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There is No Standard Model of ZFC and ZFC_2

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ABSTRACT

In this Chapter we obtain a contradictions in formal set theories under assumption that these theories have omega-models or nonstandard model with standard part. An posible generalization of Lob's theorem is considered. Main results are:

(i) $\neg Con(ZFC + \exists M_{st}^{ZFC})$,
(ii) $\neg Con(NF + \exists M_{st}^{NF})$,
(iii) $\neg Con(ZFC_2)$,
(iv) let k be an inaccessible cardinal then $\neg Con(ZFC + \exists \kappa)$,
(v) $\neg Con(ZFC + (V = L))$,
(vi) $\neg Con(ZF + (V = L))$.

Keywords: *Gödel encoding; Russell's paradox; standard model; Henkin semantics; inaccessible cardinal.*

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1 INTRODUCTION

1.1 Main Results

Let us remind that accordingly to naive set theory, any definable collection is a set. Let R be the set of all sets that are not members of themselves. If R qualifies as a member of itself, it would contradict its own definition as a set containing all sets that are not members of themselves. On the other hand, if such a set is not a member of itself, it would qualify as a member of itself by the same definition. This contradiction is Russell's paradox. In 1908, two ways of avoiding the paradox were proposed, Russell's type theory and Zermelo set theory, the first constructed axiomatic set theory. Zermelo's axioms went well beyond Frege's axioms of extensionality and unlimited set abstraction, and evolved into the now-canonical Zermelo–Fraenkel set theory ZFC.*"But how do we know that ZFC is a consistent theory, free of contradictions? The short answer is that we don't; it is a matter of faith (or of skepticism)"*- E.Nelson wrote in his paper [1]. However, it is deemed unlikely that even ZFC_2 which is significantly stronger than ZFC harbors an unsuspected contradiction; it is widely believed that if ZFC and ZFC_2 were inconsistent, that fact would have been uncovered by now. This much is certain-ZFC and ZFC_2 is immune to the classic paradoxes of naive set theory: Russell's paradox, the Burali-Forti paradox, and Cantor's paradox.

Remark 1.1.1. The inconsistency of the second order set theory ZFC2082 originally have been uncovered in [2] and officially announced in [3], see also ref. [4], [5], [6].

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Remark 1.1.2. In order to derive a contradiction in second order set theory ZFC_2 with the Henkin semantics [7], we remind the definition given in P.Cohen handbook [8], (see [8] Ch.III,sec.1,p.87). P.Cohen wroted: "A set which can be obtained as the result of a transfinite sequence of predicative definitions Godel called "constructible". His result then is that the con-structible sets are a model for ZF and that in this model GCH and AC hold.The notion of a predicative construction must be made more precise,of course, but there is essentially only one way to proceed. Another way to explain constructibility is to remark that the constructible sets are those sets which jnust occur in any model in which one admits all ordinals.The definition we now give is the one used in [9].

Definition 1.1.1. [8]. Let X be a set. The set X' is defined as the union of X and the set Y of all sets 443 for which there is a formula $A(z,t_1,...,t_k)$ in ZF such that if A_X denotes A with all bound variables restricted to X, then for some $\bar{t}_i, i=1,...,k.$ in X, $443 = \{z \in X \mid A_X(z,\bar{t}_1,...,\bar{t}_k)\}$.

Observe $X' \subsetneqq P(x) \cup X$, $\overline{\overline{X}}' = \overline{\overline{X}}$ if X is infinite (and we assume AC). It should be clear to the reader that the definition of X', as we have given it, can be done entirely within ZF

and that $Y = X'$ is a single formula $A(X,Y)$ in ZF. In general, one's intuition is that all normal definitions can be expressed in ZF, except possibly those which involve discussing the truth or falsity of an infinite sequence of statements. Since this is a very important point we shall give a rigorous proof in a later section that the construction of X' is expressible in ZF."

Remark 1.1.3. We will say that a set y is definable by the formula $A(z,t_1,...,t_k)$relative to a given set X.

Remark 1.1.4. Note that a simple generalsation of the notion of of the definability which has been by Definition1.1.1 immediately gives Russell's paradox in second order set theory ZFC_2 with the Henkin semantics [7].

Definition 1.1.2. [6]. (i) We will say that a set y is definable relative to a given set X iff there is a formula $A(z,t_1,...,t_k)$ in ZFC then for some $\bar{t}_i \in X, i=1,...,k.$ in X there exists a set z such that the condition $A(z,\bar{t}_1,...,\bar{t}_k)$ is satisfied and $y = z$ or symbolically $\exists z\left[A(z,\bar{t}_1,...,\bar{t}_k) \wedge y = z\right]$.

It should be clear to the reader that the definition of X', as we have given it, can be done entirely within second order set theory ZFC_2 with the Henkin semantics [7] denoted by ZFC_2^{Hs} and that $Y = X'$ is a single formula $A(X,Y)$ in ZFC_2^{Hs}.

(ii)We will denote the set Y of all sets 443 definable relative to a given set X by $Y \triangleq \Im_2^{Hs}$.

Definition 1.1.3. Let $\Re_2^{Hs}$ be a set of the all sets definable relative to a given set X by the first order 1-place open wff's and such that $\forall x\left(x \in \Im_2^{Hs}\right)\left[x \in \Re_2^{Hs} \iff x \notin x\right]$.

Remark 1.1.5.(a) Note that $\Re_2^{Hs} \in \Im_2^{Hs}$ since $\Re_2^{Hs}$ is a set definable by the first order 1-place open wff $\Psi\left(Z, \Im_2^{Hs}\right)$:

$$\Psi\left(Z, \Im_2^{Hs}\right) \triangleq \forall x\left(x \in \Im_2^{Hs}\right)\left[x \in Z \iff x \notin x\right], \tag{1.1.4}$$

Theorem 1.1.1. [6].Set theory ZFC_2^{Hs} is inconsistent. Proof. From (1.1.3) and Remark 1.1.2 one obtains

$$\Re_2^{Hs} \in \Re_2^{Hs} \iff \Re_2^{Hs} \notin \Re_2^{Hs}. \tag{1.1.5}$$

From (1.1.5) one obtains a contradiction $\left(\Re_2^{Hs} \in \Re_2^{Hs}\right) \wedge \left(\Re_2^{Hs} \notin \Re_2^{Hs}\right)$.

Remark 1.1.6. Note that in paper [6] we dealing by using following definability condition: a set 443 is definable if there is a formula $A(z)$ in ZFC such that

$$\exists z\left[A\left(z\right)\wedge y=z\right]. \tag{1.1.7}$$

Obviously in this case a set $Y=\Re_2^{Hs}$ is a countable set.

Definition 1.1.4. Let $\Re_2^{Hs}$ be the countable set of the all sets definable by the first order 1-place open wff's and such that

$$\forall x\left(x\in\Im_2^{Hs}\right)\left[x\in\Re_2^{Hs}\iff x\notin x\right]. \tag{1.1.8}$$

Remark 1.1.7. (a) Note that $\Re_2^{Hs}\in\Im_2^{Hs}$ since $\Re_2^{Hs}$ is a ZFC-set definable by the first order 1-place open wff

$$\Psi\left(Z,\Im_2^{Hs}\right)\triangleq\forall x\left(x\in\Im_2^{Hs}\right)\left[x\in Z\iff x\notin x\right], \tag{1.1.9}$$

one obtains a contradiction $\left(\Re_2^{Hs}\in\Re_2^{Hs}\right)\wedge\left(\Re_2^{Hs}\notin\Re_2^{Hs}\right)$. In this paper we dealing by using following definability condition.

Definition 1.1.5.(i) Let $M_{st}=M_{st}^{ZFC}$ be a standard model of ZFC. We will say that a set y is definable relative to a given standard model M_{st} of ZFC if there is a formula $A\left(z,t_1,...,t_k\right)$ in ZFC such that if $A_{M_{st}}$ denotes A with all bound variables restricted to M_{st}, then for some $\bar{t}_i\in M_{st}, i=1,...,k.$ in M_{st} there exists a set z such that the condition $A_{M_{st}}\left(z,\bar{t}_1,...,\bar{t}_k\right)$ is satisfied and $y=z$ or symbolically

$$\exists z\left[A_{M_{st}}\left(z,\bar{t}_1,...,\bar{t}_k\right)\wedge y=z\right]. \tag{1.1.10}$$

It should be clear to the reader that the definition of M'_{st}, as we have given it, can be done entirely within second order set theory ZFC_2 with the Henkin semantics.

(ii) In this paper we assume for simplicity but without loss of generality that

$$A_{M_{st}}\left(z,\bar{t}_1,...,\bar{t}_k\right)=A_{M_{st}}\left(z\right). \tag{1.1.11}$$

Remark 1.1.8. Note that in this paper we view (i) the first order set theory ZFC under the canonical first order semantics (ii) the second order set theory ZFC_2 under the Henkin semantics [7] and (iii) the second order set theory ZFC_2under the full second-order semantics [8] , [9], [10], [11], [12] but also with a proof theory bused on formal Urlogic [13].

Remark 1.1.9. Second-order logic essantially differs from the usual first-order predicate calculus in that it has variables and quantifiers not only for individuals but also for subsets of the universe and variables for n-ary relations as well [7], [8], [9], [10], [11], [12], [13]. The deductive calculus $\mathbf{DED}_2$ of second order logic is based on rules and axioms which guarantee that the quantifiers range at least over definable subsets [7]. As to the semantics, there are two tipes of models: Suppose $\mathbf{U}$ is an ordinary first-order structure and $\mathbf{S}$ is a set of subsets of the domain A of $\mathbf{U}$. The main idea is that the set-variables range over $\mathbf{S}$,i.e. $\langle\mathbf{U},\mathbf{S}\rangle\models\exists X\Phi\left(X\right)\iff\exists S\left(S\in\mathbf{S}\right)\left[\langle\mathbf{U},\mathbf{S}\rangle\models\Phi\left(S\right)\right].$

We call $\langle\mathbf{U},\mathbf{S}\rangle$ a Henkin model, if $\langle\mathbf{U},\mathbf{S}\rangle$ satisfies the axioms of $\mathbf{DED}_2$ and truth in $\langle\mathbf{U},\mathbf{S}\rangle$ is preserved by the rules of $\mathbf{DED}_2$. We call this semantics of second-order logic the Henkin semantics and second-order logic with the Henkin semantics the Henkin second-order logic. There is a special class of Henkin models, namely those $\langle\mathbf{U},\mathbf{S}\rangle$ where $\mathbf{S}$ is the set of all subsets of A.

We call these full models. We call this semantics of second-order logic the full semantics and second-order logic with the full semantics the full second-order logic.

Remark 1.1.10. We emphasize that the following facts are the main features of second-order logic:

1. The Completeness Theorem: A sentence is provable in $\mathbf{DED}_2$ if and only if it holds in all

Henkin models [7], [8], [9], [10], [11], [12], [13].

2. The Löwenheim-Skolem Theorem: A sentence with an infinite Henkin model has a countable Henkin model.

3. The Compactness Theorem: A set of sentences, every finite subset of which has a Henkin model, has itself a Henkin model.

4. The Incompleteness Theorem: Neither $\mathbf{DED}_2$ nor any other effectively given deductive calculus is complete for full models, that is, there are always sentences which are true in all full models but which are unprovable.

4. Failure of the Compactness Theorem for full models.

6. Failure of the Löwenheim-Skolem Theorem for full models.

7. There is a finite second-order axiom system $\mathbb{Z}_2$ such that the semiring $\mathbb{N}$ of natural numbers is the only full model of $\mathbb{Z}_2$ up to isomorphism.

8. There is a finite second-order axiom system RCF_2 such that the field $\mathbb{R}$ of the real numbers is the only full model of RCF_2 up to isomorphism.

Remark 1.1.11. For let second-order ZFC be, as usual, the theory that results obtained from ZFC when the axiom schema of replacement is replaced by its second-order universal closure,i.e.

$$\forall X\left[Func\left(X\right) \implies \forall u \exists \nu \forall r\left[r \in \nu \iff \exists s\left(s \in u \wedge (s,r) \in X\right)\right]\right], \qquad (1.1.12)$$

where X is a second-order variable, and where $Func\left(X\right)$ abbreviates " X is a functional relation",see [12].

Thus we interpret the wff's of ZFC_2 language with the full second-order semantics as required in [12], [13] but also wit a proof theory bused on formal Ur logic [13].

Designation 1.1.1. We will denote: (i) by ZFC_2^{Hs} set theory ZFC_2 with the Henkin semantics, (ii) by ZFC_2^{fss} set theory ZFC_2 with the full second-order semantics,(iii) by $\overline{ZFC}_2^{Hs}$ set theory $ZFC_2^{Hs} + \exists M_{st}^{ZFC_2^{Hs}}$ and (iv) by ZFC_{st} set theory $ZFC + \exists M_{st}^{ZFC}$, where M_{st}^{Th} is a standard model of the theory Th.

Remark 1.1.12. There is no completeness theorem for second-order logic with the full second-order semantics. Nor do the axioms of ZFC_2^{fss} imply a reflection principle which ensures that if a sentence Z of second-order set theory is true, then it is true in some model $M^{ZFC_2^{fss}}$ of ZFC_2^{fss} [11]. Let Z be the conjunction of all the axioms of ZFC_2^{fss}. We assume now that: Z is true,i.e. $Con\left(ZFC_2^{fss}\right)$. It is known that the existence of a model for Z requires the existence of strongly inaccessible cardinals, i.e. under ZFC it can be shown that $3ba$ is a strongly inaccessible if and only if $(H_{3ba}, \in)$ is a model of ZFC_2^{fss}. Thus

$$\neg Con(ZFC_2^{fss}) \implies \neg Con(ZFC + \exists\kappa)). \qquad (1.1.13)$$

In this paper we prove that:

(i) $ZFC_{st} \triangleq ZFC + \exists M_{st}^{ZFC}$ (ii) $\overline{ZFC}_2^{Hs} \triangleq ZFC_2^{Hs} + \exists M_{st}^{ZFC_2^{Hs}}$ and (iii) ZFC_2^{fss} is inconsistent, where M_{st}^{Th} is a standard model of the theory Th.

Axiom $\exists M^{ZFC}$. [8]. There is a set M^{ZFC} and a binary relation $\epsilon \subseteq M^{ZFC} \times M^{ZFC}$ which makes

M^{ZFC} a model for ZFC.

Remark 1.1.13. (i) We emphasize that it is well known that axiom $\exists M^{ZFC}$ a single statement in ZFC see [8], Ch.II,section 7.We denote this statement throught all this paper by symbol $Con\left(ZFC;M^{ZFC}\right)$.
The completness theorem says that $\exists M^{ZFC} \iff Con\left(ZFC\right)$.

(ii) Obviously there exists a single statement in ZFC_2^{Hs} such that $\exists M^{ZFC_2^{Hs}} \iff Con\left(ZFC_2^{Hs}\right)$.

We denote this statement throught all this paper by symbol $Con\left(ZFC_2^{Hs};M^{ZFC_2^{Hs}}\right)$ and there exists a single statement $\exists M^{Z_2^{Hs}}$ in Z_2^{Hs}. We denote this statement throught all this paper by symbol $Con\left(Z_2^{Hs};M^{Z_2^{Hs}}\right)$.

Axiom $\exists M_{st}^{ZFC}$. [8].There is a set M_{st}^{ZFC} such that if R is
$\left\{\langle x,y\rangle \,|x \in y \wedge x \in M_{st}^{ZFC} \wedge y \in M_{st}^{ZFC}\right\}$ then M_{st}^{ZFC} is a model for ZFC under the relation R.

Definition 1.1.6. [8].The model M_{st}^{ZFC} is called a standard model since the relation $\in$ used is merely the standard $\in$- relation.

Remark 1.1.14. Note that axiom $\exists M^{ZFC}$ doesn't imply axiom $\exists M_{st}^{ZFC}$,see ref. [8].

Remark 1.1.15. We remind that in Henkin semantics, each sort of second-order variable has a particular domain of its own to range over, which may be a proper subset of all sets or functions of that sort. Leon Henkin (1950) defined these semantics and proved that Gödel's completeness theorem and compactness theorem, which hold for first-order logic, carry over to second-order logic with Henkin semantics. This is because Henkin semantics are almost identical to many-sorted first-order semantics, where additional sorts of variables are added to simulate the new variables of second-order logic. Second-order logic with Henkin semantics is not more expressive than first-order logic. Henkin semantics are commonly used in the study of second-order arithmetic.Väänänen [13] argued that the choice between Henkin models and full models for second-order logic is analogous to the choice between ZFC and **V** (**V** is von Neumann universe), as a basis for set theory: "As with second-order logic, we cannot really choose whether we axiomatize mathematics using **V** or ZFC. The result is the same in both cases, as ZFC is the best attempt so far to use **V** as an axiomatization of mathematics."

Remark 1.1.16. Note that in order to deduce: (i) $\neg Con(ZFC_2^{Hs})$ from $Con(ZFC_2^{Hs})$,

(ii) $\neg Con(ZFC)$ from $Con(ZFC)$,by using Gödel encoding, one needs something more than the consistency of ZFC_2^{Hs}, e.g., that ZFC_2^{Hs} has an omega-model $M_\omega^{ZFC_2^{Hs}}$ or an standard model $M_{st}^{ZFC_2^{Hs}}$ i.e., a model in which the *integers are the standard integers and the all wff* of ZFC_2^{Hs}, ZFC,*etc. represented by standard objects.*To put it another way, why should we believe a statement just because there's a ZFC_2^{Hs}-proof of it? It's clear that if ZFC_2^{Hs} is inconsistent, then we won't believe ZFC_2^{Hs}-proofs. What's slightly more subtle is that the mere consistency of ZFC_2 isn't quite enough to get us to believe arithmetical theorems of ZFC_2^{Hs}; we must also believe that these arithmetical theorems are asserting something about the standard naturals. It is "conceivable" that ZFC_2^{Hs} might be consistent but that the only nonstandard models $M_{Nst}^{ZFC_2^{Hs}}$ it has are those in which the integers are nonstandard, in which case we might not "believe" an arithmetical statement such as "ZFC_2^{Hs} is inconsistent" even if there is a ZFC_2^{Hs}-proof of it.

Remark 1.1.17. Note that assumption $\exists M_{st}^{ZFC_2^{Hs}}$ is not necessary if nonstandard model $M_{Nst}^{ZFC_2^{Hs}}$

is a transtive or has an standard part $M_{st}^{Z_2^{Hs}} \subset M_{Nst}^{Z_2^{Hs}}$,see [14], [15].

Remark 1.1.18. Remind that if M is a transitive model, then $3c9^M$ is the standard $3c9$. This implies that the natural numbers, integers, and rational numbers of the model are also the same as their standard counterparts. Each real number in a transitive model is a standard real number, although not all standard reals need be included in a particular transitive model. Note that in any nonstandard model $M_{Nst}^{Z_2^{Hs}}$ of the second-order arithmetic Z_2^{Hs} the terms $\overline{0}$, $\mathbf{S}\overline{0} = \overline{1}, \mathbf{SS}\overline{0} = \overline{2}, \ldots$ comprise the initial segment isomorphic to $M_{st}^{Z_2^{Hs}} \subset M_{Nst}^{Z_2^{Hs}}$. This initial segment is called the standard cut of the $M_{Nst}^{Z_2^{Hs}}$. The order type of any nonstandard model of $M_{Nst}^{Z_2^{Hs}}$ is equal to $\mathbb{N} + A \times \mathbb{Z}$,see ref. [16], for some linear order A

Thus one can to choose Gödel encoding inside the standard model $M_{st}^{Z_2^{Hs}}$.

Remark 1.1.19. However there is no any problem as mentioned above in second order set theory ZFC_2 with the full second-order semantics because corresponding second order arithmetic Z_2^{fss} is categorical.

