeau constant $\widetilde{C}'=\left[\left(C'_{\varepsilon'}\right)_{\varepsilon'}\right]>0$, such that $\forall\boldsymbol{\lambda}[\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)]$, the inequality

$$\left[\left(\mathbf{E}_{\mathbf{\Omega_1}}\left[\left\|\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\varpi)-\boldsymbol{\lambda}\right\|^2\right]\right)_{\varepsilon'}\right]\leq\tilde{C}'\|\boldsymbol{U}(\varpi,t,\boldsymbol{\lambda})\|^2.$$

$\mathbf{P}_2$- a.s. is satisfied.

(**III**) For any solution

$$\left(\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\varpi)\right)_{\varepsilon'}=\left(x^{x_0,\varepsilon}_{1,t,,\varepsilon',\epsilon},\ldots,x^{x_0,\varepsilon}_{n,t,,\varepsilon',\epsilon}\right)_{\varepsilon'}$$

of a strongly$\widehat{\mathbb{R}}$–dissipative CISDE(185) and any$\mathbb{R}$-valued parameters$\lambda_1,\ldots,\lambda_n$, there exist finite Colombeau constant $\widetilde{C}'=\left[\left(C'_{\varepsilon'}\right)_{\varepsilon'}\right]>0$, such that $\forall\boldsymbol{\lambda}[\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)]$the inequality

$$\underline{\lim}_{\varepsilon\to 0}\mathbf{E}_{\mathbf{\Omega_1}}\left[\left\|\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon'=0,\epsilon=0}(\omega,\varpi)-\boldsymbol{\lambda}\right\|^2\right]\leq\tilde{C}'\|\boldsymbol{U}(\varpi,t,\boldsymbol{\lambda})\|^2 \quad (193)$$

$\mathbf{P}_2$-a.s. is satisfied. Here the vector-function $\boldsymbol{U}(\varpi,t,\boldsymbol{\lambda})=(U_1(\varpi,t,\boldsymbol{\lambda}),\ldots,U_n(\varpi,t,\boldsymbol{\lambda}))$ is the solution of the differential master equation:

$\dot{\boldsymbol{U}}(\varpi,t,\boldsymbol{\lambda})$= $\mathbf{J}[\boldsymbol{b_0}(\boldsymbol{\lambda},t,\varpi)]\boldsymbol{U}(\varpi,t,\boldsymbol{\lambda})+\boldsymbol{b_0}(\boldsymbol{\lambda},t,\varpi)$, $\boldsymbol{U}(0,\boldsymbol{\lambda},\varpi)=\boldsymbol{x_0}-\boldsymbol{\lambda}$,

where$\mathbf{J}=\mathbf{J}[\boldsymbol{b_0}(\boldsymbol{\lambda},t,\varpi)]$is a Jacobian i.e., $\mathbf{J}$ is $n\times n$-matrix:

$\mathbf{J}[\boldsymbol{b_0}(\boldsymbol{\lambda},t,\varpi)]$= $\mathbf{J}\left[\partial \boldsymbol{b_{0,i}}(\boldsymbol{x},t,\varpi)/\partial x_{\boldsymbol{j}}\right]_{\boldsymbol{x}=\boldsymbol{\lambda}}$.

Proof (**I**) we let now $\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega,\varpi)-\boldsymbol{\lambda}=\boldsymbol{y}_{\boldsymbol{t,,\varepsilon',\epsilon}}^{\boldsymbol{x_0,\varepsilon}}(\omega,\varpi)$.

Replacement $\boldsymbol{x}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega,\varpi) = \boldsymbol{y}_{\boldsymbol{t,,\varepsilon',\epsilon}}^{\boldsymbol{x_0,\varepsilon}}(\omega,\varpi) + \boldsymbol{\lambda}$ into Eq.(191) gives

$$\begin{aligned}
&\left(d\boldsymbol{y}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega,\varpi,\delta)\right)_{\varepsilon'}\\
&\quad=\left(\boldsymbol{b}_{\varepsilon',\epsilon}\left(\boldsymbol{y}_{t,,\varepsilon',\epsilon}^{x_0,\varepsilon}(\omega,\varpi,\delta)+\boldsymbol{\lambda},\boldsymbol{u}_{t,,\varepsilon',\varepsilon,\epsilon}^{\delta}(\omega,\varpi),t,\varepsilon\right)\right)_{\varepsilon'}++\sqrt{\varepsilon}d\boldsymbol{W}(t,\omega),\boldsymbol{u}_{t,,\varepsilon',\varepsilon,\epsilon}^{\delta}(\omega)\\
&\quad=\left(u_{1,t,,\varepsilon',\varepsilon,\epsilon}^{\delta}(\omega,\varpi),\dots,u_{n,t,,\varepsilon',\varepsilon,\epsilon}^{\delta}(\omega,\varpi)\right),\\
&\left(du_{i,t,,\varepsilon',\varepsilon,\epsilon}^{\delta}(\omega,\varpi)\right)_{\varepsilon'}=\epsilon\left(\left[x_{i,t,,\varepsilon',,\varepsilon,\epsilon}^{x_0,\varepsilon}(\omega,\varpi,\delta)\right]^{2l}\right)_{\varepsilon'}+\sqrt{\delta}dW_i(\omega,t),\\
&\quad i=1,\dots,n,\left(x_{0,\varepsilon',\epsilon}^{x_0,\varepsilon}\right)_{\varepsilon'}=\boldsymbol{x}_0\in\widehat{R}^{\,n},t\in[0,T],\varepsilon,\varepsilon',\epsilon,\delta\in(0,1].
\end{aligned}$$

Thus we need to estimate the quantity

$$\underline{\lim}_{\substack{\varepsilon\to0\\ \varepsilon'\to0\\ \epsilon\to0\\ \left(\frac{\varepsilon'}{\varepsilon}\right)\to0}}\ \underline{\lim}_{\delta\to0}\mathbf{E}_{\boldsymbol{\Omega_1}}\left[\left\|\boldsymbol{y}_{\boldsymbol{t,,\varepsilon',\epsilon}}^{\boldsymbol{x_0,\varepsilon}}(\omega,\varpi,\delta)\right\|^{\boldsymbol{2}}\right]$$

$\omega\in\Omega_1,\varpi\in\Omega_2$.

Application of the Theorem B.4 (see Appendix B) to Eq.(191) gives the inequality (192) directly.

(**II**) From the equality

$$\mathbf{E}_{\boldsymbol{\Omega_1}}\left[\left\|\boldsymbol{x}_{\boldsymbol{t,,\varepsilon',\epsilon}}^{\boldsymbol{x_0,\varepsilon}}(\omega,\varpi)-\boldsymbol{\lambda}\right\|^{\boldsymbol{2}}\right]=\mathbf{E}_{\boldsymbol{\Omega_1}}\left[\left\|\left[\boldsymbol{x}_{\boldsymbol{t,,\varepsilon',\epsilon}}^{\boldsymbol{x_0,\varepsilon}}(\omega,\varpi)-\boldsymbol{x}_{\boldsymbol{t,,\varepsilon',\epsilon}}^{\boldsymbol{x_0,\varepsilon}}(\omega,\varpi,\delta)\right]+\left[\boldsymbol{x}_{\boldsymbol{t,,\varepsilon',\epsilon}}^{\boldsymbol{x_0,\varepsilon}}(\omega,\varpi,\delta)-\boldsymbol{\lambda}\right]\right\|^{\boldsymbol{2}}\right],$$

by using the triangle inequality, one obtain

$$\sqrt{\mathbf{E}_{\boldsymbol{\Omega_1}}\left[\left\|\boldsymbol{x}_{\boldsymbol{t,,\varepsilon',\epsilon}}^{\boldsymbol{x_0,\varepsilon}}(\omega,\varpi)-\boldsymbol{\lambda}\right\|^{\boldsymbol{2}}\right]}\le\sqrt{\mathbf{E}_{\boldsymbol{\Omega_1}}\left[\left\|\boldsymbol{x}_{\boldsymbol{t,,\varepsilon',\epsilon}}^{\boldsymbol{x_0,\varepsilon}}(\omega,\varpi)-\boldsymbol{x}_{\boldsymbol{t,,\varepsilon',\epsilon}}^{\boldsymbol{x_0,\varepsilon}}(\omega,\varpi,\delta)\right\|^{\boldsymbol{2}}\right]}+$$

$$+\sqrt{\mathbf{E}_{\Omega_1}\left[\left\|\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\delta)-\boldsymbol{\lambda}\right\|^2\right]}.$$

Therefore statement (**II**) immediately follows fromTheoremA1 (see appendix A), Proposition 2andTheorem5.

(III) From the equality

$$\mathbf{E}_{\Omega_1}\left[\left\|\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon'=0,\epsilon=0}(\omega,\varpi)-\boldsymbol{\lambda}\right\|^2\right]=$$

$$=\mathbf{E}_{\Omega_1}\left[\left\|\left[\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon'=0,\epsilon=0}(\omega,\varpi)-\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\varpi)\right]+\left[\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\varpi)-\boldsymbol{\lambda}\right]\right\|^2\right],$$

byusingthetriangle inequality, one obtain

$$\sqrt{\mathbf{E}_{\Omega_1}\left[\left\|\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon'=0,\epsilon=0}(\omega,\varpi)-\boldsymbol{\lambda}\right\|^2\right]}\leq\sqrt{\mathbf{E}_{\Omega_1}\left[\left\|\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon'=0,\epsilon=0}(\omega,\varpi)-\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\varpi)\right\|^2\right]}$$

$$+\sqrt{\mathbf{E}_{\Omega_1}\left[\left\|\boldsymbol{x}^{x_0,\varepsilon}_{t,,\varepsilon',\epsilon}(\omega,\varpi)-\boldsymbol{\lambda}\right\|^2\right]}.$$

Therefore statement (III) immediately follows fromTheoremA2 (see appendix A), Proposition1andstatement (II).

The stochastic dynamics (183) we take now in the following form

$$\left(\dot{x}^{x_0,\varepsilon}_{t,D,\varepsilon'}(\omega,\varpi)\right)_{\varepsilon'}=\left(F_{\varepsilon'}\left(x^{x,\varepsilon}_{t,D,\varepsilon'}(\omega,\varpi),t\right)\right)_{\varepsilon'}+$$

$$+\sqrt{D}w_{\varepsilon'}(t,\varpi)+\sqrt{\varepsilon}w(t,\omega);\varepsilon,\varepsilon'\in(0,1],\omega\in\Omega_1,\varpi\in\Omega_2.$$

The force field$F_{\varepsilon'}$ is assumed to derive from a metastable potential which undergoes an arbitrary periodic modulation in time with period τ i.e,$F_{\varepsilon'}[x,t+\tau]=F_{\varepsilon'}[x,t]$.

The random time-dependent force field$F_{\varepsilon'}(x,t)$ takes the following form

$$F_{\varepsilon'}[x,t]=-\dot{V}(x)+A\sin(\Omega\cdot t)+B\cos(\Theta\cdot t)+\sqrt{D}w_{\varepsilon'}(t,\varpi)+\sqrt{\varepsilon}w(t,\omega);\varepsilon,\varepsilon'\in(0,1],\omega\in\mathit{\Omega}_1,\varpi\in\mathit{\Omega}_2.$$

As an example we consider a force field with a double well potential V(x) as cartooned in Fig.25.

The stochastic dynamics (183) takes the following form:

$$\dot{x}_{t,D}^{x_0}(\omega,\varpi) = -a\left[x_{t,D}^{x_0}(\omega,\varpi)\right]^3 + bx_{t,D}^{x_0}(\omega,\varpi) + A\sin(\Omega\cdot t) + B\cos(\Theta\cdot t) + c + \sqrt{D}w_{\varepsilon'}(t,\varpi) +$$
$$+\sqrt{\varepsilon}w(t,\omega),\ \ x_{0,D}^{x_0}(\omega,\varpi) = x_0.(194)$$

Using Theorem 9.(III) one obtain the next differential linear master equation

$$\dot{U}(t,\lambda) = -(3a\lambda^2 - b)U(t,\lambda) + A\sin(\Omega\cdot t) + B\cos(\Theta\cdot t) -$$
$$-(a\lambda^3 - b\lambda) + c + \sqrt{D}\dot{W}(t,\varpi) + \sqrt{\varepsilon}\mathrm{w}(t,\omega), U(t,\lambda) = x_0 - \lambda.$$

Solving this differential linear master equation, we obtain the next transcendental master equation

$$\left(x_0 - \lambda(t)\right)\exp[-(3a\lambda^2 - b)t] - (a\lambda^3 - b\lambda - c)\times$$
$$\times\int_0^t d\tau\exp[(3a\lambda^2 - b)(t-\tau)] +$$
$$+\int_0^t d\tau[A\sin(\Omega\cdot t) + B\cos(\Theta\cdot t)]\exp[(3a\lambda^2 - b)(t-\tau)] +$$

$$+\sqrt{D}\int_0^t \exp[(3a\lambda^2 - b)(t-\tau)]d\tau W(\tau,\varpi).$$

Note that

$$\int_0^t \exp[(3a\lambda^2 - b)(t-\tau)]d\tau W(\tau,\varpi) = W(t,\varpi) -$$
$$-(3a\lambda^2 - b)\int_0^t W(\tau,\varpi)\exp[(3a\lambda^2 - b)(t-\tau)]d\tau.$$

Finally we obtain the next transcendental master equation

$$\left(x_0 - \lambda(t)\right)\exp[-(3a\lambda^2 - b)t] - (a\lambda^3 - b\lambda - c)\times\times\int_0^t d\tau\exp[(3a\lambda^2 - b)(t-\tau)] +$$
$$+\int_0^t d\tau[A\sin(\Omega\cdot t) + B\cos(\Theta\cdot t)]\exp[(3a\lambda^2 - b)(t-\tau)] + \sqrt{D}W(t,\varpi) - \sqrt{D}(3a\lambda^2 -$$

$-b)\int_0^t W(\tau,\varpi)\exp[(3a\lambda^2-b)(t-\tau)]d\tau.$ (195)

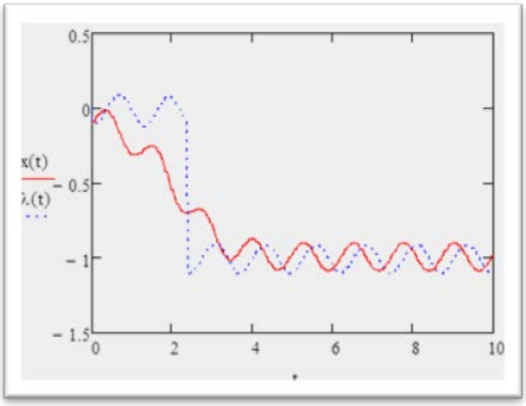

Figure 25. Comparison of: (1)dynamics(194) with $D=0$ $\varepsilon=0$(red curve) and (2) quasiclassical (blue curve) dynamics in the limit $\varepsilon\to 0$, calculated byusing SLDP.$a=1$, $b=1, c=0, A=0.5, B=0, \Omega=5, \Theta=0, D=0, x_0=-0.1$.

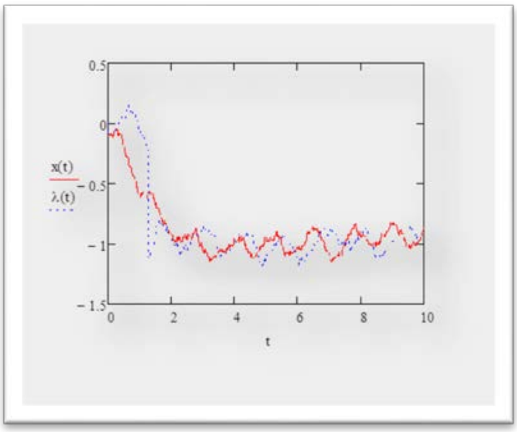

Figure 26. Comparison of: (1) stochastic dynamics (194) with $\varepsilon=0, D\neq 0$, (red curve) and (2) quasi-classical (blue curve) stochastic dynamics in the limit $\varepsilon\to 0$, calculated by using SLDP.$a=1, b=1, c=0, A=0.5, B=0, \Omega=5, \Theta=0, D=0.01, x_0=-0.1$.

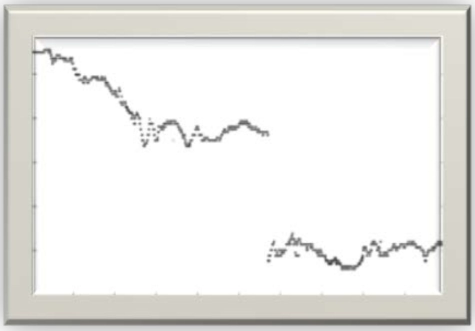

Figure27. Evolution of SLM (NYSE) February1993 [26].

Figure 27 shows the evolution of SLM over a one month period (February 1993) [26]. The price behavior over this period is clearly dominated by a large downward jump, which accounts for half of the monthly return. If we go down to an intraday scale shown in Figure 29, we see that the price moves essentially through jumps.

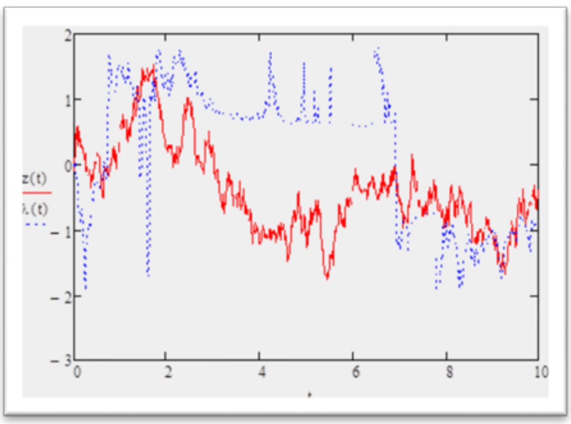

Figure 28. Comparison of: (1) stochastic dynamics (194) with $\varepsilon = 0, D \neq 0$ (red curve) and (2) quasi classical (blue curve) stochastic dynamics in the limit $\varepsilon \to 0$, calculated by using SLDP.$a = 1, b = 1, c = 0, A = 0.5, B = 0, \Omega = 5, \Theta = 0, D = 1, x_0 = -1$.

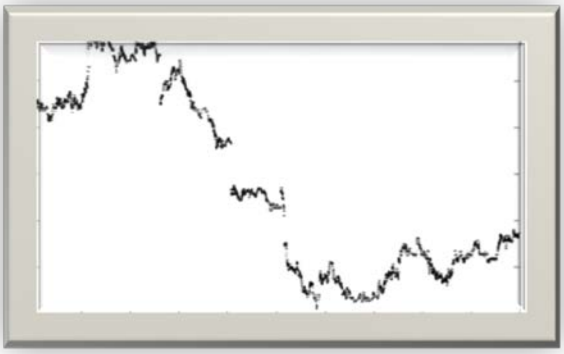

Figure 29. Evolution of SLM (NYSE) January-March 1993 [26].

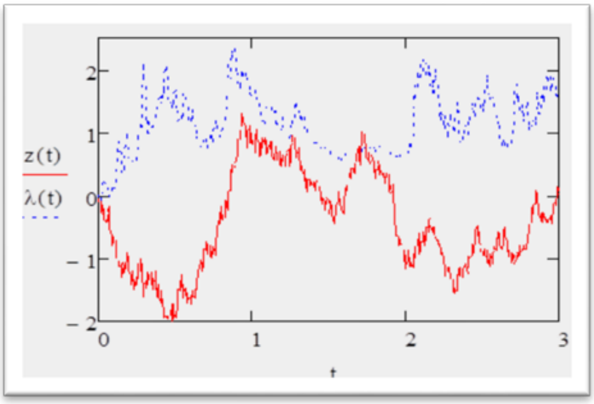

Figure 30. Comparison of stochastic dynamics (194) with $\varepsilon = 0, D \neq 0$ (red curve) and quasi-classical (blue curve) stochastic dynamics in the limit $\varepsilon \to 0$, calculated by using SLDP.$a = 1, b = 1, c = 0, A = 0.5, B = 0, \Omega = 5, \Theta = 0, D = 2, x_0 = -1$.

5. Comparison of the quasi classical stochastic dynamics obtained by using Saddle-point approximation with a non perturbative quasi classical stochastic dynamics obtained by using SLDP.

The double stochastic dynamics we take of the next form [7], [27]

$$\dot{x}(t) = F[x(t), t] + \sqrt{D}w(t, \omega) + \sqrt{\varepsilon}w(t, \varpi)$$

The force field $F[x(t),t]$ is assumed to derive from a metastable potential which undergoes an arbitrary periodic modulation in time with period τ i.e,$F[x,t+\tau] = F[x,t]$.An examples, is a static potential $V(x)$ supplemented by an additive sinusoidal and more general driving. The time-dependent force field $F[x,t]$ takesthe following form:

$$F[x,t] = -\dot{V}(x) + A\sin(\Omega \cdot t) + B\cos(\Theta \cdot t)$$

We have compared now by quantity

$$\delta(\varpi,t) = \left\| x_{t,D}^{x_0}(\varpi) - \lambda(\varpi,t) \right\| \qquad (196)$$

the above analytical predictions for the ε-limit (193)given by *master equation* (195) with very accurate numerical results for stochastic dynamics given by Ito's equation

$$x_{t,D}^{x_0}(\varpi) = x_0 + \int_0^t F\left(x_{t,D}^{x_0}(\varpi), s\right) ds + \sqrt{D}\mathrm{W}(t,\varpi) \qquad (197)$$

And we have compared now by quantity

$$\sigma(\varpi,t) = \left\| x_{t,D}^{x_0}(\varpi) - E_p(\varpi,t) \right\| \qquad (198)$$

the quasi classical analytical predictions for the ε-limit

$$\underline{\lim}_{\varepsilon \to 0} \mathbf{E}_{\Omega_1}\left[\left\| x_{t,,\varepsilon'=0,\epsilon=0}^{x_0,\varepsilon}(\omega,\varpi) \right\|^2 \right], \qquad (199)$$

given by saddle-point approximation [7],[27],denoted by $E_p(\varpi,t)$,withvery accurate numerical results for stochastic dynamics given by Ito's equation (197).

Example 5. Double well potential. As example we consider the force field with a double well potential *V*(x) as cartooned in Fig.31.

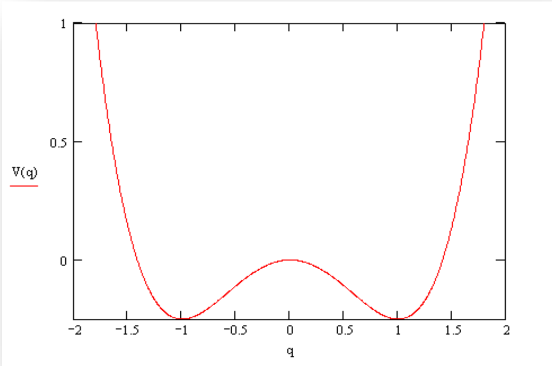

Figure 31. Double well potential $a = 1, b = 1, c = 0$.

$$V(x) = -\frac{a}{4}x^4 - \frac{b}{2}x^2 - cx, a > 0, b > 0.$$

The time-dependent force field (31) takes the following form:

$$F[x,t] = -ax^3 + bx + c + A\sin(\Omega \cdot t) + B\cos(\Theta \cdot t)$$

The stochastic dynamics (197) takes the following form:

$$\dot{x}_{t,D}^{x_0}(\omega) = -a\left[x_{t,D}^{x_0}(\omega)\right]^3 + bx_{t,D}^{x_0}(\omega) + c +$$

$$+A\sin(\Omega \cdot t) + B\cos(\varTheta \cdot t), x_{0,D}^{x_0}(\omega) = x_0.$$

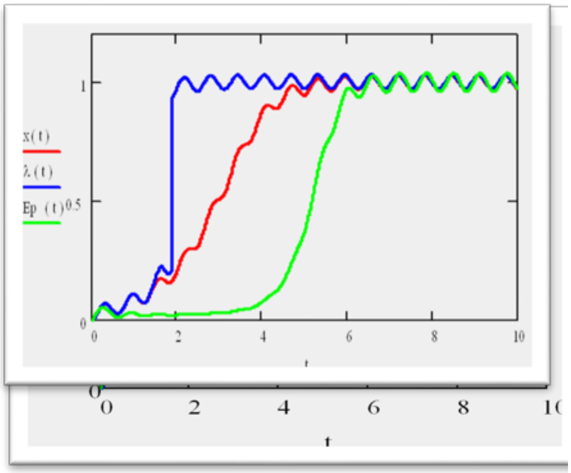

Figure 32. Comparison of: (1) classical dynamics (194) with $D = 0, \varepsilon = 0$(red curve), (2) corresponding quasi-classical dynamics with $D = 0, \varepsilon \neq 0$in the limit$\varepsilon \to 0$, calculated using SLDP(blue curve) and (3)quasi-classical dynamics in the limit $\varepsilon \to 0$, calculatedusingsaddle-point approximation[7],[22] (green curve).$a = 1, b = 1, c = 0, A = 1, B = 0, \Omega = 5, \Theta = 0, D = 0, x_0 = 0$.

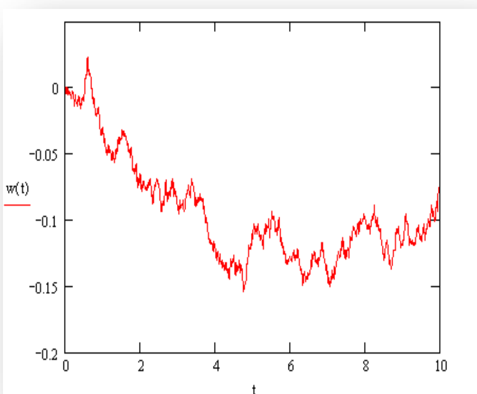

Figure 33. The realization of a Wiener process$w(t) = \sqrt{D}W(t)$, where $W(t)$is standard Wiener process, $D = 10^{-3}$.

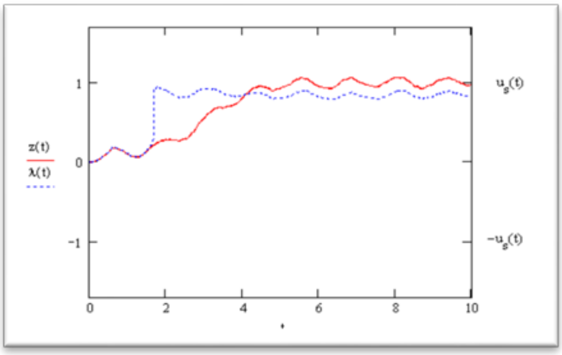

Figure34. Comparison of the quasi classical stochastic dynamics (red curve) obtained by using Saddle-point approximation [7],[22] and quasi classical stochastic dynamics (blue curve) obtained by using **SLDP**: a =1, b=1, c =0, A =0.3, B =0, $\Omega = 5$, $\Theta = 0$, D=10^{-3}, x_0 =0.

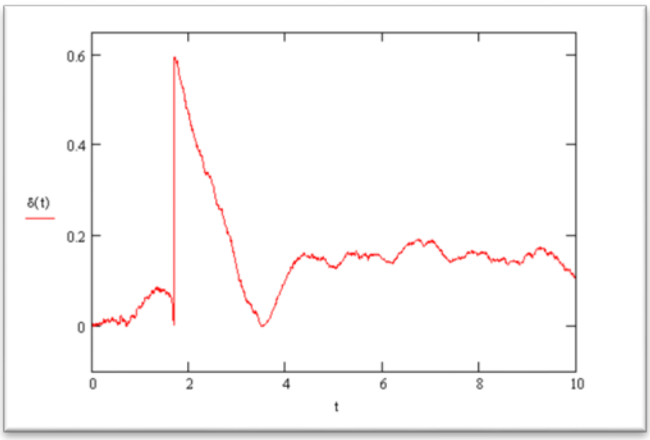

Figure 35. $\delta(\omega,t)$. Comparison of the quasi classical stochastic dynamics, obtained by using Saddle-point approximation [7] and quasi classical stochastic dynamic sobtained by using **SLDP**. a=1, b=1, $c = 0$, A =0.3, B =0, $\Omega = 5$, $\Theta = 0$, D =10^{-3}, $x_0 = 0$.

6. Strong large deviations principles of Non-Freidlin-Wentzell Type.

Definition 12. [4] Let $\mathfrak{C} = (\Omega, \Sigma. P)$ be a probability space. Let $\mathcal{E}R$ be the space of nets$\left(X_{\varepsilon'}(\omega)\right)_{\varepsilon'}$,of measurable functions on Ω. Let $\mathcal{E}R_M$ be the space of nets $(X_{\varepsilon'})_{\varepsilon'} \in \mathcal{E}R, \varepsilon' \in (0,1]$,with the property that for almost all $\omega \in \Omega$ there exist constants$r, C > 0$

and$\varepsilon_0 \in (0,1]$ such that$|(X_{\varepsilon'})_{\varepsilon'}| \leq C(\varepsilon')^{-r}, \varepsilon' \leq \varepsilon_0$.

Definition 13. Letξ be a distribution$\xi \in D'$. Distribution ξ is the generalized probability density of net $(X_{\varepsilon'})_{\varepsilon'} \in \mathcal{E}R, \varepsilon' \in (0,1]$ iff$\forall f \in D$: $(\mathbf{E}[X_{\varepsilon'}(\omega)])_{\varepsilon'} = \xi(f)$.

Let us consider now Colombeau-Ito's SDE:

$$\left(d\boldsymbol{x}_{t'',,\varepsilon'}^{x_0,\varepsilon}(\omega)\right)_{\varepsilon'} = \left(\boldsymbol{b}_{\varepsilon'}\left(\boldsymbol{x}_{t'',,\varepsilon'}^{x_0,\varepsilon}(\omega), t''\right)\right)_{\varepsilon'} + \sqrt{\varepsilon}d\boldsymbol{W}(t'',\omega) \quad (200)$$

$$\left(\mathbf{E}\left[f\left(\boldsymbol{x}_{t',\varepsilon'}^{x_0,\varepsilon}(\omega)\right)\right]\right)_{\varepsilon'} = f(\boldsymbol{x}_0), \boldsymbol{x}_0 = \boldsymbol{q}' \in \mathbb{R}^n, t'' \in [t', T], \quad (201)$$

$$\varepsilon, \varepsilon' \in (0,1], f \in C_0^\infty(\mathbb{R}^n), \boldsymbol{x}_0 \in \mathrm{supp}(f).$$

Assumption1. We assume now that there exist Colombeau constants $(C_{\varepsilon'})_{\varepsilon'}$ and $(D_{\varepsilon'})_{\varepsilon'}$ such that

(1) $(\|\boldsymbol{b}_{\varepsilon'}(\boldsymbol{x},t)\|)_{\varepsilon'} \leq ((C_{\varepsilon'})_{\varepsilon'})(1+\|\boldsymbol{x}\|)$, (202)

(2) $(\|\boldsymbol{b}_{\varepsilon'}(\boldsymbol{x},t) - \boldsymbol{b}_{\varepsilon'}(\boldsymbol{y},t)\|)_{\varepsilon'} \leq ((D_{\varepsilon'})_{\varepsilon'})\|\boldsymbol{x}-\boldsymbol{y}\|$

for all $t \in [0,\infty)$and all $\boldsymbol{x}$ and $\boldsymbol{y} \in \mathbb{R}^n$.

Here $\boldsymbol{W}(t)$ is n-dimensional Brownian motion, and $\forall t \in [0,T]$:$\left(\boldsymbol{b}_{\varepsilon'}(\boldsymbol{x},t)\right)_{\varepsilon'} \in G^n(\mathbb{R}^n)$, $\boldsymbol{b}_0(\cdot,t) \equiv \boldsymbol{b}_{\varepsilon'=0}(\cdot,t): \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial on variable $x = (x_1,\dots,x_n)$ i.e., $b_{0,i}(x,t) = \sum_{\alpha,|\alpha|\leq r} b_{0,i}^{\alpha}(t)x^\alpha, \alpha = (i_1,\dots,i_n), |\alpha| = \sum_{j=1}^n i_j, 0 \leq i_j \leq p$.

Assumption2. We assume now without loss of generality that $\boldsymbol{x}_0 \neq 0$. Colombeau-Ito's SDE (200)-(201) is well-known to be equivalent to the Colombeau-Fokker-Planck equation

$$\left(\frac{\partial p_{\varepsilon'}^{\varepsilon}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'')}{\partial t''}\right)_{\boldsymbol{\varepsilon'}} = \varepsilon \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \left(\frac{\boldsymbol{\partial^2} p_{\boldsymbol{\varepsilon'}}^{\boldsymbol{\varepsilon}}(\boldsymbol{q}',\boldsymbol{t}'|\boldsymbol{q}'',\boldsymbol{t}'')}{\partial q_i''\partial q_i''}\right)_{\boldsymbol{\varepsilon'}} -$$

$$-\sum_{i=1}^{n}\left(\frac{\partial}{\partial q_i''}\left(b_{i,\varepsilon'}(\boldsymbol{q}'',t'')p_{\varepsilon'}^{\varepsilon}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'')\right)\right)_{\boldsymbol{\varepsilon}'}. \quad (203)$$

With initial condition of the form

$$\left(p_{\varepsilon'}^{\varepsilon}(\boldsymbol{q}',t'|\boldsymbol{q}'',t')\right)_{\varepsilon'}=\left(p_{\varepsilon'}(\boldsymbol{q}''-\boldsymbol{q}')\right)_{\varepsilon'}=\delta(\boldsymbol{q}''-\boldsymbol{q}'). \quad (204)$$

Colombeau PDE (203)-(204) can be solved *formally* in terms of Feynman path integral of the form [27]-[28]:

$$\left(p_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon}}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'')\right)_{\boldsymbol{\varepsilon}'}=\lim_{\Delta t\to 0}\boldsymbol{I}_N(\boldsymbol{q}',t'|\boldsymbol{q}'',t''). \quad (205)$$

Here

$$\boldsymbol{I}_N(\boldsymbol{q}',t'|\boldsymbol{q}'',t'')=$$

$$\breve{\boldsymbol{N}}_N\int_{-\infty}^{\infty}\boldsymbol{dq}_0\int_{-\infty}^{\infty}d\boldsymbol{q}_1\ldots\int_{-\infty}^{\infty}d\boldsymbol{q}_m\ldots\int_{-\infty}^{\infty}d\boldsymbol{q}_{N-1}\times\left(\boldsymbol{p}_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}_0-\boldsymbol{q}')\right)_{\boldsymbol{\varepsilon}'}\exp\left[-\frac{1}{2\varepsilon}\left(\boldsymbol{S}_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}_0,\boldsymbol{q}_1,\ldots,\boldsymbol{q}_{N-1},\boldsymbol{q}_N,\varepsilon)\right)_{\boldsymbol{\varepsilon}'}\right] \quad (206)$$

$\boldsymbol{q}_N=\boldsymbol{q}'', d\boldsymbol{q}_m=\prod_{j=1}^{n}dq_{j,m}, m=0,\ldots,N, \Delta t=(t''-t')/N, t_m=m\Delta t,$

$$\left(\mathbf{S}_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}_0,\boldsymbol{q}_1,\ldots,\boldsymbol{q}_{N-1},\boldsymbol{q}_N,\varepsilon)\right)_{\boldsymbol{\varepsilon}'}=$$

$$=\Delta t\sum_{m=1}^{m=N}\left(\mathcal{L}_{\boldsymbol{\varepsilon}'}\left(\frac{\boldsymbol{q}_m-\boldsymbol{q}_{m-1}}{\Delta t},\frac{\boldsymbol{q}_m-\boldsymbol{q}_{m-1}}{2},t_m\right)\right)_{\boldsymbol{\varepsilon}'} \quad (207)$$

and

$$\mathcal{L}_{\boldsymbol{\varepsilon}'}\left(\frac{\boldsymbol{q}_m-\boldsymbol{q}_{m-1}}{\Delta t},\frac{\boldsymbol{q}_m-\boldsymbol{q}_{m-1}}{2},t_m\right)=\left\|\frac{\boldsymbol{q}_m-\boldsymbol{q}_{m-1}}{\Delta t}+\boldsymbol{b}_{\varepsilon'}\left(\frac{\boldsymbol{q}_m+\boldsymbol{q}_{m-1}}{2},t_m\right)\right\|^2-$$

$$-\varepsilon\sum_{i=1}^{n}b_{i,i,\varepsilon'}\left(\frac{\boldsymbol{q}_m+\boldsymbol{q}_{m-1}}{2},t_m\right), \quad (208)$$

$$b_{i,i,\varepsilon'}(\boldsymbol{q},t)=\frac{\partial b_{i,\varepsilon'}(\boldsymbol{q},t)}{\partial q_i};\;\; \varepsilon,\varepsilon'\in(0,1].$$

Here $\left(p_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon}}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'')\right)_{\boldsymbol{\varepsilon}'}$ is the generalized probability density that the system (200) will end up at q'' at time t'' if it started at q' at time t' and here $\breve{\boldsymbol{N}}_N$ is the usual overall normalization of the path integral

$$\breve{\boldsymbol{N}}_N = (2\pi\varepsilon\Delta t)^{-nN/2}. \quad (209)$$

Remark1. Note that Colombeau-Fokker-Planck equation (203)- (204) is just a Euclidean Colombeau-Schrodinger equation, and is well-known that one can transform Colombeau-Schrodinger equation

$$\left(\frac{\partial u_{\varepsilon'}}{\partial t}\right)_{\varepsilon'} = \frac{i\varepsilon}{2}(\Delta u_{\varepsilon'})_{\varepsilon'} -$$

$$-i(V_{\varepsilon'}u_{\varepsilon'})_{\varepsilon'}, \left(u_{\varepsilon'}(0)\right)_{\varepsilon'} = (\varphi_{\varepsilon'})_{\varepsilon'}, \varepsilon' \in (0,1] \quad (210)$$

into mathematically rigorous path integral by standard method using Trotter's Product Formula [29].Here Δ is the Laplace operator $\partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$, $V_{\varepsilon'}$ is a realmeasurable function on $\mathbb{R}^n$, $\varphi_{\varepsilon'}$ and each $u_{\varepsilon'}$ areelements of $L_2(\mathbb{R}^n)$ and ε is a constant. Let $\mathcal{F}$ denote the Fourier transformation, $\mathcal{F}^{-1}$ its inverse. We define now as usual [29]:

$$(\Delta\varphi_{\varepsilon'})_{\varepsilon'} = \left(\mathcal{F}^{-1}\left[(-\|\lambda\|^2)\mathcal{F}[\varphi_{\varepsilon'}]\right]\right)_{\varepsilon'}, \ \varepsilon' \in (0,1] \quad (211)$$

on the domain $D(\Delta)$ of all square-integrable $(\varphi_{\varepsilon'})_{\varepsilon'}$ such that $\left(\mathcal{F}^{-1}\left[(-\|\lambda\|^2)\mathcal{F}[\varphi_{\varepsilon'}]\right]\right)_{\varepsilon'}$ is also square-integrable.

(Here λ denotes the variable in momentum space and $\|\lambda\|^2 = \lambda_1^2 + \cdots + \lambda_n^2$.). Then Δ is self-adjoint, and

$$\left(u_{\varepsilon'}(t)\right)_{\varepsilon'} = (K_\varepsilon^t\varphi_{\varepsilon'})_{\varepsilon'}, K_\varepsilon^t = \exp\left[\frac{it\varepsilon}{2}\Delta\right] \quad (212)$$

is the solution of the Eq.(210) for $(V_{\varepsilon'})_{\varepsilon'} = 0$. The operator $V_{\varepsilon'}$ of multiplication by the function $V_{\varepsilon'}$, on the domain $D(V_{\varepsilon'})$ of all$\varphi_{\varepsilon'}$ in $L_2(\mathbb{R}^n)$suchthat$V_{\varepsilon'}\varphi_{\varepsilon'}$is also in$L_2(\mathbb{R}^n)$, is self-adjoint, and

$$\left(u_{\varepsilon'}(t)\right)_{\varepsilon'} = \left(M^t_{V_{\varepsilon'}}\varphi_{\varepsilon'}\right)_{\varepsilon'}, M^t_{V_{\varepsilon'}} = \exp[-itV_{\varepsilon'}] \qquad (213)$$

is the solution of the Eq.(210) with $\varepsilon = 0$. Kato has found conditions under which the operator$\mathcal{R}_{\varepsilon,\varepsilon'}$ is self-adjoint [29]

$$\left(\mathcal{R}_{\varepsilon,\varepsilon'}u_{\varepsilon'}\right)_{\varepsilon'} = \frac{i\varepsilon}{2}(\Delta u_{\varepsilon'})_{\varepsilon'} - i(V_{\varepsilon'}u_{\varepsilon'})_{\varepsilon'}, \qquad (214)$$

Ander these conditions if we let

$$U^t_{\varepsilon,V_{\varepsilon'}} = \exp\left[t\mathcal{R}_{\varepsilon,\varepsilon'}\right]. \qquad (215)$$

Then a theorem of Trotter [29] asserts that for all $\varphi_{\varepsilon'}$ in $L_2(\mathbb{R}^n)$

$$\left(U^t_{\varepsilon,V_{\varepsilon'}}\varphi_{\varepsilon'}\right)_{\varepsilon'} = \left(\lim_{N\to\infty}\left[\left(K^{\frac{t}{N}}_{\varepsilon}M^{\frac{t}{N}}_{V_{\varepsilon'}}\right)^N \varphi_{\varepsilon'}\right]\right)_{\varepsilon'} \qquad (216)$$

This is discussed in detail in [29] (See [29] Appendix B). Using now Eq.(211)-Eq. (215) by simple calculation one obtain [29]

$$\left(\left(K^{\frac{t}{N}}_{\tau}M^{\frac{t}{N}}_{V_{\varepsilon'}}\right)^N \varphi_{\varepsilon'}(x)\right)_{\varepsilon'} = \left(\frac{2\pi it\varepsilon}{N}\right)^{-\frac{1}{2}nN} \times \qquad (217)$$

$$\times\left(\int \dots \int d^n\boldsymbol{x}_0 \dots d^n x_{N-1}\exp[iS_{\varepsilon'}(\boldsymbol{x}_0, \dots \boldsymbol{x}_N; t)]\right)_{\varepsilon'}.$$

Here$S_{\varepsilon'}(x_0, \dots x_n; t) = \sum_{i=1}^N \left[\frac{1}{2\varepsilon}\frac{\|x_i - x_{i-1}\|^2}{(t/N)^2} - V_{\varepsilon'}(x_i)\right]\frac{t}{N}$, where we set$x_N = x$.

Theorem 10.(**Suzuki-Trotter Formula**)[31]-[33]. Let$\{A_j\}_{j=1}^{p}$ be an family of any bounded operator in an Banach algebra $\mathfrak{C}$ with a norm$\|\circ\|_{\mathfrak{C}}$. Let $\Phi_n(\{A_j\})$be a function

$$\Phi_n(\{A_j\}) = \left(\exp\left(\frac{A_1}{n}\right) \dots \exp\left(\frac{A_p}{n}\right)\right)^n. \quad (218)$$

For any bounded operators$\{A_j\}_{j=1}^{p}$in a Banach algebra $\mathfrak{C}$:

$$\lim_{n\to\infty} \left\|\Phi_n(\{A_j\}) - \exp\left(\sum_{j=1}^{p} A_j\right)\right\|_{\mathfrak{C}} = 0. \quad (219)$$

Remark2. Note that one can to transform Colombeau- Fokker- Planck equation (203)-(204)into mathematically rigorous path integral by standard method using Suzuki -Trotter's Product formula(219) (see, e.g., Ref.[33]). However path integral representation of the solutions of the Colombeau-Fokker- Planck equation (203)-(204), given by canonical Eq.(205)-Eq.(209), does not valid under canonical assumptions which is discussed above in Remark1.

Remark3. Note that formal Colombeau pseudo-differential operator [30]-[35], given by formula (see also [36]-[39])

$$\left(P_{\varepsilon'}^{t}\right)_{\varepsilon'} = \left(\exp\left(\sum_{i=1}^{i=n}\int_0^t b_{i,\varepsilon'}\left(\overset{2}{\widetilde{\boldsymbol{x}}},\tau\right) d\tau \overbrace{\frac{\partial}{\partial x_i}}^{1}\right)\right)_{\varepsilon'}, \quad (220)$$

evidently does not define any contraction Colombeau semi- group on $L_2(\mathbb{R}^n)$.Evenin the case when a functions $\left(b_{i,\varepsilon'}(x,t)\right)_{\varepsilon'}, i = 1, \dots, n$is the Colombeau constants$\left(b_{i,\varepsilon'}\right)_{\varepsilon'} \in \widetilde{\mathbb{R}}, i = 1, \dots, n$ formal Colombeau pseudo-differential operator (see[30]-[35]), given by formula

$$\left(P_{\varepsilon'}^{t}\right)_{\varepsilon'} = \left(\exp\left(t\sum_{i=1}^{i=n} b_{i,\varepsilon'}\frac{\partial}{\partial x_i}\right)\right)_{\varepsilon'}, \quad (221)$$

does not define any contraction Colombeau semi- group on $L_2(\mathbb{R}^n)$. Nevertheless formal Colombeau pseudo- differential operator (221) define an contraction Colombeau semi-group: (1) on a test space $H^{\infty}(S_R), R = (R_1 \dots R_n)$, with a members $\varphi(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$such that $\mathcal{F}[\varphi](\boldsymbol{\xi})$is supported inside region $S_R = \{\boldsymbol{\xi}||\xi_i| < R_i; i = 1, \dots, n\}$[40] and (2) on a corresponding dual space $H^{-\infty}(S_R)$. Pseudo-differential calculus on a test space $H^{\infty}(S_R)$ and on dual space $H^{-\infty}(S_R)$is discussed in detail in [40].

Theorem11. [40]. Let

$$D = (D_1, \dots, D_n), D_i = \frac{\partial}{\partial x_i} \triangleq \partial x_i \quad (222)$$

and

$$A(D) = \sum_{|\alpha|=0}^{M} a_\alpha D^\alpha, a_\alpha \in \mathbb{C}, M \leq \infty.$$

Then (1) $A(D)\varphi(\boldsymbol{x}) \in H^{\infty}(S_R)$ if $\varphi(\boldsymbol{x}) \in H^{\infty}(S_R)$,

(3) $A(D)'\Upsilon(\boldsymbol{x}) \in H^{-\infty}(S_R)$ if $'\Upsilon(\boldsymbol{x}) \in H^{-\infty}(S_R)$,

(4) Operator $A(D)$ is bounded on $H^{\infty}(S_R)$,

(5) Operator $A(D)$ is bounded on $H^{-\infty}(S_R)$.

Remark4.(I) From Theorem 10-11 one obtain, that: (1)operator$\sum_{i=1}^{i=n} b_{i,\varepsilon'}\,\partial q_i$, $\varepsilon \in \langle 0, 1]$ is bounded on $H^{\infty}(S_R)$,

(2)The generalized function$\left(u_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}'', t'')\right)_{\boldsymbol{\varepsilon}'}$ given by formula

$$\left(u_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}'', t'')\right)_{\boldsymbol{\varepsilon}'} = \left(p_{\boldsymbol{\varepsilon}'}^{\varepsilon}(\boldsymbol{q}', t'|\boldsymbol{q}'', t'')\right)_{\boldsymbol{\varepsilon}'} = \left(P_{\boldsymbol{\varepsilon}'}^{t''-t'}\varphi(\boldsymbol{q}'' - \boldsymbol{q}')\right)_{\boldsymbol{\varepsilon}'}$$

$$= \left(\exp\left(t''\sum_{i=1}^{i=n} b_{i,\varepsilon'}\,\partial q_i''\right)\varphi(\boldsymbol{q}'' - \boldsymbol{q}')\right)_{\boldsymbol{\varepsilon}'} = \quad (223)$$

$$= \left(\mathcal{F}^{-1}\left[\exp\left(-[t'' - t']\sum_{\boldsymbol{i=1}}^{\boldsymbol{i=n}} b_{\boldsymbol{i},\boldsymbol{\varepsilon}'}(i\boldsymbol{\xi}_{\boldsymbol{i}})\right)\boldsymbol{\mathcal{F}}[\varphi(\boldsymbol{q}'' - \boldsymbol{q}')](\boldsymbol{\xi})\right]\right)_{\boldsymbol{\varepsilon}'} =$$

$=\frac{\mathbf{1}}{(\boldsymbol{2\pi})^{\boldsymbol{n}}}\left(\int_{-\infty}^{\infty} \boldsymbol{d}^{\boldsymbol{n}}\boldsymbol{\xi} \exp\left(i\langle\boldsymbol{q}'', \boldsymbol{\xi}\rangle - (t'' - t')\sum_{\boldsymbol{i=1}}^{\boldsymbol{i=n}} b_{\boldsymbol{i},\boldsymbol{\varepsilon}'}(i\boldsymbol{\xi}_{\boldsymbol{i}})\right)\boldsymbol{\mathcal{F}}[\varphi(\boldsymbol{q}'' - \boldsymbol{q}')](\boldsymbol{\xi})\right)_{\boldsymbol{\varepsilon}'}$ is the solution of the Colombeau-Fokker-Planck equation (203))-(204)with initial condition $\varphi(\boldsymbol{q}'' - \boldsymbol{q}') \in$

$H^{\infty}(S_R)$, for the case: $\varepsilon = 0$ and $\left(b_{i,\varepsilon'}(\boldsymbol{q}'',t)\right)_{\varepsilon'} \equiv \left(b_{i,\varepsilon'}\right)_{\varepsilon'} i = 1,\dots,n$,

$\left(b_{i,\varepsilon'}\right)_{\varepsilon'} \in \widetilde{\mathbb{R}}, i = 1,\dots,n$ is the Colombeauconstants, and

(3) $\forall t'': u_{\varepsilon'}(\boldsymbol{q}'',t'') \in H^{\infty}(S_R)$.

(II) From Theorem 10-11 one obtain, that:

(1) Operator Δ is bounded on $H^{\infty}(S_R)$,

(2) The Colombeau generalized function $\left(u_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}'',t'')\right)_{\boldsymbol{\varepsilon}'}$ given by formula

$$\left(u_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}'',t'')\right)_{\boldsymbol{\varepsilon}'} = \left(K_{\varepsilon}^{t''-t'}\varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}''-\boldsymbol{q}')\right)_{\boldsymbol{\varepsilon}'}, \qquad (224)$$

$$K_{\varepsilon}^{t''-t'} = \exp\left[\frac{\varepsilon}{2}(t''-t')\Delta\right] \qquad (225)$$

$$\left(u_{\boldsymbol{\varepsilon}'}(\boldsymbol{q},t'')\right)_{\boldsymbol{\varepsilon}'} =$$

$$= \left(\boldsymbol{\mathcal{F}}^{-1}\left[\exp\left(-\frac{\varepsilon}{2}(t''-t')\|\boldsymbol{\xi}\|^2\right)\boldsymbol{\mathcal{F}}[\varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}''-\boldsymbol{q}')](\boldsymbol{\xi})\right]\right)_{\boldsymbol{\varepsilon}'} \quad (226)$$

is the solution of the Colombeau-Fokker-Planck equation (203))-(204)with initial condition $\varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}''-\boldsymbol{q}') \in H^{\infty}(S_R)$,for the case: $\left(b_{i,\varepsilon'}(\boldsymbol{q}'',t)\right)_{\varepsilon'} \equiv 0, i = 1,\dots,n$,and

(3) $\forall t'': u_{\varepsilon'}(\boldsymbol{q}'',t'') \in H^{\infty}(S_R)$.

(III) From Theorem 10-11 one obtain, that: operator

$\mathcal{R}_{\varepsilon,\boldsymbol{\varepsilon}'} = \sum_{i=1}^{i=n} b_{i,\varepsilon'}\frac{\partial}{\partial q_i''} + \varepsilon\Delta, \varepsilon \in \langle 0,1]$,is bounded on $H^{\infty}(S_R)$,

If we let now

$$U_{\varepsilon,\boldsymbol{\varepsilon}'}^{t''-t'} = \exp\left[(t''-t')\mathcal{R}_{\varepsilon,\boldsymbol{\varepsilon}'}\right], \qquad (227)$$

Then a Theorem10-11asserts that for all $\varphi_{\boldsymbol{\varepsilon}'}$ in $H^{\infty}(S_R)$

$$\left(U_{\varepsilon,\boldsymbol{\varepsilon}'}^{t''-t'}\varphi_{\varepsilon'}\right)_{\boldsymbol{\varepsilon}'}=\left(\lim_{N\to\infty}\left[\left(K_{\varepsilon}^{\frac{t''-t'}{N}}P_{\varepsilon'}^{\frac{t''-t'}{N}}\right)^{N}\varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}''-\boldsymbol{q}')\right]\right)_{\boldsymbol{\varepsilon}'}. \quad (228)$$

Where the limit is calculated by norm in $H^{\infty}(S_R)$.

From Eq.(224)-Eq.(228) by simple calculation one obtain

$$\left(K_{\varepsilon}^{\frac{t''-t'}{N}}P_{\varepsilon'}^{\frac{t''-t'}{N}}\right)^{N}\varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}''-\boldsymbol{q}')= \quad (229)$$

$$=\frac{1}{(2\pi)^{\frac{Nn}{2}}}\int_{-\infty}^{\infty}d\boldsymbol{q}\int_{-\infty}^{\infty}\boldsymbol{d\xi}\exp\left\{\boldsymbol{i}\sum_{m=0}^{N}\left[(\boldsymbol{q}_{m+1}-\boldsymbol{q}_{m})\boldsymbol{\xi}_{m}+\frac{t''-t'}{N}\sum_{\boldsymbol{i=1}}^{\boldsymbol{i=n}}b_{\boldsymbol{i},\boldsymbol{\varepsilon}'}\,\boldsymbol{\xi}_{i,m}\right]-\right.$$
$$\left.\frac{t''-t'}{N}\frac{\varepsilon}{2}\sum_{m=0}^{N}\|\boldsymbol{\xi_m}\|^{2}\right\}\varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}_{0}-\boldsymbol{q}').$$

Here $d\boldsymbol{q}=\boldsymbol{dq_0}\ldots\boldsymbol{dq_m}\ldots\boldsymbol{dq_N},\boldsymbol{d\xi}=\boldsymbol{d\xi_0}\ldots\boldsymbol{d\xi_m}\ldots\boldsymbol{d\xi_N},\boldsymbol{q_N}=\boldsymbol{q}''$,
$d\boldsymbol{q_m}=\prod_{\boldsymbol{j=1}}^{\boldsymbol{n}}\boldsymbol{d}q_{\boldsymbol{j},m}\,,\boldsymbol{d\xi}_m=\prod_{\boldsymbol{j=1}}^{n}\boldsymbol{d\xi_{j,m}}\,,m=0,\ldots,N.$

Integrating on variable $\boldsymbol{\xi}$ gives

$$\left(K_{\varepsilon}^{\frac{t''-t'}{N}}P_{\varepsilon'}^{\frac{t''-t'}{N}}\right)^{N}\varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}''-\boldsymbol{q}')=\boldsymbol{I}_{N,\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'')=$$

$$\frac{1}{(2\pi)^{\frac{Nn}{2}}}\int_{-\infty}^{\infty}d\boldsymbol{q}\exp\left\{\frac{\boldsymbol{1}}{2\boldsymbol{\varepsilon}}\frac{t''-t'}{N}\sum_{m=0}^{N}\sum_{i=1}^{n}\left[\frac{\left(\boldsymbol{q}_{i,m+1}-\boldsymbol{q}_{i,m}\right)}{\frac{t''-t'}{\boldsymbol{N}}}-b_{\boldsymbol{i},\boldsymbol{\varepsilon}'}\right]^{2}\right\}\varphi(\boldsymbol{q}_{0}-\boldsymbol{q}').$$

Finally we obtain

$$\left(U_{\varepsilon,\boldsymbol{\varepsilon}'}^{t''-t'}\varphi_{\boldsymbol{\varepsilon}'}\right)_{\boldsymbol{\varepsilon}'}=\lim_{N\to\infty}\left(\boldsymbol{I}_{N,\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'')\right)_{\boldsymbol{\varepsilon}'} \quad (230)$$

Remark5. Note that$\delta_{\boldsymbol{\varepsilon}'}(x)\in H^{\infty}(S_R),R=1/\varepsilon',x\in\mathbb{R},\varepsilon'\in(0,1]$, where

$$\delta_{\boldsymbol{\varepsilon}'}(x)=\frac{1}{\boldsymbol{\pi\varepsilon' x}}\sin\left(\frac{x}{\boldsymbol{\varepsilon}'}\right). \quad (231)$$

By simple calculation one obtain

$$\mathcal{F}[\delta_{\boldsymbol{\varepsilon}'}(x)] = \begin{cases} \frac{1}{2\pi}, x \in (-r,r) \\ 0, x \notin (-r,r) \end{cases} \quad (232)$$

$r = 1/\varepsilon'$[30].

Assumption 3. We assume now that

$$\left(p_{\varepsilon'}(\boldsymbol{q}'' - \boldsymbol{q}')\right)_{\varepsilon'} = \prod_{i=1}^{n}\left(\delta_{\boldsymbol{\varepsilon}'}(q_i'' - q_i')\right)_{\varepsilon'}. \quad (233)$$

Remark 6. Note that

$$\left(p_{\varepsilon'}(\boldsymbol{q_0} - \boldsymbol{q}')\right)_{\varepsilon'} = \delta(\boldsymbol{q_0} - \boldsymbol{q}'). \quad (234)$$

Definition14. We let now $n = 1$. A tagged partition of the real line$\mathbb{R} = (-\infty, +\infty)$ is a finite sequence$-\infty = x_0 < x_1 < x_2 < \cdots < x_{P-1} < x_P = +\infty$**.** This partitions the open interval$(-\infty, +\infty)$ into n sub-intervals$\boldsymbol{J}_r = [x_{r-1}, x_r]$,$r$=1,…,$p$,$\boldsymbol{J}_0 = (-\infty, x_1]$**,** $\boldsymbol{J}_P = [x_{P-1}, +\infty)$indexed by r=1,…,p. Let $b_{\boldsymbol{\varepsilon}'}(q'', t, r)$ be a quantity

$$b_{\boldsymbol{\varepsilon}'}(t,r) = \sup_{q'' \in \boldsymbol{J}_r} b_{\boldsymbol{\varepsilon}'}(q'', t) \quad (235)$$

and let $\partial q_i'' b_{\boldsymbol{\varepsilon}'}(t,r)$be a quantity

$$\partial q'' b_{\boldsymbol{\varepsilon}'}(t,r) = \sup_{q'' \in J_r} \frac{\partial b_{\boldsymbol{\varepsilon}'}(q'',t)}{\partial q''} \quad (236)$$

Let$\grave{b}_{\boldsymbol{\varepsilon}'}(q'', t)$be a function

$$\grave{b}_{\boldsymbol{\varepsilon}'}(q'', t) = \sum_{r=1}^{P} \mathbf{1}_{\boldsymbol{J}_r}(q'') b_{\boldsymbol{\varepsilon}'}(t,r). \quad (237)$$

Let $\partial\grave{b}_{\boldsymbol{\varepsilon}'}(q'',t)$be a function

$$\partial\grave{b}_{\boldsymbol{\varepsilon}'}(q'',t)=\sum_{r=0}^{P}\mathbf{1}_{\boldsymbol{J}_r}(q'')\partial b_{\varepsilon'}(t,r). \qquad (238)$$

Here $\mathbf{1}_{\boldsymbol{J}_r}(q'')$ is indicator function of a subset: $\boldsymbol{J}_r=[x_{r-1},x_r]$.