Remark 1.1.20. Note if we view second-order arithmetic Z_2 as a theory in first-order predicate calculus. Thus a model M^{Z_2} of the language of second-order arithmetic Z_2 consists of a set M (which forms the range of individual variables) together with a constant 0 (an element of M), a function S from M to M, two binary operations $+$ and $\times$ on M, a binary relation $<$ on M, and a collection D of subsets of M, which is the range of the set variables. When D is the full powerset of M, the model M^{Z_2} is called a full model. The use of full second-order semantics is equivalent to limiting the models of second-order arithmetic to the full models. In fact, the axioms of second-order arithmetic have only one full model. This follows from the fact that the axioms of Peano arithmetic with the second-order induction axiom have only one model under second-order semantics, i.e. Z_2, with the full semantics, is categorical by Dedekind's argument, so has only one model up to isomorphism. When M is the usual set of natural numbers with its usual operations, M^{Z_2} is called an $3c9$-model. In this case we may identify the model with D, its collection of sets of naturals, because this set is enough to completely determine an $3c9$-model. The unique full omega-model $M_{\omega}^{Z_2^{fss}}$, which is the usual set of natural numbers with its usual structure and all its subsets, is called the intended or standard model of second-order arithmetic.

1.2 Remarks on the Tarski's Undefinability Theorem

Theorem 1.2.1.(Tarski's undefinability theorem) Let **Th** be first order theory with formal language ,which includes negation and has a Gödel numbering $g(\circ)$ such that for every -formula $A(x)$ there is a formula B such that $B \leftrightarrow A(g(B))$ holds. Assume that **Th** has a standard model $M_{st}^{\mathbf{Th}}$ and $Con\left(\mathbf{Th}_{,st}\right)$ where

$$\mathbf{Th}_{,st} \triangleq \mathbf{Th} + \exists M_{st}^{\mathbf{Th}}. \qquad (1.2.1)$$

Let T^*be the set of Gödel numbers of -sentences true in $M_{st}^{\mathbf{Th}}$. Then there is no -formula $\mathbf{True}(n)$ (truth predicate) which defines T^*.That is, there is no -formula $\mathbf{True}(n)$ such that for every -formula A, $\mathbf{True}(g(A)) \iff [A]_{M_{st}^{\mathbf{Th}}}$ (1.2.2)

where the abbraviation $[A]_{M_{st}^{\mathbf{Th}}}$means that A holds in standard model $M_{st}^{\mathbf{Th}}$,i.e. $[A]_{M_{st}^{\mathbf{Th}}} \iff \models_{M_{st}^{\mathbf{Th}}} A$.Thus $Con\left(\mathbf{Th}_{,st}\right)$ implies that

$$\neg\exists\mathbf{True}(x)\left(\mathbf{True}(g(A)) \iff [A]_{M_{st}^{\mathbf{Th}}}\right) \qquad (1.2.3)$$

Thus Tarski's undefinability theorem reads

$$Con\left(\mathbf{Th}_{,st}\right) \implies \neg\exists\mathbf{True}(x)\ \left(\mathbf{True}(g(A)) \iff [A]_{M_{st}^{\mathbf{Th}}}\right). \quad (1.2.4)$$

Remark 1.2.2. By the other hand the Theorem 1.2.1 says that given some formal theory $\mathbf{Th}_{,st}$ that contins formal arithmetic, the concept of truth in that formal theory $\mathbf{Th}_{,st}$ is not definable using the expressive means that that arithmetic affords. This implies a major limitation on the scope of "self-representation." It is possible to define a formula $\mathbf{True}(n)$,but only by drawing on a metalanguage whose expressive power goes beyond that of .To define a truth predicate for the metalanguage would require a still higher metametalanguage, and so on.

Remark 1.2.3. In this paper under the following assumption

$$Con\left(ZFC + \exists M_{st}^{ZFC}\right) \quad (1.2.5)$$

in particular we prove that there exists countable Rassel's set $\Re_\omega$ such that the following statement holds:

$$\begin{gathered} ZFC + \exists M_{st}^{ZFC} \vdash \\ \exists \Re_\omega \left(\Re_\omega \in M_{st}^{ZFC}\right) \wedge \left(card\left(\Re_\omega\right) = \aleph_0\right) \wedge \left[\models_{M_{st}^{ZFC}} \forall x \left(x \in \Re_\omega \iff x \notin x\right)\right]. \end{gathered} \quad (1.2.6)$$

From (1.2.6) immediately follows a contradiction

$$\models_{M_{st}^{ZFC}} \ (\Re_\omega \in \Re_\omega) \wedge (\Re_\omega \notin \Re_\omega). \quad (1.2.7)$$

From (1.2.5) and (1.2.7) by *reductio ad absurdum* it follows

$$\neg Con\left(ZFC + \exists M_{st}^{ZFC}\right). \quad (1.2.8)$$

Remark 1.2.4. It follows from (1.2.8) that Tarski's undefinability theorem (Theorem 1.2.1) obviously no longer holds.

Definition 1.2.1. Let $\mathbf{Th}^{\#}$ be first order theory and $Con\left(\mathbf{Th}^{\#}\right)$. A theory $\mathbf{Th}^{\#}$ is complete if, for every formula A in the theory's language , that formula A or its negation $\neg A$ is provable in $\mathbf{Th}^{\#}$,i.e., for any wff A, always $\mathbf{Th}^{\#} \vdash A$ or $\mathbf{Th}^{\#} \vdash \neg A$.

Definition 1.2.2. Let $\mathbf{Th}$ be first order theory and $Con\left(\mathbf{Th}\right)$.We will say that a theory $\mathbf{Th}^{\#}$ is completion of the theory $\mathbf{Th}$ if (i) $\mathbf{Th} \subset \mathbf{Th}^{\#}$,(ii) a theory $\mathbf{Th}^{\#}$ is complete.

Theorem 1.2.2. [4], [5]. Assume that:$Con\left(ZFC_{st}\right)$,where $ZFC_{st} \triangleq ZFC + \exists M_{st}^{ZFC}$.Then there exists completion $ZFC_{st}^{\#}$ of the theory ZFC_{st} such that the following condtions holds:

(i) For every formula A in the language of ZFC_{st} that formula $[A]_{M_{st}^{ZFC}}$ or formula $[\neg A]_{M_{st}^{ZFC}}$is provable in $ZFC_{st}^{\#}$ i.e., for any wff A, always $ZFC_{st}^{\#} \vdash [A]_{M_{st}^{ZFC}}$ or $ZFC_{st}^{\#} \vdash [\neg A]_{M_{st}^{ZFC}}$.

(ii) $ZFC_{st}^{\#} = \cup_{m \in \mathbb{N}} \mathbf{Th}_m$,where for any m a theory $\mathbf{Th}_{m+1}$ is finite extension of the theory $\mathbf{Th}_m$.

(iii) Let $\Pr_m^{st}(y, x)$ be recursive relation such that: y is a Gödel number of a proof of the wff of the theory $\mathbf{Th}_m$ and x is a Gödel number of this wff.Then the relation $\Pr_m^{st}(y, x)$ is expressible in the theory $\mathbf{Th}_m$ by canonical Gödel encoding and really asserts provability in $\mathbf{Th}_m$.

(iv) Let $\Pr_{st}^{\#}(y, x)$ be relation such that: y is a Gödel number of a proof of the wff of the theory $ZFC_{st}^{\#}$ and x is a Gödel number of this wff.Then the relation $\Pr_{st}^{\#}(y, x)$

is expressible in the theory $ZFC_{st}^{\#}$ by the following formula

$$\Pr_{st}^{\#}(y,x) \iff \exists m \Pr_m^{st}(y,x)) \tag{1.2.9}$$

(v) The predicate $\Pr_{st}^{\#}(y,x)$ really asserts provability in the set theory $ZFC_{st}^{\#}$.

Remark 1.2.5. Note that the relation $\Pr_m^{st}(y,x)$ is expressible in the theory $\mathbf{Th}_m$ since a theory $\mathbf{Th}_m$ is an finite extension of the recursively axiomatizable theory ZFC_{st} and therefore the predicate $\Pr_m^{st}(y,x)$ exists since any theory $\mathbf{Th}_m$ is recursively axiomatizable.

Remark 1.2.6. Note that a theory $ZFC_{st}^{\#}$ obviously is not recursively axiomatizable nevertheless Gödel encoding holds by Theorem 1.2.2.

Theorem 1.2.3. Assume that:$Con(ZFC_{st})$,where $ZFC_{st} \triangleq ZFC + \exists M_{st}^{ZFC}$.Then truth predicate $\mathbf{True}(n)$ is expressible by using first order language by the following formula

$$\mathbf{True}(g(A)) \iff \exists y\,(y \in \mathbb{N})\,\exists m\,(m \in \mathbb{N}) \Pr_m^{st}(y, g(A)). \tag{1.2.10}$$

Proof.Assume that:

$$ZFC_{st}^{\#} \vdash [A]_{M_{st}^{ZFC}}. \tag{1.2.11}$$

It follows from (1.2.11) there exists $m^* = m^*(g(A))$ such that $\mathbf{Th}_{m^*} \vdash [A]_{M_{st}^{ZFC}}$ and therefore by (1.2.9) we obtain $\Pr_{st}^{\#}(y, g(A)) \iff \Pr_{m^*}^{st}(y, g(A)))$. (1.2.12)

From (1.2.12) immediately by definitions one obtains (1.2.10).

Remark 1.2.7. Note that Theorem 1.2.3 reads

$$Con(ZFC_{st}) \implies \exists \mathbf{True}(x) \left(\mathbf{True}(g(A)) \iff [A]_{M_{st}^{ZFC}}\right). \tag{1.2.13}$$

Theorem 1.2.4. $\neg Con(ZFC_{st})$.

Proof.Assume that: $Con(ZFC_{st})$.From (1.2.10) and (1.2.13) one obtains a condradiction $Con(ZFC_{st}) \wedge \neg Con(ZFC_{st})$ and therefore by *reductio ad absurdum* it follows $\neg Con(ZFC_{st})$.

Theorem 1.2.5. [4], [5]. Let M_{Nst}^{ZFC} be a nonstandard model of ZFC and let M_{st}^{PA} be a standard model of PA.We assume now that $M_{st}^{PA} \subset M_{Nst}^{ZFC}$ and denote such nonstandard model of the set theory ZFC by $M_{Nst}^{ZFC} = M_{Nst}^{ZFC}[PA]$.Let ZFC_{Nst} be the theory $ZFC_{Nst} = ZFC + M_{Nst}^{ZFC}[PA]$. Assume that:$Con(ZFC_{Nst})$,where $ZFC_{st} \triangleq ZFC + \exists M_{Nst}^{ZFC}$.Then there exists completion $ZFC_{Nst}^{\#}$ of the theory ZFC_{Nst} such that the following condtions holds:

(i) For every formula A in the language of ZFC that formula $[A]_{M_{Nst}^{ZFC}}$ or formula $[\neg A]_{M_{Nst}^{ZFC}}$is provable in $ZFC_{Nst}^{\#}$ i.e., for any wff A, always $ZFC_{Nst}^{\#} \vdash [A]_{M_{Nst}^{ZFC}}$ or $ZFC_{Nst}^{\#} \vdash [\neg A]_{M_{Nst}^{ZFC}}$.

(ii) $ZFC_{Nst}^{\#} = \cup_{m \in \mathbb{N}} \mathbf{Th}_m$,where for any m a theory $\mathbf{Th}_{m+1}$ is finite extension of the theory $\mathbf{Th}_m$.

(iii) Let $\Pr_m^{Nst}(y,x)$ be recursive relation such that: y is a Gödel number of a proof of the wff of the theory $\mathbf{Th}_m$ and x is a Gödel number of this wff.Then the relation expressible in the theory $\mathbf{Th}_m$ by canonical Gödel encoding and really asserts provability in $\mathbf{Th}_m$.

(iv) Let $\Pr_{Nst}^{\#}(y,x)$ be relation such that: y is a Gödel number of a proof of the wff of the theory

$ZFC^{\#}_{Nst}$ and x is a Gödel number of this wff.Then the relation $\Pr^{\#}_{Nst}(y,x)$ is expressible in the theory $ZFC^{\#}_{Nst}$ by the following formula

$$\Pr^{\#}_{Nst}(y,x) \iff \exists m \left(m \in M^{PA}_{st}\right) \Pr^{Nst}_{m}(y,x)) \quad (1.2.14)$$

(v) The predicate $\Pr^{\#}_{Nst}(y,x)$ really asserts provability in the set theory $ZFC^{\#}_{Nst}$.

Remark 1.2.8. Note that the relation $\Pr^{Nst}_{m}(y,x)$ is expressible in the theory $\mathbf{Th}_m$ since a theory $\mathbf{Th}_m$ is an finite extension of the recursively axiomatizable theory ZFC and therefore the predicate $\Pr^{Nst}_{m}(y,x)$ exists since any theory $\mathbf{Th}_m$ is recursively axiomatizable.

Remark 1.2.9. Note that a theory $ZFC^{\#}_{Nst}$ obviously is not recursively axiomatizable nevertheless Gödel encoding holds by Theorem 1.2.5.

Theorem 1.2.6. Assume that:$Con(ZFC_{Nst})$,where $ZFC_{Nst} \triangleq ZFC + \exists M^{ZFC}_{Nst}$.Then truth predicate $\mathbf{True}(n)$ is expressible by using first order language by the following formula

$$\mathbf{True}(g(A)) \iff \exists y \left(y \in M^{PA}_{Nst}\right) \exists m \left(m \in M^{PA}_{Nst}\right) \Pr^{Nst}_{m}(y, g(A)). \quad (1.2.15)$$

Proof.Assume that:

$$ZFC^{\#}_{Nst} \vdash [A]_{M^{ZFC}_{Nst}}. \quad (1.2.16)$$

It follows from (1.2.14) there exists $m^* = m^*(g(A))$ such that $\mathbf{Th}_{m^*} \vdash [A]_{M^{ZFC}_{Nst}}$ and therefore by (1.2.14) we obtain

$$\Pr^{\#}_{Nst}(y, g(A)) \iff \Pr^{Nst}_{m^*}(y, g(A))). \quad (1.2.17)$$

From (1.2.17) immediately by definitions one obtains (1.2.15).

Remark 1.2.10. Note that Theorem 1.2.6 reads

$$Con(ZFC_{Nst}) \implies \exists \mathbf{True}(x) \left(\mathbf{True}(g(A)) \iff [A]_{M^{\mathbf{ZFC}}_{Nst}}\right). \quad (1.2.18)$$

Theorem 1.2.7. $\neg Con(ZFC_{Nst})$.

Proof. Assume that: $Con(ZFC_{Nst})$.From (1.2.15) and (1.2.18) one obtains a condradiction $Con(ZFC_{Nst}) \wedge \neg Con(ZFC_{Nst})$ and therefore by *reductio ad absurdum* it follows $\neg Con(ZFC_{Nst})$.

2 DERIVATION OF THE INCONSISTENT DEFINABLE SET IN SET THEORY ZFC^{HS}_2 AND IN SET THEORY ZFC_{ST}

2.1 Derivation of the Inconsistent Definable Set in Set Theory $\overline{ZFC}^{Hs}_2$

In this section we obtain a contradiction in set theory $\overline{ZFC}^{Hs}_2$ by using a set of the all sets definable by first order 1-place open wff's of the set theory $\overline{ZFC}^{Hs}_2$.

We start to explain main idea from some simply formal definitions.

Definition 2.1.1. Let $M_{st} \triangleq M_{st}^{ZFC_2^{Hs}}$. Let ${}^{Hs}_{X} \triangleq {}^{Hs}_{X,M_{st}}$ be a set of the all first order 1-place open wff's $\Psi(X) = \Psi_{M_{st}}(X)$ (wff$_1$) of the set theory $\overline{ZFC}_2^{Hs}$ with all bound variables restricted to standard model M_{st} and that contains free occurrences of the first order individual variable X and quantifiers only over first order individual variables, i.e. ${}^{Hs}_{X}$ is a set of the all first order 1-place open wff's with all bound variables restricted to standard model $M_{st}^{ZFC_2^{Hs}}$.

We define now a set $\Gamma_X^{Hs} \triangleq \Gamma_{X,M_{st}}^{Hs} \subsetneqq {}^{Hs}_{X,M_{st}}$ by the following (second order) formula

$$\begin{gathered}\forall \Psi_{M_{st}}(X)\left[\Psi_{M_{st}}(X) \in \Gamma^{Hs}_{X,M_{st}} \iff \right.\\ \left.\left(\exists! X\left(X \in M_{st}^{ZFC_2^{Hs}}\right)\Psi_{M_{st}}(X)\right) \wedge \left(\Psi_{M_{st}}(X) \in {}^{Hs}_{X,M_{st}}\right)\right],\end{gathered} \tag{2.1.1}$$

or in the following equivalent form

$$\begin{gathered}\forall \Psi_{M_{st}}(X)\left[\Psi_{M_{st}}(X) \in \Gamma^{Hs}_{X,M_{st}} \iff \exists y \widehat{\mathbf{Fr}_1^{Hs}}(y,v) \searrow \right.\\ \left[\left(g_{\overline{ZFC}_2^{Hs}}\left(\Psi_{M_{st}}(X)\right) = y\right) \wedge \left(g_{\overline{ZFC}_2^{Hs}}(X) = \nu\right)\right]\\ \left.\wedge\left(\exists! X\left(X \in M_{st}^{ZFC_2^{Hs}}\right)\Psi_{M_{st}}(X)\right) \wedge \left(\Psi_{M_{st}}(X) \in {}^{Hs}_{X,M_{st}}\right)\right],\end{gathered} \tag{2.1.1.a}$$

see Remark 2.1.10 (ix) and Eq.(2.1.28). Note that there exist a set $\Gamma^{Hs}_{X,M_{st}}$ by the second order separaton axiom.

Notation 2.1.1. In this subsection we often write for short $\Psi(X), {}^{Hs}_{X}, \Gamma^{Hs}_{X}$ instead $\Psi_{M_{st}}(X)$, ${}^{Hs}_{X,M_{st}}, \Gamma^{Hs}_{X,M_{st}}$ but this should not lead to a confusion.

Assumption 2.1.1. We assume now for simplicity but without loss of generality that

$${}^{Hs}_{X,M_{st}} \in M_{st} \tag{2.1.1.b}$$

and therefore by definition of model M_{st} one obtains $\Gamma^{Hs}_{X,M_{st}} \in M_{st}$.

Definition 2.1.2. Let $\Psi_1 = \Psi_1(X) = \Psi_{1,M_{st}}(X)$ and $\Psi_2 = \Psi_2(X) = \Psi_{2,M_{st}}(X)$ be a first order 1-place open wff's of the set theory $\overline{ZFC}_2^{Hs}$ and with all bound variables restricted to standard model M_{st} .