Definition15. Let $H^{\infty}(S_R;\check{S}_U), R=(R_1\ldots R_n), U=(U_1\ldots U_n)$ be a test space with a members $\varphi(\boldsymbol{x},\boldsymbol{p}), x\in\mathbb{R}^n_{\boldsymbol{x}}, \boldsymbol{p}\in\mathbb{R}^n_{\boldsymbol{p}}$ such that $\forall\breve{\boldsymbol{x}},\breve{\boldsymbol{x}}\in\check{S}_U, \check{S}_U\subseteq S_U=\{\boldsymbol{x}||x_i|<U_i\leq\infty; i=1,\ldots,n\}$function
$\varphi(\breve{\boldsymbol{x}},\boldsymbol{p})$issupported inside region
$S_R=\{\boldsymbol{p}||p_i|<R_i; i=1,\ldots,n\}$,i.e.$\forall\breve{\boldsymbol{x}},\breve{\boldsymbol{x}}\in\boldsymbol{S_U}:\boldsymbol{\mathcal{F}^{-1}}[\varphi(\breve{\boldsymbol{x}},\boldsymbol{p})](\breve{\boldsymbol{x}},\boldsymbol{x})\in\boldsymbol{H}^{\infty}(S_{\boldsymbol{R}})$.

Definition 16.(1)We let $\boldsymbol{\mathcal{F}}^{\#}[\psi(\boldsymbol{x})](\boldsymbol{p})=\varphi(\boldsymbol{x},\boldsymbol{p})$ iff there exist an function $\varphi(\boldsymbol{x},\boldsymbol{p})\in H^{\infty}(S_R;S_U)$such that

$$\boldsymbol{\mathcal{F}^{-1}}[\varphi(\boldsymbol{x},\boldsymbol{p})](\boldsymbol{x},\boldsymbol{x})=\psi(\boldsymbol{x}). \qquad (239)$$

(2)We let now $H^{\infty}(S_R;\check{S}_{U,p})$ if $n=1$ and$\check{S}_U=S_U\backslash\{x_0,\ldots,x_p\}$.

Remark6.(**I**)From Theorem 11-12 one obtain, that: operator $(1\grave{)}\, b_{\boldsymbol{\varepsilon}'}(q'',t)\partial q=\sum_{r=1}^{r=p}\mathbf{1}_{\boldsymbol{J}_r}(q'')b_{\varepsilon'}(t,r)\,\partial q,\ \varepsilon\in\langle 0,1]$is bounded on$H^{\infty}(S_R;\check{S}_U)$.

(2)The Colombeau generalized function $\left(u_{\boldsymbol{\varepsilon}',p}(q'',t'')\right)_{\boldsymbol{\varepsilon}'}$given by formula

$$\left(u_{\boldsymbol{\varepsilon}',p}(q'',t'')\right)_{\boldsymbol{\varepsilon}'}=\left(P^{t''-t'}_{\varepsilon',p}\varphi_{\varepsilon'}(q''-q')\right)_{\boldsymbol{\varepsilon}'}=$$

$$\left(\exp\left(-[t''-t']\grave{b}_{\boldsymbol{\varepsilon}'}(q'',t)\partial q''\right)\varphi_{\varepsilon'}(q''-q')\right)_{\boldsymbol{\varepsilon}'}=$$

〖 (u_(ε^') (q^",t^"))〗

_(ε^') given by formula

$$\left(\exp\left(-[t''-t']\sum_{r=1}^{P}\mathbf{1}_{J_r}(q'')b_{\varepsilon'}(t,r)\,\partial q''\right)\varphi_{\varepsilon'}(q''-q')\right)_{\boldsymbol{\varepsilon}'}=$$

$$\left(\prod_{r=1}^{p}\exp\left(-[t''-t']1_{J_r}(q'')b_{\varepsilon'}(t,r)\partial q''\right)\varphi_{\varepsilon'}(q''-q')\right)_{\varepsilon'} \quad (240)$$

is the solution(except points$\{x_0,\dots,x_p\}$)of the Colombeau-Fokker-Planck equation (203)-(204) with initial condition $\varphi_{\varepsilon'}(q''-q')\in H^{\infty}\left(S_R;\check{S}_{U,p}\right)$, for the case:

$$\left(b_{\varepsilon'}(q'',t)\right)_{\varepsilon'}=\grave{b}_{\boldsymbol{\varepsilon}'}(q'',t)=\sum_{r=1}^{p}\mathbf{1}_{\boldsymbol{J}_r}(q'')b_{\varepsilon'}(t,r). \quad (241)$$

(3) We note that:

$$\exp\left(-[t''-t']1_{J_r}(q'')b_{\varepsilon'}(t,r)\partial q''\right)\varphi_{\varepsilon'}(q''-q')=$$

$$\mathcal{F}^{-1}\left[\exp\left(-[t''-t']1_{J_r}(q'')b_{\varepsilon'}(t,r)(i\xi)\right)\mathcal{F}^{\#}[\varphi_{\varepsilon'}(q''-q')]\right] \quad (242)$$

(4)$\forall t'': u_{\boldsymbol{\varepsilon}',p}(q'',t'')\in H^{\infty}\left(S_R;\check{S}_{U,p}\right)$.

(II) From Theorem 11-12 one obtain, that:

(1) Operator Δ is bounded on $H^{\infty}\left(S_R;\check{S}_{U,p}\right)$,

(2) The Colombeau generalized function$\left(u_{\boldsymbol{\varepsilon}',p}(\boldsymbol{q}'',t'')\right)_{\boldsymbol{\varepsilon}'}$given by formula

$$\left(u_{\boldsymbol{\varepsilon}',p}(q'',t'')\right)_{\boldsymbol{\varepsilon}'}=\left(K_{\varepsilon}^{t''-t'}\varphi_{\varepsilon'}(q''-q')\right)_{\varepsilon'}, \quad (243)$$

$$K_{\varepsilon,p}^{t''-t'}=\exp\left[\frac{\varepsilon}{2}(t''-t')\Delta\right] \quad (244)$$

$$\left(u_{\boldsymbol{\varepsilon}',p}(q,t'')\right)_{\boldsymbol{\varepsilon}'}=$$

$$=\left(\boldsymbol{\mathcal{F}}^{-1}\left[\exp\left(-\frac{\varepsilon}{2}(t''-t')\|\boldsymbol{\xi}\|^2\right)\boldsymbol{\mathcal{F}}^{\#}[\varphi_{\boldsymbol{\varepsilon}'}(q''-q')](\boldsymbol{\xi})\right]\right)_{\boldsymbol{\varepsilon}'} \quad (245)$$

is the solution of the Colombeau-Fokker-Planck equation (203))-(204)with initial condition $\varphi_{\boldsymbol{\varepsilon}'}(q''-q') \in H^{\infty}(S_R;\check{S}_{U,p})$,for the case:$\left(b_{\varepsilon'}(q'',t)\right)_{\varepsilon'} \equiv 0$,and

(3) $\forall t'': u_{\boldsymbol{\varepsilon}',p}(q'',t'') \in H^{\infty}(S_R;\check{S}_{U,p})$.

(III)) From Theorem 11-12 one obtain, that: operator

$\mathcal{R}_{\varepsilon,\boldsymbol{\varepsilon}',p} = \sum_{r=1}^{r=p} \mathbf{1}_{J_r}(q'') b_{\varepsilon'}(t,r) \frac{\partial}{\partial q''} + \varepsilon\Delta, \varepsilon' \in \langle 0,1]$, is bounded on $H^{\infty}(S_R;\check{S}_{U,p})$,

If we let now

$$U_{\varepsilon,\boldsymbol{\varepsilon}',p}^{t''-t'} = \exp\left[(t''-t')\mathcal{R}_{\varepsilon,\boldsymbol{\varepsilon}',p}\right], \qquad (246)$$

then Theorem 10-11asserts that for all $\varphi_{\boldsymbol{\varepsilon}'}$ in $H^{\infty}(S_R;\check{S}_{U,p})$

$$\left(U_{\varepsilon,\boldsymbol{\varepsilon}',\boldsymbol{p}}^{t''-t'} \varphi_{\boldsymbol{\varepsilon}'}\right)_{\boldsymbol{\varepsilon}'} = \left(\lim_{N\to\infty}\left[\left(K_{\varepsilon,p}^{\frac{t''-t'}{N}} P_{\varepsilon',p}^{\frac{t''-t'}{N}}\right)^N \varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}''-\boldsymbol{q}')\right]\right)_{\boldsymbol{\varepsilon}'}. \qquad (247)$$

Where the limit is calculated by norm in $H^{\infty}(S_R;\check{S}_{U,p})$.

From Eq.(244)-Eq.(247) by simple calculation one obtain

$$\left(K_{\varepsilon,p}^{\frac{t''-t'}{N}} P_{\varepsilon',p}^{\frac{t''-t'}{N}}\right)^N \varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}''-\boldsymbol{q}') = \qquad (248)$$

$$= \frac{1}{(2\pi)^{\frac{Nn}{2}}} \int_{-\infty}^{\infty} d\boldsymbol{q} \int_{-\infty}^{\infty} \boldsymbol{d\xi} \exp\left\{\boldsymbol{i}\sum_{m=0}^{N}\left[(\boldsymbol{q}_{m+1}-\boldsymbol{q}_m)\xi_m + \frac{t''-t'}{N}\sum_{\boldsymbol{r=1}}^{r=p} \mathbf{1}_{J_r}(q_m) b_{\boldsymbol{\varepsilon}'}(t,r)\xi_m\right] - \frac{t''-t'}{N}\frac{\varepsilon}{2}\sum_{m=0}^{N}\xi_m^2\right\} \varphi(q_0-q').$$

Here $d\boldsymbol{q} = dq_0 \dots dq_m \dots dq_{N-1}$, $d\boldsymbol{\xi} = d\xi_0 \dots d\xi_m \dots d\xi_N$,$q_N = q''$, $m = 0,\dots,N$.

Integrating on variable $\boldsymbol{\xi}$ gives

$$\left(K_{\varepsilon,p}^{\frac{t''-t'}{N}} P_{\varepsilon',p}^{\frac{t''-t'}{N}}\right)^N \varphi_{\boldsymbol{\varepsilon}'}(q''-q') = \boldsymbol{I}_{N,\boldsymbol{\varepsilon}',p}(q',t'|q'',t'') =$$

$$\frac{1}{(2\pi)^{\frac{Nn}{2}}}\int_{-\infty}^{\infty} d\boldsymbol{q}\exp\left\{\frac{1}{2\varepsilon}\frac{t''-t'}{N}\sum_{m=0}^{N}\left[\frac{(\boldsymbol{q}_{m+1}-\boldsymbol{q}_m)}{\frac{t''-t'}{N}} - -\sum_{r=1}^{r=p}\mathbf{1}_{J_r}(q_m)b_{\boldsymbol{\varepsilon}'}(t,r)\right]^2\right\}\varphi_{\boldsymbol{\varepsilon}'}(q_0 - q') \quad (249)$$

From Eq.(249) we obtain

$$\left(u_{\boldsymbol{\varepsilon}',p}(q'',t'')\right)_{\boldsymbol{\varepsilon}'} = \left(\lim_{N\to\infty}\boldsymbol{I}_{N,\boldsymbol{\varepsilon}',p}(q',t'|q'',t'')\right)_{\boldsymbol{\varepsilon}'}. \quad (250)$$

Here the limit is calculated by norm in$H^{\infty}\left(S_R;\check{S}_{U,p}\right)$.

Let$\delta_p$bethe quantity$\delta_p = \max_{1\leq r\leq p}\{\delta_r|r=1,\dots,p\}$,$\delta_r = |x_{r-1},x_r|$.We assume that:

(1)$\delta_p \to 0$ if $p\to\infty$,(2) $x_0\to\infty$ if $p\to\infty$,(3)$x_p\to\infty$ if $p\to\infty$.

Finally we obtain

$$\left(u_{\boldsymbol{\varepsilon}'}(q'',t'')\right)_{\boldsymbol{\varepsilon}'} = \left(\lim_{p\to\infty}u_{\varepsilon',p}(q'',t'')\right)_{\boldsymbol{\varepsilon}'} =$$

$$= \left(\lim_{p\to\infty}U_{\varepsilon,\boldsymbol{\varepsilon}',p}^{t''-t'}\varphi_{\boldsymbol{\varepsilon}'}(q''-q')\right)_{\boldsymbol{\varepsilon}'} \quad (251)$$

Here the limit is calculated by norm$\|\cdot\|_{W^{2,2}(\mathbb{R},\omega)}$of the weighted Sobolev space $W^{2,2}(\mathbb{R},\omega)$[41].

Theorem 13. Assume that$\varphi_{\boldsymbol{\varepsilon}'}(q''-q') \in H^{\infty}(S_R)\cap W^{2,2}(\mathbb{R},\omega)$, $\varepsilon' \in \langle 0,1]$,$\omega=\omega(q'')$.Then:

(1) $\forall\varepsilon' \in \langle 0,1]$there exist p_0 such that$\forall p_1\forall p_2[(p_1\geq p_0)\wedge(p_2\geq p_0)]$ the inequality

$$\|u_{\boldsymbol{\varepsilon}',p_1}(q',t'|q'',t'') - u_{\boldsymbol{\varepsilon}',p_1p_2}(q',t'|q'',t'')\|_{W^{2,2}(\mathbb{R},\omega)}$$
$$\leq ((t''-t')/(p_1))C_1\exp[C_2(t''-t')]\|\varphi_{\boldsymbol{\varepsilon}'}\|_{W^{2,2}(\mathbb{R},\omega)}$$

holds for each $t'' \in [t', \infty)$.

(2) The Colombeau generalized function $\left(u_{\varepsilon'}(q'', t'')\right)_{\varepsilon'}$ given by formula(251) is the solution of the Colombeau-Fokker-Planck equation (203))-(204)(except a set Lebesgue measure zero) with initial condition $\left(\varphi_{\varepsilon'}(q'' - q')\right)_{\varepsilon'}$ such that $\forall \varepsilon' \in \langle 0, 1]: \varphi_{\varepsilon'}(q'' - q') \in H^{\infty}(S_R)$.

Let us consider now n-dimensional case. We shall be working with *rectangular* parallelograms in $\mathbb{R}^n$, those parallelograms whose edges are mutually orthogonal. Actually, we shall be even more restrictive, and consider only those whose edges are in the directions of the coordinate axes.

Definition.17. We call them *special rectangles*. Each of these may be expressed as a Cartesian product of intervals in $\mathbb{R}$.:

$$\boldsymbol{I} = [a_1, b_1] \times \ldots \times [a_n, b_n] = \{q'' | q'' \in \mathbb{R}^n, a_i \leq q_i'' \leq b_i, i = 1, \ldots, n\}$$

and we let $\boldsymbol{I}_\infty = \mathbb{R}^n \setminus \boldsymbol{I}$.

Definition18. We define a *partition* of $\boldsymbol{I}$ to be a collection of non-overlapping special rectangles $\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots, \boldsymbol{I}_r, \ldots, \boldsymbol{I}_P$ whose union is $\boldsymbol{I}$. "Non-overlapping" requires that the *interiors* of these rectangles are mutually disjoint.

Definition.19. Let $b_{\boldsymbol{i},\boldsymbol{\varepsilon}'}(, t, r), i = 1, \ldots, n$ be a quantities

$$b_{\boldsymbol{i},\boldsymbol{\varepsilon}'}(t, r) = \sup_{\boldsymbol{q}'' \in \boldsymbol{I}_r} b_{\boldsymbol{i},\boldsymbol{\varepsilon}'}(\boldsymbol{q}'', t) \qquad (252)$$

and let $\partial q_j b_{\boldsymbol{i},\boldsymbol{\varepsilon}'}(t, r), i = 1, \ldots, n, j = 1, \ldots, n$ be a quantities

$$\partial q_j b_{\boldsymbol{i},\boldsymbol{\varepsilon}'}(t, r) = \sup_{\boldsymbol{q}'' \in \boldsymbol{I}_r} \frac{\partial b_{i,\varepsilon'}(\boldsymbol{q}'', t)}{\partial q_j}. \qquad (253)$$

Definition20. Let $\check{b}_{\boldsymbol{i},\boldsymbol{\varepsilon}'}(\boldsymbol{q}'',t,r), i = 1,\dots,n$ be a functions

$$\grave{b}_{\boldsymbol{i},\boldsymbol{\varepsilon}'}(\boldsymbol{q}'',t) = \sum_{r=1}^{P} \mathbf{1}_{\boldsymbol{I}_r}(\boldsymbol{q}'')b_{i,\varepsilon'}(t,r). \qquad (254)$$

Let $\partial q_j \grave{b}_{\boldsymbol{i},\boldsymbol{\varepsilon}'}(\boldsymbol{q}'',t)$be a function

$$\partial q_j \grave{b}_{\boldsymbol{i},\boldsymbol{\varepsilon}'}(\boldsymbol{q}'',t) = \sum_{r=0}^{P} \mathbf{1}_{\boldsymbol{I}_r}(\boldsymbol{q}'')\partial q_j b_{i,\varepsilon'}(t,r). \qquad (255)$$

Here $\mathbf{1}_{\boldsymbol{I}_r}(\boldsymbol{q}'')$is indicator function of a subset$\boldsymbol{I}_r \subset \boldsymbol{I}$.

Definition21. We let now $H_n^{\infty}(S_R;\check{S}_{U,p})$ if$\check{S}_U = S_U \backslash \{\partial I_0,\dots,\partial I_p\}$.

Remark 7.(**I**)From Theorem 11-12 one obtain, that:

(1)Operator$\grave{b}_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}'',t)\partial\boldsymbol{q}'' = \sum_{\boldsymbol{i}=\mathbf{1}}^{n} \tilde{b}_{\boldsymbol{i},\boldsymbol{\varepsilon}'}(\boldsymbol{q}'',t)\partial q_i'' = \sum_{i=1}^{n}\sum_{r=1}^{r=p} \mathbf{1}_{\boldsymbol{I}_r}(\boldsymbol{q}'')b_{i,\varepsilon'}(t,r)\,\partial q_i''$,$\boldsymbol{\varepsilon}' \in \langle 0,1]$is bounded on$H_n^{\infty}(S_R;\check{S}_{U,p})$.

(2)The Colombeau generalized function $\left(u_{\boldsymbol{\varepsilon}',p}(\boldsymbol{q}'',t'')\right)_{\boldsymbol{\varepsilon}'}$ given by formula

$$\left(u_{\boldsymbol{\varepsilon}',p}(\boldsymbol{q}'',t'')\right)_{\boldsymbol{\varepsilon}'} = \left(P_{\varepsilon',p}^{t''-t'}\varphi_{\varepsilon'}(\boldsymbol{q}''-\boldsymbol{q}')\right)_{\boldsymbol{\varepsilon}'} =$$

$$\left(\exp\left(-[t''-t']\grave{b}_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}'',t)\partial\boldsymbol{q}''\right)\varphi_{\varepsilon'}(\boldsymbol{q}''-\boldsymbol{q}')\right)_{\boldsymbol{\varepsilon}'} =$$

〖 (u_(ε^') (q^",t^"))〗

_(ε^') given by formula

$$\left(\exp\left(-[t''-t']\sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{n}}\sum_{r=1}^{P}\mathbf{1}_{\boldsymbol{I}_r}(\boldsymbol{q}'')b_{i,\varepsilon'}(t,r)\,\partial q_i''\right)\varphi_{\varepsilon'}(\boldsymbol{q}''-\boldsymbol{q}')\right)_{\boldsymbol{\varepsilon}'}$$

=

$\left(\prod_{\mathrm{i}=1}^{\mathrm{n}}\prod_{r=1}^{p}\exp\left(-[t''-t']1_{\boldsymbol{I}_r}(\boldsymbol{q}'')b_{i,\varepsilon'}(t,r)\partial q_i''\right)\varphi_{\varepsilon'}(\boldsymbol{q}''-\right.$

$$\boldsymbol{q}')\big)_{\boldsymbol{\varepsilon}'} \quad (256)$$

is the solution(except points$\boldsymbol{q}'' \in \bigcup_{r=1}^{p} \partial \boldsymbol{I}_r$) of the Colombeau-Fokker-Planck equation (203)-(204) with initial condition$\big(\varphi_{\varepsilon'}(\boldsymbol{q}'' - \boldsymbol{q}')\big)_{\boldsymbol{\varepsilon}'}$such that $\forall \varepsilon' \in \langle 0, 1]: \varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}'' - \boldsymbol{q}') \in H_n^{\infty}\big(S_R; \check{S}_{U,p}\big)$,for the case:$\varepsilon = 0$ and

$$\Big(b_{i,\varepsilon'}(\boldsymbol{q}'', t)\Big)_{\varepsilon'} = \grave{b}_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}'', t) = \sum_{r=1}^{p} \mathbf{1}_{\boldsymbol{I}_r}(\boldsymbol{q}'') b_{i,\varepsilon'}(t, r), \quad (257)$$

$$i = 1, \dots, n.$$

(3) We note that:

$$\exp\big(-[t'' - t'] \mathbf{1}_{\boldsymbol{I}_r}(\boldsymbol{q}'') b_{i,\varepsilon'}(t, r) \partial q_i''\big) \varphi_{\varepsilon'}(\boldsymbol{q}'' - \boldsymbol{q}') =$$

$$\mathcal{F}^{-1}\Big[\exp\Big(-[t'' - t'] 1_{\boldsymbol{I}_r}(\boldsymbol{q}'') b_{i,\varepsilon'}(t, r)(i\xi_i)\Big)\ \mathcal{F}^{\#}[\varphi_{\varepsilon'}(\boldsymbol{q}'' - \boldsymbol{q}')]\Big] (258)$$

(4)$\forall t'': u_{\boldsymbol{\varepsilon}',p}(\boldsymbol{q}'', t'') \in H_n^{\infty}\big(S_R; \check{S}_{U,p}\big)$.

(II) From Theorem 11-12 one obtain, that:

(1) Operator Δ is bounded on $H_n^{\infty}\big(S_R; \check{S}_{U,p}\big)$,

(2) The Colombeau generalized function$\Big(u_{\boldsymbol{\varepsilon}',p}(\boldsymbol{q}'', t'')\Big)_{\boldsymbol{\varepsilon}'}$given by formula

$$\Big(u_{\varepsilon',p}(\boldsymbol{q}'', t'')\Big)_{\boldsymbol{\varepsilon}'} = \Big(K_{\varepsilon,p}^{t''-t'} \varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}'' - \boldsymbol{q}')\Big)_{\boldsymbol{\varepsilon}'}, \quad (259)$$

$$K_{\varepsilon,p}^{t''-t'} = \exp\Big[\frac{\varepsilon}{2}(t'' - t')\Delta\Big] \quad (260)$$

$$\Big(u_{\boldsymbol{\varepsilon}',p}(\boldsymbol{q}'', t'')\Big)_{\boldsymbol{\varepsilon}'} =$$

$$= \left(\mathcal{F}^{-1}\left[\exp\left(-\frac{\varepsilon}{2}(t''-t')\|\boldsymbol{\xi}\|^2\right)\mathcal{F}^{\#}[\varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}''-\boldsymbol{q}')](\boldsymbol{\xi})\right]\right)_{\boldsymbol{\varepsilon}'} \quad (261)$$

is the solution of the Colombeau-Fokker-Planck equation (203))-(204)with initial condition $\left(\varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}''-\boldsymbol{q}')\right)_{\boldsymbol{\varepsilon}'}$such that

$\forall \varepsilon' \in \langle 0,1]: \varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}''-\boldsymbol{q}') \in H_n^{\infty}(S_R;\check{S}_{U,p})$, for the case: $\left(b_{i,\boldsymbol{\varepsilon}'}(\boldsymbol{q}'',t)\right)_{\varepsilon'} \equiv 0, i=1,\ldots,n$ and

(3) $\forall t'': u_{\boldsymbol{\varepsilon}',p}(\boldsymbol{q}'',t'') \in H_n^{\infty}(S_R;\check{S}_{U,p})$.

(III)) From Theorem 11-12 one obtain, that: operator

$\mathcal{R}_{\varepsilon,\boldsymbol{\varepsilon}',p} = \sum_{i=1}^{n}\sum_{r=1}^{r=p} b_{i,\varepsilon'}\frac{\partial}{\partial q_i''} + \varepsilon\Delta, \varepsilon \in \langle 0,1]$, is bounded on $H_n^{\infty}(S_R;\check{S}_{U,p})$.

If we let now

$$U_{\varepsilon,\boldsymbol{\varepsilon}',p}^{t''-t'} = \exp\left[(t''-t')\mathcal{R}_{\varepsilon,\boldsymbol{\varepsilon}',p}\right], \quad (262)$$

then Theorem 10-11asserts that for all $\varphi_{\boldsymbol{\varepsilon}'}$ in $H_n^{\infty}(S_R;\check{S}_{U,p})$

$$\left(U_{\varepsilon,\boldsymbol{\varepsilon}',\boldsymbol{p}}^{t''-t'}\varphi_{\varepsilon'}\right)_{\boldsymbol{\varepsilon}'} = \left(\lim_{N\to\infty}\left[\left(K_{\varepsilon,p}^{\frac{t''-t'}{N}} P_{\varepsilon',p}^{\frac{t''-t'}{N}}\right)^{N}\varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}''-\boldsymbol{q}')\right]\right)_{\boldsymbol{\varepsilon}'}. \quad (263)$$

Where the limit is calculated by norm in $H_n^{\infty}(S_R;\check{S}_{U,p})$.

From Eq.(256)-Eq.(263) by simple calculation one obtain

$$\left(K_{\varepsilon,p}^{\frac{t''-t'}{N}} P_{\varepsilon',p}^{\frac{t''-t'}{N}}\right)^{N}\varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}''-\boldsymbol{q}') = \quad (264)$$

$$= \frac{1}{(2\pi)^{\frac{Nn}{2}}}\int_{-\infty}^{\infty} d\boldsymbol{q}\int_{-\infty}^{\infty} \boldsymbol{d\xi}\exp\left\{\boldsymbol{i}\sum_{m=0}^{N}\left[(\boldsymbol{q}_{m+1}-\boldsymbol{q}_m)\boldsymbol{\xi}_m + \frac{t''-t'}{N}\sum_{\boldsymbol{i=1}}^{n}\sum_{\boldsymbol{r=1}}^{r=p}\boldsymbol{1}_{J_r}(\boldsymbol{q}_m)b_{\boldsymbol{i},\varepsilon'}(t,r)\boldsymbol{\xi}_{\boldsymbol{i},m}\right] -\right.$$

$\frac{t''-t'}{N}\frac{\varepsilon}{2}\sum_{m=0}^{N}\boldsymbol{\xi}_{m}^{2}\Big\}\varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q_0}-\boldsymbol{q}')$.

Here $d\boldsymbol{q}=d\boldsymbol{q_0}\dots d\boldsymbol{q_m}\dots d\boldsymbol{q_{N-1}}, d\boldsymbol{\xi}=d\boldsymbol{\xi}_0\dots d\boldsymbol{\xi}_m\dots d\boldsymbol{\xi}_N, \boldsymbol{q}_N=\boldsymbol{q}''$,
$d\boldsymbol{q_m}=\prod_{\boldsymbol{i}=1}^{\boldsymbol{n}}\boldsymbol{d}q_{\boldsymbol{i},m}$, $\boldsymbol{d\xi}_m=\prod_{\boldsymbol{i}=1}^{n}\boldsymbol{d}\xi_{\boldsymbol{i},\boldsymbol{m}}$, $m=0,\dots,N$.
Integrating on variable $\boldsymbol{\xi}$ gives

$$\left(K_{\varepsilon,p}^{\frac{t''-t'}{N}}P_{\varepsilon',p}^{\frac{t''-t'}{N}}\right)^{N}\varphi_{\boldsymbol{\varepsilon}'}(q''-q')=\boldsymbol{I}_{N,\boldsymbol{\varepsilon}',p}(q',t'|q'',t'')=$$

$$\frac{1}{(2\pi)^{\frac{Nn}{2}}}\int_{-\infty}^{\infty}d\boldsymbol{q}\exp\left\{\frac{1}{2\boldsymbol{\varepsilon}}\frac{t''-t'}{N}\sum_{m=0}^{N}\sum_{i=1}^{n}\left[\frac{(q_{i,m+\mathbf{1}}-q_{i,m})}{\frac{t''-t'}{\boldsymbol{N}}}-\sum_{\boldsymbol{r}=\mathbf{1}}^{r=p}\mathbf{1}_{I_r}(\boldsymbol{q}_m)b_{\boldsymbol{i},\boldsymbol{\varepsilon}'}(t,r)\right]^2\right\}\varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q_0}-\boldsymbol{q}')\qquad(265)$$

From Eq.(265) we obtain

$$\left(u_{\boldsymbol{\varepsilon}',p}(\boldsymbol{q}'',t'')\right)_{\boldsymbol{\varepsilon}'}=\lim_{N\to\infty}\boldsymbol{I}_{N,\boldsymbol{\varepsilon}',p}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'').$$

Here the limit is calculated by norm in $H_n^{\infty}(S_R;\check{S}_{U,p})$.