(i) We define now the equivalence relation $(\cdot \sim_X \cdot) \triangleq (\cdot \sim_{X,M_{st}} \cdot) \subset \Gamma^{Hs}_{X,M_{st}} \times \Gamma^{Hs}_{X,M_{st}}$ by the following formula

$$\begin{gathered}\forall \Psi_1 \forall \Psi_2 \left(\Psi_1 \sim_X \Psi_2\right) \iff \forall \Psi_1(X) \forall \Psi_2(X) \left\{\left[\Psi_1(X) \sim_X \Psi_2(X)\right]\right.\\ \left.\iff \forall X\left(X \in M_{st}^{ZFC_2^{Hs}}\right)\left[\Psi_1(X) \iff \Psi_2(X)\right]\right\} \iff\\ \forall \Psi_1(X) \forall \Psi_2(X)\left\{\left[\Psi_1(X) \sim_X \Psi_2(X)\right] \iff\right.\\ \left.\left[\forall X\left(X \in M_{st}^{ZFC_2^{Hs}}\right)\Psi_1(X) \iff \forall X\left(X \in M_{st}^{ZFC_2^{Hs}}\right)\Psi_2(X)\right]\right\}.\end{gathered} \tag{2.1.2}$$

or in the following equivalent form

$$\begin{array}{c}\forall\Psi_1\forall\Psi_2\left(\Psi_1\sim_X\Psi_2\right)\iff\forall\Psi_1(X)\forall\Psi_2(X)\{[\Psi_1(X)\sim_X\Psi_2(X)]\\ \iff\exists y_1\widehat{\mathbf{Fr}_1^{Hs}}(y_1,v)\exists y_2\widehat{\mathbf{Fr}_1^{Hs}}(y_2,v)\forall X\left(X\in M_{st}^{ZFC_2^{Hs}}\right)[\Psi_1(X)\iff\Psi_2(X)]\Big\}\iff\\ \forall\Psi_1(X)\forall\Psi_2(X)\{[\Psi_1(X)\sim_X\Psi_2(X)]\iff\exists y_1\widehat{\mathbf{Fr}_1^{Hs}}(y_1,v)\exists y_2\widehat{\mathbf{Fr}_1^{Hs}}(y_2,v)\searrow\\ \left[\left(g_{\overline{ZFC_2^{Hs}}}(\Psi_1(X))=y_1\right)\wedge\left(g_{\overline{ZFC_2^{Hs}}}(\Psi_2(X))=y_2\right)\wedge\left(g_{\overline{ZFC_2^{Hs}}}(X)=\nu\right)\right]\wedge\\ \left[\forall X\left(X\in M_{st}^{ZFC_2^{Hs}}\right)\Psi_1(X)\iff\forall X\left(X\in M_{st}^{ZFC_2^{Hs}}\right)\Psi_2(X)\right]\Big\}.\end{array}\quad(2.1.2.a)$$

(ii) Note that the equivalence relation $(\cdot\sim_X\cdot)$ well defined as a set of ordered pair $Z_{1,2}$

such that

$$\begin{array}{c}Z_{1,2}=\left\{(\Psi_1,\Psi_2)\,|\left[(\Psi_1,\Psi_2)\in\Gamma_X^{Hs}\times\Gamma_X^{Hs}\right]\wedge[\Theta(\Psi_1,\Psi_2)]\right\},\\ \Theta(\Psi_1,\Psi_2)\triangleq\ \models_{M_{st}^{ZFC_2^{Hs}}}\forall X[\Psi_1(X)\iff\Psi_2(X)],\end{array}\quad(2.1.3)$$

or in the following equivalent form

$$\begin{array}{c}Z_{1,2}=\left\{(\Psi_1,\Psi_2)\,|\left[(\Psi_1,\Psi_2)\in\Gamma_X^{Hs}\times\Gamma_X^{Hs}\right]\wedge\exists y_1\widehat{\mathbf{Fr}_1^{Hs}}(y_1,v)\exists y_2\widehat{\mathbf{Fr}_1^{Hs}}(y_2,v)\searrow\right.\\ \left[\left(g_{\overline{ZFC_2^{Hs}}}(\Psi_1(X))=y_1\right)\wedge\left(g_{\overline{ZFC_2^{Hs}}}(\Psi_2(X))=y_2\right)\wedge\left(g_{\overline{ZFC_2^{Hs}}}(X)=\nu\right)\right]\wedge\\ \left.[\Theta(\Psi_1,\Psi_2)]\right\},\\ \Theta(\Psi_1,\Psi_2)\triangleq\ \models_{M_{st}^{ZFC_2^{Hs}}}\forall X[\Psi_1(X)\iff\Psi_2(X)],\end{array}\quad(2.1.3.a)$$

i.e. $(\Psi_1,\Psi_2)\in Z_{1,2}$ if and only if the sentence $\forall X[\Psi_1(X)\iff\Psi_2(X)]$ holds in standard model $M_{st}^{ZFC_2^{Hs}}$.Note that the relation $\models_{M_{st}^{ZFC_2^{Hs}}}\forall X[\Psi_1(X)\iff\Psi_2(X)]$ is expressible in ZFC_2^{Hs} by a single formula $\Theta(\Psi_1,\Psi_2)$ of the set theory ZFC_2^{Hs}, since there exists a single statement $Con\left(ZFC_2^{Hs},M^{ZFC_2^{Hs}}\right)$ in ZFC_2^{Hs} such that $Con\left(ZFC_2^{Hs};M^{ZFC_2^{Hs}}\right)\iff\exists M^{ZFC_2^{Hs}}\iff$ $\iff Con\left(ZFC_2^{Hs}\right)$ .see Remark 1.1.4(ii).

(iii) It follows from the statement (ii) and Axiom schema of separation that $Z_{1,2}$ is a set in the sense of the set theory ZFC_2^{Hs}.

(iv) A subset Λ_X^{Hs} of Γ_X^{Hs} such that $\Psi_1(X)\sim_X\Psi_2(X)$ holds for all $\Psi_1(X)$ and $\Psi_2(X)$ in Λ_X^{Hs}, and never for $\Psi_1(X)$ in Λ_X^{Hs} and $\Psi_2(X)$ outside Λ_X^{Hs}, is called an equivalence class of Γ_X^{Hs} by $\sim_X$.

(v) A set of the all possible equivalence classes of a set Γ_X^{Hs} devided by $\tilde{}_X$, will by denoted by $\Gamma_X^{Hs}/\sim_X$

$$\Gamma_X^{Hs}/\sim_X\triangleq\left\{[\Psi(X)]_{Hs}\,|\Psi(X)\in\Gamma_X^{Hs}\right\},\quad(2.1.4)$$

is the quotient set of a set Γ_X^{Hs} devided by the equivalence relation $\sim_X$.

(vi) For any $\Psi(X)\in\Gamma_X^{Hs}$ by symbol $[\Psi(X)]_{Hs}\triangleq\left\{\Phi(X)\in\Gamma_X^{Hs}|\Psi(X)\sim_X\Phi(X)\right\}$ we denote the equivalence class to which $\Psi(X)$ belongs. All elements of Γ_X^{Hs} that equivalent to each other are also elements of the same equivalence class.

Definition 2.1.3. We define now the operations join $\vee$,meet $\wedge$,and complementation, denoted $[\Phi(X)]'$on $\Gamma_X^{Hs}/\sim_X$ by :

(1) $[\Phi(X)]\vee[\Psi(X)]=[\Phi(X)\vee\Psi(X)]$,

(2) $[\Phi(X)] \wedge [\Psi(X)] = [\Phi(X) \wedge \Psi(X)]$,

(3) $[\Phi(X)]' = [\neg\Phi(X)]$.

The resulting bulean algebra $\mathbf{B}_X$ is the Lindenbaum-Tarski algebra of the second order

language $_2^{Hs}$ of $\overline{ZFC}_2^{Hs}$ and it may be shown that

$$\begin{aligned} {}_{t\in T}\,[\Phi(t)] &= \left[\exists X \left(X \in M_{st}^{ZFC_2^{Hs}}\right) \Phi(X)\right], \\ {}_{t\in T}\,[\Phi(t)] &= \left[\forall X \left(X \in M_{st}^{ZFC_2^{Hs}}\right) \Phi(X)\right], \end{aligned} \quad (2.1.5)$$

where $t \in T = M_{st}^{ZFC_2^{Hs}}$ and T is the set of all terms in the language $_2^{Hs}$.

Remark 2.1.1. Note that in bulean notations definition (2.1.2) reads

$$\begin{aligned} Z_{1,2} &= \left\{(\Psi_1, \Psi_2) \mid \left((\Psi_1, \Psi_2) \in \Gamma_X^{Hs} \times \Gamma_X^{Hs}\right) \wedge \left([\Omega(\Psi_1, \Psi_2)] = \mathbf{1}_{\mathbf{B}_X}\right)\right\}, \\ \Omega(\Psi_1, \Psi_2) &\triangleq \forall X \left(X \in M_{st}^{ZFC_2^{Hs}}\right) [\Psi_1(X) \iff \Psi_2(X)] \end{aligned} \quad (2.1.6)$$

Definition 2.1.4. [17]. Let Th be any theory in the recursive language $_{Th} \supset_{PA}$, where $_{PA}$ is a language of Peano arithmetic.We say that a number-theoretic relation $R(x_1, ..., x_n)$ of n arguments is expressible in Th if and only if there is a wff $\widehat{R}(x_1, ..., x_n)$ of Th with the free variables $x_1, ..., x_n$ such that,for any natural numbers $k_1, ..., k_n$, the following holds:

(i) If $R(k_1, ..., k_n)$ is true, then $\vdash_{Th} \widehat{R}\left(\overline{k}_1, ..., \overline{k}_n\right)$.

(ii) If $R(k_1, ..., k_n)$ is false, then $\vdash_{Th} \neg\widehat{R}\left(\overline{k}_1, ..., \overline{k}_n\right)$.

Remark 2.1.2. Recoll that any recursive language $_{Th}$ except logical connectives and quantifiers contains the following sets of symbols (see for example ref. [17], p.51):

(i) a set of symbols $\Delta_0 = \{(,), ,, \neg, \implies, \forall\}$ and we will identify these symbols with a

1-tuples $\widehat{\Delta}_0 = \{\{(\}, \{)\}, \{,\}, \{\neg\}, \{\implies, \{\forall\}\}\}$ by using a one-one function$\wp_{\Delta_0}$:

$$\begin{gathered} \wp_{\Delta_0}(\{(\}) = (, \wp_{\Delta_0}(\{)\}) =), \wp_{\Delta_0}(\{,\}) = ,, \wp_{\Delta_0}(\{\neg\}) = \neg, \wp_{\Delta_0}(\{\implies\}) = \implies, \\ \wp_{\Delta_0}(\{\forall\}) = \forall, \\ \wp_{\Delta_0}^{-1}(() = \{(\}, \wp_{\Delta_0}^{-1}()) = \{)\}, \wp_{\Delta_0}^{-1}(,) = \{,\}, \wp_{\Delta_0}^{-1}(\neg) = \{\neg\}, \wp_{\Delta_0}^{-1}(\implies) = \{\implies\}, \\ \wp_{\Delta_0}^{-1}(\forall) = \{\forall\}, \end{gathered} \quad (2.1.7)$$

and we will be often abbreviate

$$\widehat{(} = \{(\}, \widehat{)} = \{)\}, \widehat{,} = \{,\}, \widehat{\neg} = \{\neg\}, \widehat{\implies} = \{\implies\}, \widehat{\forall} = \{\forall\}; \quad (2.1.8)$$

(ii) a set of the first order individual variables: $\Delta_1 = \{x_1, x_2, ..., x_n, ...\}$ and we will identify these individual variables with a 1-tuples $\widehat{\Delta}_1 = \{\{x_1\}, \{x_2\}, ..., \{x_n\}, ...\}$ by using a one-one function$\wp_{\Delta_1}$:

$$\wp_{\Delta_1}(\{x_n\}) = x_n, \wp_{\Delta_1}^{-1}(x_n) = \{x_n\}, \quad (2.1.9)$$

and we will be often abbreviate $\widehat{x}_n = \{x_n\}$;

(iii) a set of the second order individual variables: $\widetilde{\Delta} = \{y_1, y_2, ..., y_n, ...\}$

(iv) a set of the individual constants: $\Delta_2 = \{a_1, a_2, ..., a_n, ...\}$ and we will identify these individual constants with a 1-tuples $\Delta_2 = \{\{a_1\}, \{a_2\}, ..., \{a_n\}, ...\}$ by using a one-one function$\wp_{\Delta_2}(\{a_n\})$

$$\wp_{\Delta_2}(\{a_n\}) = a_n, \wp_{\Delta_2}^{-1}(a_n) = \{a_n\}, \quad (2.1.10)$$

and we will be often abbreviate $\widehat{a}_n = \{a_n\}$;

(v) for every integer $n \geq 0$ there is a set of n-ary, or n-place, predicate symbols:

$\Delta_3 = \{A_k^n\}_{n,k\in\mathbb{N}}$ and we will identify these predicate symbols with a 1-tuples

$\widehat{\Delta}_3 = \{\{A_k^n\}\}_{n,k\in\mathbb{N}}$ by using a one-one function $\wp_{\Delta_3}$:

$$\wp_{\Delta_3}(\{A_k^n\}) = A_k^n, \wp_{\Delta_3}^{-1}(A_k^n) = \{A_k^n\}; \quad (2.1.11)$$

and we will be often abbreviate $\widehat{A_k^n} = \{A_k^n\}$;

(vi) for every integer $n \geq 0$ there is a set of n-ary, or n-place, function symbols:

$\Delta_4 = \{f_k^n\}_{n,k\in\mathbb{N}}$ and we will identify these predicate symbols with a 1-tuples

$\widehat{\Delta}_4 = \{\{f_k^n\}\}_{n,k\in\mathbb{N}}$ by using a one-one function $\wp_{\Delta_4}$:

$$\wp_{\Delta_4}(\{f_k^n\}) = f_k^n, \wp_{\Delta_4}^{-1}(f_k^n) = \{f_k^n\}, \quad (2.1.12)$$

and we will be often abbreviate $\widehat{f_k^n} = \{f_k^n\}$;

(vii) A theory Th is said to have a primitive recursive vocabulary (or a recursive vocabulary) if the following predicates are primitive recursive (or recursive)

(a) $\mathbf{IC^{Hs}}(x)$: x is the Godel number of an individual constant of $\overline{ZFC}_2^{Hs}$,

(b) $\mathbf{FL^{Hs}}(x)$: x is the Godel number of a function letter of $\overline{ZFC}_2^{Hs}$,

(c) $\mathbf{PL^{Hs}}(x)$: x is the Godel number of a predicate letter of $\overline{ZFC}_2^{Hs}$.

Remark 2.1.3. (i) Note that in fact it was alwais implicitly assumed that these sets $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ are a sets in a sense of ZFC (ZFC-set),see ref.[8],[17].

(ii) we will write for short A is a ZFC-set instead A is a set in a sense of ZFC,etc.

Remark 2.1.4.(a) Recoll that the function symbols applied to the variables and individual constants inductively generate a full ZFC-set of the terms [8], [17]:

(1) Variables and individual constants are terms.

(1.a) First order variables and individual constants are first order terms.

(1.b) Second order variables are second order terms.

(2) If ${f_k}^n$ is a function symbol and $t_1, t_2, ..., t_n$, are terms, then ${f_k}^n(t_1, t_2, ..., t_n)$ is a term.

(2.a) If ${f_k}^n$ is a function symbol and $t_1, t_2, ..., t_n$, are first order terms, then ${f_k}^n(t_1, t_2, ..., t_n)$ is a first order term.

(2.b) If f_k^n is a function symbol and sequence $t_1, t_2, ..., t_n$, contain at least one second order term, then $f_k^n(t_1, t_2, ..., t_n)$ is a second order term.

(2.b) If f_k^n is a function symbol and sequence $t_1, t_2, ..., t_n$, contain at least one second order term, then $f_k^n(t_1, t_2, ..., t_n)$ is a second order term.

(3) An expression is a term only iff it can be shown to be a term on the basis of conditions (1) and (2).

(3.a) An expression is a first order term only iff it can be shown to be a term on the basis of conditions (1.a) and (2.a).

(3.b) An expression is a second order term only iff it can be shown to be a term on the basis of conditions (1.b) and (2.b).

(3.c) We will be identify the first order terms with ordered $n+3$-tuples

$$\{\{f_k^n\}, \{(\}, \{t_1\}, \{t_2\}, ..., \{t_n\}, \{)\}\} \qquad (2.1.13)$$

by using a one-one function $\wp_\tau$:

$$\begin{array}{l}\wp_\tau(\{\{f_k^n\}, \{(\}, \{t_1\}, \{t_2\}, ..., \{t_n\}, \{)\}\}) = f_k^n(t_1, t_2, ..., t_n), \\ \wp_\tau^{-1}(f_k^n(t_1, t_2, ..., t_n)) = \{\{f_k^n\}, \{(\}, \{t_1\}, \{t_2\}, ..., \{t_n\}, \{)\}\}.\end{array} \qquad (2.1.14)$$

It follows from Remark 2.1.2 and Remark 2.1.4 that there is a ZFC-set of the all first order terms Υ_1:

$$\Upsilon_1 = \Delta_1 \cup \Delta_2 \cup \{f_k^n(t_1, t_2, ..., t_n)\}_{n,k\in\mathbb{N}}. \qquad (2.1.15)$$

(3.d) We will be denoted the image $\wp_\tau^{-1}(\Upsilon_1)$ by

$$\wp_\tau^{-1}(\Upsilon_1) = \widehat{\Upsilon}_1. \qquad (2.1.16)$$

(4) Recoll that the predicate symbols applied to terms yield the <u>atomic formulas</u>; that is, if A_n^k is a predicate letter and $t_1, t_2, ..., t_n$, are terms, then $A_n^k(t_1, t_2, ..., t_n)$ is an atomic formula.

(4.a) The predicate symbols applied to the first order terms yield the <u>first order</u> <u>atomic formulas</u>; that is, if A_n^k is a predicate letter and $t_1, t_2, ..., t_n$, are first order terms, then $A_n^k(t_1, t_2, ..., t_n)$ is an first order atomic formula.

(4.b) The predicate symbols applied to the second order terms yield the <u>second order</u> <u>atomic formulas</u>; that is, if A_n^k is a predicate letter and $t_1, t_2, ..., t_n$, are second order terms, then $A_n^k(t_1, t_2, ..., t_n)$ is an second order atomic formula.

(4.c) We will be identify the <u>first order atomic formulas</u> with ordered $n+3$-tuples

$\{A_k^n, (, t_1, t_2, ..., t_n,)\}$ by using a one-one function $\wp_\pi$:

$$\begin{array}{l}\wp_\pi(\{A_k^n, (, t_1, t_2, ..., t_n,)\}) = A_k^n(t_1, t_2, ..., t_n), \\ \wp_\pi^{-1}(A_k^n(t_1, t_2, ..., t_n)) = \{A_k^n, (, t_1, t_2, ..., t_n,)\}.\end{array} \qquad (2.1.17)$$

It follows from Remark 2.1.2 and Remark 2.1.4 that there is a ZFC-set of the all

<u>first order atomic formulas</u> Σ_1:

$$\Sigma_1 = \{A_k^n(t_1, t_2, ..., t_n)\}_{n,k\in\mathbb{N}}. \qquad (2.1.18)$$

(4.d)We will be denoted the image $\wp_{\pi}^{-1}(\Sigma)$ by

$$\wp_{\pi}^{-1}(\Sigma_1) = \widehat{\Sigma}_1. \quad (2.1.19)$$

(5) We introduce now a one-one function $\wp_{\tau,\pi}$ such that

$$\wp_{\tau,\pi}|_{\Upsilon_1} = \wp_{\tau},\ \wp_{\tau,\pi}|_{\Sigma_1} = \wp_{\pi}. \quad (2.1.20)$$

(6) Recoll that the well-formed formulas (wff's) of quantification theory are defined inductively as follows [8], [17], [18]:

(6.a) Every atomic formula is a wff.

(6.b) Every first order atomic formula is a <u>first order wff</u>.

(6.c) Every second order atomic formula is a <u>second order wff</u>.

(6.d) If B and C are wff's and y is a variable, then $(\neg B), (B \implies C)$ and $((\forall y) B)$ are wff's.

(6.e) An expression is a wff only if it can be shown to be a wff on the basis of conditions (6.a) and (6.d).

(7) If B and C are <u>first order</u> wff's and y is a <u>first order</u> variable, then $(\neg B), (B \implies C)$ and $((\forall y) B)$ are <u>first order wff's</u>.

(7.a) An expression is a <u>first order</u> wff only if it can be shown to be a wff on the basis of conditions (6.b) and (6.d).

Remark 2.1.5. It follows from Remark 2.1.1-Remark 2.1.3 that there is a ZFC-set Ξ_1 of the all <u>first order</u> wff's and in partcular

$$\Upsilon_1 \cup \Sigma_1 \subset \Xi_1. \quad (2.1.21)$$

We extend now one-one function$\wp_{\tau,\pi}$ up one-one function $\wp_{\Xi_1}$ by natural way,i.e.,

$\wp_{\Xi}\ |_{\Upsilon\cup\Sigma} = \wp_{\tau,\pi}$ and we will be denoted the image $\wp_{\Xi_1}^{-1}(\Xi_1)$ by

$$\wp_{\Xi_1}^{-1}(\Xi_1) = \widehat{\Xi}_1. \quad (2.1.22)$$

Remark 2.1.6. Recoll that for an arbitrary second-order theory Th, we correlate with each symbol u of Th an odd positive integer $g(u)$, called the Godel number of u, in the following rules [10] [[10]]:1.$g(()=3$,2.$g())=5$,3.$g(,)=7$,4.$g(\neg)=9$,5.$g(\implies)=11$,6.$g(\forall)=13$,

7.$g(x_k)=13+8k$,8.$g(a_k)=7+8k$,9.$g(f_k^n)=1+8(2^n3^k)$,10.$g(A_k^n)=3+8(2^n3^k)$,

11.$g(y_k)=15+8k$, where $k,n\geq 1$.

Example 2.1.1.$g(x_2)=29; g(a_4)=39; g(f_1{}^2)=97; g(A_2^1)=147$.

Remark 2.1.7. Note that g is a bijection and therefore there exist a functon g^{-1}such that

1.$g^{-1}(3)=($;2.$g^{-1}(5)=)$;3.$g^{-1}(7)=,$;4.$g^{-1}(9)=\neg$;5.$g^{-1}(11)=\implies$,6.$g^{-1}(13)=\forall$;

7.$g(13+8k)=x_k$;8.$g(7+8k)=a_k$;9.$g(1+8(2^n3^k))=f_k^n$;10.$g(3+8(2^n3^k))=A_k^n$,where $k,n\geq 1$.

Example 2.1.2.$g^{-1}(29) = x_2; g^{-1}(39) = a_4; g^{-1}(97) = {f_1}^2; g^{-1}(147) = A_2^1$.

Remark 2.1.8. Note that $g \circ g^{-1}(x) = x$.

Remark 2.1.9. Given an expression $u_0u_1..u_j...u_r$,where each u_j is a symbol of Th, i.e., each $u_j \in \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$ ($\Delta_0 = \{(,),,,\neg, \Longrightarrow, \forall\}$) we define its Godel number $g_{\overline{ZFC}_2^{Hs}}(u_0u_1..u_j...u_r) \triangleq \breve{\mathbf{g}}(u_0u_1..u_j...u_r)$ by the formula:

$$g \triangleq \breve{\mathbf{g}}(u_0u_1..u_j...u_r) = \breve{\mathbf{g}}(u_0) \cdot \breve{\mathbf{g}}(u_1) \cdot ... \cdot \breve{\mathbf{g}}(u_j) \cdot ... \cdot \breve{\mathbf{g}}(u_r) = \\ = 2^{g(u_0)} \cdot 3^{g(u_1)} \cdot ... \cdot p_j^{g(u_j)} \cdot ... \cdot p_r^{g(u_r)}, \tag{2.1.23}$$

where $\breve{\mathbf{g}}(u_j) = p_j^{g(u_j)}$and where p_j denotes the j-th prime number and we assume that $p_0 = 2$.

Example 2.1.3. $g\left(A_2^1(x_1, x_2)\right) = 2^{g\left(A_2^1\right)} \cdot 3^{g(()} \cdot 5^{g(x_1)} \cdot 7^{g(,)} \cdot 11^{g(x_2)} \cdot 13^{g())} = 2^{99} \cdot 3^3 \cdot 5^{21} \cdot$ $\cdot 7^7 \cdot 11^{29} \cdot 13^5$.

Definition 2.1.5. Given any natural number $k \in \mathbb{N}$ wich has representation of the form

$k = 2^{g(u_0)} \cdot 3^{g(u_1)} \cdot ... \cdot p_j^{g(u_j)} \cdot ... \cdot p_r^{g(u_r)}$ for some sequance of a symbols $u_0, u_1, .., u_j, ..., u_r$,

where each u_j is a symbol of Th, we define a function $\breve{\mathbf{g}}^{-1} : \mathbb{N} \to \Xi_1$ by the following formula

$$\breve{\mathbf{g}}^{-1}(k) = \breve{\mathbf{g}}^{-1}\left(2^{g(u_0)}\right) \cdot ... \cdot \breve{\mathbf{g}}^{-1}\left(p_j^{g(u_j)}\right) \cdot ... \cdot \breve{\mathbf{g}}^{-1}\left(p_r^{g(u_r)}\right) = \\ = u_0u_1..u_j...u_r \in \Xi_1, \tag{2.1.24}$$

where $\breve{\mathbf{g}}^{-1}\left(p_j^{g(u_j)}\right) = g^{-1}(g(u_j)) = u_j \in \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$.

Definition 2.1.6. [10] Thus g is one-one function from the set $\mathbf{S}^{\#} = \cup_{n \in \mathbb{N}} \mathbf{S}^n$,where $\mathbf{S} = \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$,of symbols of Th, first order expressions of Th and finite sequences of first order expressions of Th into the set of positive integers.

The following conditions are to be satisfied by the function g : (1) g is effectively computable, (2) there is an effective procedure that determines whether any given positive integer m is in the range of g and, if m is in the range of g, the procedure: finds the object $\digamma \in \cup_{n \in \mathbb{N}} \Xi_1^n = \Omega_1$ such that $g(\digamma) = m$.

We extend now one-one function$\wp_{\Xi_1}$ up one-one function $\wp_{\Omega_1}$ by natural way,i.e.,

$$\wp_{\Omega_1} \mid_{\Xi_1} = \wp_{\Xi_1}, \wp_{\Omega_1} \mid_{\Xi_1^n} = \wp_{\Xi_1} \underbrace{\times ... \times}_{n} \wp_{\Xi_1}, \tag{2.1.25}$$

$n \in \mathbb{N}$, and we will be denoted the image $\wp_{\Omega_1}^{-1}(\Omega_1)$ by

$$\wp_{\Omega_1}^{-1}(\Omega_1) = \widehat{\Omega}_1. \tag{2.1.26}$$

Proposition 2.1.1. Let Th be a theory with a primitive recursive (or recursive) vocabulary.

Then withe following relations and functions (1-11) are primitive recursive (or recursive).

In each case, we give first the notation and <u>intuitive definition</u> for the relation or function, and then an equivalent formula from which its primitive recursiveness (or recursiveness) can be deduced.

(1) $\mathbf{EVbl}_1(x)$: x is the Godel number of an expression consisting of a first order variable,

$(\exists z)_{z<x}(1 \leq z \wedge x = 2^{13+8z})$. By [17], Proposition 3.18, this is primitive recursive.

$\mathbf{EVbl}_2(x)$: x is the Godel number of an expression consisting of a second order variable,

$(\exists z)_{z<x}(1 \leq z \wedge x = 2^{15+8z})$. By [17], Proposition 3.18, this is primitive recursive.

$\mathbf{EVbl}_{1\vee 2}(x)$: x is the Godel number of an expression consisting of a variable,

$(\exists z)_{z<x}(1 \leq z \wedge \left(x = 2^{13+8z}\right) \vee \left(x = 2^{15+8z}\right))$. By [17], Proposition 3.18, this is primitive recursive.

$\mathbf{EIC}(x)$: x is the Godel number of an expression consisting of an individual constant, $(\exists y)_{y<x}(\mathbf{IC}(y) \wedge x = 2^y)$ [17], Proposition 3.18).

$\mathbf{EFL}(x)$: x is the Godel number of an expression consisting of a function letter, $(\exists y)_{y<x}(\mathbf{FL}(y) \wedge x = 2^y)$ [17], Proposition 3.18.

$\mathbf{EPL}(x)$: x is the Godel number of an expression consisting of a predicate letter,

$(\exists y)_{y<x}(\mathbf{PL}(y) \wedge x = 2^y)$ [17], Proposition 3.18.

(2) $\mathbf{Arg_T}(x) = (qt(8, x \dot{-} 1))_0$: If x is the Godel·number of a function letter $f_j{}^n$, then $\mathbf{Arg_T}(x) = n$. $\mathbf{Arg_T}(x)$ is primitive recursive [17], Proposition 3.18.

$\mathbf{Arg_P}(x) = (qt(8, x \dot{-} 3))_0$: If x is the Godel number of a predicate letter $A_j{}^n$, then

$\mathbf{Arg_P}(x) = n.\mathbf{Arg_P}(x)$ is primitive recursive [17], Proposition 3.18.

(3) $\mathbf{Gd}_1(x)$: x is the Godel number of an first order expression of Th,

$\mathbf{EVbl}_1(x) \vee \mathbf{EIC}(x) \vee \mathbf{EFL}(x) \vee \mathbf{EPL}(x) \vee x = 2^3 \vee x = 2^5 \vee x = 2^7 \vee x = 2^9 \vee x =$

$2^{11} \vee x = 2^{13} \vee (\exists u)_{u<x}(\exists v)_{v<x}(x = u * v \wedge \mathbf{Gd}_1(u) \wedge \mathbf{Gd}_1(v))$.

$\mathbf{Gd}_{1\vee 2}(x)$: x is the Godel number of an expression of Th,

$\mathbf{EVbl}_{1\vee 2}(x) \vee \mathbf{EIC}(x) \vee \mathbf{EFL}(x) \vee \mathbf{EPL}(x) \vee x = 2^3 \vee x = 2^5 \vee x = 2^7 \vee x = 2^9 \vee x =$

$2^{11} \vee x = 2^{13} \vee x = 2^{15} \vee (\exists u)_{u<x}(\exists v)_{v<x}(x = u * v \wedge \mathbf{Gd}_{1\vee 2}(u) \wedge \mathbf{Gd}_{1\vee 2}(v))$.

(4) $\mathbf{MP}_1(x, y, z)$: The first order expression with Godel number z is a direct consequence of the first order expressions with Godel numbers x and y by modus ponens, $y = 2^3 * x * 2^{11} * z * 2^5 \wedge \mathbf{Gd}_1(x) \wedge \mathbf{Gd}_1(z)$.

$\mathbf{MP}_{1\vee 2}(x, y, z)$: The expression with Godel number z is a direct consequence of the expressions with Godel numbers x and y by modus ponens,

$y = 2^3 * x * 2^{11} * z * 2^5 \wedge \mathbf{Gd}_{1\vee 2}(x) \wedge \mathbf{Gd}_{1\vee 2}(z)$.

(5) $\mathbf{Gen}_1(x, y)$: The first order expression with Godel number y comes from the first order expression with Godel number x by the generalization rule:

$(\exists v)_{v<y}(\mathbf{EVbl}_1(v) \wedge y = 2^3 * 2^3 * 2^{13} * v * 2^5 * x * 2^5 \wedge \mathbf{Gd}_1(x))$.

$\mathbf{Gen}_{1\vee 2}(x,y)$: The expression with Godel number y comes from the an expression with Godel number x by the generalization rule:

$(\exists v)_{v<y}(\mathbf{EVbl}_{1\vee 2}(v) \wedge y = 2^3 * 2^3 * 2^{13} * v * 2^5 * x * 2^5 \wedge \mathbf{Gd}_{1\vee 2}(x))$.

(6) $\mathbf{Trm}_1(x)$: x is the Godel number of an first order term of Th.

$\mathbf{Trm}_1(x)$ is equivalent to the following relation: '

$\mathbf{EVbl}_1(x) \vee \mathbf{EIC}(x) \vee (\exists y)_{y<(p_x!)^x}[x = (y)_{lh(y)\dot{-}1} \wedge$

$lh(y) = \mathbf{Arg_T}((x)_0) + 1 \wedge \mathbf{FL}(((y)_0)_0) \wedge ((y)_0)_1 = 3 \wedge$

$lh((y)_0) = 2 \wedge (\forall u)_{u<lh(y)\dot{-}1}(\exists v)_{v<x}((y)_{u+1} = (y)_u * v * 2^7 \wedge \mathbf{Trm}_1(v)) \wedge$

$(\exists v)_{v<x}((y)_{lh(y)\dot{-}1} = (y)_{lh(y)\dot{-}2} * v * 2^5 \wedge \mathbf{Trm}_1(v))]$.

(7) $\mathbf{Atfml}_1(x)$: x is the Godel number of an atomic first order wff of Th.

$\mathbf{Atfml}_1(x)$ is equivalent to the following:

$(\exists y)_{y<(p_x!)^x}[x = (y)_{lh(y)\dot{-}1} \wedge lh(y) = \mathbf{Arg_P}((x)_0) + 1 \wedge$

$\mathbf{PL}(((y)_0)_0) \wedge ((y)_0)_1 = 3 \wedge lh((y)_0) = 2 \wedge$

$(\forall u)_{u<lh(y)\dot{-}2}(\exists v)_{v<x}((y)_{u+1} = (y)_u * v * 2^7 \wedge \mathbf{Trm}_1(v)) \wedge$

$(\exists v)_{v<x}((y)_{lh(y)\dot{-}1} = (y)_{u<lh(y)\dot{-}2} * v * 2^5 \wedge \mathbf{Trm}_1(v))]$.

(8)$\mathbf{Fml}_1(y)$: y is the Godel number of an first order formula of Th:

$\mathbf{Atfml}_1(y) \vee (\exists z)_{z<y}[(\mathbf{Fml}_1(z) \wedge = 2^3 * 2^9 * z * 2^5) \vee$

$(\mathbf{Fml}_1((z)_0) \wedge \mathbf{Fml}_1((z)_1 \wedge y = 2^3 * (z)_0 * 2^{11} * (z)_1 * 2^5) \vee$

$(\mathbf{Fml}_1((z)0) \wedge \mathbf{EVbl}_1((z)_1) \wedge y = 2^3 * 2^3 * 2^{13} * ((z)_1 * 2^5 * (z)_0 * 2^5)]$.

(9) $\mathbf{Subst}_1(x,y,u,v)$: x is the Godel number of the result of substituting in the first order expression with Godel number y the first order term with Godel number u for all free occurrences of the variable with Godel number v.

(10) $\mathbf{Sub}_1(y,u,v)$: the Godel number of the result of substituting the first order term with Godel number u for all free occurrences in the first order expression with Godel number y of the variable with Godel number v :

$\mathbf{Sub}_1(y,u,v) = \mu x_{x<(p_{uy}!)^{uy}} \mathbf{Subst}_1(u,y,u,v)$.

(11) $\mathbf{Fr}_1(y,v)$: y is the Godel number of the first order wff or the first order term of Th that contains free occurrences of the variable with Godel number v :

$(\mathbf{Fml}_1(y) \vee \mathbf{Trm}_1(y)) \wedge \mathbf{EVbl}_1(2^v) \wedge \neg\mathbf{Subst}_1(y,y,2^{13+8v},v)$.

Remark 2.1.10. Note that in order to obtain completely formal definitions of the first order predicates $\mathbf{EVbl}_1^{Hs}(x), \mathbf{EIC}_1^{Hs}(x), \mathbf{EFL}_1^{Hs}(x), ..., \mathbf{Fr}_1^{Hs}(y,v)$ one needs the following second order predicates:

(i) $\mathbf{EVbl}_1^{Hs}(x,\alpha)$: x is the Godel number of the first order expression $\alpha \in \Delta_1$ consisting of a first order variable;

(ii) $\mathbf{EIC}_1^{Hs}(x,\beta)$: x is the Godel number of the first order expression $\beta \in \Delta_2$ consisting of individual constant;

(iii) $\mathbf{EFL}_1^{Hs}(x,\gamma)$: x is the Godel number of the first order expression $\gamma \in \Delta_3$ consisting of function letter;

(iv) $\mathbf{EPL}_1^{Hs}(x,\delta)$: x is the Godel number of the first order expression $\delta \in \Delta_4$ consisting of predicate letter;

(v) $\mathbf{Gd}_1^{Hs}(x,\zeta)$: x is the Godel number of the first order expression $\zeta \in \mathbf{S}^n$, $n \in \mathbb{N}$ of the $\overline{ZFC}_2^{Hs}$;

(vi) $\mathbf{Trm}_1^{Hs}(x,\tau)$: x is the Godel number of the first order term $\tau \in \Upsilon_1$ of the $\overline{ZFC}_2^{Hs}$.

(vii) $\mathbf{Atfml}_1^{Hs}(x,\pi)$: x is the Godel number of the <u>first order</u> atomic wff $\pi \in \Sigma_1$ of the $\overline{ZFC}_2^{Hs}$.

(viii) $\mathbf{Fml}_1^{Hs}(y,\varphi)$: y is the Godel number of the the <u>first order</u> wff formula $\varphi \in \Xi_1$ of the $\overline{ZFC}_2^{Hs}$.

(ix) $\mathbf{Fr}_1^{Hs}(y,v,\varpi)$: y is the Godel number of the the <u>first order</u> wff ϖ or the the <u>first order</u> term $\varpi \in \Xi_1$ of the $\overline{ZFC}_2^{Hs}$ that contains free occurrences of the variable with Godel number v.