Let$\delta_p=\max_{1\le r\le p}\{\delta_r|r=1,\dots,p\}$,$\delta_r=\operatorname{diam}(\boldsymbol{I_r})$, where
$\operatorname{diam}(I_r)=\sup_{x,y\in\boldsymbol{I_r}}\|x-y\|$We assume that: (1)$\delta_p\to 0$ if $p\to\infty$,(2) $a_i\to-\infty, i=1,\dots,n$ if $p\to\infty$,(3)$b_i\to\infty, i=1,\dots,n$ if $p\to\infty$.
Finally we obtain

$$\left(u_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}'',t'')\right)_{\boldsymbol{\varepsilon}'}=\left(\lim_{p\to\infty}u_{\varepsilon',p}(\boldsymbol{q}'',t'')\right)_{\boldsymbol{\varepsilon}'}=$$

$$=\left(\lim_{p\to\infty}U_{\varepsilon,\varepsilon',p}^{t''-t'}\varphi_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}''-\boldsymbol{q}')\right)_{\boldsymbol{\varepsilon}'}\qquad(266)$$

Here the limit is calculated by norm $\|\cdot\|_{W^{2,2}(\mathbb{R}^n,\omega)}$ of the weighted Sobolev space $(\mathbb{R}^n,\omega)$[41].

Theorem14. Assume that $\varphi_{\boldsymbol{\varepsilon'}}(\boldsymbol{q''}-\boldsymbol{q'}) \in H_n^{\infty}(S_R)\cap W^{2,2}(\mathbb{R}^n,\omega)$, $\varepsilon' \in \langle 0,1],\omega=\omega(q'')$. Then:

(1) $\forall\varepsilon' \in \langle 0,1]$there exist p_0 such that$\forall p_1 \forall p_2[(p_1 \geq p_0)\wedge(p_2 \geq p_0)]$the inequality

〖∀ε〗 ^'∈⟨0,├ 1]┤ there

exist p_0 such that $\forall p_1 \forall p_2[(p_1 \geq p_0)\wedge(p_2 \geq p_0)]$ the inequality

〖∀ε〗 ^'∈⟨0,├ 1]┤ there

exist p_0 such that $\forall p_1 \forall p_2[(p_1 \geq p_0)\wedge(p_2 \geq p_0)]$ the inequality

$$\|u_{\boldsymbol{\varepsilon'},p_1}(\boldsymbol{q'},t'|\boldsymbol{q''},t'')-u_{\boldsymbol{\varepsilon'},p_1p_2}(\boldsymbol{q'},t'|\boldsymbol{q''},t'')\|_{W^{2,2}(\mathbb{R}^n,\omega)}$$
$$\leq ((t''-t')/(p_1))C_1\exp[C_2(t''-t')]\|\varphi_{\boldsymbol{\varepsilon'}}\|_{W^{2,2}(\mathbb{R}^n,\omega)}$$

holds for each $t'' \in [t',\infty)$.

(2) The Colombeau generalized function$\left(u_{\boldsymbol{\varepsilon'}}(\boldsymbol{q''},t'')\right)_{\boldsymbol{\varepsilon'}}$ given by formula (266) is the solution of the Colombeau-Fokker-Planck equation (203))-(204) (except a set Lebesgue measure zero) with initial condition $\left(\varphi_{\boldsymbol{\varepsilon'}}(\boldsymbol{q''}-\boldsymbol{q'})\right)_{\boldsymbol{\varepsilon'}}$such that

$$\forall\varepsilon' \in \langle 0,1]: \varphi_{\boldsymbol{\varepsilon'}}(\boldsymbol{q''}-\boldsymbol{q'}) \in H_n^{\infty}(S_R).$$

Remark 8.The continuous-space-time conditional probability when $p\to\infty$in (266) is symbolically indicated by the path-integral expression [22-23]:

$$\left(p_{\boldsymbol{\varepsilon'}}^{\boldsymbol{\varepsilon}}(\boldsymbol{q'},t'|\boldsymbol{q''},t'')\right)_{\boldsymbol{\varepsilon'}} =$$

$$=\int_{\boldsymbol{q}(t')=\boldsymbol{q'}}^{\boldsymbol{q}(t'')=\boldsymbol{q''}}[\boldsymbol{Dq}(t)]\exp\left[\left(-\frac{1}{2\boldsymbol{\varepsilon}}\boldsymbol{S}_{\boldsymbol{\varepsilon'}}(\dot{\boldsymbol{q}},\boldsymbol{q},t;\varepsilon)\right)_{\boldsymbol{\varepsilon'}}\right]. \qquad (267)$$

Here

$$\boldsymbol{S}_{\boldsymbol{\varepsilon}'}(\dot{\boldsymbol{q}},\boldsymbol{q},t,\varepsilon) = \int_{t'}^{t''} \mathcal{L}_{\boldsymbol{\varepsilon}'}(\dot{\boldsymbol{q}},\boldsymbol{q},t;\varepsilon)dt \quad (268)$$

is the continuous-time limit of the discrete action(207)with

$$\mathcal{L}_{\boldsymbol{\varepsilon}'}(\dot{\boldsymbol{q}},\boldsymbol{q},t;\varepsilon) =$$

$$\|\dot{\boldsymbol{q}}(t) - \boldsymbol{b}_{\varepsilon'}(\boldsymbol{q}(t),t;\varepsilon)\|^2 - \varepsilon \sum_{\boldsymbol{i}=\boldsymbol{1}}^{n} b_{\boldsymbol{i},\boldsymbol{i},\boldsymbol{\varepsilon}'}(\boldsymbol{q}(t),t;\varepsilon) \quad (269)$$

$$b_{\boldsymbol{i},\boldsymbol{i},\boldsymbol{\varepsilon}'}(\boldsymbol{q}(t),t;\varepsilon) = \frac{\partial b_{\boldsymbol{i}}(\boldsymbol{q}(t),t;\varepsilon)}{\partial q_i};\ \varepsilon,\varepsilon' \in (0,1], \quad (270)$$

as the Lagrangian. From Eq.**(**267) one obtain

$$\left(\mathbf{E}\left[\boldsymbol{x}_{t'',\varepsilon'}^{\boldsymbol{q}',\varepsilon}(\omega)\right]^{\mathbf{2}}\right)_{\boldsymbol{\varepsilon}'} = \int_{-\infty}^{\infty} \boldsymbol{dq}''\, \|\boldsymbol{q}''\|^{\mathbf{2}} \left[\left(\boldsymbol{p}_{\boldsymbol{\varepsilon}'}^{\varepsilon}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'')\right)_{\boldsymbol{\varepsilon}'}\right] =$$

$$= \left(\lim_{p\to\infty}\ \lim_{\Delta t\to 0} \breve{\boldsymbol{I}}_{N,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',p}(\boldsymbol{q}',t',t'')\right)_{\boldsymbol{\varepsilon}'}. \quad (271)$$

Here

$$\breve{\boldsymbol{I}}_{N,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',p}(\boldsymbol{q}',t',t'') = \int_{-\infty}^{\infty} \boldsymbol{dq}''\, \|\boldsymbol{q}''\|^{\mathbf{2}} \boldsymbol{I}_{N,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',p}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'') \quad (272)$$

Remark9. 1. Note that for any fixed N in the limit $\varepsilon \to 0$ only one unique minimizing path $\{\breve{\boldsymbol{q}}_0,\breve{\boldsymbol{q}}_1,\dots,\breve{\boldsymbol{q}}_{N-1},\breve{\boldsymbol{q}}_N\}$ significantly contribute to the multiple integral $\breve{\boldsymbol{I}}_{N,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}',p}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'')$given by expression (272).The extremality conditions for this minimizing path is

$$\overline{\nabla}\boldsymbol{q}(t_m) = \boldsymbol{b}_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}(t_{m-1}),t_m), m = 1,2,\dots,N. \quad (273)$$

With a boundary condition

$$\boldsymbol{q}(t') = \breve{\boldsymbol{q}}_{\boldsymbol{0}} = \boldsymbol{q}'. \quad (274)$$

Here $\bar{\nabla}$ is a conjugate of the difference operator ∇ defined by formulae [45]:

$$\nabla \boldsymbol{q}(t_m) = \frac{\boldsymbol{q}(t_{m+1})-\boldsymbol{q}(t_m)}{\Delta t}, N \geq m \geq 0, \quad (275)$$

$$\bar{\nabla} \boldsymbol{q}(t_m) = \frac{\boldsymbol{q}(t_m)-\boldsymbol{q}(t_{m-1})}{\Delta t}, N+1 \geq m \geq 1. \quad (276)$$

2. However we note that as that was shown in [7] the canonical Laplace approximation [27] is not a valid asymptotic approxi- mation in the limit $\varepsilon \to 0$ for a path-integral (271), see also [42].

From Eq.(267)-Eq.(268) one obtain

$$\left(\mathbf{E}\left[\boldsymbol{x}_{t'',\varepsilon'}^{\boldsymbol{q}',\varepsilon}(\omega)\right]^{\mathbf{2}}\right)_{\boldsymbol{\varepsilon}'} = \int_{-\infty}^{\infty} \boldsymbol{dq}'' \, \|\boldsymbol{q}''\|^{\mathbf{2}} \left[\left(\boldsymbol{p}_{\boldsymbol{\varepsilon}'}^{\varepsilon}(\boldsymbol{q}',\boldsymbol{t}'|\boldsymbol{q}'',t'')\right)_{\boldsymbol{\varepsilon}'}\right]$$

$$= \int_{-\infty}^{\infty} \boldsymbol{dq}'' \, \|\boldsymbol{q}''\|^{2} \int_{\boldsymbol{q}(t')=\boldsymbol{q}'}^{\boldsymbol{q}(t'')=\boldsymbol{q}''} [D\boldsymbol{q}(t)] \exp\left[\left(-\frac{1}{2\varepsilon}\int_{t'}^{t''} \mathcal{L}_{\boldsymbol{\varepsilon}'}(\dot{\boldsymbol{q}},\boldsymbol{q},t;\varepsilon)dt\right)_{\boldsymbol{\varepsilon}'}\right] =$$

$$\int_{\boldsymbol{q}(\boldsymbol{t}')=\boldsymbol{q}'} [\boldsymbol{Dq}(t)] \|\boldsymbol{q}(t'')\|^{2} \exp\left[\left(-\frac{1}{2\varepsilon}\int_{\boldsymbol{t}'}^{\boldsymbol{t}''} \mathcal{L}_{\varepsilon'}(\dot{\boldsymbol{q}},\boldsymbol{q},t;\varepsilon)dt\right)_{\boldsymbol{\varepsilon}'}\right]. \quad (277)$$

Let us consider now the quantity

$$\left(\boldsymbol{p}_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon}}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'';L,m')\right)_{\boldsymbol{\varepsilon}'} = \left(\lim_{\Delta \mathbf{t}\to \mathbf{0}} \boldsymbol{I}_{m',N,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}^{L}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'')\right)_{\boldsymbol{\varepsilon}'}. \quad (278)$$

Here

$$\boldsymbol{I}_{m',N,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}^{L}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'') =$$

$$= \breve{\boldsymbol{N}}_N \int_{-\boldsymbol{L}}^{\boldsymbol{L}} \boldsymbol{dq}_{\mathbf{1}} \dots \int_{-\boldsymbol{L}}^{\boldsymbol{L}} \boldsymbol{dq}_{m'} \int_{-\infty}^{\infty} \boldsymbol{dq}_{m'+1} \dots \int_{-\infty}^{\infty} \boldsymbol{dq}_{N-1} \times$$

$$\exp\left[-\frac{1}{2\boldsymbol{\varepsilon}}\left(S_{\boldsymbol{\varepsilon}'}\left(\boldsymbol{q_0},\boldsymbol{q}_1,\dots,\boldsymbol{q}_{m'},\boldsymbol{q}_{m'+1,}\dots,\boldsymbol{q_{N-1}},\boldsymbol{q}_N,\varepsilon\right)\right)_{\boldsymbol{\varepsilon}'}\right], \quad (279)$$

and $m' \ll N, L \gg 1$.

The quantity defined by Eq.(278)-Eq. (279) is symbolically indicated by the path-integral expression

$$\left(\boldsymbol{p}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'';L,m')\right)_{\boldsymbol{\varepsilon}'} =$$

$$= \int_{\boldsymbol{q}(\boldsymbol{t}')=\boldsymbol{q}'}^{\boldsymbol{q}(\boldsymbol{t}'')=\boldsymbol{q}''}[D\boldsymbol{q}(t;L,m')]\exp\left[\left(-\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\varepsilon}}S_{\boldsymbol{\varepsilon}'}(\dot{\boldsymbol{q}},\boldsymbol{q},t;\varepsilon)\right)_{\boldsymbol{\varepsilon}'}\right]. \quad (280)$$

Using Eq.(278)-Eq.(280) we define the quantity

$$\left(\mathbf{E}_{L,m'}\left[\boldsymbol{x}^{q',\varepsilon}_{t'',\varepsilon'}(\omega)\right]^{\mathbf{2}}\right)_{\boldsymbol{\varepsilon}'} =$$

$$= \int_{-\infty}^{\infty}\boldsymbol{dq}''\,\|\boldsymbol{q}''\|^2\left[\left(\boldsymbol{p}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'';L,m')\right)_{\boldsymbol{\varepsilon}'}\right] \quad (281)$$

$$= \lim_{\Delta\boldsymbol{t}\to 0}\breve{\boldsymbol{I}}_{N,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t',t'';L,m').$$

Here

$$\breve{\boldsymbol{I}}_{\boldsymbol{N},\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t',t'';L,m') =$$

$$\int_{-\infty}^{\infty}\boldsymbol{dq}''\,\|\boldsymbol{q}''\|^2\boldsymbol{I}_{N,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'';L,m'). \quad (282)$$

The quantity (281) is symbolically indicated by the path-integral expression

$$\left(\mathbf{E}_{L,m'}\left[\boldsymbol{x}^{q',\varepsilon}_{t'',\varepsilon'}(\omega)\right]^2\right)_{\boldsymbol{\varepsilon}'}$$

$$= \int_{-\infty}^{\infty} \boldsymbol{dq}'' \, \|\boldsymbol{q}''\|^2 \int_{\boldsymbol{q}(\boldsymbol{t}')=\boldsymbol{q}'}^{\boldsymbol{q}(\boldsymbol{t}'')=\boldsymbol{q}''} [D\boldsymbol{q}(t;L,m')] \times \exp\left[\left(-\frac{1}{2\varepsilon}\int_{\boldsymbol{t}'}^{\boldsymbol{t}''} \boldsymbol{\mathcal{L}}_{\boldsymbol{\varepsilon}'}(\dot{\boldsymbol{q}},\boldsymbol{q},t)dt\right)_{\boldsymbol{\varepsilon}'}\right] =$$

$$=\int_{\boldsymbol{q}(\boldsymbol{t}')=\boldsymbol{q}'} [D\boldsymbol{q}(t;L,m')] \|\boldsymbol{q}(t'')\|^2 \exp\left[\left(-\tfrac{1}{2\varepsilon}\int_{\boldsymbol{t}'}^{\boldsymbol{t}''} \boldsymbol{\mathcal{L}}_{\boldsymbol{\varepsilon}'}(\dot{\boldsymbol{q}},\boldsymbol{q},t)dt\right)_{\boldsymbol{\varepsilon}'}\right]. \quad (283)$$

From Eq.(205) and Eq.(279) we obtain

$$\boldsymbol{I}_{N,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'') = \boldsymbol{I}^{\boldsymbol{L}}_{\boldsymbol{m}',\boldsymbol{N},\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}(\boldsymbol{q}',\boldsymbol{t}'|\boldsymbol{q}'',\boldsymbol{t}'') + \boldsymbol{\Theta}^{\boldsymbol{L}}_{\boldsymbol{m}',\boldsymbol{N},\boldsymbol{\varepsilon},,\boldsymbol{\varepsilon}'}(\boldsymbol{q}',\boldsymbol{t}'|\boldsymbol{q}'',\boldsymbol{t}''). \quad (284)$$

Here

$$\boldsymbol{\Theta}^{L}_{m',N,\boldsymbol{\varepsilon},,\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'') =$$

$$\breve{\boldsymbol{N}}_{\boldsymbol{N}} \int_{\mathbb{R}^{\boldsymbol{n}}\setminus\boldsymbol{\varpi}_{\boldsymbol{L}}} \boldsymbol{dq}_1 \dots \int_{\mathbb{R}^{\boldsymbol{n}}\setminus\boldsymbol{\varpi}_{\boldsymbol{L}}} \boldsymbol{dq}_{m'} \int_{-\infty}^{\infty} \boldsymbol{dq}_{m'+1} \dots \int_{-\infty}^{\infty} \boldsymbol{dq}_{N-1} \times$$

$$\exp\left[-\tfrac{1}{2\varepsilon}\left(\boldsymbol{S}_{\varepsilon'}(\boldsymbol{q}_0,\boldsymbol{q}_1,\dots,\boldsymbol{q}_{m'},\boldsymbol{q}_{m'+1},\dots,\boldsymbol{q}_{N-1},\boldsymbol{q}_N,t,\varepsilon)\right)_{\boldsymbol{\varepsilon}'}\right]. \quad (285)$$

Here

$$\boldsymbol{\varpi}_{\boldsymbol{L}}=[-L,L]^n. \quad (286)$$

From Eq.(282) and Eq.(284) we obtain

$$\breve{\boldsymbol{I}}_{\boldsymbol{N},\boldsymbol{\varepsilon},,\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t',t'') = \breve{\boldsymbol{I}}^{\boldsymbol{L}}_{m',N,\boldsymbol{\varepsilon},,\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t',t'') + \breve{\boldsymbol{\Theta}}^{\boldsymbol{L}}_{m',N,\boldsymbol{\varepsilon},,\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t',t''). \quad (287)$$

Here

$$\breve{\boldsymbol{\Theta}}^{\boldsymbol{L}}_{m,N,\boldsymbol{\varepsilon},,\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t',t'') = \breve{\boldsymbol{N}}_{\boldsymbol{N}} \int_{\mathbb{R}^{\boldsymbol{n}}\setminus\boldsymbol{\varpi}_{\boldsymbol{L}}} \boldsymbol{dq}_{\boldsymbol{1}} \dots \int_{\mathbb{R}^{\boldsymbol{n}}\setminus\boldsymbol{\varpi}_{\boldsymbol{L}}} \boldsymbol{dq}_{m'} \int_{-\infty}^{\infty} \boldsymbol{dq}_{m'+1} \dots \int_{-\infty}^{\infty} \boldsymbol{dq}_{\boldsymbol{N}-1} \int_{-\infty}^{\infty} \boldsymbol{dq}_{N} \times$$

$$\|\boldsymbol{q}_N\|^2 \exp\left[-\frac{1}{2\varepsilon}\left(\boldsymbol{S}_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}_0, \boldsymbol{q}_1, \dots, \boldsymbol{q}_{m'}, \boldsymbol{q}_{m'+1} \dots, \boldsymbol{q}_{N-1}, \boldsymbol{q}_N, t, \varepsilon)\right)_{\boldsymbol{\varepsilon}'}\right] \quad (288)$$

Remark. We note that: $\forall \boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}' \neq 0$ there exist parameter $L = L(\boldsymbol{\varepsilon}')$ such that$\forall \boldsymbol{q}_m(\|\boldsymbol{q}_m\| \geq L)$the inequality

$$\left\|\boldsymbol{b}_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}_m, t_m)\right\|^2 \leq \|q_m\|^{-q}, q \geq 2 \qquad (289)$$

is satisfied.

Lemma1.

$$\breve{\boldsymbol{\Theta}}^{\boldsymbol{L}}_{m,N,\boldsymbol{\varepsilon},,\boldsymbol{\varepsilon}'}(\boldsymbol{q}', t', t'') \leq O\left(\exp(-L)\right). \qquad (290)$$

Proof. Using inequality (289),we willing to choose parameter L such that the equality

$$S_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}_0, \boldsymbol{q}_1, \dots, \boldsymbol{q}_m, \dots, \boldsymbol{q}_{N-1}, \boldsymbol{q}_N, t, \varepsilon) = \Delta t \sum_{m=1}^{N} \left\|\frac{\boldsymbol{q}_m - \boldsymbol{q}_{m-1}}{\Delta t}\right\|^2 +$$

$$+O\left((t'' - t')L^{-q}\right) \cong \frac{1}{\Delta t}\sum_{m=1}^{N}\|\boldsymbol{q}_m - \boldsymbol{q}_{m-1}\|^2 \qquad (291)$$

is satisfied. From Eq.(288) and Eq.(291) we obtain

$$\breve{\boldsymbol{\Theta}}^{L}_{m',N,\boldsymbol{\varepsilon},,\boldsymbol{\varepsilon}'}(\boldsymbol{q}', t', t'') =$$

$$= \breve{\boldsymbol{N}}_{\boldsymbol{N}} \int_{\mathbb{R}^{\boldsymbol{n}}\setminus\boldsymbol{\varpi}_{\boldsymbol{L}}} d\boldsymbol{q}_1 \dots \int_{\mathbb{R}^{\boldsymbol{n}}\setminus\boldsymbol{\varpi}_{\boldsymbol{L}}} d\boldsymbol{q}_{m'} \int_{-\infty}^{\infty} d\boldsymbol{q}_{m'+1} \dots \int_{-\infty}^{\infty} d\boldsymbol{q}_{N-1} \int_{-\infty}^{\infty} d\boldsymbol{q}_N$$

$$\times \|\boldsymbol{q}_N\|^2 \exp\left[-\frac{1}{2\varepsilon}\left(S_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}_0, \boldsymbol{q}_1, \dots, \boldsymbol{q}_m, \boldsymbol{q}_{m'+1} \dots, \boldsymbol{q}_{N-1}, \boldsymbol{q}_N, \varepsilon)\right)_{\boldsymbol{\varepsilon}'}\right] \cong$$

$$\cong \breve{\boldsymbol{N}}_{\boldsymbol{N}} \int_{\mathbb{R}^{\boldsymbol{n}}\setminus\boldsymbol{\varpi}_{\boldsymbol{L}}} d\boldsymbol{q}_1 \dots \int_{\mathbb{R}^{\boldsymbol{n}}\setminus\boldsymbol{\varpi}_{\boldsymbol{L}}} d\boldsymbol{q}_{m'} \int_{-\infty}^{\infty} d\boldsymbol{q}_{m'+1} \dots \int_{-\infty}^{\infty} d\boldsymbol{q}_{N-1} \int_{-\infty}^{\infty} d\boldsymbol{q}_N \times$$

$$\times \|\boldsymbol{q}_N\|^2 \exp\left[-\frac{1}{2\varepsilon\Delta t}\sum_{m=1}^{N}\|\boldsymbol{q}_m - \boldsymbol{q}_{m-1}\|^2\right]. \qquad (292)$$

From Eq.(292) we obtain the inequality

$$\breve{\mathbf{\Theta}}^{L}_{m',N,\boldsymbol{\varepsilon},,\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t',t'') \leq$$

$$\breve{\boldsymbol{N}}_{\boldsymbol{N}} \int_{-\infty}^{\infty} d\boldsymbol{q}_1 \dots \int_{-\infty}^{\infty} d\boldsymbol{q}_{m-1} \int_{\mathbb{R}^n \setminus \boldsymbol{\varpi}_L} d\boldsymbol{q}_{m'} \int_{-\infty}^{\infty} d\boldsymbol{q}_{m'+1} \dots \quad (293)$$

$$\dots \int_{-\infty}^{\infty} d\boldsymbol{q}_{N-1} \int_{-\infty}^{\infty} \boldsymbol{dq}_N \times \|\boldsymbol{q}_N\|^2 \exp\left[-\frac{1}{2\varepsilon\Delta t} \sum_{m=1}^{N} \|\boldsymbol{q}_m - \boldsymbol{q}_{m-1}\|^2\right].$$

Let $\mathfrak{C}^{L}_{m',N,\boldsymbol{\varepsilon},,\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t',t'')$be the multiple integral:

$$\mathfrak{C}^{L}_{m',N,\boldsymbol{\varepsilon},,\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t',t'') == \breve{\boldsymbol{N}}_N \int_{-\infty}^{\infty} d\boldsymbol{q}_1 \dots \int_{-\infty}^{\infty} d\boldsymbol{q}_{m'-1} \int_{\mathbb{R}^n \setminus \boldsymbol{\varpi}_L} d\boldsymbol{q}_{m'} \int_{-\infty}^{\infty} d\boldsymbol{q}_{m'+1} \dots$$

$$\dots \int_{-\infty}^{\infty} d\boldsymbol{q}_{N-1} \int_{-\infty}^{\infty} \boldsymbol{dq}_N \, \|\boldsymbol{q}_N\|^2 \exp\left[-\frac{1}{2\varepsilon\Delta t} \sum_{m=1}^{N} \|\boldsymbol{q}_m - \boldsymbol{q}_{m-1}\|^2\right] =$$

$$= \breve{\boldsymbol{N}}_N \int_{-\infty}^{\infty} d\boldsymbol{q}_1 \dots \int_{-\infty}^{\infty} d\boldsymbol{q}_{m'-1} \exp\left[\frac{1}{2\varepsilon\Delta t} \sum_{m=1}^{m'} \|\boldsymbol{q}_m - \boldsymbol{q}_{m-1}\|^2\right] \times$$

$$\times \int_{\mathbb{R}^n \setminus \boldsymbol{\varpi}_L} d\boldsymbol{q}_{m'} \int_{-\infty}^{\infty} d\boldsymbol{q}_{m'+1} \dots \int_{-\infty}^{\infty} d\boldsymbol{q}_{N-1} \int_{-\infty}^{\infty} \boldsymbol{dq}_N \times$$

$$\times \|\boldsymbol{q}_{\boldsymbol{N}}\|^2 \exp\left[\frac{1}{2\varepsilon\Delta t} \sum_{m=m'+1}^{N} \|\boldsymbol{q}_m - \boldsymbol{q}_{m-1}\|^2\right]. \quad (294)$$

By simple canonical observation (see [44] chapt.3) we obtain

$$\int_{-\infty}^{\infty} d\boldsymbol{q}_1 \dots \int_{-\infty}^{\infty} d\boldsymbol{q}_{m'-1} \exp\left[-\frac{1}{2\varepsilon\Delta t} \sum_{m=1}^{m'} \|\boldsymbol{q}_m - \boldsymbol{q}_{m-1}\|^2\right] =$$

$$(2\pi\varepsilon\Delta t)^{n(m'-1)/2}\exp\left[\frac{1}{2\varepsilon(m'-1)\Delta t}\|\boldsymbol{q}_{m'}-\boldsymbol{q}_0\|^2\right]. \qquad (295)$$

Substitution Eq.(295) into Eq.(294)gives

$$\mathfrak{C}^{L}_{m',N,\boldsymbol{\varepsilon},,\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t',t'') =$$

$$= \breve{\boldsymbol{N}}_{N,m'}\int_{\mathbb{R}^{\boldsymbol{n}}\backslash\boldsymbol{\varpi}_{L}} d\boldsymbol{q}_{m'}\exp\left[-\frac{1}{2\varepsilon(m'-1)\Delta t}\|\boldsymbol{q}_{m'}-\right.$$

$$\left.-\,\boldsymbol{q}_0\|^2\right]\int_{-\infty}^{\infty} d\boldsymbol{q}_{m'+1}\cdots\int_{-\infty}^{\infty} d\boldsymbol{q}_{N-1}\int_{-\infty}^{\infty} \boldsymbol{dq}_N\,\|\boldsymbol{q}_N\|^2\times$$

$$\times\exp\left[-\frac{1}{2\varepsilon\Delta t}\sum_{m=m'+1}^{N}\|\boldsymbol{q}_m-\boldsymbol{q}_{m-1}\|^2\right]. \qquad (296)$$

Here

$$\breve{\boldsymbol{N}}_{N,m'} = (2\pi\varepsilon\Delta t)^{n(m'-1)/2}\breve{\boldsymbol{N}}_{N}. \qquad (297)$$

From Eq.(296) we obtain

$$\mathfrak{C}^{L}_{m',N,\boldsymbol{\varepsilon},,\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t',t'') =$$

$$\breve{\boldsymbol{N}}_{N,m'}(2\pi\varepsilon\Delta t)^{\frac{n(N-m'-1)}{2}}\int_{\mathbb{R}^{n}\backslash\boldsymbol{\varpi}_{L}} d\boldsymbol{q}_{m'}\times\exp\left[\frac{1}{2\varepsilon(m'-1)\Delta t}\|\boldsymbol{q}_{m'}-\boldsymbol{q}_0\|^2\right]\int_{-\infty}^{\infty}\boldsymbol{dq}_N\,\|\boldsymbol{q}_N\|^2\times$$

$$\exp\left[\frac{1}{2\varepsilon(N-m'+1)\Delta t}\|\boldsymbol{q}_N-\boldsymbol{q}_{m'}\|^2\right] =$$

$$\breve{\boldsymbol{N}}_{N,m'}(2\pi\varepsilon\Delta t)^{n(N-m'+1)/2}\times\int_{\mathbb{R}^{\boldsymbol{n}}\backslash\boldsymbol{\varpi}_{L}}\boldsymbol{dq}_{m'}\|\boldsymbol{q}_{m'}\|^2\exp\left[\frac{\mathbf{1}}{\mathbf{2}\varepsilon(m'-1)\Delta t}\|\boldsymbol{q}_{m'}-\right.$$

$$\left.\boldsymbol{q}_0\|^2\right]. \qquad (298)$$

Assumption. We assume now that$\boldsymbol{q}_0 \notin \mathbb{R}^{\boldsymbol{n}}\backslash\boldsymbol{\varpi}_{\boldsymbol{L}}$.