Thus finally we obtain:

$$\begin{gathered}
\mathbf{EVbl}_1^{Hs}(x) \iff \exists\alpha\,(\alpha \in \Delta_1)\,\mathbf{EVbl}_1^{Hs}(x,\alpha) \iff (\exists z)_{z<x}(1 \leq z \wedge x = 2^{13+8z}),\\
\mathbf{EIC}_1^{Hs}(x) \iff \exists\beta\,(\beta \in \Delta_2)\,\mathbf{EIC}_1^{Hs}(x,\beta) \iff (\exists y)_{y<x}(\mathbf{IC}^{Hs}(y) \wedge x = 2^y),\\
\mathbf{EFL}_1^{Hs}(x) \iff \exists\gamma\,(\gamma \in \Delta_3)\,\mathbf{EFL}_1^{Hs}(x,\gamma) \iff (\exists y)_{y<x}(\mathbf{FL}^{Hs}(y) \wedge x = 2^y),\\
\mathbf{EPL}_1^{Hs}(x) \iff \exists\delta\,(\delta \in \Delta_4)\,\mathbf{EPL}_1^{Hs}(x,\delta) \iff (\exists y)_{y<x}(\mathbf{PL}^{Hs}(y) \wedge x = 2^y),
\end{gathered}$$

$$\begin{gathered}
\mathbf{Gd}_1^{Hs}(x) \iff \exists\zeta\exists n\,(\zeta \in \mathbf{S}^n)\,\mathbf{Gd}_1^{Hs}(x,\zeta) \iff\\
\mathbf{EVbl}_1^{Hs}(x) \vee \mathbf{EIC}_1^{Hs}(x) \vee \mathbf{EFL}_1^{Hs}(x) \vee \mathbf{EPL}_1^{Hs}(x) \vee\\
x = 2^7 \vee x = 2^9 \vee x = 2^{11} \vee x = 2^{13} \vee\\
(\exists u)_{u<x}(\exists v)_{v<x}(x = u * v \wedge \mathbf{Gd}_2^{Hs}(u) \wedge \mathbf{Gd}_2^{Hs}(v)).
\end{gathered}$$

$$\begin{gathered}
\mathbf{Trm}_1^{Hs}(x) \iff \exists\tau\,(\tau \in \Upsilon)\,\mathbf{Trm}_1^{Hs}(x,\tau) \iff\\
\mathbf{EVbl}(x) \vee \mathbf{EIC}(x) \vee (\exists y)_{y<(p_x!)^x}[x = (y)_{lh(y)\dot{-}1} \wedge\\
lh(y) = \mathbf{Arg_T}((x)_0) + 1 \wedge \mathbf{FL}(((y)_0)_0) \wedge ((y)_0)_1 = 3 \wedge\\
lh((y)_0) = 2 \wedge (\forall u)_{u<lh(y)\dot{-}1}(\exists v)_{v<x}((y)_{u+1} = (y)_u * v * 2^7 \wedge \mathbf{Trm}_1^{Hs}(v)) \wedge\\
(\exists v)_{v<x}((y)_{lh(y)\dot{-}1} = (y)_{lh(y)\dot{-}2} * v * 2^5 \wedge \mathbf{Trm}_1^{Hs}(v))].
\end{gathered} \tag{2.1.27}$$

$$\begin{gathered}
\mathbf{Atfml}_1^{Hs}(x) \iff \exists\pi\,(\pi \in \Sigma_1)\,\mathbf{Atfml}_1^{Hs}(x,\pi) \iff\\
(\exists y)_{y<(p_x!)^x}[x = (y)_{lh(y)\dot{-}1} \wedge lh(y) = \mathbf{Arg}_P((x)_0) + 1 \wedge\\
\mathbf{PL}_1^{Hs}(((y)_0)_0) \wedge ((y)_0)_1 = 3 \wedge lh((y)_0) = 2 \wedge\\
(\forall u)_{u<lh(y)\dot{-}2}(\exists v)_{v<x}((y)_{u+1} = (y)_u * v * 2^7 \wedge \mathbf{Trm}_1^{Hs}(v)) \wedge\\
(\exists v)_{v<x}((y)_{lh(y)\dot{-}1} = (y)_{u<lh(y)\dot{-}2} * v * 2^5 \wedge \mathbf{Trm}_1^{Hs}(v))].
\end{gathered}$$

$$\begin{gathered}
\mathbf{Fml}_1^{Hs}(y) \iff \exists\,(\varphi \in \Xi_1)\,\mathbf{Fml}_1^{Hs}(y,\varphi) \iff\\
\mathbf{Atfml}_1^{Hs}(y) \vee (\exists z)_{z<y}[(\mathbf{Fml}_1^{Hs}(z) \wedge = 2^3 * 2^9 * z * 2^5) \vee\\
(\mathbf{Fml}_1^{Hs}((z)_0) \wedge \mathbf{Fml}_1^{Hs}((z)_1 \wedge y = 2^3 * (z)_0 * 2^{11} * (z)_1 * 2^5) \vee\\
(\mathbf{Fml}_1^{Hs}((z)_0) \wedge \mathbf{EVbl}_1^{Hs}((z)_1) \wedge y = 2^3 * 2^3 * 2^{13} * ((z)_1 * 2^5 * (z)_0 * 2^5)].
\end{gathered}$$

$$\begin{gathered}
\mathbf{Fr}_1^{Hs}(y,v) \iff \exists\varpi\,[(\varpi \in \Xi_1) \vee (\varpi \in \Upsilon_1)]\,\mathbf{Fr}_1^{Hs}(y,v,\varpi) \iff\\
(\mathbf{Fml}_1^{Hs}(y) \vee \mathbf{Trm}_1^{Hs}(y)) \wedge \mathbf{EVbl}_1^{Hs}(2^v) \wedge \neg\mathbf{Subst}_1^{Hs}(y,y,2^{13+8v},v).
\end{gathered}$$

Designation 2.1.2. (i) Let $g_{ZFC_2^{Hs}}(u)$ be a Gödel number of an given expression u of the set theory $\overline{ZFC}_2^{Hs} \triangleq \quad ZFC_2^{Hs} + \exists M_{st}^{ZFC_2^{Hs}}$.

(ii) Let $\mathbf{Fr}_1^{Hs}(y,v)$ be the relation : y is the Gödel number of a first order wff of the set theory $\overline{ZFC}_2^{Hs}$ that contains free occurrences of the variable X with Gödel number v,see Remark 2.1.10(ix).

(iii) Note that the relation $\mathbf{Fr}_1^{Hs}(y,v)$ is recursive and thus an equivalent from which it recursiveness can be deduced,i.e. the relation $\mathbf{Fr}_1^{Hs}(y,v)$ is expressible in $\overline{ZFC}_2^{Hs}$ by a wff $\widehat{\mathbf{Fr}_1^{Hs}}(y,v)$:

$$\widehat{\mathbf{Fr}_1^{Hs}}(y,v) \equiv \left(\mathbf{Fml}_1^{Hs}(y) \vee \mathbf{Trm}_1^{Hs}(y)\right)\mathbf{Evbl}_1^{Hs}(2^\nu) \wedge \neg\mathbf{Subst}_1^{Hs}\left(y,y,2^{13+8\nu},\nu\right). \tag{2.1.28}$$

(iv) Note that for any $y, v \in \mathbb{N}$ by the definition of the relation $\mathbf{Fr}_1^{Hs}(y, v)$ follows that

$$\widehat{\mathbf{Fr}_1^{Hs}}(y, v) \iff \exists! \Psi(X) \left[\left(g_{\overline{ZFC}_2^{Hs}} (\Psi(X)) = y \right) \wedge \left(g_{\overline{ZFC}_2^{Hs}} (X) = \nu \right) \right], \quad (2.1.29)$$

where $\Psi(X)$ is a unique wff of $\overline{ZFC}_2^{Hs}$ which contains free occurrences of the variable X with Gödel number v.We will be often denote the unique wff $\Psi(X)$ defined by using

equivalence (2.1.29) by the symbol $\Psi_{y,\nu}(X)$,i.e.

$$\widehat{\mathbf{Fr}_1^{Hs}}(y, v) \iff \exists! \Psi_{y,\nu}(X) \left[\left(g_{\overline{ZFC}_2^{Hs}} (\Psi_{y,\nu}(X)) = y \right) \wedge \left(g_{\overline{ZFC}_2^{Hs}} (X) = \nu \right) \right]. \quad (2.1.30)$$

Remark 2.1.11. (i) Note that a function $g_{\overline{ZFC}_2^{Hs}} (\Psi_{y,\nu}(X)) = y$ is expressible in set theory $\overline{ZFC}_2^{Hs}$ by a wff of the set theory $\overline{ZFC}_2^{Hs}$ (see Remark 2.1.13) that contains free occurrences of the variable $y \in \mathbb{N}$.Note that formula $\Psi_{y,\nu}(X)$ is given by an expression $u_0 u_1 .. u_j ... u_r$,i.e.$\Psi_{y,\nu}(X) = u_0 u_1 .. u_j ... u_r$,
where each u_j is a symbol of $\overline{ZFC}_2^{Hs}$.

(ii) Note that in order to obtain Gödel encoding (2.1.23) rigorously without any refference to non formal notion of the expression $u_0 u_1 .. u_j ... u_r$ and by using only notion of ZFC-set Ξ_1(see Remark 2.1.5) we remind that $\Psi_{y,\nu}(X) = u_0 u_1 .. u_j ... u_r \in \Xi_1$ and therefore $\widehat{\Psi}_{y,\nu}(X) = \widehat{u}_0 \widehat{u}_1 .. \widehat{u}_j ... \widehat{u}_r \in \widehat{\Xi}_1$.

(iii) In order to obtain Gödel encoding as required above in Remark 2.1.11(ii) we introduce now a countale secuence of the functions

$$[\Psi_{y,\nu}(X); j] : \Xi_1 \times \mathbb{N} \to \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4, j = 0, 1, ... \quad (2.1.31)$$

which are defined by the following formulas

$$[\Psi_{y,\nu}(X); j] = u_j, j = 0, 1, ..., \quad (2.1.32)$$

and we revrite now the expression $u_0 u_1 .. u_j ... u_r \in \Xi_1$ in the following equivalent form

$$[\Psi_{y,\nu}(X); 0] [\Psi_{y,\nu}(X); 1] ... [\Psi_{y,\nu}(X); j] ... [\Psi_{y,\nu}(X); r]. \quad (2.1.33)$$

By definitions are given above (see Remark 2.1.11(i)-(ii)) we obtain that

$$g_{\overline{ZFC}_2^{Hs}} (\Psi_{y,\nu}(X)) = y \iff$$
$$y = 2^{g([\Psi_{y,\nu}(X);0])} \cdot 3^{g([\Psi_{y,\nu}(X);1])} \cdot ... \cdot p_j^{g([\Psi_{y,\nu}(X);j])} \cdot ... \cdot p_r^{g([\Psi_{y,\nu}(X);r])}. \quad (2.1.34)$$

Let us denote by $(y)_j$ (see ref.[10] [[10]]) the exponent $g([\Psi_{y,\nu}(X); j])$ in this factorization:

$$y = 2^{g([\Psi_{y,\nu}(X);0])} \cdot 3^{g([\Psi_{y,\nu}(X);1])} \cdot ... \cdot p_j^{g([\Psi_{y,\nu}(X);j])} \cdot ... \cdot p_r^{g([\Psi_{y,\nu}(X);r])}. \quad (2.1.35)$$

Recoll that every positive integer y has a unique factorization into prime powers:

$$y = p_0^{a_0} \cdot p_1^{a_1} \cdot ... \cdot p_j^{a_j} \cdot ... \cdot p_r^{a_r} \quad (2.1.36)$$

Let us denote by $(y)_j$ the exponent a_j in this factorization (2.1.29).If $y = 1, (y)_j = 1$ for all j.

If $y = 0$, we arbitrarily let $(y)_j = 0$ for all j. Then the functions $(y)_j, j = 0, 1, ...$ is primitive recursive, since $(y)_j = \mu_{z<y} \left(p_j^z | y \wedge \neg p_j^{z+1} | y \right)$,see [17], p.181.

Remark 2.1.12. Thus the functions $(y)_j = g([\Psi(X); j]), j = 0, 1, ...$ are expressible in set theory $\overline{ZFC}_2^{Hs}$ by the formulas denoted below for a short by the symbol $\lambda_j(y, g([\Psi(X); j]))$.

For $y > 0$, let $lh(y)$ be the number of non-zero exponents in the factorization of y into powers of primes, or, equivalently, the number of distinct primes that divide y. Let $lh(0) = 0$, then $lh(y)$ is primitive recursive.

Remark 2.1.13. (i) Note that a function $\left(g_{\overline{ZFC}_2^{Hs}}\left(\Psi\left(X\right)\right) = y\right) \wedge \left(g_{\overline{ZFC}_2^{Hs}}\left(X\right) = \nu\right)$ is expressible in set theory $\overline{ZFC}_2^{Hs}$ by the following formula $\widetilde{\Xi}_1\left(\Psi\left(X\right), y, \nu\right)$:

$$\widetilde{\Xi}_1\left(\Psi\left(X\right), y, \nu\right) \iff \left(y \in \mathbb{N}\right) \wedge \left(\nu \in \mathbb{N}\right) \widehat{\mathbf{Fr}_1^{Hs}}(y, v) \wedge \left[{}_{j \leq lh(y)} \lambda_j\left(y, g\left(\left[\Psi\left(X\right); j\right]\right)\right)\right], \quad (2.1.37)$$

where $\Psi\left(X\right)$ is 1-open first order wff of the set theory $\overline{ZFC}_2^{Hs}$.

(ii) Note that the length of the formula (2.1.37) depend on numerals $\overline{y}, \overline{\nu}$ but nevertheless $\widetilde{\Xi}\left(\Psi\left(X\right), y, \nu\right)$ is a single 3-open wiff of $\overline{ZFC}_2^{Hs}$.

(iii) Note that

$$g_{\overline{ZFC}_2^{Hs}}\left(\Psi_{y,\nu}\left(X\right)\right) = y \iff \widetilde{\Xi}\left(\breve{\mathbf{g}}^{-1}\left(y\right), y, \nu\right). \quad (2.1.38)$$

Definition 2.1.7. $Let_{X,\nu}^{Hs}$ be a set of the all 1-place open wff's $\Psi\left(X\right)$ of the set theory $\overline{ZFC}_2^{Hs}$ that contains free occurrences of the individual variable X with Gödel number v and quantifiers only over individual variables.We define now a set $\Gamma_{X,\nu}^{Hs} \subsetneqq_{X,\nu}^{Hs}$ by the following formula

$$\forall\Psi\left(X\right)\left[\Psi\left(X\right) \in \Gamma_{X,\nu}^{Hs} \iff \left(\exists! X\left(X \in M_{st}^{ZFC_2^{Hs}}\right)\Psi\left(X\right)\right) \wedge \left(\Psi\left(X\right) \in_{X,\nu}^{Hs}\right)\right]. \quad (2.1.39)$$

Remark 2.1.14. Let $g_{\overline{ZFC}_2^{Hs}}\left(X\right) = \nu$.We define now a set $\Gamma_\nu^{Hs} \subsetneqq \mathbb{N}$ by the following formula

$$\Gamma_\nu^{Hs} = \left\{y \in \mathbb{N} | \left(\langle y, \nu\rangle \in \mathbf{Fr}_1^{Hs}(y, v)\right) \wedge \breve{\mathbf{g}}^{-1}\left(y\right) \in \Gamma_{X,\nu}^{Hs}\right\}, \quad (2.1.40)$$

or in the following equivalent form:

$$\forall y\left(y \in \mathbb{N}\right)\left[y \in \Gamma_\nu^{Hs} \iff \left(y \in \mathbb{N}\right) \wedge \widehat{\mathbf{Fr}_1^{Hs}}(y, v) \wedge \breve{\mathbf{g}}^{-1}\left(y\right) \in \Gamma_{X,\nu}^{Hs}\right]. \quad (2.1.41)$$

Definition 2.1.8. Let $\Xi_{1,X}$ be a ZFC-set of the all first order 1-open wiff's of the set theory $\overline{ZFC}_2^{Hs}$,see Remark 2.1.4, then we abbreviate $\Xi_{1,X} \triangleq \mathbf{Wff}_{1,X}\left[\overline{ZFC}_2^{Hs}\right]$.

Remark 2.1.15.(a) Note that a ZFC-set $\mathbf{Wff}_1\left[\overline{ZFC}_2^{Hs}\right]$ in canonical handbooks always considered as an standard set in the sense of the set theory ZFC,see ref. [8].

See for example the proof of the Gödel Completness Theorem, ref. [8] Theorem 2, sect.4,p.13.

(b) Note that from statement (a) (see also Remark 2.1.4) and from the axiom of separation it follows directly that Γ_ν^{Hs} is a standard set in the sense of the set theory $\overline{ZFC}_2^{Hs}$,

(441) note that the collections Γ_X^{Hs} and $\Gamma_{X,\nu}^{Hs}$ in fact can be considered as a standard set directly

without any reference to Gödel number, since a countable collection of the all first order wff's of the set theory $\overline{ZFC}_2^{Hs}$ is a set in the sense of the set theory ZFC.

Definition 2.1.9.(i) We define now the equivalence relation

$$(\cdot \sim_\nu \cdot) \subset \Gamma_\nu^{Hs} \times \Gamma_\nu^{Hs} \quad (2.1.42)$$

in the sense of the set theory $\overline{ZFC}_2^{Hs}$ by the following formula

$$\forall y_1 \forall y_1 \left[y_1 \sim_\nu y_2 \iff \left(\forall X \left(X \in M_{st}^{ZFC_2^{Hs}}\right)[\Psi_{y_1,\nu}(X) \iff \Psi_{y_2,\nu}(X)]\right)\right]. \quad (2.1.43)$$

Remark 2.1.16. Note that (2.1.43) by using second order lenguage of the set theory

$\overline{ZFC}_2^{Hs}$ can be written in the following equivalent form

$$\begin{gathered} y_1 \sim_\nu y_2 \iff \\ \mathbf{Fr}_1^{Hs}(y_1, v) \wedge \mathbf{Fr}_1^{Hs}(y_2, v) \wedge \exists \Psi_{y_1,\nu}(X)\left(g_{\overline{ZFC}_2^{Hs}}(\Psi_{y_1,\nu}(X)) = y_1\right) \wedge \\ \exists \Psi_{y_2,\nu}(X)\left(g_{\overline{ZFC}_2^{Hs}}(\Psi_{y_2,\nu}(X)) = y_2\right) \wedge \left(g_{\overline{ZFC}_2^{Hs}}(X) = \nu\right) \wedge \\ \left[\forall X \left(X \in M_{st}^{ZFC_2^{Hs}}\right)[\Psi_{y_1,\nu}(X) \iff \Psi_{y_2,\nu}(X)]\right]. \end{gathered} \quad (2.1.44)$$

Remark 2.1.17. Note that from the axiom of separation it follows directly that the equivalence relation $(\cdot \sim_\nu \cdot)$ is a relation in the sense of the set theory $\overline{ZFC}_2^{Hs}$.

(i) A subset Λ_ν^{Hs} of Γ_ν^{Hs} such that $y_1 \sim_\nu y_2$ holds for all y_1 and y_1 in Λ_ν^{Hs}, and never for y_1 in Λ_ν^{Hs} and y_2 outside Λ_ν^{Hs}, is an equivalence class of Γ_ν^{Hs}.

(iii) For any $y \in \Gamma_\nu^{Hs}$ by symbol $[y]_{Hs} \triangleq \{z \in \Gamma_\nu^{Hs} | y \sim_\nu z\}$ we denote the equivalence class to which y belongs. All elements of Γ_ν^{Hs} equivalent to each other are also elements of the same equivalence class.

(iii)The collection of all possible equivalence classes of Γ_ν^{Hs} by $\tilde{}_\nu$, denoted by symbol

$\Gamma_\nu^{Hs}/\sim_\nu$:

$$\Gamma_\nu^{Hs}/\sim_\nu \triangleq \{[y]_{Hs} | y \in \Gamma_\nu^{Hs}\}. \quad (2.1.45)$$

Remark 2.1.18. Note that from the axiom of separation it follows directly that $\Gamma_\nu^{Hs}/\sim_\nu$is a set in the sense of the set theory $\overline{ZFC}_2^{Hs}$.

Definition 2.1.10. Let $\Im_2^{Hs}$ be the countable ZFC-set of the all sets definable by the first order 1-place open wff's of the set theory $\overline{ZFC}_2^{Hs}$, i.e.

$$\forall Y \left\{Y \in \Im_2^{Hs} \iff \exists \Psi(X)\left[\left([\Psi(X)]_{Hs} \in \Gamma_X^{Hs}/\sim_X\right) \wedge [\exists! X [\Psi(X) \wedge Y = X]]\right]\right\}. \quad (2.1.46)$$

Definition 2.1.11. We rewrite now (2.1.46) in the following equivalent form

$$\forall Y \left\{Y \in \Im_2^{Hs} \iff \exists \Psi(X)\left[\left([\Psi(X)]_{Hs} \in \Gamma_X^{*Hs}/\sim_X\right) \wedge (Y = X)\right]\right\}, \quad (2.1.47)$$

where the countable set $\Gamma_X^{*Hs}/\sim_X$ is defined by the following formula

$$\forall \Psi(X) \left\{[\Psi(X)] \in \Gamma_X^{*Hs}/\sim_X \iff \left[\left([\Psi(X)] \in \Gamma_X^{Hs}/\sim_X\right) \wedge \exists! X \Psi(X)\right]\right\} \quad (2.1.48)$$

Definition 2.1.12. Let $\Re_2^{Hs}$ be the countable set of the all sets definable by the first order 1-place open wff's and such that

$$\forall X\left\{\left(X\in M_{st}\wedge X\in\Im_2^{Hs}\right)\left[X\in\Re_2^{Hs}\iff X\notin X\right]\right\}. \quad (2.1.49)$$

Remark 2.1.19.(a) Note that $\Re_2^{Hs}\in\Im_2^{Hs}$ since $\Re_2^{Hs}$ is a ZFC-set definable by the first order 1-place open wff

$$\Psi_{M_{st}}\left(Z,\Im_2^{Hs}\right)\triangleq\Psi\left(Z,\Im_2^{Hs}\right)\triangleq\forall X\left\{\left(X\in M_{st}\wedge X\in\Im_2^{Hs}\right)[X\in Z\iff X\notin X]\right\}, \quad (2.1.50)$$

and obviously $\Psi_{M_{st}}\left(Z,\Im_2^{Hs}\right)\in\mathbf{Wff}_{1,Z}\left[\overline{ZFC}_2^{Hs}\right]$.

From (2.1.47)-(2.1.50) one obtains

$$\Re_2^{Hs}\in\Re_2^{Hs}\iff\Re_2^{Hs}\notin\Re_2^{Hs}. \quad (2.1.51)$$

But (2.1.51) immediately gives a contradiction

$$\left(\Re_2^{Hs}\in\Re_2^{Hs}\right)\wedge\left(\Re_2^{Hs}\notin\Re_2^{Hs}\right). \quad (2.1.52)$$

(b) Note that the contradiction (2.1.52) that is a contradiction inside $\overline{ZFC}_2^{Hs}$ for the reason that the countable set $\Im_2^{Hs}$ is a standard set in a sense of the set theory $\overline{ZFC}_2^{Hs}$,

see Remark 2.1.15 (a)-(c) and Remark 2.1.4.

Theorem 2.1.1. Let $\overline{ZFC}_2^{Hs}$ be a theory $\overline{ZFC}_2^{Hs}\triangleq ZFC_2^{Hs}+\exists M_{st}^{ZFC_2^{Hs}}$ and $\mathbf{Wff}_1\left[\overline{ZFC}_2^{Hs}\right]\in$

$\in M_{st}^{ZFC_2^{Hs}}$.Then set theory $\overline{ZFC}_2^{Hs}$ is inconsistent.

Proof. Immediately from (2.1.52).

Remark 2.1.20. In order to obtain a contradiction inside $\overline{ZFC}_2^{Hs}$**,** in more general case,i.e., without any refference to Assumption 2.1.1 we introduce the following definitions.

Definition 2.1.13. We define now the countable set $\Gamma_\nu^{*Hs}/\sim_\nu$ by the following formula

$$\forall y\left\{[y]_{Hs}\in\Gamma_\nu^{*Hs}/\sim_\nu\iff\left([y]_{Hs}\in\Gamma_\nu^{Hs}/\sim_\nu\right)\wedge\widehat{\mathbf{Fr}_2^{Hs}}(y,v)\wedge[\exists!X\Psi_{y,\nu}(X)]\right\}. \quad (2.1.53)$$

Remark 2.1.21. Note that from the axiom of separation it follows directly that $\Gamma_\nu^*/$ is a set in the sense of the set theory $\overline{ZFC}_2^{Hs}$.

Definition 2.1.14. We define now the countable set $\Im_2^{*Hs}$ by the following formula

$$\forall Y\left\{Y\in\Im_2^{*Hs}\iff\exists y\left[\left([y]\in\Gamma_\nu^{*Hs}/\sim_\nu\right)\right]\right\}. \quad (2.1.54)$$

Note that from the axiom schema of replacement (1.1.1) it follows directly that $\Im_2^{*Hs}$ is a set in a sense of the set theory $\overline{ZFC}_2^{Hs}$.

Definition 2.1.15. We define now the countable set $\Re_2^{*Hs}$ by formula

$$\forall X\left\{\left[\left(X\in\Im_2^{*Hs}\right)\wedge(X\in M_{st})\right]\wedge\left[X\in\Re_2^{*Hs}\iff X\notin X\right]\right\}. \quad (2.1.55)$$

Note that from the axiom schema of separation it follows directly that $\Re_2^{*Hs}$ is a set in the sense of the set theory $\overline{ZFC}_2^{Hs}$.

Remark 2.1.22. Note that $\Re_2^{*Hs} \in \Im_2^{*Hs}$ since $\Re_2^{*Hs}$ is definable by the following formula

$$\Psi^*_{M_{st}}\left(Z, \Im_2^{*Hs}\right) \triangleq \Psi^*\left(Z, \Im_2^{*Hs}\right) \triangleq \forall X \left\{\left(X \in \Im_2^{*Hs} \wedge X \in M_{st}\right)[X \in Z \iff X \notin X]\right\}, \quad (2.1.56)$$

where obviously $\Psi^*\left(Z, \Im_2^{*Hs}\right) \in \mathbf{Wff}_{1,X}\left[\overline{ZFC}_2^{Hs}\right]$.

Theorem 2.1.2. Set theory $\overline{ZFC}_2^{Hs}$ is inconsistent.

Proof. From (2.1.55) and Remark 2.1.22 we obtain

$$\Re_2^{*Hs} \in \Re_2^{*Hs} \iff \Re_2^{*Hs} \notin \Re_2^{*Hs}. \quad (2.1.57)$$

From (2.1.57) one obtains a contradiction

$$\left(\Re_2^{*Hs} \in \Re_2^{*Hs}\right) \wedge \left(\Re_2^{*Hs} \notin \Re_2^{*Hs}\right). \quad (2.1.58)$$

Definition 2.1.16. Let $\widetilde{ZFC}_2^{Hs}$ be a set theory $\widetilde{ZFC}_2^{Hs} \triangleq \quad ZFC_2^{Hs} + \exists M_{Nst}^{ZFC_2^{Hs}}$.
We assume now that:$\exists M_{st}^{ZFC_2^{Hs}}$ such that $M_{st}^{ZFC_2^{Hs}} \subset M_{Nst}^{ZFC_2^{Hs}}$.Then we will say that $M_{st}^{ZFC_2^{Hs}}$ is a standard

part of $M_{Nst}^{ZFC_2^{Hs}}$.

Theorem 2.1.3. Set theory $\widetilde{ZFC}_2^{Hs}$ is inconsistent.

Proof. Similarly to proof of the Theorem 2.1.2 but with quantifiers bounded on standard part $M_{st}^{ZFC_2^{Hs}}$ of $M_{Nst}^{ZFC_2^{Hs}}$.

Definition 2.1.17. Let Δ be an standard set in the sense of the set theory ZFC.We will say that:

(i) a set Δ is admissible relative to model $M_{Nst}^{ZFC_2^{Hs}}$ iff

$$Con\left(\widetilde{ZFC}_2^{Hs}\right) \Longrightarrow Con\left(\widetilde{ZFC}_2^{Hs} + \left(\Delta \in M_{Nst}^{ZFC_2^{Hs}}\right)\right). \quad (2.1.59)$$

(ii) a set Δ is not admissible relative to model $M_{Nst}^{ZFC_2^{Hs}}$ iff

$$Con\left(\widetilde{ZFC}_2^{Hs}\right) \Longrightarrow \neg Con\left(\widetilde{ZFC}_2^{Hs} + \left(\Delta \in M_{Nst}^{ZFC_2^{Hs}}\right)\right). \quad (2.1.60)$$

(iii) a set Δ is absolute not admissible iff Δ is not admissible relative to any model $M_{Nst}^{ZFC_2^{Hs}}$.

Definition 2.1.18. Let $\Xi_{1,X}$ by a ZFC-set of the all the first order 1-place open wff's of the set theory $\overline{ZFC}_2^{Hs}$, then we abbreviate $\Xi_{1,X} \triangleq \mathbf{Wff}_{1,X}\left[\widetilde{ZFC}_2^{Hs}\right]$.

Theorem 2.1.4.(1) Set theory $\widetilde{ZFC}_2^{Hs} + \left(\mathbf{Wff}_{1,X}\left[\widetilde{ZFC}_2^{Hs}\right] \in M_{Nst}^{ZFC_2^{Hs}}\right)$ is inconsistent.

(2) A set $\mathbf{Wff}_{1,X}\left[\widetilde{ZFC}_2^{Hs}\right]$ absolute is not admissible.

Proof. Similarly to proof of the Theorem 2.1.2 since canonical Gödel encoding holds by property $\mathbb{N} \in M_{Nst}^{ZFC_2^{Hs}}$.

Proof. (2) Immediately from (1) and Definition 2.1.17.

2.2 Derivation of the Inconsistent Definable Set in Set Theory ZFC_{st}

In this section we obtain a contradiction in the set theory $ZFC_{st} \triangleq ZFC + \exists M_{st}^{ZFC}$. by using a set of the all sets definable by 1-place open wff's of the set theory ZFC_{st}.

Definition 2.2.1. Let $^{st}_X$ be a set of the all 1-place open wff's $\Psi(X)$ of the set theory ZFC_{st} with all bound variables restricted to standard model M_{st} that contains free occurrences of the individual variable X with Godel number v and we will be denoted these

wff's by $\Psi(X) = \Psi_{M_{st}}(X), \Psi_X = \Psi_{X,M_{st}}, \Psi_{y,\nu}(X) = \Psi_{y,\nu,M_{st}}(X), y, \nu \in \mathbb{N}$.We define now a set $\Gamma_X^{st} \subsetneqq^{st}_X$ by the following second order formula

$$\forall \Psi(X) \left[\Psi(X) \in \Gamma_X^{st} \iff \left(\exists! X \left(X \in M_{st}^{ZFC} \right) \Psi(X) \right) \wedge \left(\Psi(X) \in^{st}_X \right) \right]. \quad (2.2.1)$$

or in the following equivalent form

$$\begin{gathered} \forall \Psi(X) \Big[\Psi(X) \in \Gamma_X^{st} \iff \exists y \widehat{\mathbf{Fr}^{ZFC}}(y, v) \searrow \\ \left[\left(g_{\overline{ZFC}}(\Psi(X)) = y \right) \wedge \left(g_{\overline{ZFC}}(X) = \nu \right) \right] \\ \wedge \left(\exists! X \left(X \in M_{st}^{ZFC} \right) \Psi(X) \right) \wedge \left(\Psi(X) \in^{st}_X \right) \Big], \end{gathered} \quad (2.2.1.a)$$

see Remark 2.2.2 (ix) and Eq.(2.2.). Note that there exist a set Γ_X^{st} by the second order separaton axiom of ZFC_2^{Hs}.

Notation 2.2.1. In this subsection we often write for short $\Psi(X), \Psi_X, \Psi_{y,\nu}(X)$ instead

$\Psi_{M_{st}}(X), \Psi_{X,M_{st}}, \Psi_{y,\nu,M_{st}}(X)$ but this should not lead to a confusion.

Assumption 2.2.1. We assume now for simplicity but without loss of generality that

$$^{st}_X \in M_{st} \quad (2.2.1.b)$$

and therefore by definition of model M_{st} one obtains $\Gamma_X^{st} \in M_{st}$.

Definition 2.2.2. Let $\Xi_{1,X}$ be a ZFC-set of the all 1-open wiff's of the set theory ZFC_{st},then we abbreviate $\Xi_{1,X} \triangleq \mathbf{Wff}_{1,X}[ZFC_{st}]$.

Definition 2.2.3. Let$\widetilde{\ }^{st}_X$ be a set $\widetilde{\ }^{st}_X = \wp_{\Xi_1}^{-1}\left(^{st}_X\right)$,and $\widehat{\Psi}(X) \triangleq \widehat{\Psi}_X = \wp_{\Xi_1}^{-1}(\Psi(X))$ where one-one function $\wp_{\Xi_1}^{-1}$ defined in sec.2.1,see Remark 2.1.5 and Eq.(2.1.22).

Remark 2.2.1.(i)We define now a set $\widehat{\Gamma}_X^{st} = \wp_{\Xi_1}^{-1}\left(\Gamma_X^{st}\right), \widehat{\Gamma}_X^{st} \subsetneqq \widetilde{\ }^{st}_X$ by the following first order formula with quantifiers over first order individual varables $\widehat{\Psi}_X$ and X :

$$\forall \widehat{\Psi}_X \left[\widehat{\Psi}_X \in \widehat{\Gamma}_X^{st} \iff \exists! X \left(X \in M_{st}^{ZFC} \right) \Psi_X \wedge \left(\widehat{\Psi}_X \in \widetilde{\ }^{st}_X \right) \right], \quad (2.2.2)$$

(where we write Ψ_X instead $\Psi(X)$) or in the following equivalent form

$$\forall \widehat{\Psi}_X \left[\widehat{\Psi}_X \in \widehat{\Gamma}_X^{st} \iff \exists y \widehat{\mathbf{Fr}^{ZFC}}(y,v) \searrow \right.$$
$$\left[\left(\widehat{g}_{\overline{ZFC}} \left(\widehat{\Psi}_X \right) = y \right) \wedge \left(\widehat{g}_{\overline{ZFC}} \left(\widehat{X} \right) = \nu \right) \right] \quad (2.2.2.a)$$
$$\left. \wedge \left(\exists ! X \left(X \in M_{st}^{ZFC} \right) \Psi_X \right) \wedge \left(\widehat{\Psi}_X \in \widehat{\gamma}_X^{st} \right) \right],$$

where one-one function $\widehat{g}_{\overline{ZFC}} \left(\widehat{\Psi}_X \right) = y$ is defined below by Eq.(2.2.4), see Remark 2.2.3.

Note that there exist a set $\widehat{\Gamma}_X^{st}$ by the (first order) separaton axiom of ZFC.

(ii) Note that second order definition (2.2.1) and first order definition (2.2.2) are equivalent.

We abbreviate now:

(a) $\mathbf{IC}^{ZFC}(x)$: x is the Godel number of an individual constant of $\overline{ZFC}$,

(b) $\mathbf{FL}^{ZFC}(x)$: x is the Godel number of a function letter of $\overline{ZFC}$,

(c) $\mathbf{PL}^{ZFC}(x)$: x is the Godel number of a predicate letter of $\overline{ZFC}$.

Remark 2.2.2. Note that in order to obtain by using only first order logic the formal definitions of the first order predicates $\mathbf{EVbl}^{ZFC}(x), \mathbf{EIC}^{ZFC}(x), \mathbf{EFL}^{ZFC}(x), ..., \mathbf{Fr}^{ZFC}(y,v)$

from the first order predicates $\mathbf{IC}^{ZFC}(x), \mathbf{FL}^{ZFC}(x), \mathbf{PL}^{ZFC}(x)$ one needs the following first order predicates:

(i) $\mathbf{EVbl}^{ZFC}(x, \widehat{\alpha})$: x is the Godel number of the 1-tuple $\widehat{\alpha} \in \widehat{\Delta}_1, \widehat{\Delta}_1 = \wp_{\Delta_1}^{-1}(\Delta_1)$

correspondinging to the individual variable $\alpha \in \Delta_1$, $\alpha = \wp_{\Delta_1}(\widehat{\alpha})$,see Remark 2.1.2 (i).

(ii) $\mathbf{EIC}^{ZFC}(x, \widehat{\beta})$: x is the Godel number of the 1-tuple $\widehat{\beta} \in \widehat{\Delta}_2, \widehat{\Delta}_2 = \wp_{\Delta_2}^{-1}(\Delta_2)$

correspondinging to the individual constant $\beta \in \Delta_2$, $\beta = \wp_{\Delta_2}\left(\widehat{\beta}\right)$,see Remark 2.1.2 (ii).

(iii) $\mathbf{EFL}^{ZFC}(x, \widehat{\gamma})$: x is the Godel number of the 1-tuple $\widehat{\gamma} \in \widehat{\Delta}_3, \widehat{\Delta}_3 = \wp_{\Delta_3}^{-1}(\Delta_3)$

correspondinging to the function letter $\gamma \in \Delta_3, \gamma = \wp_{\Delta_3}(\widehat{\gamma})$,see Remark 2.1.2 (iii).

(iv) $\mathbf{EPL}^{ZFC}(x, \widehat{\delta})$: x is the Godel number of the 1-tuple $\widehat{\delta} \in \widehat{\Delta}_4, \widehat{\Delta}_4 = \wp_{\Delta_4}^{-1}(\Delta_4)$

correspondinging to the predicate letter $\delta \in \Delta_4, \delta = \wp_{\Delta_4}\left(\widehat{\delta}\right)$,see Remark 2.1.2 (iv).

(v) $\mathbf{Gd}^{ZFC}(x, \widehat{\zeta})$: x is the Godel number of the element $\widehat{\zeta} \in \widehat{\mathbf{S}^{\#}}$ of the set $\widehat{\mathbf{S}^{\#}} = \cup_{n\in\mathbb{N}} \widehat{\mathbf{S}}^n$, $\widehat{\mathbf{S}} = \widehat{\Delta}_1 \cup \widehat{\Delta}_2 \cup \widehat{\Delta}_3 \cup \widehat{\Delta}_4$, correspondinging to the expression $\zeta \in \mathbf{S}^{\#}$, of ZFC,where

$\mathbf{S}^{\#} = \cup_{n\in\mathbb{N}} \mathbf{S}^n, \mathbf{S} = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$,see Definition 2.1.6.

(vi) $\mathbf{Trm}^{ZFC}(x, \widehat{\tau})$: x is the Godel number of the element $\widehat{\tau} \in \widehat{\Upsilon}_1$ of the set

$\widehat{\Upsilon} = \wp_{\tau}^{-1}(\Upsilon_1)$,corres-pondinging to the term $\tau = \wp_{\tau}(\widehat{\tau})$ of ZFC.

(vii) $\mathbf{Atfml}^{ZFC}(x, \widehat{\pi})$: x is the Godel number of the element $\widehat{\pi} \in \widehat{\Sigma}_1$ of the set $\widehat{\Sigma}_1 = \wp_{\pi}^{-1}(\Sigma_1)$,corres-ponding-ing to atomic wff $\pi = \wp_{\pi}(\widehat{\pi})$ of ZFC.

(viii) $\mathbf{Fml}^{ZFC}(y,\widehat{\varphi})$: y is the Godel number of the element $\widehat{\varphi} \in \widehat{\Xi}_1$ of the set $\widehat{\Xi}_1 = \wp_{\Xi_1}^{-1}(\Xi_1)$,corres-pondinging to the wff formula $\varphi = \wp_{\Xi_1}(\widehat{\varphi})$ of ZFC.

(ix) $\mathbf{Fr}^{ZFC}(y,v,\widehat{\varpi})$: y is the Godel number of the element $\widehat{\varpi} \in \widehat{\Xi}_{1,\nu}$ of the set $\widehat{\Xi}_{1,\nu} = \wp_{\Xi_{1,\nu}}^{-1}(\Xi_{1,\nu})$,correspondinging to the wff formula or term $\varpi = \wp_{\Xi_{1,\nu}}(\widehat{\varpi})$

of ZFC that contains free occurrences of the variable with Godel number v.