From Eq.(298) using Laplace approximation [43] we obtain

$$\mathfrak{C}^{L}_{m',N,\boldsymbol{\varepsilon},,\boldsymbol{\varepsilon}'}(\boldsymbol{q}',t',t'') \leq O\big(\exp(-L)\big). \qquad (299)$$

Theorem 15.(Hölder's inequality) Let $r_1 = p, r_2 = q \in [1,\infty]$,with $1/p + 1/q = 1$, and let $\Upsilon, \boldsymbol{\mathfrak{C}_i} : \left(C^1([t',t''])\right)^n \to \mathbb{R}, i = 1,2,3$ be an functional such that $\Upsilon = \Upsilon(\dot{\boldsymbol{q}}, \boldsymbol{q}, t', t''), \mathfrak{C}_{\boldsymbol{i}} = \mathfrak{C}_i(\dot{\boldsymbol{q}}, \boldsymbol{q}, t', t'') i = 1,2,3, \mathfrak{C}_3 = \mathfrak{C}_{\boldsymbol{1}}\mathfrak{C}_{\boldsymbol{2}}, \Upsilon(\dot{\boldsymbol{q}}, \boldsymbol{q}, t', t'') > 0$. Let $\|\mathfrak{C}_{\boldsymbol{i}}(\boldsymbol{q}', t', t''; L, m')\|_{r_i}^{\Upsilon}$, $i = 1,2,3$ be the path integral

$$\|\mathfrak{C}_i(\boldsymbol{q}', t', t''; L, m')\|_{r_i}^{\Upsilon} =$$

$$= \left(\int_{-\infty}^{\infty} d\boldsymbol{q}'' \int_{\boldsymbol{q}(t')=\boldsymbol{q}'}^{\boldsymbol{q}(t'')=\boldsymbol{q}''} [\boldsymbol{Dq}(t; L, m')] \Upsilon(\dot{\boldsymbol{q}}, \boldsymbol{q}, t', t'') \mathfrak{C}_{\boldsymbol{i}}^{r_i} |(\dot{\boldsymbol{q}}, \boldsymbol{q}, t', t'')|^{r_i} \right)^{\boldsymbol{1/r_i}} < \infty.$$

Then

$$\|\mathfrak{C}_{\boldsymbol{3}}(\boldsymbol{q}', t', t''; L, m')\|_1^{\Upsilon} \leq$$

$$\leq \|\mathfrak{C}_{\boldsymbol{1}}(\boldsymbol{q}', t', t''; L, m')\|_p^{\Upsilon} \|\mathfrak{C}_{\boldsymbol{2}}(\boldsymbol{q}', t', t''; L, m')\|_q^{\Upsilon}. \quad (300)$$

From Theorem 5 we obtain.

Corollary1. Assume that: (1) $\frac{1}{p} + \frac{1}{q} = 1$,

(2) $\boldsymbol{I}_p = \boldsymbol{I}_p(\boldsymbol{q}', t', t''; L, m') = \int_{-\infty}^{\infty} \boldsymbol{dq}'' \|\boldsymbol{q}''\|^2 \int_{\boldsymbol{q}(\boldsymbol{t}')=\boldsymbol{q}'}^{\boldsymbol{q}(\boldsymbol{t}'')=\boldsymbol{q}''} [\boldsymbol{Dq}(t; L, m')] \exp\left[-\int_{\boldsymbol{t}'}^{\boldsymbol{t}''} [\dot{\boldsymbol{q}}(\boldsymbol{t})]^2 dt\right] \times$
$\exp\left[p \int_{\boldsymbol{t}'}^{\boldsymbol{t}''} \boldsymbol{G}_{\boldsymbol{1}}(\dot{\boldsymbol{q}}, \boldsymbol{q}, t) dt\right] < \infty, \quad (301)$

(3) $\boldsymbol{I}_{\boldsymbol{q}} = \boldsymbol{I}_q(t', t''; L, m') = \int_{-\infty}^{\infty} \boldsymbol{dq}'' \|\boldsymbol{q}''\|^2 \int_{\boldsymbol{q}(\boldsymbol{t}')=\boldsymbol{q}'}^{\boldsymbol{q}(\boldsymbol{t}'')=\boldsymbol{q}''} [\boldsymbol{Dq}(t; L, m')] \exp\left[-\int_{\boldsymbol{t}'}^{\boldsymbol{t}''} [\dot{\boldsymbol{q}}(\boldsymbol{t})]^2 dt\right] \times$
$\exp\left[q \int_{\boldsymbol{t}'}^{\boldsymbol{t}''} \boldsymbol{G}_{\boldsymbol{2}}(\dot{\boldsymbol{q}}, \boldsymbol{q}, t) dt\right] < \infty. \quad (302)$

Then inequality

$$\int_{-\infty}^{\infty} \boldsymbol{dq}'' \|\boldsymbol{q}''\|^2 \int_{\boldsymbol{q}(\boldsymbol{t}')=\boldsymbol{q}'}^{\boldsymbol{q}(\boldsymbol{t}'')=\boldsymbol{q}''} [\boldsymbol{Dq}(t; L, m')] \exp\left[-\int_{\boldsymbol{t}'}^{\boldsymbol{t}''} [\dot{\boldsymbol{q}}(\boldsymbol{t})]^2 \boldsymbol{dt}\right] \times$$

$$\exp\left[\int_{\boldsymbol{t}'}^{\boldsymbol{t}''} \boldsymbol{G}_{\boldsymbol{1}}(\dot{\boldsymbol{q}}, \boldsymbol{q}, t) dt\right] \exp\left[\int_{\boldsymbol{t}'}^{\boldsymbol{t}''} \boldsymbol{G}_{\boldsymbol{2}}(\dot{\boldsymbol{q}}, \boldsymbol{q}, t) dt\right] \leq [\boldsymbol{I}_p]^{\frac{1}{p}} \times [\boldsymbol{I}_q]^{\frac{1}{q}} (303)$$

is satisfied.

Theorem 16.(1)Let $\boldsymbol{b}(\boldsymbol{x},t) = \left(b_1(\boldsymbol{x},t),\ldots,b_n(\boldsymbol{x},t)\right)$ an be an vector function, where$b_i(\boldsymbol{x},t), i = 1,\ldots,n$ is a polynomial on variable $\boldsymbol{x}$.Let $\widehat{\boldsymbol{b}}(\boldsymbol{x},t)$be the linear part of thevector function $\boldsymbol{b}(\boldsymbol{x},t)$ i. e.,

$$\widehat{b}_{\mathrm{i}}(\boldsymbol{x},t) = \sum_{\alpha,|\alpha|\leq 1} b_i^{\alpha}(t)\boldsymbol{x}^{\alpha}, i = 1,\ldots,n. \quad (304)$$

Let$\hat{\mathcal{L}}(\dot{\boldsymbol{q}},\boldsymbol{q},t)$be the Lagrangian

$$\hat{\mathcal{L}}(\dot{\boldsymbol{q}},\boldsymbol{q},t) = \left\|\dot{\boldsymbol{q}}(\boldsymbol{t}) - \widehat{\boldsymbol{b}}(\boldsymbol{q}(t),t)\right\|^{\mathbf{2}} - \varepsilon\sum_{\boldsymbol{i}=1}^{n}\widehat{b}_{\boldsymbol{i},\boldsymbol{i}}(\boldsymbol{q}(t),t). \quad (305)$$

Here

$$\widehat{b}_{\boldsymbol{i},\boldsymbol{i}}(\boldsymbol{q}(t),t) = \frac{\boldsymbol{\partial}\widehat{b}_i(q(t),t)}{\boldsymbol{\partial q_i}}. \quad (306)$$

and let $\boldsymbol{\mathcal{L}}_{\boldsymbol{\varepsilon}'}(\dot{\boldsymbol{q}},\boldsymbol{q},t)$be

$$\boldsymbol{\mathcal{L}}_{\varepsilon'}(\dot{\boldsymbol{q}},\boldsymbol{q},t) = \|\dot{\boldsymbol{q}}(t) - \boldsymbol{b}(\boldsymbol{q}_{\boldsymbol{\varepsilon}'}(t),t)\|^2 - \varepsilon\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} b_{\boldsymbol{i},\boldsymbol{i}}(\boldsymbol{q}_{\boldsymbol{\varepsilon}'}(t),t). \quad (307)$$

Here $\boldsymbol{q}_{\boldsymbol{\varepsilon}'}(t) = \left(q_{\varepsilon',1}(t),\ldots,\boldsymbol{q}_{\boldsymbol{\varepsilon}',\boldsymbol{i}}(t),\ldots,\boldsymbol{q}_{\boldsymbol{\varepsilon}',n}(t)\right)$ and

$$q_{\varepsilon',i}(t) = \frac{q_i(t)}{1+(\boldsymbol{\varepsilon}')^{\boldsymbol{l}}\left[q_i^2(t)+\int_{t'}^{t''} q_i^2(t)dt\right]^l} \quad (308)$$

$$l \geq 3.$$

(2) Let $\boldsymbol{x}_{t'',,\varepsilon'}^{x_0,\varepsilon}(\omega)$ be the solution of the Colombeau-Ito's SDE (200)-(201).

Then there exist Colombeau constant $\widetilde{C}' = \left[\left(C_{\boldsymbol{\varepsilon}'}'\right)_{\boldsymbol{\varepsilon}'}\right] > 0$, such that the inequalities:

(1) $\left(\liminf_{\varepsilon\to 0}\mathbf{E}\left[\left\|\boldsymbol{x}^{x_0,\varepsilon}_{t'',\varepsilon'}(\omega)\right\|^2\right]\right)_{\varepsilon'} \leq \left(C'_{\varepsilon'}\right)_{\varepsilon'}\left(\boldsymbol{I}^{\varepsilon}_{\varepsilon'}\right)_{\varepsilon'}$ (309)

(2) $\left[\left(\liminf_{\varepsilon\to 0}\mathbf{E}\left[\left\|\boldsymbol{x}^{x_0,\varepsilon}_{t'',\varepsilon'}(\omega)\right\|^2\right]\right)_{\varepsilon'}\right] \leq \tilde{C}'\left[\left(\boldsymbol{I}^{\varepsilon}_{\varepsilon'}\right)_{\varepsilon'}\right],$

where

$\left(\boldsymbol{I}^{\varepsilon}_{\varepsilon'}\right)_{\varepsilon'} = \int_{-\infty}^{\infty}\boldsymbol{dq}''\,\|\boldsymbol{q}''\|^2\int_{\boldsymbol{q}(\boldsymbol{t}')=\boldsymbol{q}'}^{\boldsymbol{q}(\boldsymbol{t}'')=\boldsymbol{q}''}[\boldsymbol{Dq}(t)]\exp\left[\left(-\frac{1}{2\varepsilon}\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\hat{\mathcal{L}}_{\varepsilon'}(\dot{\boldsymbol{q}},\boldsymbol{q},t)dt\right)_{\varepsilon'}\right]$ (310)

and

(3) $\liminf_{\varepsilon\to 0}\mathbf{E}\left[\boldsymbol{x}^{x_0,\varepsilon}_{t'',\varepsilon'}(\omega)\right]^2 \leq \tilde{C}'\|\boldsymbol{U}(t'')\|^2$ (311)

is satisfies. Here a vector function $\boldsymbol{U}(t) = (U_1(\mathrm{t}),\ldots,U_{\boldsymbol{n}}(\mathrm{t}))$ is the solution of the differential master equation

$$\dot{\boldsymbol{U}}(t)=\mathbf{J}_{\boldsymbol{t}}\boldsymbol{U}(t)+\boldsymbol{b}(t),\ \boldsymbol{U}(0)=\boldsymbol{x}_0=\boldsymbol{q}'. \quad (312)$$

Here $\boldsymbol{b}(t)=\boldsymbol{b}(\boldsymbol{0},t)$ and $\mathbf{J}_t=\mathbf{J}(t)$ is Jacobian i.e., $\boldsymbol{J}_t$ is $n\times n$-matrix:

$$\mathbf{J}(t)=\left[\partial b_i(\boldsymbol{x},t)/\partial x_j\right]_{\boldsymbol{x}=0}. \quad (313)$$

Proof. For short, we will be considered proof only for the case of the 1-dimensionalColombeau-Ito'sSDE,withoutloss of generality.
Let us consider Feynman's path integral(283) corresponding to 1-dimensional Colombeau-Ito's SDE i.e.,

$$\left(\boldsymbol{I}^{\varepsilon}_{\varepsilon'}(L,m')\right)_{\varepsilon'} = \left(\mathbf{E}_{L,m'}\left[x^{x_0,\varepsilon}_{t'',\varepsilon'}(\omega)\right]\right)_{\varepsilon'} = \int_{-\infty}^{\infty}dq''\,\|q''\|^2\times$$

$$\times\int_{\boldsymbol{q}(\boldsymbol{t}')=\boldsymbol{q}'}^{\boldsymbol{q}(\boldsymbol{t}'')=\boldsymbol{q}''}[\boldsymbol{D}q(t;L,m')]\exp\left[\left(-\frac{1}{2\varepsilon}\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\mathcal{L}_{\varepsilon'}(\dot{q},q,t)dt\right)_{\varepsilon'}\right]. \quad (314)$$

Let us rewrite now Lagrangian $\mathcal{L}_{\varepsilon'}(\dot{q},q,t)$in the next equivalent form

$$\mathcal{L}_{\varepsilon'}(\dot{q},q,t)=\mathcal{L}'_{\varepsilon'}(\dot{q},q,t)-2\varepsilon b_{1,1}(q_{\varepsilon'}(t),t), \qquad (315)$$

Here

$$\mathcal{L}'_{\varepsilon'}(\dot{q},q,t)=[\dot{q}(t)-b(q_{\varepsilon'}(t),t)]^2=$$

$$=\left[\dot{q}(t)-\hat{b}(q_{\varepsilon'}(t),t)-b_2(q_{\varepsilon'}(t),t)\right]^2, \qquad (316)$$

$$\hat{b}(x,t)=b^0(t)+xb^1(t) \qquad (317)$$

and

$$b_2(x,t)=\sum_{\alpha,2\leq|\alpha|\leq r} b^{\alpha}(t)x^{\alpha}. \qquad (318)$$

Let us rewrite now Eq.(318) in the next form

$$b_2(x,t)=b_{2,2}(x,t)+b_{2,3}(x,t). \qquad (319)$$

Here

$$b_{2,2}(x,t)=\sum_{\alpha} b^{\alpha}(t)x^{\alpha}, |\alpha|=2, \qquad (320)$$

$$b_{2,3}(x,t)=\sum_{\alpha} b^{\alpha}(t)x^{\alpha}, |\alpha|\geq 3. \qquad (321)$$

From Eq.(316)-Eq.(318) we obtain

$$\mathcal{L}'_{\varepsilon'}(\dot{q},q,t)=\left[\left[\dot{q}(t)-\hat{b}\ (q_{\varepsilon'}(t),t)\right]-b_2(q_{\varepsilon'}(t),t)\right]^2=$$
$$\left[\dot{q}(t)-\hat{b}(q_{\varepsilon'}(t),t)\right]^{\mathbf{2}}-2b_2(q_{\varepsilon'}(t),t)\left[\dot{q}(t)-\hat{b}(q(t),t)\right]+$$
$$+b_2^2(q_{\varepsilon'}(t),t)=[\dot{q}(t)]^2-2\dot{q}(t)\hat{b}(q_{\varepsilon'}(t),t)+\hat{b}^2(q_{\varepsilon'}(t),t)-$$
$$-2\dot{q}(t)b_2(q_{\varepsilon'}(t),t)+2b_2(q_{\varepsilon'}(t),t)\hat{b}\ (q(t),t). \qquad (322)$$

Substitution Eq.(320) and Eq.(321), into Eq.(322) gives

$$\mathcal{L}'_{\varepsilon'}(\dot{q},q,t) = [\dot{q}(t)]^2 - 2\dot{q}(t)b^0(t) - 2\dot{q}(t)q_{\boldsymbol{\varepsilon}'}(t)b^1(t) +$$

$$+\left[b^0(t)\right]^{\mathbf{2}} + 2q_{\varepsilon'}(t)b^0(t)b^1(t) + \left[q_{\boldsymbol{\varepsilon}'}(t)b^1(t)\right]^2 -$$

$$-2\dot{q}(t)b_2(q_{\varepsilon'}(t),t) + 2b_2(q_{\varepsilon'}(t),t)\hat{b}\ \ (q(t),t) =$$

$$= [\dot{q}(t)]^2 - 2\dot{q}(t)b^0(t) - 2\dot{q}(t)q_{\boldsymbol{\varepsilon}'}(t)b^1(t) +$$

$$\left[b^0(t)\right]^{\mathbf{2}} + 2q_{\varepsilon'}(t)b^0(t)b_0^1(t) + \left[q_{\boldsymbol{\varepsilon}'}(t)b^1(t)\right]^2 -$$

$$-2\dot{q}(t)b_2(q_{\varepsilon'}(t),t) + 2b_2(q_{\varepsilon'}(t),t)b^0(t) +$$

$$+2q_{\varepsilon'}(t)b_2(q_{\varepsilon'}(t),t)b^1(t) =$$

$$= [\dot{q}(t)]^2 - 2\dot{q}(t)b^0(t) - 2\dot{q}(t)q_{\boldsymbol{\varepsilon}'}(t)b^1(t) +$$

$$+\left[b^0(t)\right]^{\mathbf{2}} + 2q_{\varepsilon'}(t)b^0(t)b_0^1(t) + \left[q_{\boldsymbol{\varepsilon}'}(t)b^1(t)\right]^2 -$$

$$-2\dot{q}(t)b_{2,2}(q_{\varepsilon'}(t),t) - 2\dot{q}(t)b_{2,3}(q_{\varepsilon'}(t),t) +$$

$$+2b_{2,2}(q_{\varepsilon'}(t),t)b^0(t) + 2b_{2,3}(q_{\varepsilon'}(t),t)b^0(t) +$$

$$+2q_{\varepsilon'}(t)b_2(q_{\varepsilon'}(t),t)b^1(t). \qquad (323)$$

Substitution Eq. (323) into Eq. (314) gives

$$\left(\boldsymbol{I}^{\varepsilon}_{\boldsymbol{\varepsilon}'}(q',t',t'';L,m')\right)_{\boldsymbol{\varepsilon}'} == \exp\left[-\frac{1}{2\varepsilon}\int_{t'}^{t''}\left[b^0(t)\right]^2 dt\right]\int_{-\infty}^{\infty} dq''\,\|q''\|^2 \int_{q(t')=\boldsymbol{q}'}^{q(\boldsymbol{t}'')=q''}[\boldsymbol{D}q(t;L,m')] \times$$

$$\times$$

$$\exp\Big[-\tfrac{\mathbf{1}}{\mathbf{2}\boldsymbol{\varepsilon}}\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\Big\{[\dot{q}(t)]^2 - 2\dot{q}(t)b^0(t) - 2\dot{q}(t)q_{\boldsymbol{\varepsilon}'}(t)b^1(t) + +2q_{\varepsilon'}(t)b^0(t)b^1(t) +$$

$$\left[q_{\boldsymbol{\varepsilon}'}(t)b^1(t)\right]^2 - 2\dot{q}(t)b_{2,2}(q_{\varepsilon'}(t),t) — 2\dot{q}(t)b_{2,3}(q_{\varepsilon'}(t),t) + 2b_{2,2}(q_{\varepsilon'}(t),t)b^0(t) +$$

$$2b_{2,3}(q_{\varepsilon'}(t),t)b^0(t) + 2q_{\varepsilon'}(t)b_2(q_{\varepsilon'}(t),t)b^1(t)\Big\}dt\Big]. \qquad (324)$$

Using replacement $q(t) = p(t)\sqrt{2\varepsilon}$ into Feynman path integral (324), we obtain

$$\left(\boldsymbol{I}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'}(q',t',t'';L,m')\right)_{\boldsymbol{\varepsilon}'} =$$

$$= \exp\left[-\frac{1}{2\varepsilon}\int_{t'}^{t''}\left[b^0\,(t)\right]^2 dt\right]\int_{-\infty}^{\infty} dq''\,\|q''\|^2 \int_{p(t')=\frac{q'}{\sqrt{2\varepsilon}}}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}}\left[\boldsymbol{D}p\left(t;\frac{L}{\sqrt{2\varepsilon}},m'\right)\right]$$

$\times \exp\Big[-\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\Big\{[\dot{p}(t)]^{\mathbf{2}} - \frac{2}{\sqrt{2\varepsilon}}\dot{p}(t)b^0\,(t) - 2\dot{p}(t)p_{\varepsilon'}(t)b^1\,(t) + +\frac{2}{\sqrt{2\varepsilon}}p_{\varepsilon'}(t)b^0\,(t)b^1\,(t) +$

$\left[p_{\varepsilon'}(t)b^1\,(t)\right]^{\mathbf{2}} - 2\sqrt{2\varepsilon}\dot{p}(t)\breve{b}_2(p_{\varepsilon'}(t),t) +$

$+2\breve{b}_2(p_{\varepsilon'}(t),t)\hat{b}\ \ \left(\sqrt{2\varepsilon}p_{\varepsilon'}(t),t\right)\Big\}dt\Big].$ (325)

Here

$\breve{b}_2(p_{\varepsilon'}(t),t) = \sum_\alpha\left(\sqrt{2\varepsilon}\right)^{|\alpha|-2} b_0^\alpha\,(t)\left(p_{\varepsilon'}(t)\right)^\alpha, 2 \le |\alpha| \le r$ (326)

and

$$p_{\varepsilon'}(t) = \frac{p(t)}{1+2(\varepsilon')^l\varepsilon^l\left[p^2\,(\boldsymbol{t})+\int_{\boldsymbol{t}'}^{\boldsymbol{t}''} p^2\,(\boldsymbol{t})dt\right]^l}\ , l \ge 1. \qquad (327)$$

Let us rewrite now Feynman path integral (325)in the next equivalent form

$$\left(\boldsymbol{I}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'}(q',t',t'';L,m')\right)_{\boldsymbol{\varepsilon}'} =$$

$$= \ \exp\left[-\frac{1}{2\varepsilon}\int_{t'}^{t''}\left[b^0\,(t)\right]^2 dt\right]\int_{-\infty}^{\infty} dq''\,\|q''\|^2 \int_{p(t')=\frac{q'}{\sqrt{2\varepsilon}}}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}}\left[\boldsymbol{D}p\left(t;\frac{L}{\sqrt{2\varepsilon}},m'\right)\right]\times$$

$$\exp\left[-\int_{t'}^{t''}[\dot{p}(t)]^2 dt\right]\left\{\exp\left[-\int_{t'}^{t''}\left[-\frac{2}{\sqrt{2\varepsilon}}\dot{p}(t)b^0\,(t) - \ 2\dot{p}(t)p_{\varepsilon'}(t)b^1\,(t)\right.\right.\right.$$

$$\left.\left.\left. + \frac{2}{\sqrt{2\varepsilon}}p_{\varepsilon'}(t)b^0\,(t)b^1\,(t)+\left[p_{\varepsilon'}(t)b^1\,(t)\right]^2\right]dt\right]\right\}$$

$\exp\left[\int_{t'}^{t''}\left[2\sqrt{2\varepsilon}\dot{p}(t)\breve{b}_2(p_{\varepsilon'}(t),t) -\right.\right.$

$\left.\left.2\breve{b}_2(p_{\varepsilon'}(t),t)\hat{b}\ \ \left(\sqrt{2\varepsilon}p(t),t\right)\right]dt\right]$ (328)

Assume that $1/p + 1/q = 1$ and $q = 1/\varepsilon$. Then

$$p = \frac{1}{1-\varepsilon} = 1 + \varepsilon + o(\varepsilon). \qquad (329)$$

Using now Corollary1, from Eq.(328) we obtain

$$\left(\boldsymbol{I}^{\varepsilon}_{\boldsymbol{\varepsilon}'}(q',t',t'';L,m')\right)_{\boldsymbol{\varepsilon}'} \leq$$

$$\leq \exp\left[-\frac{1}{2\varepsilon}\int_{t'}^{t''}\left[b^0\,(t)\right]^2 dt\right]\left[\int_{-\infty}^{\infty} dq''\,\|q''\|^2 \int_{p(t')=\frac{q'}{\sqrt{2\varepsilon}}}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}}\left[\boldsymbol{D}p\left(t;\frac{L}{\sqrt{2\varepsilon}},m'\right)\right]\right.$$

$$\times \exp\left[-\int_{t'}^{t''}[\dot{p}(t)]^2 dt\right]\times$$

$$\times\left\{\exp\left[-(1+\varepsilon)\int_{t'}^{t''}\left[-\frac{2}{\sqrt{2\varepsilon}}\dot{p}(t)b^0\,(t) - 2\dot{p}(t)p_{\varepsilon'}(t)b^1\,(t) + \frac{2}{\sqrt{2\varepsilon}}p_{\varepsilon'}(t)b^0_0(t)b^1_0(t)\right.\right.\right.$$

$$\left.\left.\left.\left.+\left[p_{\varepsilon'}(t)b^1\,(t)\right]^2\right]dt\right]\right\}\right]^{1-\varepsilon}\times$$

$$\times\left[\int_{-\infty}^{\infty} dq'' \int_{p(\boldsymbol{t}')=\frac{q'}{\sqrt{2\varepsilon}}}^{p(\boldsymbol{t}'')=\frac{q''}{\sqrt{2\varepsilon}}}\left[\boldsymbol{D}p\left(t;\frac{L}{\sqrt{2\varepsilon}},m'\right)\right]\exp\left[-\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}[\dot{p}(t)]^2 dt\right]\times\right.$$

$$\left\{\exp\left[\frac{1}{\varepsilon}\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\left[2\sqrt{2\varepsilon}\dot{p}(t)\breve{b}_2(p_{\varepsilon'}(t),t) - \right.\right.\right.$$

$$\left.\left.\left.\left.2\breve{b}_2(p_{\varepsilon'}(t),t)\hat{b}\ \left(\sqrt{2\varepsilon}\,p(t),t\right)\right]dt\right]\right\}\right]^{\varepsilon}. \qquad (330)$$

Therefore

$$\left(\boldsymbol{I}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'}(q',t',t'';L,m')\right)_{\boldsymbol{\varepsilon}'} \leq \left(\left(\left[\boldsymbol{I}^{\varepsilon,1}_{\boldsymbol{\varepsilon}'}(q',t',t'';L,m')\right]^{1-\boldsymbol{\varepsilon}}\right)_{\boldsymbol{\varepsilon}'}\right)\times\left(\left(\left[\boldsymbol{I}^{\varepsilon,2}_{\boldsymbol{\varepsilon}'}(q',t',t'';L,m')\right]^{\boldsymbol{\varepsilon}}\right)_{\boldsymbol{\varepsilon}'}\right). \qquad (331)$$

Here

$$\left(\boldsymbol{I}^{\varepsilon,1}_{\boldsymbol{\varepsilon}'}(q',t',t'';L,m')\right)_{\boldsymbol{\varepsilon}'} = \left(\exp\left[-\frac{1}{2\varepsilon(1+\varepsilon)}\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\left[b^0\,(t)\right]^2 dt\right]\times\right.$$

$$\times \int_{-\infty}^{\infty} dq'' \, \|q''\|^2 \int_{p(t')=\frac{q'}{\sqrt{2\varepsilon}}}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}} \left[\boldsymbol{D}p\left(t; \frac{L}{\sqrt{2\varepsilon}}, m'\right)\right] \exp\left[-\int_{t'}^{t''} [\dot{p}(t)]^2 dt\right]$$

$\times$

$\exp\Big[-(1+$

$\varepsilon) \int_{t'}^{t''} \Big[-\frac{2}{\sqrt{2\varepsilon}} \dot{p}(t) b^0(t) - 2\dot{p}(t) p_{\varepsilon'}(t) b^1(t) + \frac{2}{\sqrt{2\varepsilon}} p_{\varepsilon'}(t) b^0(t) b^1(t) +$

$\left[p_{\varepsilon'}(t) b^1(t)\right]^2\Big] dt\Big]\Big)_{\varepsilon'}$. (332)

And

$$\left(\boldsymbol{I}_{\varepsilon'}^{\varepsilon,2}\left((q',t',t'';L,m')\right)\right)_{\varepsilon'} =$$

$$= \int_{-\infty}^{\infty} dq'' \int_{p(t')=\frac{q'}{\sqrt{2\varepsilon}}}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}} \left[\boldsymbol{D}p\left(t; \frac{L}{\sqrt{2\varepsilon}}, m'\right)\right] \exp\left[-\int_{t'}^{t''} [\dot{p}(t)]^2 dt\right] \times$$

$\Big\{\exp\Big[\frac{1}{\varepsilon}\int_{t'}^{t''} \big[2\sqrt{2\varepsilon}\dot{p}(t)\breve{b}_2(p_{\varepsilon'}(t),t) -$

$-2\breve{b}_2(p_{\varepsilon'}(t),t)\hat{b}\ \ \left(\sqrt{2\varepsilon}\, p(t),t\right)\big] dt\Big]\Big\}$. (333)

(I) Let us evaluate now path integral $\left(\boldsymbol{I}_{\varepsilon'}^{\varepsilon,1}(q',t',t'';L,m')\right)_{\varepsilon'}$. From Eq.(332)using replacement $p(t) = \frac{q(t)}{\sqrt{2\varepsilon}}$ into Feynman path integral in the RHS of the Eq. (332), we obtain

$$\left(\boldsymbol{I}_{\varepsilon'}^{\varepsilon,1}(q',t',t'';L,m')\right)_{\varepsilon'} == \exp\left[-\frac{1}{2\varepsilon}\int_{t'}^{t''} \left[b^0(t)\right]^2 dt\right] \int_{-\infty}^{\infty} dq'' \, \|q''\|^2 \int_{q(t')=q'}^{q(t'')=q''} [\boldsymbol{D}q(t;L,m')]$$

$$\exp\left[-\frac{1}{2\varepsilon}\int_{t'}^{t''}\left\{[\dot{q}(t)]^2 - 2\dot{q}(t)b^0(t) - 2\dot{q}(t)q_{\varepsilon'}(t)b^1(t) + 2q_{\varepsilon'}(t)b^0(t)b_0^1(t) + \left[q_{\varepsilon'}(t)b^1(t)\right]^2\right\}dt\right] =$$

$$= \int_{-\infty}^{\infty} dq''\,\|q''\|^2 \int_{q(t')=q'}^{q(t'')=q''}[\boldsymbol{D}q(t;L,m')] \times$$

$$\times \exp\left[-\frac{1}{2\varepsilon}\int_{t'}^{t''}\{[\dot{q}(t) - b_0(q(t),t)]^2 + O(\varepsilon'/\varepsilon)\}dt\right]. \quad (334)$$

We estimate now path integral in the RHS of the Eq. (334), using canonical perturbation expansion of anharmonic systems (see [45] chapter3, subsection15). Denoting the global minimum of the action

$$\hat{S} = \int_{t'}^{t''}[\dot{q}(t) - b_0(q(t),t)]^2 dt \quad (335)$$

by $\breve{q}(t)$, it follows that it satisfies the extremality conditions for the minimizing path $\breve{q}(t)$ is

$$\dot{\breve{q}}(t) - b_0(\breve{q}(t),t) = 0, \breve{q}(t') = q'. \quad (336)$$

In the limit $\varepsilon \to 0, \varepsilon' \to 0, \frac{\varepsilon'}{\varepsilon} \to 0$ from Eq. (334) we obtain

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon'\to 0\\ \frac{\varepsilon'}{\varepsilon}\to 0}} \boldsymbol{I}_{\varepsilon'}^{\varepsilon,1} = |\breve{q}(t)|^2. \quad (338)$$

Or in the following equivalent form: for any $\varepsilon \approx 0, \varepsilon' \approx 0, \frac{\varepsilon'}{\varepsilon} \approx 0$

$$\left(\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\varepsilon,1}\right)_{\boldsymbol{\varepsilon}'} \approx |\breve{q}(t)|^2 + O\left(\frac{\varepsilon'}{\varepsilon}\right). \quad (339)$$

(II) Let us evaluate now path integral $\left(\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon},\boldsymbol{2}}(L,m')\right)_{\boldsymbol{\varepsilon}'}$. Let us rewrite Eq.(326) in the following form

$$\breve{b}_2(p_{\varepsilon'}(t),t) = \breve{b}_{2,2}(p_{\varepsilon'}(t),t) + \breve{b}_{2,3}(p_{\varepsilon'}(t),t), \quad (340)$$

$$\breve{b}_{2,2}(p_{\varepsilon'}(t),t) = b_0^2(t)p_{\varepsilon'}^2(t), \quad (341)$$

$$\breve{b}_{2,3}(p_{\varepsilon'}(t),t) =$$

$$= \sum_\alpha \left(\sqrt{2\varepsilon}\right)^{|\alpha|-2} b_0^\alpha(t)\left(p_{\varepsilon'}(t)\right)^\alpha, 3 \le |\alpha| \le r. \quad (342)$$

Substitution Eq.(340)-Eq.(342) into Eq.(333) gives

$$\left(\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\varepsilon,2}(L,m')\right)_{\boldsymbol{\varepsilon}'} = \int_{-\infty}^{\infty} dq'' \int_{p(t')=\frac{q'}{\sqrt{2\varepsilon}}}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}} \left[\boldsymbol{D}p\left(t;\frac{L}{\sqrt{2\varepsilon}},m'\right)\right] \times$$

$$\exp\left[-\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\left\{[\dot{p}(t)]^{\boldsymbol{2}} + \frac{1}{\varepsilon}b^2(t)b^0(t)p_{\varepsilon'}^2(t) + \frac{2\sqrt{2}}{\sqrt{\varepsilon}}b^2(t)\dot{p}(t)p_{\varepsilon'}^2(t) + \frac{2\sqrt{2}}{\sqrt{\varepsilon}}b^3(t)b^0(t)p_{\varepsilon'}^3(t) + O\left(\dot{p}(t)p_{\varepsilon'}^3(t)\right) + O\left(p_{\varepsilon'}^4(t)\right) + o\left(\sqrt{\varepsilon}\right)\right\}dt\right]. \quad (343)$$

We let now that

$$\sup_{t\in[\boldsymbol{t}',\boldsymbol{t}'']}\left|b^2(t)b^0(t)\right| = \mu. \quad (344)$$

From Eq.(343) and Eq.(344) one obtain the inequality

$$\left(I_{\varepsilon'}^{\varepsilon,2}(L,m')\right)_{\boldsymbol{\varepsilon}'} \leq \left(\int_{-\infty}^{\infty} dq'' \int_{p(t')=\frac{q'}{\sqrt{2\varepsilon}}}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}} \left[\boldsymbol{D}p\left(t;\frac{L}{\sqrt{2\varepsilon}},m'\right)\right] \times\right.$$

$\times$

$$\mathbf{e}\mathrm{xp}\left[-\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\left\{[\dot{p}(t)]^{\mathbf{2}} - \frac{\mu}{\varepsilon}p^2(t) + \frac{2\sqrt{2}}{\sqrt{\varepsilon}}b_0^2(t)\dot{p}(t)p_{\varepsilon'}^2(t) + \frac{2\sqrt{2}}{\sqrt{\varepsilon}}b_0^3(t)b_0^0(t)p_{\varepsilon'}^3(t) + O\left(\dot{p}(t)p_{\varepsilon'}^3(t)\right) + O\left(p_{\varepsilon'}^4(t)\right) + o\left(\sqrt{\varepsilon}\right)\right\}dt\right]\Bigg)_{\boldsymbol{\varepsilon}'} = \left(\breve{\boldsymbol{I}}_{\boldsymbol{\varepsilon}'}^{\varepsilon,2}(L,m')\right)_{\boldsymbol{\varepsilon}'} \quad (345)$$

Let us estimate now path integral$\left(\breve{\boldsymbol{I}}_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon},\boldsymbol{2}}(L,m')\right)_{\boldsymbol{\varepsilon}'}$ in the RHS of the inequality. (345), using canonical perturbation expansion of an- harmonic systems (see [44] chapter3, subsection15). Denoting the critical path of the action

$$\hat{S} = \int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\left[[\dot{p}(t)]^{\mathbf{2}} - \frac{\mu}{\varepsilon}p^2(t) + O\left(\sqrt{\varepsilon}p_{\varepsilon'}^3(t)\right) + O\left(\sqrt{\varepsilon}\dot{p}(t)p_{\varepsilon'}^2(t)\right) + \cdots\right]dt \quad (346)$$

by $p_{\mathrm{cr},\varepsilon'}(t)$, it follows that it satisfies theEuler equation for the critical path $p_{\mathrm{cr},\varepsilon'}(t)$ is

$$\omega^{-2}\ddot{p}_{\mathrm{cr},\varepsilon'}(t) + p_{\mathrm{cr},\varepsilon'}(t) + O\left(\varepsilon\sqrt{\varepsilon}\dot{p}_{\mathrm{cr},\varepsilon'}(t)\right) +$$

$$+O\left(\varepsilon\sqrt{\varepsilon}p_{\mathrm{cr},\varepsilon'}^2(t)\right) + \cdots = 0, \quad (347)$$

$$\omega^2 = \omega^2(\varepsilon) = \frac{\mu}{\varepsilon} \quad (348)$$

$$p(t') = \frac{q'}{\sqrt{2\varepsilon}} = \tilde{q}', \; p(t'') = \frac{q''}{\sqrt{2\varepsilon}} = \tilde{q}''. \quad (349)$$

Therefore [44]-[45]:

$$p_{\mathrm{cr},\varepsilon'}(t) = \frac{[\tilde{q}''\sin\omega(t-t') + \tilde{q}'\sin\omega(t''-t)]}{\sin\omega(t''-t')} + O(\varepsilon'\varepsilon^{\gamma}), \gamma \geq 1.5. \quad (350)$$

Let $\hat{S}_2$ be

$$\hat{S}_2 = -\int_{t'}^{t''} [[\dot{p}(t)]^2 - \omega^2 p^2(t)]dt. \quad (351)$$

Substitution Eq.(350) into Eq.(351) gives

$$\widehat{S}_2 = -\frac{\omega}{2\sin(\omega T)}[(\tilde{q}'^2 + \tilde{q}''^2)\cos(\omega T) - 2\tilde{q}'\tilde{q}''], T = t'' - t'. (352)$$

Assumption. We assume now that:$\mathrm{ctg}(\omega T) > 0$.

Remark. Let$\tilde{q}_s''$ be a saddle point of the polynomial $\widehat{S}_2(\tilde{q}', \tilde{q}''; T)$ on variable $\tilde{q}''$.Note that a saddle point $\tilde{q}_s''$ of the polynomial $\widehat{S}_2$
is:

$$\tilde{q}_s'' = \frac{\tilde{q}'}{\cos(\omega T)}. \quad (353)$$

Substitution Eq.(353) into Eq.(352) gives

$$\widehat{S}_2(\tilde{q}', \tilde{q}_s''; \omega, T)\Big|_{\tilde{q}_s''=\tilde{q}'/\cos(\omega T)} \triangleq \widehat{S}_2^{\#}(\tilde{q}'; \omega, T)=$$

$$=-\frac{\tilde{q}'^2\omega}{2\sin(\omega T)}[\cos(\omega T) - \cos^{-1}(\omega T)]=\frac{\tilde{q}'^2\omega\mathrm{tg}(\omega T)}{2}. \quad (354)$$

Assumption. We assume now that:$\cos(\omega T) \cong 1, \sin(\omega T) \cong 0$ such that the condition

$$\tilde{q}'^2\omega\mathrm{tg}(\omega T) = q'\omega^2\mathrm{tg}(\omega T)/4\mu \cong 0, \quad (355)$$

is satisfied, and so

$$\exp\left[\widehat{S}_2^{\#}(\tilde{q}';\omega,T)\right] = O(1). \quad (356)$$

Remark. We note that

$$\int_{t'}^{t''} p_{cr,\varepsilon'}^2(t;\tilde{q}',\tilde{q}_s'')dt = O\big(T/\sin(\omega T)\big). \quad (357)$$

We are dealing now with the finite Fourier series [39]:

$$\boldsymbol{q}_n = \boldsymbol{q}(t_n) =$$

$$\boldsymbol{q}_0 + \sum_{m=1}^{N} \sqrt{2/(N+1)}\sin[\nu_m(t_n - t')]\boldsymbol{q}(\nu_m) \quad (358)$$

Here

$$\nu_m = \frac{\pi m}{T} = \frac{\pi m}{(N+1)\epsilon}, \epsilon = \frac{T}{(N+1)}. \quad (359)$$

Inserting now the expansion (358) into the time-sliced action (207), yields

$$\big(\mathbf{S}_{\varepsilon'}(\boldsymbol{q}_0,\boldsymbol{q}_1,\dots,\boldsymbol{q}_{N-1},\boldsymbol{q}_N,\varepsilon)\big)_{\varepsilon'} =$$

$$= \Delta t \sum_{n=1}^{N}\left(\mathcal{L}_{\varepsilon'}\left(\frac{\boldsymbol{q}_n - \boldsymbol{q}_{n-1}}{\Delta t}, \frac{\boldsymbol{q}_n - \boldsymbol{q}_{n-1}}{2}, t_n\right)\right)_{\varepsilon'} =$$

$$\epsilon \sum_{n=1}^{N}\left(\frac{\boldsymbol{q}_n(\{\boldsymbol{q}(\nu_m)\}_{m=1}^{N}) - \boldsymbol{q}_{n-1}(\{\boldsymbol{q}(\nu_m)\}_{m=1}^{N})}{\epsilon},\right.$$

$$\left.\frac{\boldsymbol{q}_n(\{\boldsymbol{q}(\nu_m)\}_{m=1}^{N}) - \boldsymbol{q}_{n-1}(\{\boldsymbol{q}(\nu_m)\}_{m=1}^{N})}{2}, t_n, \varepsilon\right)_{\varepsilon'} =$$

$$\epsilon\sum_{n=1}^{N}\left(\left\|\frac{\boldsymbol{q}_n(\{\boldsymbol{q}(\nu_m)\}_{m=1}^{N})-\boldsymbol{q}_{n-1}(\{\boldsymbol{q}(\nu_m)\}_{m=1}^{N})}{\epsilon}\right.\right.$$

$$\left.\left.+\boldsymbol{b}_{\varepsilon'}\left(\frac{\boldsymbol{q}_n(\{\boldsymbol{q}(\nu_m)\}_{m=1}^{N})-\boldsymbol{q}_{n-1}(\{\boldsymbol{q}(\nu_m)\}_{m=1}^{N})}{2},t_m,\varepsilon\right)\right\|^2\right)_{\varepsilon'}=$$

$$=\left(\boldsymbol{S}_{\varepsilon'}(\boldsymbol{q}_0,\boldsymbol{q}(\nu_1),\dots,\boldsymbol{q}(\nu_{N-1}),\boldsymbol{q}(\nu_N),\boldsymbol{q}_{N+1},\varepsilon)\right)_{\varepsilon'} \quad (360)$$

Before performing the integral (206), we must transform the measure of integration from the local variables$\boldsymbol{q}_n$, to the Fourier components$\boldsymbol{q}(\nu_m)$. Due to the orthogonality relation [45], the transformation (358) has a unit determinant implying that

$$\prod_{n=1}^{N} d\,\boldsymbol{q}_n=\prod_{m=1}^{N} d\boldsymbol{q}(\nu_m)=\prod_{n=1}^{N} d\boldsymbol{p}_m. \quad (361)$$

Substitution Eq.(360) and Eq.(361) into Eq.(205)-Eq.(206), yields

$$\left(p_{\varepsilon'}^{\varepsilon}(\boldsymbol{q}',t'|\boldsymbol{q}'',t'')\right)_{\varepsilon'}=\lim_{\epsilon\to 0}\boldsymbol{I}_N(\boldsymbol{q}',t'|\boldsymbol{q}'',t''). \quad (362)$$

Here

$$\boldsymbol{I}_N(\boldsymbol{q}',t'|\boldsymbol{q}'',t'')=$$

$$=\breve{\boldsymbol{N}}_N\int_{-\infty}^{\infty}\boldsymbol{dq}_0\int_{-\infty}^{\infty}d\boldsymbol{q}(\nu_1)\dots\int_{-\infty}^{\infty}d\boldsymbol{q}(\nu_m)\dots\int_{-\infty}^{\infty}d\boldsymbol{q}(\nu_{N-1})\quad\times$$

$$\left(\boldsymbol{p}_{\varepsilon'}(\boldsymbol{q}_0-\boldsymbol{q}')\right)_{\varepsilon'}\times$$

$$\times\exp\left[-\frac{1}{2\varepsilon}\left(\boldsymbol{S}_{\varepsilon'}(\boldsymbol{q}_0,\boldsymbol{q}(\nu_1),\dots,\boldsymbol{q}(\nu_{N-1}),\boldsymbol{q}(\nu_N),\boldsymbol{q}_{N+1},t_m,\varepsilon)\right)_{\varepsilon'}\right] \quad (363)$$

$\boldsymbol{q}_{N+1}=\boldsymbol{q}'',d\boldsymbol{q}(\nu_m)=\prod_{j=1}^{n} dq_j(\nu_m)\,,m=0,\dots,N,\epsilon=(t''-t')/N+1,t_m=m\Delta t.$
The quantity defined by Eq.(362)-Eq. (363) is symbolically indicated by the

path-integral expression

$$\left(\boldsymbol{p}^{\varepsilon}_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}', t'|\boldsymbol{q}'', t'')\right)_{\boldsymbol{\varepsilon}'} =$$

$$= \int_{\boldsymbol{q}(t')=\boldsymbol{q}'}^{\boldsymbol{q}(t'')=\boldsymbol{q}''} [D\boldsymbol{q}(\{\nu_m\}_{m=1}^{\infty})] \exp\left[\left(-\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\varepsilon}} S_{\boldsymbol{\varepsilon}'}(\dot{\boldsymbol{q}}, \boldsymbol{q}, t', t''\varepsilon)\right)_{\boldsymbol{\varepsilon}'}\right]. (364)$$

Let us consider now the quantity

$$\left(\boldsymbol{p}^{\varepsilon}_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}', t'|\boldsymbol{q}'', t''; P, m')\right)_{\boldsymbol{\varepsilon}'} = \left(\lim_{\epsilon\to 0} \boldsymbol{I}^{P}_{m',N,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}(\boldsymbol{q}', t'|\boldsymbol{q}'', t'')\right)_{\boldsymbol{\varepsilon}'}. (365)$$

Here

$$\boldsymbol{I}^{P}_{m',N,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}(\boldsymbol{q}', t'|\boldsymbol{q}'', t'') = \boldsymbol{I}_{N,\varepsilon,\varepsilon'}(\boldsymbol{q}', t'|\boldsymbol{q}'', t''; P, m') =$$

$$= \breve{\boldsymbol{N}}_N \int_{-P}^{P} d\boldsymbol{q}(\nu_1) \ldots \int_{-P}^{P} d\boldsymbol{q}(\nu_{m'}) \int_{-\infty}^{\infty} d\boldsymbol{q}(\nu_{m'+1}) \ldots \int_{-\infty}^{\infty} d\boldsymbol{q}(\nu_{N-1}) \times$$

$$\int_{-\infty}^{\infty} d\boldsymbol{q}(\nu_N) \exp\left[-\frac{1}{2\varepsilon} (S_{\boldsymbol{\varepsilon}'} (\boldsymbol{q_0}, \boldsymbol{q}(\nu_1), \ldots, \boldsymbol{q}(\nu_{m'}),\right.$$

$$\left.\boldsymbol{q}(\nu_{m'+1}) \ldots, \boldsymbol{q}(\nu_{N-1}), \boldsymbol{q}(\nu_N), \boldsymbol{q}_{N+1}, \varepsilon)\right)_{\boldsymbol{\varepsilon}'} (366)$$

and $m' \ll N, P \gg 1$.

The quantity defined by Eq.(365)-Eq.(366) is symbolically indicated by the path-integral expression

$$\left(\boldsymbol{p}^{\varepsilon}_{\boldsymbol{\varepsilon}'}(\boldsymbol{q}', t'|\boldsymbol{q}'', t''; P, m')\right)_{\boldsymbol{\varepsilon}'} =$$

$$= \int_{\boldsymbol{q}(\boldsymbol{t}')=\boldsymbol{q}'}^{\boldsymbol{q}(\boldsymbol{t}'')=\boldsymbol{q}''} [D\boldsymbol{q}(\{\nu_m\}_{m=1}^{\infty}; P, m')] \exp\left[\left(-\frac{\mathbf{1}}{\mathbf{2}\boldsymbol{\varepsilon}} S_{\boldsymbol{\varepsilon}'}(\dot{\boldsymbol{q}}, \boldsymbol{q}, t', t''\varepsilon)\right)_{\boldsymbol{\varepsilon}'}\right]. (367)$$

Using Eq.(362)-Eq.(367) we define the quantity

$$\left(\mathbf{E}_{P,m'}\left[\boldsymbol{x}_{t'',\varepsilon'}^{\boldsymbol{q}',\varepsilon}(\omega)\right]^{\mathbf{2}}\right)_{\boldsymbol{\varepsilon}'} =$$

$$= \int_{-\infty}^{\infty} \boldsymbol{dq}'' \|\boldsymbol{q}''\|^{\mathbf{2}} \left[\left(\boldsymbol{p}_{\varepsilon'}^{\boldsymbol{\varepsilon}}(\boldsymbol{q}', t'|\boldsymbol{q}'', t''; P, m')\right)_{\boldsymbol{\varepsilon}'}\right] (368)$$

$$= \lim_{\epsilon\to 0} \breve{\boldsymbol{I}}_{N,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}(\boldsymbol{q}', t', t''; P, m').$$

Here

$$\breve{\boldsymbol{I}}_{\boldsymbol{N},\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}(\boldsymbol{q}', t', t''; P, m') =$$

$$\int_{-\infty}^{\infty} \boldsymbol{dq}'' \|\boldsymbol{q}''\|^{2} \boldsymbol{I}_{N,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}'}(\boldsymbol{q}', t'|\boldsymbol{q}'', t''; P, m'). \qquad (369)$$

The quantity (368) is symbolically indicated by the path-integral expression

$$\left(\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon}}(P, m')\right)_{\boldsymbol{\varepsilon}'} = \left(\mathbf{E}_{P,m'}\left[\boldsymbol{x}_{t'',\varepsilon'}^{\boldsymbol{q}',\varepsilon}(\omega)\right]^{\mathbf{2}}\right)_{\boldsymbol{\varepsilon}'} = \int_{-\infty}^{\infty} \boldsymbol{dq}'' \|\boldsymbol{q}''\|^{\mathbf{2}} \int_{\boldsymbol{q}(\boldsymbol{t}')=\boldsymbol{q}'}^{\boldsymbol{q}(\boldsymbol{t}'')=\boldsymbol{q}''} [D\boldsymbol{q}(\{\nu_m\}_{m=1}^{\infty}; P, m')] \times$$

$$\times \exp\left[\left(-\frac{1}{2\varepsilon}\int_{\boldsymbol{t}'}^{\boldsymbol{t}''} \boldsymbol{\mathcal{L}}_{\boldsymbol{\varepsilon}'}(\dot{\boldsymbol{q}}, \boldsymbol{q}, t) dt\right)_{\boldsymbol{\varepsilon}'}\right]. \ (370)$$

Substitution Eq. (323) into Eq. (370) gives

$$\left(\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\varepsilon}(q', t', t''; P, m')\right)_{\boldsymbol{\varepsilon}'} = \exp\left[-\frac{1}{2\varepsilon}\int_{t'}^{t''} \left[b^{0}(t)\right]^{2} dt\right] \int_{-\infty}^{\infty} dq'' \|q''\|^{2} \times$$

$$\int_{q(\boldsymbol{t}')=\boldsymbol{q}'}^{q(\boldsymbol{t}'')=q''} [\boldsymbol{D}q(\{\nu_m\}_{m=1}^{\infty}; P, m')] \times$$

$\times$

$\exp\left[-\frac{\mathbf{1}}{\mathbf{2\varepsilon}}\int_{\mathbf{t'}}^{\mathbf{t''}}\left\{[\dot{q}(t)]^2-2\dot{q}(t)b^0(t)-2\dot{q}(t)q_{\boldsymbol{\varepsilon}'}(t)b^1(t)++2q_{\boldsymbol{\varepsilon}'}(t)b^0(t)b^1(t)+\right.\right.$

$\left[q_{\boldsymbol{\varepsilon}'}(t)b^1(t)\right]^2-2\dot{q}(t)b_{2,2}(q_{\varepsilon'}(t),t)—2\dot{q}(t)b_{2,3}(q_{\varepsilon'}(t),t)+2b_{2,2}(q_{\varepsilon'}(t),t)b^0(t)+$

$\left.\left.2b_{2,3}(q_{\varepsilon'}(t),t)b^0(t)+2q_{\varepsilon'}(t)b_2(q_{\varepsilon'}(t),t)b^1(t)\right\}dt\right].$ (371)

Using replacement $q(t)=p(t)\sqrt{2\varepsilon}$into Feynman path integral (371), we obtain

$$\left(\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon}}(q',t',t'';P,m')\right)_{\boldsymbol{\varepsilon}'}=\exp\left[-\frac{1}{2\varepsilon}\int_{t'}^{t''}\left[b^0(t)\right]^2dt\right]\int_{-\infty}^{\infty}dq''\,\|q''\|^2\times$$

$$\int_{p(t')=\frac{q'}{\sqrt{2\varepsilon}}}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}}\left[\boldsymbol{D}p\left(\{\nu_m\}_{m=1}^{\infty};\frac{P}{\sqrt{2\varepsilon}},m'\right)\right]\times$$

$\times\exp\left[-\int_{\mathbf{t'}}^{\mathbf{t''}}\left\{[\dot{p}(t)]^{\mathbf{2}}-\frac{2}{\sqrt{2\varepsilon}}\dot{p}(t)b^0(t)-2\dot{p}(t)p_{\varepsilon'}(t)b^1(t)++\frac{2}{\sqrt{2\varepsilon}}p_{\varepsilon'}(t)b^0(t)b^1(t)+\right.\right.$

$\left[p_{\varepsilon'}(t)b^1(t)\right]^{\mathbf{2}}-2\sqrt{2\varepsilon}\dot{p}(t)\breve{b}_2(p_{\varepsilon'}(t),t)+$

$\left.\left.+2\breve{b}_2(p_{\varepsilon'}(t),t)\hat{b}\;\left(\sqrt{2\varepsilon}p_{\varepsilon'}(t),t\right)\right\}dt\right].$ (372)

Let us rewrite now Feynman path integral (372) in the next equivalent form

$$\left(\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon}}(q',t',t'';P,m')\right)_{\boldsymbol{\varepsilon}'}=\exp\left[-\frac{1}{2\varepsilon}\int_{t'}^{t''}\left[b^0(t)\right]^2dt\right]\int_{-\infty}^{\infty}dq''\,\|q''\|^2\times$$

$\int_{p(t')=\frac{q'}{\sqrt{2\varepsilon}}}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}}\left[\boldsymbol{D}p\left(\{\nu_m\}_{m=1}^{\infty};\frac{P}{\sqrt{2\varepsilon}},m'\right)\right]\times$(373)

$$\exp\left[-\int_{t'}^{t''}[\dot{p}(t)]^2dt\right]\left\{\exp\left[-\int_{t'}^{t''}\left[-\frac{2}{\sqrt{2\varepsilon}}\dot{p}(t)b^0(t)-2\dot{p}(t)p_{\varepsilon'}(t)b^1(t)\right.\right.\right.$$

$$\left.\left.\left.+\frac{2}{\sqrt{2\varepsilon}}p_{\varepsilon'}(t)b^0(t)b^1(t)+\left[p_{\varepsilon'}(t)b^1(t)\right]^2\right]dt\right]\right\}$$

$$\exp\left[\int_{t'}^{t''}\left[2\sqrt{2\varepsilon}\dot{p}(t)\breve{b}_2(p_{\varepsilon'}(t),t)-\breve{b}_2(p_{\varepsilon'}(t),t)\hat{b}\;\left(\sqrt{2\varepsilon}p(t),t\right)\right]dt\right]$$

Assume that $1/p+1/q=1$ and $q=1/\varepsilon$. Then

$$p = \frac{1}{1-\varepsilon} = 1 + \varepsilon + o(\varepsilon). \quad (374)$$

Using now Corollary1, from Eq.(373) we obtain

$$\left(\boldsymbol{I}^{\varepsilon}_{\boldsymbol{\varepsilon}'}(q',t',t'';P,m')\right)_{\boldsymbol{\varepsilon}'} \leq \exp\left[-\frac{1}{2\varepsilon}\int_{t'}^{t''}\left[b^{0}(t)\right]^{2}dt\right] \times$$

$$\left[\int_{-\infty}^{\infty} dq'' \|q''\|^{2} \int_{p(t')=\boldsymbol{q}'}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}}\left[\boldsymbol{D}p\left(\{\nu_m\}_{m=1}^{\infty};\frac{P}{\sqrt{2\varepsilon}},m'\right)\right] \times \exp\left[-\int_{t'}^{t''}[\dot{p}(t)]^{2}dt\right] \times\right.$$

$$\times\left\{\exp\left[-(1+\varepsilon)\int_{t'}^{t''}\left[-\frac{2}{\sqrt{2\varepsilon}}\dot{p}(t)b^{0}(t) - 2\dot{p}(t)p_{\varepsilon'}(t)b^{1}(t) + \frac{2}{\sqrt{2\varepsilon}}p_{\varepsilon'}(t)b_{0}^{0}(t)b_{0}^{1}(t)\right.\right.\right.$$

$$\left.\left.\left.\left. + \left[p_{\varepsilon'}(t)b^{1}(t)\right]^{2}\right]dt\right]\right\}\right]^{1-\varepsilon} \times$$

$$\times\left[\int_{-\infty}^{\infty} dq'' \int_{p(\boldsymbol{t}')=q'}^{p(\boldsymbol{t}'')=\frac{q''}{\sqrt{2\varepsilon}}}\left[\boldsymbol{D}p\left(\{\nu_m\}_{m=1}^{\infty};\frac{P}{\sqrt{2\varepsilon}},m'\right)\right]\exp\left[-\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}[\dot{p}(t)]^{2}dt\right] \times\right.$$

$$\left\{\exp\left[\frac{1}{\varepsilon}\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\left[2\sqrt{2\varepsilon}\dot{p}(t)\breve{b}_{2}(p_{\varepsilon'}(t),t)\breve{b}_{2}(p_{\varepsilon'}(t),t) - \right.\right.\right.$$

$$\left.\left.\left.\left.\breve{b}_{2}(p_{\varepsilon'}(t),t)\hat{b}\ \left(\sqrt{2\varepsilon}\,p(t),t\right)\right]dt\right]\right\}\right]^{\varepsilon}. \quad (375)$$