Thus finally we obtain:

$$\begin{aligned}
\mathbf{EVbl}^{ZFC}(x) &\iff \exists\widehat{\alpha}\left(\widehat{\alpha}\in\widehat{\Delta}_1\right)\mathbf{EVbl}^{ZFC}(x,\widehat{\alpha}) \iff (\exists z)_{z<x}(1\le z\wedge x=2^{13+8z}),\\
\mathbf{EIC}^{ZFC}(x) &\iff \exists\widehat{\beta}\left(\widehat{\beta}\in\widehat{\Delta}_2\right)\mathbf{EIC}^{ZFC}(x,\widehat{\beta}) \iff (\exists y)_{y<x}(\mathbf{IC}^{ZFC}(y)\wedge x=2^{y}),\\
\mathbf{EFL}^{ZFC}(x) &\iff \exists\widehat{\gamma}\left(\widehat{\gamma}\in\widehat{\Delta}_3\right)\mathbf{EFL}^{ZFC}(x,\widehat{\gamma}) \iff (\exists y)_{y<x}(\mathbf{FL}^{ZFC}(y)\wedge x=2^{y}),\\
\mathbf{EPL}^{ZFC}(x) &\iff \exists\widehat{\delta}\left(\widehat{\delta}\in\widehat{\Delta}_4\right)\mathbf{EPL}^{ZFC}(x,\widehat{\delta}) \iff (\exists y)_{y<x}(\mathbf{PL}^{ZFC}(y)\wedge x=2^{y}),
\end{aligned}$$

$$\begin{gathered}
\mathbf{Gd}^{ZFC}(x) \iff \exists\widehat{\zeta}\exists n\left(\widehat{\zeta}\in\widehat{\mathbf{S}}^n\right)\mathbf{Gd}^{ZFC}(x,\widehat{\zeta}) \iff\\
\mathbf{EVbl}^{ZFC}(x)\vee\mathbf{EIC}^{ZFC}(x)\vee\mathbf{EFL}^{ZFC}(x)\vee\mathbf{EPL}^{ZFC}(x)\vee\\
x=2^7\vee x=2^9\vee x=2^{11}\vee x=2^{13}\vee\\
(\exists u)_{u<x}(\exists v)_{v<x}(x=u*v\wedge\mathbf{Gd}^{ZFC}(u)\wedge\mathbf{Gd}^{ZFC}(v)).
\end{gathered}$$

$$\begin{gathered}
\mathbf{Trm}^{ZFC}(x) \iff \exists\widehat{\tau}\left(\widehat{\tau}\in\widehat{\Upsilon}\right)\mathbf{Trm}^{ZFC}(x,\widehat{\tau}) \iff\\
\mathbf{EVbl}^{ZFC}(x)\vee\mathbf{EIC}^{ZFC}(x)\vee(\exists y)_{y<(p_x!)^x}[x=(y)_{lh(y)\dot{-}1}\wedge\\
lh(y)=\mathbf{Arg_T}((x)_0)+1\wedge\mathbf{FL}^{ZFC}(((y)_0)_0)\wedge((y)_0)_1=3\wedge\\
lh((y)_0)=2\wedge(\forall u)_{u<lh(y)\dot{-}1}(\exists v)_{v<x}((y)_{u+1}=(y)_u*v*2^7\wedge\mathbf{Trm}^{ZFC}(v))\wedge\\
(\exists v)_{v<x}((y)_{lh(y)\dot{-}1}=(y)_{lh(y)\dot{-}2}*v*2^5\wedge\mathbf{Trm}^{ZFC}(v))].
\end{gathered} \qquad (2.2.3)$$

$$\begin{gathered}
\mathbf{Atfml}^{ZFC}(x) \iff \exists\widehat{\pi}\left(\widehat{\pi}\in\widehat{\Sigma}\right)\mathbf{Atfml}^{ZFC}(x,\widehat{\pi}) \iff\\
(\exists y)_{y<(p_x!)^x}[x=(y)_{lh(y)\dot{-}1}\wedge lh(y)=\mathbf{Arg}_P((x)_0)+1\wedge\\
\mathbf{PL}^{ZFC}(((y)_0)_0)\wedge((y)_0)_1=3\wedge lh((y)_0)=2\wedge\\
(\forall u)_{u<lh(y)\dot{-}2}(\exists v)_{v<x}((y)_{u+1}=(y)_u*v*2^7\wedge\mathbf{Trm}^{ZFC}(v))\wedge\\
(\exists v)_{v<x}((y)_{lh(y)\dot{-}1}=(y)_{u<lh(y)\dot{-}2}*v*2^5\wedge\mathbf{Trm}^{ZFC}(v))].
\end{gathered}$$

$$\begin{gathered}
\mathbf{Fml}^{ZFC}(y) \iff \exists\widehat{\varphi}\left(\widehat{\varphi}\in\widehat{\Xi}\right)\mathbf{Fml}^{ZFC}(y,\widehat{\varphi}) \iff\\
\mathbf{Atfml}^{ZFC}(y)\vee(\exists z)_{z<y}[(\mathbf{Fml}^{ZFC}(z)\wedge=2^3*2^9*z*2^5)\vee\\
(\mathbf{Fml}^{ZFC}((z)_0)\wedge\mathbf{Fml}^{ZFC}((z)_1\wedge y=2^3*(z)_0*2^{11}*(z)_1*2^5)\vee\\
(\mathbf{Fml}^{ZFC}((z)_0)\wedge\mathbf{EVbl}^{ZFC}((z)_1)\wedge y=2^3*2^3*2^{13}*((z)_1*2^5*(z)_0*2^5)].
\end{gathered}$$

$$\begin{gathered}
\mathbf{Fr}^{ZFC}(y,v) \iff \exists\varpi\left[(\varpi\in\Xi_{1,\nu})\vee\left(\varpi\in\widehat{\Upsilon}_1\right)\right]\mathbf{Fr}^{ZFC}(y,v,\varpi) \iff\\
(\mathbf{Fml}^{ZFC}(y)\vee\mathbf{Trm}^{ZFC}(y))\wedge\mathbf{EVbl}^{ZFC}(2^v)\wedge\neg\mathbf{Subst}^{ZFC}(y,y,2^{13+8v},v).
\end{gathered}$$

Remark 2.2.3. $(i)Let g_{ZFC_{st}}(u)$ be a Gödel number of given an expression $u\in\Omega$ of the language of the set theory $ZFC_{st}\triangleq ZFC+\exists M_{st}^{ZFC}$. Recall that $\wp_{\Omega_1}^{-1}(\Omega_1)=\widehat{\Omega}_1$ see Definition 2.1.6. We set now

$$\widehat{g}_{ZFC_{st}}(u)=g_{ZFC_{st}}(u). \qquad (2.2.4)$$

(ii) Let $\mathbf{Fr}^{ZFC}(y,v)$ be the relation : y is the Gödel number of a wff of the set theoryZFC_{st} that contains free occurrences of the variable X with Gödel number v,see Eq.(2.2.3).

(iii) Note that the relation $\mathbf{Fr}^{ZFC}(y,v)$ is expressible in ZFC_{st} by a wff $\widehat{\mathbf{Fr}^{ZFC}}(y,v)$

(iv) Note that for any $y,v\in\mathbb{N}$ by definition of the relation $\mathbf{Fr}^{ZFC}(y,v)$ follows that

$$\begin{aligned}\widehat{\mathbf{Fr}^{ZFC}}(y,v) &\iff \exists!\Psi_X\left[(g_{ZFC_{st}}(\Psi_X)=y)\wedge(g_{ZFC_{st}}(X)=\nu)\right]\iff\\ &\exists!\widehat{\Psi}_X\left[\left(\widehat{g}_{ZFC_{st}}\left(\widehat{\Psi}_X\right)=y\right)\wedge\left(\widehat{g}_{ZFC_{st}}\left(\widehat{X}\right)=\nu\right)\right],\end{aligned}\tag{2.2.5}$$

where $\Psi_X=\Psi(X)$ is a unique wff of ZFC_{st} which contains free occurrences of the variable X with Gödel number v.We denote such unique wff $\Psi(X)$ defined by equivalence (2.2.5) by symbol $\Psi_{y,\nu}(X)$,i.e.

$$\begin{aligned}\widehat{\mathbf{Fr}^{ZFC}}(y,v) &\iff \exists!\Psi_{y,\nu}(X)\left[(g_{ZFC_{st}}(\Psi_{y,\nu}(X))=y)\wedge(g_{ZFC_{st}}(X)=\nu)\right]\iff\\ &\exists!\widehat{\Psi_{y,\nu}(X)}\left[\left(g_{ZFC_{st}}\left(\widehat{\Psi_{y,\nu}(X)}\right)=y\right)\wedge\left(g_{ZFC_{st}}\left(\widehat{X}\right)=\nu\right)\right].\end{aligned}\tag{2.2.6}$$

Remark 2.2.4. Note that a function $g_{ZFC_{st}}(\Psi_{y,\nu}(X))=y$ is expressible in set theory ZFC_{st} by a wff of the set theory ZFC_{st} that contains free occurrences of the variable $y\in\mathbb{N}$.

Note that any formula $\Psi_{y,\nu}(X)$ is given by an expression $u_0u_1..u_j...u_r$,i.e. $\Psi_{y,\nu}(X)=:u_0u_1..u_j...u_r$, where each u_j is a symbol of ZFC_{st}. We introduce now a functions $[\Psi_{y,\nu}(X);j]:\Psi_{y,\nu}(X)\to u_j, j=0,1,...$,i.e. $[\Psi_{y,\nu}(X);j]=:u_j$ and revrite expression $u_0u_1..u_j...u_r$ in the following equivalent form

$$[\Psi_{y,\nu}(X);0]\,[\Psi_{y,\nu}(X);1]\ldots[\Psi_{y,\nu}(X);j]\ldots[\Psi_{y,\nu}(X);r].\tag{2.2.7}$$

By definitions we obtain that

$$\begin{gathered}g_{ZFC_{st}}(\Psi_{y,\nu}(X))=y\\ \iff y=2^{g([\Psi_{y,\nu}(X);0])}\cdot3^{g([\Psi_{y,\nu}(X);1])}\cdot\ldots\cdot p_j^{g([\Psi_{y,\nu}(X);j])}\cdot\ldots\cdot p_r^{g([\Psi_{y,\nu}(X);r])}.\end{gathered}\tag{2.2.8}$$

and

$$\begin{gathered}\widehat{g}_{ZFC_{st}}\left(\widehat{\Psi_{y,\nu}(X)}\right)=y\\ \iff y=2^{\widehat{g}\left(\left[\widehat{\Psi_{y,\nu}(X)};0\right]\right)}\cdot3^{\widehat{g}\left(\left[\widehat{\Psi_{y,\nu}(X)};1\right]\right)}\cdot\ldots\cdot p_j^{\widehat{g}\left(\left[\widehat{\Psi_{y,\nu}(X)};j\right]\right)}\cdot\ldots\cdot p_r^{\widehat{g}\left(\left[\widehat{\Psi_{y,\nu}(X)};r\right]\right)}.\end{gathered}\tag{2.2.9}$$

correspondingly. Let us denote by $(y)_j$ the exponent $g([\Psi_{y,\nu}(X);j])$ in this factorization

$$y=2^{g([\Psi_{y,\nu}(X);0])}\cdot3^{g([\Psi_{y,\nu}(X);1])}\cdot\ldots\cdot p_j^{g([\Psi_{y,\nu}(X);j])}\cdot\ldots\cdot p_r^{g([\Psi_{y,\nu}(X);r])}.\tag{2.2.10}$$

If $y=1,(y)_j=1$ for all j. If $x=0$, we arbitrarily let $(y)_j=0$ for all j. Then the functions $(y)_j, j=0,1,...$ is primitive recursive, since $(y)_j=\mu_{z<y}\left(p_j^z|y\wedge\neg p_j^{z+1}|y\right)$,is primitive recursive.

Thus the function $(y)_j$ is expressible in set theory ZFC_{st} by formula denoted below by $\lambda_j(y,g([\Psi_{y,\nu}(X);j]))$.

For $y>0$, let $lh(y)$ be the number of non-zero exponents in the factorization of y into powers of primes, or, equivalently, the number of distinct primes that divide y. (i) Let $lh(0)=0$, then $lh(y)$ is primitive recursive. Thus function $g_{ZFC_{st}}(\Psi_{y,\nu}(X))=y$ is expressible in set theory ZFC_{st} by the following formula $\widetilde{\Xi}(\Psi_{y,\nu}(X),y)$

$$\widetilde{\Xi}(\Psi_{y,\nu}(X),y)\iff{}_{j\leq lh(y)}\lambda_j(y,g([\Psi_{y,\nu}(X);j])).\tag{2.2.11}$$

(ii) function $g_{ZFC_{st}}(\Psi_{y,\nu}(X)) = y$ is expressible in set theory ZFC_{st} by the following formula $\widetilde{\Xi}(\Psi_{y,\nu}(X), y)$

$$\widetilde{\Xi}\left(\widehat{\Psi_{y,\nu}(X)}, y\right) \iff {}_{j\leq lh(y)}\lambda_j\left(y, \widehat{g}\left(\left[\widehat{\Psi_{y,\nu}(X)}; j\right]\right)\right). \quad (2.2.12)$$

Definition 2.2.4. Let ${}_{\nu}^{st}$ be a set of the all Gödel numbers of the 1-place open wff's of the set theory ZFC_{st} that contains free occurrences of the variable X with Gödel number v,i.e.

$${}_{\nu}^{st} = \left\{y \in \mathbb{N} | \langle y, \nu\rangle \in \mathbf{Fr}^{ZFC}(y, v)\right\}, \quad (2.2.13)$$

or in the following equivalent form:

$$\forall y\,(y \in \mathbb{N})\left[y \in_{\nu}^{st} \iff (y \in \mathbb{N}) \wedge \widehat{\mathbf{Fr}^{ZFC}}(y, v)\right]. \quad (2.2.14)$$

We define now a set $\Gamma_{\nu}^{st} \subsetneqq_{\nu}^{Hs}$ by the following <u>first order</u> formula

$$\begin{gathered}\forall y\,(y \in \mathbb{N})\left\{y \in \Gamma_{\nu}^{st} \iff \left(y \in_{\nu}^{st}\right)\right.\\ \left.\wedge \exists \widehat{\Psi_X}\left[\left(\widehat{g}_{ZFC_{st}}\left(\widehat{\Psi_X}\right) = y\right) \wedge \left(\widehat{g}_{\overline{ZFC_2^{Hs}}}\left(\widehat{X}\right) = \nu\right)\left(\exists! X\left(X \in M_{st}^{ZFC}\right)\Psi_X\right)\right]\right\}\end{gathered} \quad (2.2.15)$$

where $\Psi_X = \Psi(X)$ is a unique wff of ZFC_{st} which contains free occurrences of the variable X with Gödel number v. or in the following equivalent form

$$\begin{gathered}\forall y\,(y \in \mathbb{N})\left[y \in \Gamma_{\nu}^{st} \iff \left(y \in_{\nu}^{st}\right) \wedge \exists y \widehat{\mathbf{Fr}^{ZFC}}(y, v) \searrow\right.\\ \left.\wedge \exists \widehat{\Psi_X}\left[\left(\widehat{g}_{\overline{ZFC}}\left(\widehat{\Psi_X}\right) = y\right) \wedge \left(\widehat{g}_{\overline{ZFC_2^{Hs}}}\left(\widehat{X}\right) = \nu\right)\right] \wedge \left(\exists! X\left(X \in M_{st}^{ZFC}\right)\Psi_X\right)\right],\end{gathered} \quad (2.2.16)$$

Remark 2.2.5. Note that from the axiom of separation it follows directly that Γ_{ν}^{st} is a set in the sense of the set theory ZFC_{st}.

Definition 2.2.5. Let $\Psi_1 = \Psi_1(X)$ and $\Psi_2 = \Psi_2(X)$ be 1-place open wff's of the set theory ZFC.

(i) We define now the equivalence relation $\left(\cdot \sim_{\widehat{X}} \cdot\right) \subset \widehat{\Gamma}_X^{st} \times \widehat{\Gamma}_X^{st}$ by

$$\begin{gathered}\widehat{\Psi_1(X)} \sim_{\widehat{X}} \widehat{\Psi_2(X)} \iff\\ \Psi_1(X) \sim_X \Psi_2(X) \iff \left(\forall X\left(X \in M_{st}^{ZFC}\right)[\Psi_1(X) \iff \Psi_2(X)]\right)\end{gathered} \quad (2.2.17)$$

or more precisely

$$\begin{gathered}\forall\widehat{\Psi}_1\forall\widehat{\Psi}_2\left(\widehat{\Psi}_1 \sim_{\widehat{X}} \widehat{\Psi}_2\right) \iff \forall\widehat{\Psi_1(X)}\forall\widehat{\Psi_2(X)}\left\{\left[\widehat{\Psi_1(X)} \sim_{\widehat{X}} \widehat{\Psi_2(X)}\right]\right.\\ \left.\iff \forall\widehat{\Psi_1(X)}\forall\widehat{\Psi_2(X)}\left[\forall X\left(X \in M_{st}^{ZFC_2^{Hs}}\right)[\Psi_1(X) \iff \Psi_2(X)]\right]\right\} \iff\\ \forall\widehat{\Psi_1(X)}\forall\widehat{\Psi_2(X)}\left\{\left[\widehat{\Psi_1(X)} \sim_{\widehat{X}} \widehat{\Psi_2(X)}\right] \iff\right.\\ \left.\forall\widehat{\Psi_1(X)}\forall\widehat{\Psi_2(X)}\left[\forall X\left(X \in M_{st}^{ZFC_2^{Hs}}\right)\Psi_1(X) \iff \forall X\left(X \in M_{st}^{ZFC_2^{Hs}}\right)\Psi_2(X)\right]\right\}.\end{gathered} \quad (2.2.18)$$

or in the following equivalent form

$$\begin{gathered}\forall\widehat{\Psi}_1\forall\widehat{\Psi}_2\left(\widehat{\Psi}_1 \sim_{\widehat{X}} \widehat{\Psi}_2\right) \iff \forall\widehat{\Psi_1(X)}\forall\widehat{\Psi_2(X)}\left[\widehat{\Psi_1(X)} \sim_{\widehat{X}} \widehat{\Psi_2(X)}\right] \iff\\ \forall\widehat{\Psi_1(X)}\forall\widehat{\Psi_2(X)}\left\{\left[\widehat{\Psi_1(X)} \sim_X \widehat{\Psi_2(X)}\right] \iff \exists y_1\widehat{\mathbf{Fr}_1^{Hs}}(y_1, v)\exists y_2\widehat{\mathbf{Fr}_1^{Hs}}(y_2, v) \searrow\right.\\ \left[\left(\widehat{g}_{\overline{ZFC}}\left(\widehat{\Psi_1(X)}\right) = y_1\right) \wedge \left(\widehat{g}_{\overline{ZFC}}\left(\widehat{\Psi_2(X)}\right) = y_2\right) \wedge \left(\widehat{g}_{\overline{ZFC}}\left(\widehat{X}\right) = \nu\right)\right] \wedge\\ \left.\left[\forall X\left(X \in M_{st}^{ZFC}\right)\Psi_1(X) \iff \forall X\left(X \in M_{st}^{ZFC}\right)\Psi_2(X)\right]\right\}.\end{gathered} \quad (2.2.19)$$

(ii) A subset $\widehat{\Lambda}_X^{st}$ of $\widehat{\Gamma}_X^{st}$ such that $\widehat{\Psi_1(X)} \sim_{\widehat{X}} \widehat{\Psi_2(X)}$ holds for all $\widehat{\Psi_1(X)}$ and $\widehat{\Psi_2(X)}$ in $\widehat{\Lambda}_X^{st}$, and never for $\widehat{\Psi_1(X)}$ in $\widehat{\Lambda}_X^{st}$ and $\widehat{\Psi_2(X)}$ outside $\widehat{\Lambda}_X^{st}$, is an equivalence class of $\widehat{\Gamma}_X^{st}$.

(iii) For any $\widehat{\Psi(X)} \in \widehat{\Gamma}_X^{st}$ let $\left[\widehat{\Psi(X)}\right]_{st} \triangleq \left\{\widehat{\Phi(X)} \in \widehat{\Gamma}_X^{st} \middle| \widehat{\Psi(X)} \sim_{\widehat{X}} \widehat{\Phi(X)}\right\}$ denote the equivalence class to which $\widehat{\Psi(X)}$ belongs. All elements of $\widehat{\Gamma}_X^{st}$ equivalent to each other are also elements of the same equivalence class.