Therefore

$$\left(\boldsymbol{I}^{\boldsymbol{\varepsilon}}_{\boldsymbol{\varepsilon}'}(q',t',t'';P,m')\right)_{\boldsymbol{\varepsilon}'} \leq$$

$$\left(\left(\left[\boldsymbol{I}^{\boldsymbol{\varepsilon},1}_{\boldsymbol{\varepsilon}'}(q',t',t'';P,m')\right]^{1-\boldsymbol{\varepsilon}}\right)_{\boldsymbol{\varepsilon}'}\right) \times$$

$$\left(\left(\left[\boldsymbol{I}^{\boldsymbol{\varepsilon},\boldsymbol{2}}_{\boldsymbol{\varepsilon}'}(q',t',t'';P,m')\right]^{\boldsymbol{\varepsilon}}\right)_{\boldsymbol{\varepsilon}'}\right). \quad (376)$$

Here

$$\left(\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon},1}(q',t',t'';P,m')\right)_{\boldsymbol{\varepsilon}'} = \left(\exp\left[-\frac{1}{2\varepsilon(1+\varepsilon)}\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\left[b^0\,(t)\right]^2 dt\right]\times\right.$$

$$\times\int_{-\infty}^{\infty} dq''\,\|q''\|^2\int_{p(t')=0}^{p(\boldsymbol{t}'')=\frac{q''}{\sqrt{2\varepsilon}}}\left[\boldsymbol{D}p\left(\{\nu_m\}_{m=1}^{\infty};\frac{P}{\sqrt{2\varepsilon}},m'\right)\right]\times\exp\left[-\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}[\dot{p}(t)]^2 dt\right]\times$$

$\times$

$\exp\Big[-(1+$

$\varepsilon)\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\Big[-\frac{2}{\sqrt{2\varepsilon}}\dot{p}(t)b^0\,(t) - 2\dot{p}(t)p_{\varepsilon'}(t)b^1\,(t) + \frac{2}{\sqrt{2\varepsilon}}p_{\varepsilon'}(t)b^0\,(t)b^1\,(t) +$

$\left.\left[p_{\varepsilon'}(t)b^1\,(t)\right]^2\Big]\,dt\Big]\right)_{\boldsymbol{\varepsilon}'}$ (377)

and

$$\left(\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon},2}\left((q',t',t'';P,m')\right)\right)_{\boldsymbol{\varepsilon}'} =$$

$$= \int_{-\infty}^{\infty} dq''\int_{p(t')=0}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}}\left[\boldsymbol{D}p\left(\{\nu_m\}_{m=1}^{\infty};\frac{P}{\sqrt{2\varepsilon}},m'\right)\right]\exp\left[-\int_{t'}^{t''}[\dot{p}(t)]^2 dt\right]\times$$

$\Big\{\exp\Big[\frac{1}{\varepsilon}\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\big[2\sqrt{2\varepsilon}\dot{p}(t)\breve{b}_2(p_{\varepsilon'}(t),t) -$

$-\breve{b}_2(p_{\varepsilon'}(t),t)\hat{b}\ \ \left(\sqrt{2\varepsilon}\,p(t),t\right)\big]dt\Big]\Big\}.$ (378)

Let us evaluate now path integral$\left(\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\varepsilon,1}(q',t',t'';L,m')\right)_{\boldsymbol{\varepsilon}'}$. From Eq.(377)using replacement $p(t) = \frac{q(t)}{\sqrt{2\varepsilon}}$ into Feynman path integral in the RHS of the Eq. (377), we obtain

$$\left(\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\varepsilon,1}(q',t',t'';P,m')\right)_{\boldsymbol{\varepsilon}'} == \exp\left[-\frac{1}{2\varepsilon}\int_{t'}^{t''}\left[b^0\,(t)\right]^2 dt\right]\int_{-\infty}^{\infty} dq''\,\|q''\|^2\times$$

$$\int_{q(t')=0}^{q(\boldsymbol{t}'')=q''}[\boldsymbol{D}q(\{\nu_m\}_{m=1}^{\infty};P,m')]\times$$

$$\exp\left[-\frac{1}{2\varepsilon}\int_{t'}^{t''}\left\{[\dot{q}(t)]^2-2\dot{q}(t)b^0(t)-2\dot{q}(t)q_{\varepsilon'}(t)b^1(t)+2q_{\varepsilon'}(t)b^0(t)b_0^1(t)+\left[q_{\varepsilon'}(t)b^1(t)\right]^2\right.\right.$$

$$\left.\left.+O(\varepsilon)\right\}dt\right]=$$

$$=\int_{-\infty}^{\infty}dq''\,\|q''\|^2\int_{q(t')=q'}^{q(t'')=q''}[\boldsymbol{D}q(\{\nu_m\}_{m=1}^{\infty};P,m')]\times$$

$$\times\exp\left[-\frac{1}{2\varepsilon}\int_{t'}^{t''}\left\{\left[\dot{q}(t)-\hat{b}(q_{\varepsilon'}(t),t)\right]^2+O(\varepsilon)\right\}dt\right]. \quad (379)$$

In the limit $\varepsilon\to 0,\boldsymbol{\varepsilon}'\to 0,\boldsymbol{\varepsilon}'/\boldsymbol{\varepsilon}\to 0$ by simple calculation, one obtain

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon'\to 0\\ \varepsilon'/\varepsilon\to 0}}\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\varepsilon,1}(q',t',t'';P,m')\leq[\breve{q}(t)]^{\mathbf{2}}, \quad (380)$$

where

$$\dot{\breve{q}}(t)-b_0(\breve{q}(t),t)=0,\breve{q}(t')=q'. \quad (381)$$

Or in the following equivalent form

$$\left(\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\varepsilon,1}(q',t',t'';L,m')\right)_{\boldsymbol{\varepsilon}'}\leq(\mathrm{C})_{\boldsymbol{\varepsilon}'}([\breve{q}(t,\varepsilon')]^2)_{\varepsilon'}. \quad (382)$$

Here $\varepsilon\approx 0,\boldsymbol{\varepsilon}'\approx 0,\boldsymbol{\varepsilon}'/\boldsymbol{\varepsilon}\approx 0$ and

$$\dot{\breve{q}}(t,\varepsilon')-\hat{b}(\breve{q}_{\varepsilon'}(t,\varepsilon'),t)=0,\breve{q}(t',\varepsilon')=q'. \quad (383)$$

Let us evaluate now path integral $\left(\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon},\mathbf{2}}(P,m')\right)_{\boldsymbol{\varepsilon}'}$. Let us rewrite Eq.(326) in the following form

$$\breve{b}_2(p_{\varepsilon'}(t),t)=\breve{b}_{2,2}(p_{\varepsilon'}(t),t)+\breve{b}_{2,3}(p_{\varepsilon'}(t),t), \quad (384)$$

$$\breve{b}_{2,2}(p_{\varepsilon'}(t),t) = b_0^2(t)p_{\varepsilon'}^2(t), \quad (385)$$

$$\breve{b}_{2,3}(p_{\varepsilon'}(t),t) =$$

$$= \sum_\alpha \left(\sqrt{2\varepsilon}\right)^{|\alpha|-2} b_0^\alpha\,(t)\left(p_{\varepsilon'}(t)\right)^\alpha, 3 \leq |\alpha| \leq r. \quad (386)$$

Substitution Eq.(384)-Eq.(386) into Eq.(377) gives

$$\left(\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon},\boldsymbol{2}}(P,m')\right)_{\boldsymbol{\varepsilon}'} = \int_{-\infty}^{\infty} dq'' \int_{p(t')=\frac{q'}{\sqrt{2\varepsilon}}}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}} \left[\boldsymbol{D}p\left(\{\nu_m\}_{m=1}^\infty; \frac{P}{\sqrt{2\varepsilon}}, m'\right)\right] \times$$

$$\exp\left[-\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\left\{[\dot{p}(t)]^{\boldsymbol{2}} + \frac{1}{\varepsilon}b^2\,(t)b^0\,(t)p_{\varepsilon'}^2(t) + \frac{2\sqrt{2}}{\sqrt{\varepsilon}}b^2\,(t)\dot{p}(t)p_{\varepsilon'}^2(t) + \frac{2\sqrt{2}}{\sqrt{\varepsilon}}b^3\,(t)b^0\,(t)p_{\varepsilon'}^3(t) + O\left(\dot{p}(t)p_{\varepsilon'}^3(t)\right) + O\left(p_{\varepsilon'}^4(t)\right) + o\left(\sqrt{\varepsilon}\right)\right\}dt\right]. \quad (387)$$

We let now that

$$\sup_{t\in[\boldsymbol{t}',\boldsymbol{t}'']}|b_0^2(t)b_0^0(t)| = \mu. \quad (388)$$

From Eq.(327) and Eq.(387)-Eq.(388) one obtain the inequality

$$\left(\boldsymbol{I}_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon},\boldsymbol{2}}(P,m')\right)_{\boldsymbol{\varepsilon}'} \leq \left(\int_{-\infty}^{\infty} dq'' \int_{p(t')=\frac{q'}{\sqrt{2\varepsilon}}}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}} \left[\boldsymbol{D}p\left(\{\nu_m\}_{m=1}^\infty; \frac{P}{\sqrt{2\varepsilon}}, m'\right)\right]\right.$$

$$\times$$

$$\left.\mathbf{exp}\left[-\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\left\{[\dot{p}(t)]^{\boldsymbol{2}} - \frac{\mu}{\varepsilon}p^2(t) + \frac{2\sqrt{2}}{\sqrt{\varepsilon}}b_0^2(t)\dot{p}(t)p_{\varepsilon'}^2(t) + \frac{2\sqrt{2}}{\sqrt{\varepsilon}}b_0^3(t)b_0^0(t)p_{\varepsilon'}^3(t) + O\left(\dot{p}(t)p_{\varepsilon'}^3(t)\right) + O\left(p_{\varepsilon'}^4(t)\right) + o\left(\sqrt{\varepsilon}\right)\right\}dt\right]\right)_{\boldsymbol{\varepsilon}'} = \left(\breve{\boldsymbol{I}}_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon},\boldsymbol{2}}\left(\frac{P}{\sqrt{2\varepsilon}}, m'\right)\right)_{\boldsymbol{\varepsilon}'} \quad (389)$$

Let us estimate now path integral$\left(\breve{\boldsymbol{I}}_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon},\mathbf{2}}\left(\frac{P}{\sqrt{2\varepsilon}},m'\right)\right)_{\boldsymbol{\varepsilon}'}$ in the RHS of the inequality (389). Denoting the critical path of the action

$$\hat{S}=\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\left[[\dot{p}(t)]^{\mathbf{2}}-\frac{\mu}{\varepsilon}p^2(t)+O\left(\sqrt{\varepsilon}p_{\varepsilon'}^3(t)\right)+O\left(\sqrt{\varepsilon}\dot{p}(t)p_{\varepsilon'}^2(t)\right)+\cdots\right]dt$$
(390)

by $p_{\mathrm{cr}}(t)$, it follows that it satisfies the Euler equation for the critical path $p_{\mathrm{cr},\varepsilon'}(t)$ is

$$\omega^{-2}\ddot{p}_{\mathrm{cr},\varepsilon'}(t)+p_{\mathrm{cr},\varepsilon'}(t)+o\left(\sqrt{\varepsilon}\dot{p}_{\mathrm{cr},\varepsilon'}(t)\right)+$$

$$+O\left(\sqrt{\varepsilon}p_{\mathrm{cr},\varepsilon'}^2(t)\right)+\cdots=0, \qquad (391)$$

where

$$\omega^2=\ \omega^2(\varepsilon)=\frac{\mu}{\varepsilon}, \qquad (392)$$

$$p(t')=\frac{q'}{\sqrt{2\varepsilon}}=\tilde{q}',\ p(t'')=\frac{q''}{\sqrt{2\varepsilon}}=\tilde{q}''. \qquad (393)$$

We estimate now path integral$\left(\breve{\boldsymbol{I}}_{\boldsymbol{\varepsilon}'}^{\boldsymbol{\varepsilon},\mathbf{2}}\left(\frac{P}{\sqrt{2\varepsilon}},m'\right)\right)_{\boldsymbol{\varepsilon}'}$ using canonical perturbation expansion of anharmonic systems (see [45]
chapter3,sect.15). Let us rewrite action (390) in the following form

$$\hat{S}=\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\left[[\dot{p}(t)]^{\mathbf{2}}-\frac{\mu}{\varepsilon}p^2(t)+O\left(\sqrt{\varepsilon}p_{\varepsilon'}^3(t)\right)+O\left(\sqrt{\varepsilon}\dot{p}(t)p_{\varepsilon'}^2(t)\right)+\cdots\right]dt$$
$$=-\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}[[\dot{p}(t)]^{\mathbf{2}}-\omega^2p^2(t)]dt$$
$$+\int_{\boldsymbol{t}'}^{\boldsymbol{t}''}\left[O\left(\sqrt{\varepsilon}p_{\varepsilon'}^3(t)\right)+O\left(\sqrt{\varepsilon}\dot{p}(t)p_{\varepsilon'}^2(t)\right)+\cdots\right]dt=$$

$$= \widehat{\boldsymbol{S}}_2 + \int_{\boldsymbol{t'}}^{\boldsymbol{t''}} V_{\varepsilon'}[p(t),\dot{p}(t),t]dt'. \quad (394)$$

Thus corresponding perturbation expansion of the path integral$\left(\breve{\boldsymbol{I}}_{\boldsymbol{\varepsilon'}}^{\boldsymbol{\varepsilon},\boldsymbol{2}}\left(\frac{P}{\sqrt{2\varepsilon}},m'\right)\right)_{\boldsymbol{\varepsilon'}}$ **is**

$$\left(\breve{\boldsymbol{I}}_{\boldsymbol{\varepsilon'}}^{\boldsymbol{\varepsilon},\boldsymbol{2}}\left(\frac{P}{\sqrt{2\varepsilon}},m'\right)\right)_{\boldsymbol{\varepsilon'}} = \left(\int_{-\infty}^{\infty}dq'' \int_{p(t')=\frac{q'}{\sqrt{2\varepsilon}}}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}}\left[\boldsymbol{D}p\left(\{\nu_m\}_{m=1}^{\infty};\frac{P}{\sqrt{2\varepsilon}},m'\right)\right]\right.$$

$$\times \exp\left[\widehat{\boldsymbol{S}}_2\right]\Big)_{\boldsymbol{\varepsilon'}} + \left(\int_{-\infty}^{\infty}dq'' \int_{p(t')=\frac{q'}{\sqrt{2\varepsilon}}}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}}\left[\boldsymbol{D}p\left(\{\nu_m\}_{m=1}^{\infty};\frac{P}{\sqrt{2\varepsilon}},m'\right)\right]\times\right.$$

$$\times \int_{\boldsymbol{t'}}^{\boldsymbol{t''}} V_{\varepsilon'}[p(t),\dot{p}(t),t]dt' \exp\left[\widehat{\boldsymbol{S}}_2\right]\Big)_{\boldsymbol{\varepsilon'}} + \cdots \quad (395)$$

Let us consider now Gaussian path integral

$$\boldsymbol{I}_{\boldsymbol{\omega}}^{\boldsymbol{2}}(L,m') = \int_{-\infty}^{\infty}dq'' \int_{p(t')=\frac{q'}{\sqrt{2\varepsilon}}}^{p(t'')=\frac{q''}{\sqrt{2\varepsilon}}}\left[\boldsymbol{D}p\left(t;\frac{P}{\sqrt{2\varepsilon}},m'\right)\right]\exp\left(\hat{S}_{2,N}\right) =$$

$$= \lim_{\boldsymbol{N}\to\infty}\boldsymbol{I}_N^{\omega}(P,m')(396)$$

$$\boldsymbol{I}_N^{\omega}(P,m') = \breve{\boldsymbol{N}}_N \int_{-P/\sqrt{2\boldsymbol{\varepsilon}}}^{P/\sqrt{2\boldsymbol{\varepsilon}}}dp_1 \ldots \int_{-P/\sqrt{2\boldsymbol{\varepsilon}}}^{P/\sqrt{2\boldsymbol{\varepsilon}}}dp_{m'} \int_{-\infty}^{\infty}dp_{m'+1} \times \ldots \int_{-\infty}^{\infty}dp_{N-1} \times$$

$$\times \exp\left[\widehat{\boldsymbol{S}}_{2,N}\right]. \quad (397)$$

Here

$$\widehat{\boldsymbol{S}}_{2,N} = -\frac{\epsilon}{2}\sum_{m=1}^{N+1}\left[\left(\overline{\nabla}p_m\right)^2 - \omega^2 p_m\right]. \quad (398)$$

Note that: $\sum_{m=1}^{N+1}\left(\overline{\nabla}p_m\right)^2=-\sum_{m=1}^{N+1}p_m\nabla\overline{\nabla}p_m$[45].

We now turn to the fluctuation factor [45] of the path integral $\boldsymbol{I}_{\boldsymbol{N}}^{\boldsymbol{\omega}}(P,m')$. With the matrix notation for the lattice fluctuation operator $\nabla\overline{\nabla}+\omega^2$, we have to solve the multiple integral

$$\mathcal{F}_N^{\omega}(T)=\frac{1}{\sqrt{2\pi\epsilon}}\prod_{m=1}^{m'}\left[\int_{-\boldsymbol{P}/\sqrt{2\boldsymbol{\varepsilon}}}^{\boldsymbol{P}/\sqrt{2\boldsymbol{\varepsilon}}}\frac{dp_m}{\sqrt{2\pi\epsilon}}\right]\prod_{m=m'+1}^{N}\left[\int_{-\infty}^{\infty}\frac{dp_m}{\sqrt{2\pi\epsilon}}\right]\times$$

$$\exp\left\{\epsilon\sum_{m=1}^{N}\delta p_m\left[\nabla\overline{\nabla}+\omega^2\right]_{mm''}\right\}\delta p_{m''}. \qquad (397)$$

The eigenvalues of the fluctuation operator $\nabla\overline{\nabla}+\omega^2$ are [45]

$$\sigma_m(\omega)=\omega^2-\Omega_m\overline{\Omega}_m=\boldsymbol{\omega^2}-\frac{\boldsymbol{1}}{\boldsymbol{\epsilon^2}}[2-2\cos(\epsilon\nu_m)], \qquad (398)$$

$$\nu_m=\frac{\pi m}{T}=\frac{\pi m}{(N+1)\epsilon},\epsilon=\frac{T}{(N+1)}. \qquad (399)$$

Thus

$$\sigma_m(\omega)=\omega^2-\frac{\boldsymbol{1}}{\boldsymbol{\epsilon^2}}\left[2-2\cos\left(\frac{\pi m}{N+1}\right)\right]. \qquad (400)$$

We set now

$$m'=\sup_{m\geq 1}\left\{m\,|\omega^2-2\left(\frac{\pi m}{T}\right)^2\geq 0 \text{ and } \pi m/(N+1)<1\right\}.(401)$$

Therefore for all $m\leq m'$one obtain

$$\sigma_m(\omega)=\omega^2-\frac{1}{\epsilon^2}\left[2-2\left(1-\left(\frac{\pi m}{N+1}\right)^2+O(\epsilon^4)\right)\right]=$$

$$= \omega^2 - 2\left(\frac{\pi m}{T}\right)^2 + O(\epsilon^2). \qquad (402)$$

Thus for all $m \leq m'$the inequality

$$\sigma_m(\omega) = \omega^2 - 2\left(\frac{\pi m}{T}\right)^2 + O(\epsilon^2) \geq 0 \qquad (403)$$

is satisfied and consequently (1) the all eigenvalues $\sigma_m(\omega)$with $m \leq m'$are positive and (2) the all eigenvalues $\sigma_m(\omega)$with $m \gg m' + 1$ are negative. We have choose now number $r = m'$in Eq.(368) such that the inequalities

$$\sigma_r(\omega) = \omega^2 - 2\left(\frac{\pi r}{T}\right)^2 + O(\epsilon^2) \geq 0 \qquad (404)$$

and

$$\sigma_{r+1}(\omega) = \omega^2 - 2\left(\frac{\pi(r+1)}{T}\right)^2 + O(\epsilon^2) < 0 \qquad (405)$$

is satisfied and therefore $m' = O(\omega)$.We have choose now the number $\epsilon \in \mathbb{R}_+$such that the equalities

$$\rho(m) = \left(P/\sqrt{2\varepsilon}\right)\sqrt{\epsilon\sigma_m(\omega)} = O(1), m \leq m' \qquad (406)$$

is satisfied. From Eq.((397), inequalities (404)-(405) and
Eq.(406)one obtain

$$\mathcal{F}_N^{\omega}(T) == \frac{\mathbf{1}}{\sqrt{2\pi\epsilon}}\prod_{m=\mathbf{1}}^{m'}\left[\int_{-P/\sqrt{\mathbf{2\varepsilon}}}^{P/\sqrt{\mathbf{2\varepsilon}}}\frac{dp_m}{\sqrt{2\pi\epsilon}}\right]\prod_{m=m'+\mathbf{1}}^{\boldsymbol{N}}\left[\int_{-\infty}^{\infty}\frac{dp_m}{\sqrt{2\pi\epsilon}}\right]\times$$

$$\times\ \exp\left\{-\epsilon\sum_{m=1}^{N}(\delta p_m)^2\sigma_m(\omega)\right\} =$$

$$= \frac{1}{\sqrt{2\pi\epsilon}}\prod_{m=1}^{m'}\left[\int_{-\boldsymbol{\rho}(m)}^{\boldsymbol{\rho}(m)}\frac{dp_m}{\sqrt{2\pi|\sigma_m(\omega)|\epsilon^{\mathbf{2}}}}\right]\exp\left\{\sum_{m=1}^{m'}(\delta p_m)^{\mathbf{2}}\right\}$$

$$\times\prod_{m=m'+\mathbf{1}}^{N}\left[\int_{-\infty}^{\infty}\frac{dp_m}{\sqrt{2\pi\sigma_m(\omega)\epsilon^2}}\right]\exp\left\{\sum_{m'+1}^{N}-(\delta p_m)^2\right\}$$

$$= \frac{1}{\sqrt{2\pi\epsilon}}\prod_{m=1}^{m'}\left[\frac{1}{\sqrt{2\pi|\sigma_m(\omega)|\epsilon^2}}\right]\prod_{m=m'+\mathbf{1}}^{\boldsymbol{N}}\left[\frac{1}{\sqrt{2\pi\boldsymbol{\epsilon}^{\mathbf{2}}\sigma_m(\omega)}}\right]\times$$

$\times \exp\big(O(m')\big)$.(407)

Note that $\exp\big(O(m')\big) \simeq \exp\big(O(\omega^2)\big)$. From Eq.(407) we obtain

$$\mathcal{F}_N^{\omega}(T) =$$

$$\frac{\mathbf{1}}{\sqrt{2\pi\epsilon}}\prod_{m=\mathbf{1}}^{m'}\left[\frac{1}{\sqrt{2\pi\epsilon^2|\sigma_m(\omega)|}}\right]\prod_{m=m'+\mathbf{1}}^{\boldsymbol{N}}\left[\frac{1}{\sqrt{2\pi\epsilon^2\sigma_m(\omega)}}\right]\times$$

$$\exp\big(O(\omega^2)\big) =$$

$$= \frac{\exp\big(O(\omega^2)\big)}{\sqrt{2\pi\epsilon}}\prod_{m=\mathbf{1}}^{\boldsymbol{N}}\frac{1}{\sqrt{2\pi\boldsymbol{\epsilon}^{\mathbf{2}}|\Omega_m\overline{\Omega}_m-\omega^2|}}. \qquad (408)$$

The product of these eigenvalues, as well known [45] is found by introducing an auxiliary frequency ϖ satisfying

$$\sin\frac{\epsilon\varpi}{2} = \frac{\epsilon\omega}{2}. \qquad (409)$$

Now we decompose the product as [46]:

$$\prod_{m=1}^{N}\left[\epsilon^2\left|\Omega_m\overline{\Omega}_m-\omega^2\right|\right]=\prod_{m=\mathbf{1}}^{\boldsymbol{N}}\left[\boldsymbol{\epsilon}^{\mathbf{2}}\left|\Omega_m\overline{\Omega}_m\right|\right]\prod_{m=1}^{N}\left[\frac{\epsilon^2\left|\Omega_m\overline{\Omega}_m-\omega^2\right|}{\epsilon^2\left|\Omega_m\overline{\Omega}_m\right|}\right]$$

$$\prod_{m=1}^{\boldsymbol{N}}\left[\boldsymbol{\epsilon}^{\mathbf{2}}\left|\Omega_m\overline{\Omega}_m\right|\right]\prod_{m=1}^{N}\left[\left|1-\frac{\sin^2\left(\frac{\epsilon\varpi}{2}\right)}{\sin^2\left(\frac{m\pi}{2(N+1)}\right)}\right|\right]=\frac{\sin(\varpi T)}{\sin(\epsilon\varpi)}. \quad (410)$$

From Eq.(381) and Eq.(382)we obtain [46]:

$$\mathcal{F}_N^{\omega}(T)=\frac{1}{\sqrt{2\pi\epsilon}}\sqrt{\frac{\sin(\epsilon\varpi)}{|\sin(\varpi T)|}}\exp\big(O(\omega^2)\big). \quad (411)$$

In the limit $\epsilon\to 0$ finally we obtain [45]:

$$\mathcal{F}^{\omega}(T)=\lim_{N\to\infty}\mathcal{F}_N^{\omega}(T)=\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\omega}{|\sin(\omega T)|}}\exp\big(O(\omega^2)\big). \quad (412)$$

Remark. For a given ε we can to choose some real T, such that the inequality

$$\mathfrak{C}_T=\frac{(\varepsilon')\varepsilon^{2l}}{\sin(\omega T)}\gg 1 \quad (413)$$

where $\omega=\omega(\varepsilon)$, is satisfied.

From Eq.(395) and inequality (413)we obtain

$$\left(\breve{\boldsymbol{I}}_{\boldsymbol{\varepsilon}'}^{\varepsilon,\mathbf{2}}\left(\frac{P}{\sqrt{2\varepsilon}},m'\right)\right)_{\boldsymbol{\varepsilon}'}=\boldsymbol{I}_N^{\omega}(P,m')+O(1/\mathfrak{C}_T). \quad (414)$$

From inequality (389) and Eq.(414) we obtain

$$\left(\boldsymbol{I}_{\varepsilon'}^{\varepsilon,2}(P,m')\right)_{\varepsilon'} \leq \boldsymbol{I}_N^{\omega}(P,m') + O(1/\mathfrak{C}_T). \qquad (415)$$

From Eq.(412) and Eq.(352)-Eq.(356) we obtain

$$\boldsymbol{I}_N^{\omega}(P,m') = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\omega}{|\sin(\omega T)|}}\,\exp\left(O(\omega^2)\right) \times$$

$$\times \int_{-\infty}^{\infty} \mathrm{d}\tilde{q}''\exp\left[\widehat{S}_2(\tilde{q}',\tilde{q}'')\right] =$$

$$\frac{1}{\sqrt{2\pi}}\exp[\tilde{q}'^2\omega\mathrm{tg}(\omega T)/2]\exp\left(O(\omega^2)\right)\sqrt{\frac{\omega}{|\sin(\omega T)|}}\Big/\sqrt{\frac{\partial^2\widehat{S}_2(\tilde{q}',\tilde{q}'')}{\partial\tilde{q}''^2}} =$$

$$= \exp\left(O(\omega^2)\right). \qquad (416)$$

Substitution equality (416) into inequality (415) gives

$$\left(\boldsymbol{I}_{\varepsilon'}^{\varepsilon,2}(L,m')\right)_{\varepsilon'} \leq \exp\left(O(\omega^2)\right). \qquad (417)$$

Substitution equality (382) and inequality (417)into inequality (376) gives

$$\left(\boldsymbol{I}_{\varepsilon'}^{\varepsilon}(q',t',t'';P,m')\right)_{\varepsilon'} \leq \left(\left(\left[\boldsymbol{I}_{\varepsilon'}^{\varepsilon,1}(q',t',t'';P,m')\right]^{1-\varepsilon}\right)_{\varepsilon'}\right) \times$$

$$\times \left(\left(\left[\boldsymbol{I}_{\varepsilon'}^{\varepsilon,2}(q',t',t'';P,m')\right]^{\varepsilon}\right)_{\varepsilon'}\right) \leq$$

$$\leq (\mathrm{C})_{\varepsilon'}\left([\breve{q}(t)]^{2(1-\varepsilon)}\right)_{\varepsilon'}\exp\left(\varepsilon \times O(\omega^2)\right). \qquad (418)$$

Here $\varepsilon \approx 0, \varepsilon' \approx 0, \varepsilon'/\varepsilon \approx 0$ and

$$\dot{\breve{q}}(t) - b_0(\breve{q}(t),t) = 0, \breve{q}(t') = q'.$$

Inequality (418) completed the proof.

7. Conclusions.

1. We pointed out that the canonical Laplace approximation [27] is not a valid asymptotic approximation in the limit $\varepsilon \to 0$ for a path-integral (271), see also [42]

2. Supporting technical analysis. Let us consider optimal control problem from Example.1. Corresponding Bellman equation is:

$$\min_{\alpha_{1\in[-\rho,\rho]}}\left(\max_{\alpha_{1\in[-\rho,\rho]}}\left[\frac{\partial V}{\partial t}+x_2\frac{\partial V}{\partial x_1}+(-x_2^3+\alpha_1+\alpha_2)\frac{\partial V}{\partial x_2}\right]\right)=0$$

$$V(T,x_1,x_2)=x_1^2+x_2^2, t\in[0,T]. \qquad (419)$$

Complete constructing the exact analytical solution for PDE (27) is a complicated unresolved classical problem, because PDE (27) is not amenable to analytical treatments. Even the theorem of existence classical solution for boundary Problems such (27) is not proved. Thus, even for simple cases a problem of construction feedback optimal control by Bellman equation (419) complicated numerical technology or principal simplification is needed [46]. However as one can see complete constructing feedback optimal control from Theorem 6 is simple. In this paper, the generic imperfect dynamics models of air-to-surface missiles are given in addition to the related simple guidance law.

Appendix A

Proposition A1.[16].Assume that (1)$\varphi(t),\alpha(t)\in L_1([0,T])$, $\sup_{t\in[0,T]}|\varphi(t)|<\infty, \sup_{t\in[0,T]}|\alpha(t)|<\infty$ and (2) the inequality

$$\varphi(t)\le\alpha(t)+L\int_0^T\varphi(s)ds(1)$$

is satisfied. Then the inequality

$$\varphi(t) \leq \alpha(t) + L \int_0^T e^{L(t-s)} \alpha(s) ds \qquad (2)$$

is satisfied.

TheoremA1.(I) Assume that: (1)let $\boldsymbol{x}_{t,n}(\omega), n = 1,2, \dots$ be the solutions of the Ito's SDE's

$$d\boldsymbol{x}_{t,n} = \boldsymbol{b}_n(\boldsymbol{x}_{t,n}, t)dt + \boldsymbol{\sigma}_n(\boldsymbol{x}_{t,n}, t)d\boldsymbol{W}(t,\omega), \qquad (3)$$

$$\boldsymbol{x}_{0,n} = \boldsymbol{x}(\omega), x \in \mathbb{R}^d.$$

And let$\widetilde{\boldsymbol{x}}_{t,n}(t), n = 1,2, \dots$ be the solutions of the Ito's SDE's

$$d\widetilde{\boldsymbol{x}}_{t,n} = \widetilde{\boldsymbol{b}}_n(\widetilde{\boldsymbol{x}}_{t,n}, t)dt + \widetilde{\boldsymbol{\sigma}}_n(\widetilde{\boldsymbol{x}}_{t,n}, t)d\boldsymbol{W}(t,\omega), \qquad (4)$$

$$\widetilde{\boldsymbol{x}}_{0,n} = \boldsymbol{x}(\omega), x \in \mathbb{R}^d.$$

Here

$$\boldsymbol{\sigma}_n(\boldsymbol{x}_{t,n}, t)d\boldsymbol{W}(t,\omega) = \sum_{r=1}^{k} \sigma_{r,l,n}(\boldsymbol{x}_{t,n}, t)dW_r(t,\omega),$$

$$\widetilde{\boldsymbol{\sigma}}_n(\widetilde{\boldsymbol{x}}_{t,n}, t)d\boldsymbol{W}(t,\omega) = \sum_{r=1}^{k} \widetilde{\sigma}_{r,l,n}(\boldsymbol{x}_{t,n}, t)dW_r(t,\omega),$$

$$l = 1,2, \dots,, d.$$

(2) The inequalities

$$\|\boldsymbol{b}_n(\boldsymbol{x}, t)\|^2 + \|\boldsymbol{\sigma}_n(\boldsymbol{x}, t)\|^2 \quad \leq K_n(1 + \|\boldsymbol{x}\|^2), \qquad (5)$$

$$\|\boldsymbol{b}_n(\boldsymbol{x},t)-\boldsymbol{b}_n(\boldsymbol{y},t)\|+\|\boldsymbol{\sigma}_n(\boldsymbol{x},t)-\boldsymbol{\sigma}_n(\boldsymbol{x},t)\|\le K_n\|\boldsymbol{x}-\boldsymbol{y}\|, \qquad (6)$$

$$\left\|\widetilde{\boldsymbol{b}}_n(\boldsymbol{x},t)\right\|^2+\|\widetilde{\boldsymbol{\sigma}}_n(\boldsymbol{x},t)\|^2 \quad \le K_n(1+\|\boldsymbol{x}\|^2), \qquad (7)$$

$$\left\|\widetilde{\boldsymbol{b}}_n(\boldsymbol{x},t)-\widetilde{\boldsymbol{b}}_n(\boldsymbol{y},t)\right\|+\|\widetilde{\boldsymbol{\sigma}}_n(\boldsymbol{x},t)-\widetilde{\boldsymbol{\sigma}}_n(\boldsymbol{x},t)\|\le K_n\|\boldsymbol{x}-\boldsymbol{y}\|, \qquad (8)$$

$$\left\|\boldsymbol{b}_n(\boldsymbol{x},t)-\widetilde{\boldsymbol{b}}_n(\boldsymbol{x},t)\right\|\le \delta_{1,n}\|\boldsymbol{x}\|, \qquad (9)$$

$$\|\boldsymbol{\sigma}_n(\boldsymbol{x},t)-\widetilde{\boldsymbol{\sigma}}_n(\boldsymbol{x},t)\|\le \delta_{2,n}\|\boldsymbol{x}\|\;, \qquad (10)$$

where $0\le t\le T$,is satisfied.

Then the inequality

$$\sup_{0\le t\le T}\mathbf{E}\left[\left\|\boldsymbol{x}_{t,n}-\widetilde{\boldsymbol{x}}_{t,n}\right\|^2\right]\le$$

$$e^{L_n}\left(T\delta_{1,n}^2+\delta_{2,n}^2\right)\mathbf{E}\left[\int_0^T\left\|\widetilde{\boldsymbol{x}}_{t,n}\right\|^2dt\right] \qquad (11)$$

with $L_n=3(1+T)K_n$, is satisfied.

(**II**)Let $\tau_{U_n}(\omega)=\tau_n(\omega)$ be the random variable equal to the time at which the sample function of the process $\widetilde{\boldsymbol{x}}_{t,n}$first leaves thebounded neighborhood$U_n\ni 0$,and let$\tau_n(\omega,t)=\min(\tau_n(\omega),t)$.

Assume that: (1)$\forall n$:$U_n\subset U_{n+1}$,$\cup\, U_n=\mathbb{R}^d$, (2)

$$\sup_{\mathrm{n}\in\mathrm{N}}\left(\mathbf{E}\left[\int_0^T\left\|\widetilde{\boldsymbol{x}}_{t,n}\right\|^2dt\right]\right)<\infty. \qquad (12)$$

Then the inequality

$$\sup_{0\leq t\leq T}\left(\mathbf{E}\left[\left\|\boldsymbol{x}_{\tau_n(\omega,t),n}-\widetilde{\boldsymbol{x}}_{\tau_n(\omega,t),n}\right\|^2\right]\right)\leq$$

$$\leq e^{L_n}\left(T\delta_{1,n}^2+\delta_{2,n}^2\right)\sup_{\mathrm{n}\in\mathrm{N}}\left(\mathbf{E}\left[\int_0^T\left\|\widetilde{\boldsymbol{x}}_{t,n}\right\|^2dt\right]\right) \qquad (13)$$

with $L_n = 3(1+T)K_n$ is satisfied.

Proof.(**I**) From Eq.(3) and Eq.(4) one obtain

$$\boldsymbol{x}_{t,n}-\widetilde{\boldsymbol{x}}_{t,n}=\boldsymbol{\xi}_n(t)+\int_0^t\left[\boldsymbol{b}_n\left(\boldsymbol{x}_{s,n},s\right)-\widetilde{\boldsymbol{b}}_n\left(\widetilde{\boldsymbol{x}}_{s,n},s\right)\right]ds+$$

$$+\int_0^t\left[\boldsymbol{\sigma}_n\left(\boldsymbol{x}_{s,n},s\right)-\widetilde{\boldsymbol{\sigma}}_n\left(\widetilde{\boldsymbol{x}}_{s,n},s\right)\right]d\boldsymbol{W}(s). \qquad (14)$$

Here

$$\boldsymbol{\xi}_n(t)=\int_0^t\left[\boldsymbol{b}_n\left(\widetilde{\boldsymbol{x}}_{s,n},s\right)-\widetilde{\boldsymbol{b}}_n\left(\widetilde{\boldsymbol{x}}_{s,n},s\right)\right]ds+$$

$$+\int_0^t\left[\boldsymbol{\sigma}_n\left(\widetilde{\boldsymbol{x}}_{s,n},s\right)-\widetilde{\boldsymbol{\sigma}}_n\left(\widetilde{\boldsymbol{x}}_{s,n},s\right)\right]d\boldsymbol{W}(s). \qquad (15)$$

From Eq.(15) and inequalities (6) and(8) one obtain the inequality

$$\mathbf{E}\left[\left\|\boldsymbol{x}_{t,n}-\widetilde{\boldsymbol{x}}_{t,n}\right\|^2\right]\leq 3\mathbf{E}[\|\boldsymbol{\xi}_n(t)\|^2]+L_n\int_0^t\mathbf{E}\left[\left\|\boldsymbol{x}_{s,n}-\widetilde{\boldsymbol{x}}_{s,n}\right\|^2\right]ds, \qquad (16)$$

with$L_n = 3(1+T)K_n$.Using Proposition 1, from inequality (16) one obtain the inequality

$$\mathbf{E}\left[\left\|\boldsymbol{x}_{t,n}-\widetilde{\boldsymbol{x}}_{t,n}\right\|^2\right]\leq 3\mathbf{E}[\|\boldsymbol{\xi}_n(t)\|^2]+$$

$$+L_n\int_0^t \mathrm{e}^{L_n(t-s)}\mathbf{E}\left[\left\|\boldsymbol{x}_{s,n}-\widetilde{\boldsymbol{x}}_{s,n}\right\|^2\right]ds. \qquad (17)$$

From inequality (9) one obtain the inequality

$$\sup_{0\leq t\leq T}\left\|\mathbf{E}\left[\int_0^t [\boldsymbol{b}_n(\widetilde{\boldsymbol{x}}_{s,n},s)-\widetilde{\boldsymbol{b}}_n(\widetilde{\boldsymbol{x}}_{s,n},s)]ds\right]\right\|^{\mathbf{2}} \leq$$

$$T\int_0^T \mathbf{E}\left[\left\|\boldsymbol{b}_n(\widetilde{\boldsymbol{x}}_{s,n},s)-\widetilde{\boldsymbol{b}}_n(\widetilde{\boldsymbol{x}}_{s,n},s)\right\|^{\mathbf{2}}\right]ds \leq T\delta_{1,n}^2\int_0^T \mathbf{E}\left[\left\|\widetilde{\boldsymbol{x}}_{s,n}\right\|^{\mathbf{2}}\right]ds. (18)$$

From inequality (10) one obtain the inequality

$$\mathbf{E}\left[\sup_{0\leq t\leq T}\left\|\int_{\mathbf{0}}^{\boldsymbol{t}}[\boldsymbol{\sigma}_n(\boldsymbol{x}_{s,n},s)-\widetilde{\boldsymbol{\sigma}}_n(\widetilde{\boldsymbol{x}}_{s,n},s)]d\boldsymbol{W}(s)\right\|^{\mathbf{2}}\right] \leq$$

$$4\mathbf{E}\left[\int_{\mathbf{0}}^{\boldsymbol{T}}\left[\left\|\boldsymbol{\sigma}_n(\boldsymbol{x}_{s,n},s)-\widetilde{\boldsymbol{\sigma}}_n(\widetilde{\boldsymbol{x}}_{s,n},s)\right\|^{\mathbf{2}}\right]ds\right] \leq$$

$$\leq \delta_{2,n}^2\int_0^T \mathbf{E}\left[\left\|\widetilde{\boldsymbol{x}}_{s,n}\right\|^{\mathbf{2}}\right]ds. (19)$$

From Eq.(15) and inequalities (18)-(19) one obtain the inequality

$$\sup_{0\leq t\leq T}\mathbf{E}[\|\boldsymbol{\xi}_n(t)\|^{\mathbf{2}}] \leq \left(T\delta_{1,n}^2+\delta_{2,n}^2\right)\int_0^T \mathbf{E}\left[\left\|\widetilde{\boldsymbol{x}}_{s,n}\right\|^{\mathbf{2}}\right]ds. \quad (20)$$

Substitution the inequality (20) into inequality (17) gives

$$\sup_{0\leq t\leq T}\mathbf{E}\left[\left\|\boldsymbol{x}_{t,n}-\widetilde{\boldsymbol{x}}_{t,n}\right\|^2\right] \leq e^{L_n}(T\delta_{1,n}^2+\delta_{2,n}^2)\mathbf{E}\left[\int_0^T\left\|\widetilde{\boldsymbol{x}}_{t,n}\right\|^2 dt\right]. (21)$$

The inequality (21) completed the proof.

Proof.(**II**) Similarity to proof of the statement (**I**).

Let $\mathfrak{C}_i = (\Omega_i, \boldsymbol{\Sigma}_i, \mathbf{P}_i), i = 1,2$ be a probability spaces such that: $\Omega_1 \cap \Omega_2 = \emptyset$.Let $\boldsymbol{W}(t,\omega)$be a Wiener process on $\mathfrak{C}_1$and let
$\boldsymbol{W}(t,\varpi)$be a Wiener process on $\mathfrak{C}_2$.

Proposition A2.Assume that (1)$\varphi(t,\varpi), \alpha(t,\varpi) \in L_1([0,T])\mathbf{P}_2 - \text{o.s.}$,
$\sup_{t\in[0,T]}|\varphi(t,\varpi)| < \infty \mathbf{P}_2 - \text{o.s.}, \sup_{t\in[0,T]}|\alpha(t,\varpi)| < \infty$ $\mathbf{P}_2 - \text{o.s.}$, and (2) the inequality

$$\varphi(t,\varpi) \leq \alpha(t,\varpi) + L_\varpi \int_0^T \varphi(s,\varpi)ds \qquad (22)$$

$\mathbf{P}_2 - \text{o.s.}$ is satisfied. Then the inequality

$$\varphi(t,\varpi) \leq \alpha(t,\varpi) + L_\varpi \int_0^T e^{L_\varpi(t-s)}\alpha(s,\varpi)ds \qquad (23)$$

$\mathbf{P}_2 - \text{o.s.}$is satisfied.

Theorem A2..(I) Assume that: (1) let $\boldsymbol{x}_{t,n} = \boldsymbol{x}_{t,n}(\omega,\varpi), n = 1,2,\ldots$be the solutions of the Ito's SDE's

$$d\boldsymbol{x}_{t,n} = \boldsymbol{b}_n(\boldsymbol{x}_{t,n},t,\varpi)dt + \boldsymbol{\sigma}_n(\boldsymbol{x}_{t,n},t,\varpi)d\boldsymbol{W}(t,\omega), \qquad (24)$$

$$x_{0,n} = x(\omega,\varpi), x \in \mathbb{R}^d.$$

And let $\tilde{x}_{t,n}(t) = \tilde{x}_{t,n}(\omega,\varpi), n = 1,2,\ldots$ be the solutions of the Ito's SDE's

$$d\tilde{\boldsymbol{x}}_{t,n} = \tilde{\boldsymbol{b}}_n(\tilde{x}_{t,n},t,\varpi)dt + \tilde{\boldsymbol{\sigma}}_n(\tilde{x}_{t,n},t,\varpi)d\boldsymbol{W}(t,\omega), \qquad (25)$$

$$\tilde{\boldsymbol{x}}_{\mathbf{0},\boldsymbol{n}} = \boldsymbol{x}(\omega,\varpi), \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{d}}.$$

Here

$$\boldsymbol{\sigma_n}\big(\boldsymbol{x_{t,n}}, t, \varpi\big)\boldsymbol{dW}(t, \boldsymbol{\omega}) = \sum_{\boldsymbol{r=1}}^{\boldsymbol{k}} \boldsymbol{\sigma_{r,l,n}}\big(\boldsymbol{x_{t,n}}, t, \varpi\big)\boldsymbol{dW_r}(t, \omega),$$

$$\widetilde{\boldsymbol{\sigma}}_{\boldsymbol{n}}\big(\widetilde{\boldsymbol{x}}_{\boldsymbol{t,n}}, t, \varpi\big)\boldsymbol{dW}(t, \omega) = \sum_{\boldsymbol{r=1}}^{\boldsymbol{k}} \widetilde{\boldsymbol{\sigma}}_{\boldsymbol{r,l,n}}\big(\boldsymbol{x_{t,n}}, t, \varpi\big)\boldsymbol{dW_r}(\boldsymbol{t}, \omega),$$

$$l = 1,2, \dots, , d.$$

(2) The inequalities

$$\|\boldsymbol{b}_n(\boldsymbol{x}, t, \varpi)\|^2 + \|\boldsymbol{\sigma}_n(\boldsymbol{x}, t, \varpi)\|^2 \quad \le K_{n,\varpi}(1 + \|\boldsymbol{x}\|^{\mathbf{2}}) \ \ \mathbf{P}_2 - \text{o. s.,} \quad (26)$$

$$\|\boldsymbol{b}_n(\boldsymbol{x}, t, \varpi) - \boldsymbol{b}_n(\boldsymbol{y}, t, \varpi)\| +$$

$$\|\boldsymbol{\sigma}_n(\boldsymbol{x}, t, \varpi) - \boldsymbol{\sigma}_n(\boldsymbol{x}, t, \varpi)\| \le K_{n,\varpi}\|\boldsymbol{x} - \boldsymbol{y}\| \ \ \mathbf{P}_2 - \text{o. s.,} \qquad (27)$$

$$\left\|\widetilde{\boldsymbol{b}}_n(\boldsymbol{x}, t, \varpi)\right\|^2 + \|\widetilde{\boldsymbol{\sigma}}_n(\boldsymbol{x}, t, \varpi)\|^2 \quad \le K_{n,\varpi}(1 + \|\boldsymbol{x}\|^{\mathbf{2}}) \ \mathbf{P}_2 - \text{o. s.,} \ (28)$$

$$\left\|\widetilde{\boldsymbol{b}}_n(\boldsymbol{x}, t, \varpi) - \widetilde{\boldsymbol{b}}_n(\boldsymbol{y}, t, \varpi)\right\| +$$

$$\|\widetilde{\boldsymbol{\sigma}}_n(\boldsymbol{x}, t, \varpi) - \widetilde{\boldsymbol{\sigma}}_n(\boldsymbol{x}, t, \varpi)\| \le K_{n,\varpi}\|\boldsymbol{x} - \boldsymbol{y}\| \ \mathbf{P}_2 - \text{o. s.,} \qquad (29)$$

$$\left\|\boldsymbol{b}_n(\boldsymbol{x}, t, \varpi) - \widetilde{\boldsymbol{b}}_n(\boldsymbol{x}, t, \varpi)\right\| \le \delta_{1,n}\|\boldsymbol{x}\| \ \ \mathbf{P}_2 - \text{o. s.,} \qquad (30)$$

$$\|\boldsymbol{\sigma}_n(\boldsymbol{x}, t, \varpi) - \widetilde{\boldsymbol{\sigma}}_n(\boldsymbol{x}, t, \varpi)\| \le \delta_{2,n}\|\boldsymbol{x}\| \ \mathbf{P}_2 - \text{o. s.,} \qquad (31)$$

where $0 \le t \le T$,is satisfied. Then the inequality

$$\sup_{0\le t\le T} \mathbf{E}_{\Omega_1}\left[\left\|\boldsymbol{x}_{t,n}(\omega, \varpi) - \widetilde{\boldsymbol{x}}_{t,n}(\omega, \varpi)\right\|^2\right] \le$$

$$\leq e^{L_{n,\varpi}}\left(T\delta_{1,n}^2+\delta_{2,n}^2\right)\mathbf{E}_{\Omega_1}\left[\int_0^T\left\|\widetilde{\boldsymbol{x}}_{t,n}(\omega,\varpi)\right\|^2dt\right] \quad (32)$$

with $L_{n,\varpi}=3(1+T)K_{n,\varpi}$ $\mathbf{P}_2-\text{o.s.}$ is satisfied.

(**II**)Let $\tau_{U_n}(\omega,\varpi)=\tau_n(\omega,\varpi)$ be the random variable equal to the time at which the sample function of the process $\widetilde{\boldsymbol{x}}_{t,n}$first leaves the bounded neighborhood$U_n\ni 0$,and let$\tau_n(\omega,,\varpi,t)=\min(\tau_n(\omega,\varpi),t)$.

Assume that: (1)$\forall n$:$U_n\subset U_{n+1}$,$\cup\, U_n=\mathbb{R}^d$, (2)

$$\sup_{n\in\mathrm{N}}\left(\mathbf{E}_{\Omega_1}\left[\int_0^T\left\|\widetilde{\boldsymbol{x}}_{t,n}(\omega,\varpi)\right\|^2dt\right]\right)<\infty\ \ \mathbf{P}_2-\text{o.s.} \quad (33)$$

Then the inequality

$$\sup_{0\leq t\leq T}\left(\mathbf{E}_{\Omega_1}\left[\left\|\boldsymbol{x}_{\tau_n(\omega,\varpi,t),n}-\widetilde{\boldsymbol{x}}_{\tau_n(\omega,\varpi,t),n}\right\|^2\right]\right)\leq e^{L_{n,\varpi}}\left(T\delta_{1,n}^2+\delta_{2,n}^2\right)$$

$$\times\sup_{n\in\mathrm{N}}\left(\mathbf{E}_{\Omega_1}\left[\int_0^T\left\|\widetilde{\boldsymbol{x}}_{t,n}(\omega,\varpi)\right\|^2dt\right]\right)\mathbf{P}_2-\text{o.s.} \quad (34)$$

with $L_{n,\varpi}=3(1+T)K_{n,\varpi}$ is satisfied.

Proof.(**I**) From Eq.(24) and Eq.(25) one obtain

$$\boldsymbol{x}_{t,n}(\omega,\varpi)-\widetilde{\boldsymbol{x}}_{t,n(\omega,\varpi)}=\boldsymbol{\xi}_n(t,\omega,\varpi)+\int_0^t\left[\boldsymbol{b}_n\left(\boldsymbol{x}_{s,n},s,\varpi\right)-\widetilde{\boldsymbol{b}}_n\left(\widetilde{\boldsymbol{x}}_{s,n},s,\varpi\right)\right]ds+$$
$$\int_0^t\left[\boldsymbol{\sigma}_n\left(\boldsymbol{x}_{s,n},s,\varpi\right)-\widetilde{\boldsymbol{\sigma}}_n\left(\widetilde{\boldsymbol{x}}_{s,n},s,\varpi\right)\right]d\boldsymbol{W}(s). \quad (35)$$

Here

$$\boldsymbol{\xi}_n(t,\omega,\varpi)=\int_0^t\left[\boldsymbol{b}_n\left(\widetilde{\boldsymbol{x}}_{s,n},s,\varpi\right)-\widetilde{\boldsymbol{b}}_n\left(\widetilde{\boldsymbol{x}}_{s,n},s,\varpi\right)\right]ds+$$

$$+\int_0^t[\boldsymbol{\sigma}_n(\tilde{\boldsymbol{x}}_{s,n},s,\varpi)-\tilde{\boldsymbol{\sigma}}_n(\tilde{\boldsymbol{x}}_{s,n},s,\varpi)]d\boldsymbol{W}(s). \quad (36)$$

From Eq.(36) and inequalities (27) and(28) one obtain that the inequality

$$\mathbf{E}_{\Omega_1}\left[\left\|\boldsymbol{x}_{t,n}-\tilde{\boldsymbol{x}}_{t,n}\right\|^{2}\right]\leq 3\mathbf{E}[\|\boldsymbol{\xi}_n(t,\omega,\varpi)\|^{2}]+$$

$$+L_n\int_0^t\mathbf{E}_{\Omega_1}\left[\left\|\boldsymbol{x}_{s,n}(t,\omega,\varpi)-\ \tilde{\boldsymbol{x}}_{s,n}(t,\omega,\varpi)\right\|^{2}\right]ds, \quad (37)$$

with $L_{n,\varpi}=3(1+T)K_{n,\varpi}$. $\mathbf{P}_2-\mathrm{o.s.}$ is satisfied. Using now Proposition 2, from inequality (23) one obtain the inequality

$$\mathbf{E}_{\Omega_1}\left[\left\|\boldsymbol{x}_{t,n}(\omega,\varpi)-\tilde{\boldsymbol{x}}_{t,n}(\omega,\varpi)\right\|^{2}\right]\leq 3\mathbf{E}_{\Omega_1}[\|\boldsymbol{\xi}_n(t,\omega,\varpi)\|^{2}]+$$

$$+L_{n,\varpi}\int_0^t\mathrm{e}^{L_{n,\varpi}(t-s)}\mathbf{E}_{\Omega_1}\left[\left\|\boldsymbol{x}_{s,n}(\omega,\varpi)-\tilde{\boldsymbol{x}}_{s,n}(\omega,\varpi)\right\|^{2}\right]ds\ \ \mathbf{P}_2-\mathrm{o.s.}\ (38)$$

From inequality (30) one obtain the inequality

$$\sup_{0\leq t\leq T}\left\|\mathbf{E}_{\Omega_1}\left[\int_0^t[\boldsymbol{b}_n(\tilde{\boldsymbol{x}}_{s,n},s,\varpi)-\tilde{\boldsymbol{b}}_n(\tilde{\boldsymbol{x}}_{s,n},s,\varpi)]ds\right]\right\|^{2}\leq$$

$$T\int_0^T\mathbf{E}_{\Omega_1}\left[\left\|\boldsymbol{b}_n(\tilde{\boldsymbol{x}}_{s,n},s,\varpi)-\tilde{\boldsymbol{b}}_n(\tilde{\boldsymbol{x}}_{s,n},s,\varpi)\right\|^{2}\right]ds\leq$$

$$T\delta_{1,n}^2\int_0^T\mathbf{E}_{\Omega_1}\left[\left\|\tilde{\boldsymbol{x}}_{s,n}\right\|^{2}\right]ds\ \ \mathbf{P}_2-\mathrm{o.s.} \quad (39)$$

From inequality (31) one obtain the inequality

$$\mathbf{E}_{\Omega_1}\left[\sup_{0\le t\le T}\left\|\int_{\mathbf{0}}^{\boldsymbol{t}}\left[\boldsymbol{\sigma}_n\left(\boldsymbol{x}_{s,n},s,\varpi\right)-\widetilde{\boldsymbol{\sigma}}_n\left(\widetilde{\boldsymbol{x}}_{s,n},s,\varpi\right)\right]d\boldsymbol{W}(s)\right\|^{\mathbf{2}}\right]\le$$

$$4\mathbf{E}_{\Omega_1}\left[\int_{\mathbf{0}}^{\boldsymbol{T}}\left[\left\|\boldsymbol{\sigma}_n\left(\boldsymbol{x}_{s,n},s\right)-\widetilde{\boldsymbol{\sigma}}_n\left(\widetilde{\boldsymbol{x}}_{s,n},s\right)\right\|^{\mathbf{2}}\right]ds\right]\le$$

$$\le\delta_{2,n}^2\int_0^T\mathbf{E}_{\Omega_1}\left[\left\|\widetilde{\boldsymbol{x}}_{s,n}\right\|^{\mathbf{2}}\right]ds\ \ \mathbf{P}_2-\text{o.s.} \qquad (40)$$

From Eq.(15) and inequalities (18)-(19) one obtain that the inequality

$$\sup_{0\le t\le T}\mathbf{E}_{\Omega_1}[\|\boldsymbol{\xi}_n(t)\|^{\mathbf{2}}]\le\left(T\delta_{1,n}^2+\delta_{2,n}^2\right)\int_0^T\mathbf{E}\left[\left\|\widetilde{\boldsymbol{x}}_{s,n}\right\|^{\mathbf{2}}\right]ds. \quad (41)$$

$\mathbf{P}_2-\text{o.s.}$ is satisfied. Substitution the inequality (41) into inequality (39) gives:

$$\sup_{0\le t\le T}\mathbf{E}_{\Omega_1}\left[\left\|\boldsymbol{x}_{t,n}-\widetilde{\boldsymbol{x}}_{t,n}\right\|^2\right]\le$$

$$\le e^{L_n}\left(T\delta_{1,n}^2+\delta_{2,n}^2\right)\mathbf{E}_{\Omega_1}\left[\int_0^T\left\|\widetilde{\boldsymbol{x}}_{t,n}\right\|^2dt\right]\mathbf{P}_2-\text{o.s} \qquad (42)$$

The inequality (42) completed the proof.

Proof.(**II**) Similarity to proof of the statement (**I**).

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