(iv) The set of all possible equivalence classes of $\widehat{\Gamma}_X^{st}$ by $\tilde{}_{\widehat{X}}$, denoted $\widehat{\Gamma}_X^{st}/\sim_{\widehat{X}}$

$$\widehat{\Gamma}_X^{st}/\sim_{\widehat{X}} \triangleq \left\{\left[\widehat{\Psi(X)}\right]_{st} \middle| \widehat{\Psi(X)} \in \widehat{\Gamma}_X^{st}\right\}. \qquad (2.2.20)$$

Definition 2.2.6.(i)We define now the equivalence relation $(\cdot \sim_\nu \cdot) \subset \widehat{\Gamma}_\nu^{st} \times \widehat{\Gamma}_\nu^{st}$ in the sense of the set theory ZFC_{st} by

$$y_1 \sim_\nu y_2 \iff \left[\widehat{\Psi_{y_1,\nu}(X)} \sim_{\widehat{X}} \widehat{\Psi_{y_2,\nu}(X)}\right] \qquad (2.2.21)$$

Note that from the axiom of separation it follows directly that the equivalence relation $(\cdot \sim_\nu \cdot)$ is a relation in the sense of the set theory ZFC_{st}.

(ii) A subset $\widehat{\Lambda}_\nu^{st}$ of $\widehat{\Gamma}_\nu^{st}$ such that $y_1 \sim_\nu y_2$ holds for all y_1 and y_1 in $\widehat{\Lambda}_\nu^{st}$,and never for y_1 in $\widehat{\Lambda}_\nu^{st}$ and y_2 outside $\widehat{\Lambda}_\nu^{st}$, is an equivalence class of $\widehat{\Gamma}_\nu^{st}$.

(iii) For any $y \in \widehat{\Gamma}_\nu^{st}$ let $[y]_{st} \triangleq \left\{z \in \widehat{\Gamma}_\nu^{st} | y \sim_\nu z\right\}$ denote the equivalence class to which y belongs. All elements of $\widehat{\Gamma}_\nu^{st}$ equivalent to each other are also elements of the same equivalence class.

(iv)The set of all possible equivalence classes of $\widehat{\Gamma}_\nu^{st}$ by $\tilde{}_\nu$, denoted $\widehat{\Gamma}_\nu^{st}/\sim_\nu$

$$\widehat{\Gamma}_\nu^{st}/\sim_\nu \triangleq \left\{[y]_{st} | y \in \widehat{\Gamma}_\nu^{st}\right\}. \qquad (2.2.22)$$

Remark 2.2.6. Note that from the axiom of separation it follows directly that $\widehat{\Gamma}_\nu^{st}/\sim_\nu$ is a set in the sense of the set theory ZFC_{st}.

Definition 2.2.7. Let $\Im_{st}$ be the countable set of the all sets definable by 1-place open wff of the set theory ZFC_{st}, i.e. by using second order lenguage correspondng definition reads

$$\forall Y \left\{Y \in \Im_{st} \iff \exists \Psi(X) \left[\left([\Psi(X)]_{st} \in \Gamma_X^{st}/\sim_X\right) \wedge [\exists! X [\Psi(X) \wedge Y = X]]\right]\right\}. \qquad (2.2.23)$$

We rewrite now (2.2.23) by using first order lenguage of the set theory ZFC_{st} in the following equvalent form

$$\forall Y \left\{Y \in \Im_{st} \iff \exists \widehat{\Psi(X)} \left[\left(\left[\widehat{\Psi(X)}\right]_{st} \in \widehat{\Gamma}_X^{st}/\sim_{\widehat{X}}\right) \wedge [\exists! X [\Psi(X) \wedge Y = X]]\right]\right\}. \qquad (2.2.24)$$

Remark 2.2.7. Note that from the axiom of replacement it follows directly that $\Gamma_\nu^{st}/\sim_\nu$ is a set in the sense of the set theory ZFC_{st}.

Definition 2.2.8. We rewrite now (2.2.24) in the following equivalent form

$$\forall Y \left\{Y \in \Im_{st} \iff \exists \widehat{\Psi(X)} \left[\left(\left[\widehat{\Psi(X)}\right]_{st} \in \widehat{\Gamma}_X^{*st}/\sim_{\widehat{X}}\right) \wedge (Y = X)\right]\right\}, \qquad (2.2.25)$$

where the countable set $\Gamma_X^{*st}/\sim_X$ is defined by

$$\forall \widehat{\Psi(X)} \left\{ \left[\widehat{\Psi(X)} \right]_{st} \in \widehat{\Gamma}_X^{*st}/\sim_{\widehat{X}} \iff \left[\left(\left[\widehat{\Psi(X)} \right]_{st} \in \widehat{\Gamma}_X^{st} \Big/ \sim_X \right) \wedge \exists! X \Psi(X) \right] \right\} \quad (2.2.26)$$

Definition 2.2.9. Let $\Re_{st}$ be the countable set of the all sets such that

$$\forall X (X \in \Im_{st}) [X \in \Re_{st} \iff X \notin X]. \quad (2.2.27)$$

Remark 2.2.8. Note that $\Re_{st} \in \Im_{st}$ since $\Re_{st}$ is a set definable by 1-place open wff

$$\Psi(Z, \Im_{st}) \triangleq \forall X (X \in \Im_{st}) [X \in Z \iff X \notin X]. \quad (2.2.28)$$

From (2.2.27) and Remark 2.2.8 one obtains directly

$$\Re_{st} \in \Re_{st} \iff \Re_{st} \notin \Re_{st}. \quad (2.2.29)$$

But (2.2.29) immediately gives a contradiction

$$(\Re_{st} \in \Re_{st}) \wedge (\Re_{st} \notin \Re_{st}). \quad (2.2.30)$$

The contradiction (2.2.30) it is a true contradiction inside ZFC_{st} for the reason that the countable set $\Im_{st}$ is a set in the sense of the set theory ZFC_{st}.

Definition 2.2.10. Let $\Xi_{1,X}$ be a ZFC-set of the all first order 1-open wiff's of the set theory ZFC_{st},then we abbreviate $\Xi_{1,X} \triangleq \mathbf{Wff}_{1,X}[ZFC_{st}]$.

Theorem 2.2.1. Let ZFC_{st}^* be a theory $ZFC_{st}^* \triangleq ZFC + \exists M_{st}^{ZFC}$and $\mathbf{Wff}_{1,X}[ZFC_{st}] \in M_{st}^{ZFC}$.

Then set theory ZFC_{st}^* is inconsistent.

Proof. Immediately from (2.2.29).

Remark 2.2.9. In order to obtain a contradiction inside $\overline{ZFC}_2^{Hs}$**,** in more general case,i.e., without any refference to Assumption 2.2.1 we introduce the following definitions.

Definition 2.2.11. We define now countable set $\widehat{\Gamma}_\nu^{*st}/\sim_\nu$by the following formula

$$\forall y \left\{ [y]_{st} \in \widehat{\Gamma}_\nu^{*st}/\sim_\nu \iff \left([y]_{st} \in \widehat{\Gamma}_\nu^{st}/\sim_\nu \right) \wedge \widehat{\mathbf{Fr}}_{st}(y, v) \wedge [\exists! X \Psi_{y,\nu}(X)] \right\}. \quad (2.2.31)$$

Remark 2.2.10. Note that from the axiom of separation it follows directly that $\widehat{\Gamma}_\nu^{*st}/\sim_\nu$ is a set in the sense of the set theory ZFC_{st}.

Definition 2.2.12. We define now the countable set $\Im_{st}^*$ by the following formula

$$\forall Y\left\{Y\in\Im^*_{st}\iff\exists y\left[\left([y]_{st}\in\widehat{\Gamma}^{*st}_{\nu}/\sim_{\nu}\right)\wedge\left(\widehat{g}_{ZFC_{st}}(X)=\nu\right)\wedge Y=X\right]\right\}. \quad (2.2.32)$$

Note that from the axiom schema of replacement it follows directly that $\Im^*_{st}$ is a set in the sense of the set theory ZFC_{st}.

Definition 2.2.13. We define now the countable set $\Re^*_{st}$ by the following formula

$$\forall X\left(X\in\Im^*_{st}\right)\left[X\in\Re^*_{st}\iff X\notin X\right]. \quad (2.2.33)$$

Note that from the axiom schema of separation it follows directly that $\Re^*_{st}$ is a set in the sense of the set theory ZFC_{st}.

Remark 2.2.11. Note that $\Re^*_{st}\in\Im^*_{st}$ since $\Re^*_{st}$ is a definable by the following formula

$$\Psi^*(Z)\triangleq\forall X\left(X\in\Im^*_{st}\right)\left[X\in Z\iff X\notin X\right]. \quad (2.2.34)$$

Theorem 2.2.2. Set theory ZFC_{st} is inconsistent.

Proof. From (2.2.34) and Remark 2.2.11 we obtain

$$\Re^*_{st}\in\Re^*_{st}\iff\Re^*_{st}\notin\Re^*_{st}\ . \quad (2.2.35)$$

From (2.2.34) immediately one obtains a contradiction $\left(\Re^*_{st}\in\Re^*_{st}\right)\wedge\left(\Re^*_{st}\notin\Re^*_{st}\right)$.

2.3 Derivation of the Inconsistent Definable Set in ZFC_{Nst}

Definition 2.3.1. Let $\overline{PA}$ be a first order theory which contain usual postulates of Peano arithmetic [17] and recursive defining equations for every primitive recursive function as desired.So for any $(n+1)$-place function f defined by primitive recursion over any n-place base function g and $(n+2)$-place iteration function h there would be the defining equations:

(i) $f(0,y_1,...,y_n)=g(y_1,...,y_n)$,(ii) $f(x+1,y_1,...,y_n)=h(x,f(x,y_1,...,y_n),y_1,...,y_n)$.

Designation 2.3.1.(i) Let M^{ZFC}_{Nst} be a nonstandard model of ZFC and let $M^{\overline{PA}}_{st}$ be a standard model of $\overline{PA}$.We assume now that $M^{\overline{PA}}_{st}\subset M^{ZFC}_{Nst}$ and denote such nonstandard model of the set theory ZFC by $M^{ZFC}_{Nst}\left[\overline{PA}\right]$.(ii) Let ZFC_{Nst} be the theory

$$ZFC_{Nst}=ZFC+M^{ZFC}_{Nst}\left[\overline{PA}\right]. \quad (2.3.1)$$

Designation 2.3.2.(i) Let $g_{ZFC_{Nst}}(u)$ be a Gödel number of given an expression u of the set theory $ZFC_{Nst}\triangleq ZFC+\exists M^{ZFC}_{Nst}\left[\overline{PA}\right]$.

(ii) Let $\mathbf{Fr}_{Nst}(y,v)$ be the relation : y is the Gödel number of a wff of the set theory ZFC_{Nst} that contains free occurrences of the variable X with Gödel number v,see Remark 2.3.2.

(iii) Note that the relation $\mathbf{Fr}_{Nst}(y,v)$ is expressible in ZFC_{Nst} by a wff $\widehat{\mathbf{Fr}}_{Nst}(y,v)$

(iv) Note that for any $y,v\in\mathbb{N}$ by definition of the relation $\mathbf{Fr}_{Nst}(y,v)$ follows that

$$\widehat{\mathbf{Fr}}_{Nst}(y,v)\iff\exists!\Psi(X)\left[\left(g_{ZFC_{Nst}}(\Psi(X))=y\right)\wedge\left(g_{ZFC_{Nst}}(X)=\nu\right)\right], \quad (2.3.2)$$

where $\Psi(X)$ is a unique wff of ZFC_{st} which contains free occurrences of the variable X with Gödel number v.We denote a unique wff $\Psi(X)$ defined by using equivalence (2.3.2)

by symbol $\Psi_{y,\nu}(X)$,i.e.

$$\widehat{\mathbf{Fr}}_{Nst}(y,v) \iff \exists! \Psi_{y,\nu}(X)\left[\left(g_{ZFC_{Nst}}(\Psi_{y,\nu}(X))=y\right)\wedge\left(g_{ZFC_{Nst}}(X)=\nu\right)\right], \qquad (2.3.3)$$

Definition 2.3.2. Let ${}^{Nst}_{X}$ be a set of the all 1-place open wff's $\Psi(X)$ (with all bound variables restricted to nonstandard model M^{ZFC}_{Nst} of the set theory ZFC) that contains free occurrences of the individual variable X with Godel number v and we will be denoted these wff's by $\Psi(X), \Psi_X, \Psi_{y,\nu}(X), y, \nu \in \mathbb{N}$. We define now a set $\Gamma^{Nst}_X \subsetneqq {}^{Nst}_{X}$ by the following second order formula

$$\forall \Psi(X)\left[\Psi(X) \in \Gamma^{Nst}_X \iff \left(\exists! X\left(X \in M^{ZFC}_{Nst}\right)\Psi(X)\right)\wedge\left(\Psi(X) \in {}^{st}_{X}\right)\right]. \qquad (2.3.4)$$

or in the following equivalent form

$$\begin{array}{l}\forall \Psi(X)\Big[\Psi(X) \in \Gamma^{Nst}_X \iff \left(\exists y \in M^{\overline{PA}}_{st}\right)\mathbf{Fr}^{\widehat{ZFC_{Nst}}}(y,v) \searrow \\ \quad\left[\left(g_{ZFC_{Nst}}(\Psi(X))=y\right)\wedge\left(g_{ZFC_{Nst}}(X)=\nu\right)\right] \\ \quad\wedge\left(\exists! X\left(X \subset M^{ZFC}_{Nst}\right)\Psi(X)\right)\wedge\left(\Psi(X) \in {}^{Nst}_{X}\right)\Big],\end{array} \qquad (2.3.4.a)$$

Note that there exist a set Γ^{Nst}_X by the second order separaton axiom of ZFC^{Hs}_2.

Assumption 2.3.1. We assume now for simplicity but without loss of generality that

$${}^{Nst}_{X} \in M_{Nst} \qquad (2.2.1.b)$$

and therefore by definition of model M^{ZFC}_{Nst} one obtains $\Gamma^{Nst}_X \in M^{ZFC}_{Nst}$.

Definition 2.3.3. Let $\Xi_{1,X}$ be a ZFC-set of the all 1-open wiff's of the set theory ZFC_{Nst}, then we abbreviate $\Xi_{1,X} \triangleq \mathbf{Wff}_{1,X}[ZFC_{Nst}]$.

Definition 2.3.4. Let $\widehat{\ }^{Nst}_{X}$ be a set $\widehat{\ }^{Nst}_{X} = \wp^{-1}_{\Xi_1}\left({}^{Nst}_{X}\right)$,and $\widehat{\Psi}(X) \triangleq \widehat{\Psi}_X = \wp^{-1}_{\Xi_1}(\Psi(X))$ where one-one function $\wp^{-1}_{\Xi_1}$ defined in sec.2.1,see Remark 2.1.5 and Eq.(2.1.22).

Remark 2.3.1. (i)We define now a set $\widehat{\Gamma}^{Nst}_X = \wp^{-1}_{\Xi_1}\left(\Gamma^{Nst}_X\right), \widehat{\Gamma}^{Nst}_X \subsetneqq \widehat{\ }^{Nst}_{X}$ by the following first order formula with quantifiers over first order individual varables $\widehat{\Psi}_X$ and X :

$$\forall \widehat{\Psi}_X\left[\widehat{\Psi}_X \in \widehat{\Gamma}^{Nst}_X \iff \exists! X\left(X \in M^{ZFC}_{Nst}\right)\Psi_X \wedge\left(\widehat{\Psi}_X \in \widehat{\ }^{Nst}_{X}\right)\right], \qquad (2.3.5)$$

(where we write Ψ_X instead $\Psi(X)$) or in the following equivalent form

$$\begin{array}{l}\forall \widehat{\Psi}_X\Big[\widehat{\Psi}_X \in \widehat{\Gamma}^{st}_X \iff \left(\exists y \in M^{\overline{PA}}_{st}\right)\mathbf{Fr}^{\widehat{ZFC_{Nst}}}(y,v) \searrow \\ \quad\left[\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{\Psi}_X\right)=y\right)\wedge\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{X}\right)=\nu\right)\right] \\ \quad\wedge\left(\exists! X\left(X \in M^{ZFC}_{Nst}\right)\Psi_X\right)\wedge\left(\widehat{\Psi}_X \in \widehat{\ }^{Nst}_{X}\right)\Big],\end{array} \qquad (2.3.5.a)$$

where one-one function $\widehat{g}$ where one-one function $\widehat{g}_{ZFC_{Nst}}\left(\widehat{\Psi}_X\right) = y$ is defined below by Eq.(2.3.), see Remark 2.3.. Note that there exist a set $\widehat{\Gamma}^{Nst}_X$ by the (first order) separaton axiom of ZFC.

(ii) Note that second order definition (2.3.4) and first order definition (2.3.5) are equivalent.

We abbreviate now:

(a) $\mathbf{IC}^{ZFC_{Nst}}(x)$: x is the Godel number of an individual constant of ZFC_{Nst},

(b) $\mathbf{FL}^{ZFC_{Nst}}(x)$: x is the Godel number of a function letter of ZFC_{Nst},

(c) $\mathbf{PL}^{ZFC_{Nst}}(x)$: x is the Godel number of a predicate letter of ZFC_{Nst}.

Remark 2.3.2. Note that in order to obtain by using only first order logic the formal definitions of the first order predicates $\mathbf{EVbl}^{ZFC_{Nst}}(x), \mathbf{EIC}^{ZFC_{Nst}}(x), \mathbf{EFL}^{ZFC_{Nst}}(x), ..., \quad \mathbf{Fr}^{ZFC_{Nst}}(y, v)$ from the first order predicates $\mathbf{IC}^{ZFC_{Nst}}(x), \mathbf{FL}^{ZFC_{Nst}}(x), \mathbf{PL}^{ZFC_{Nst}}(x)$ one needs the following first order predicates:

(i) $\mathbf{EVbl}^{ZFC_{Nst}}(x, \widehat{\alpha})$: x is the Godel number of the 1-tuple $\widehat{\alpha} \in \widehat{\Delta}_1, \widehat{\Delta}_1 = \wp_{\Delta_1}^{-1}(\Delta_1)$ correspondinging to the individual variable $\alpha \in \Delta_1$, $\alpha = \wp_{\Delta_1}(\widehat{\alpha})$,see Remark 2.1.2 (i).

(ii) $\mathbf{EIC}^{ZFC_{Nst}}(x, \widehat{\beta})$: x is the Godel number of the 1-tuple $\widehat{\beta} \in \widehat{\Delta}_2, \widehat{\Delta}_2 = \wp_{\Delta_2}^{-1}(\Delta_2)$ correspondinging to the individual constant $\beta \in \Delta_2$, $\beta = \wp_{\Delta_2}\left(\widehat{\beta}\right)$,see Remark 2.1.2 (ii).

(iii) $\mathbf{EFL}^{ZFC_{Nst}}(x, \widehat{\gamma})$: x is the Godel number of the 1-tuple $\widehat{\gamma} \in \widehat{\Delta}_3, \widehat{\Delta}_3 = \wp_{\Delta_3}^{-1}(\Delta_3)$ correspondinging to the function letter $\gamma \in \Delta_3, \gamma = \wp_{\Delta_3}(\widehat{\gamma})$,see Remark 2.1.2 (iii).

(iv) $\mathbf{EPL}^{ZFC_{Nst}}(x, \widehat{\delta})$: x is the Godel number of the 1-tuple $\widehat{\delta} \in \widehat{\Delta}_4, \widehat{\Delta}_4 = \wp_{\Delta_4}^{-1}(\Delta_4)$ correspondinging to the predicate letter $\delta \in \Delta_4, \delta = \wp_{\Delta_4}\left(\widehat{\delta}\right)$,see Remark 2.1.2 (iv).

(v) $\mathbf{Gd}^{ZFC_{Nst}}(x, \widehat{\zeta})$: x is the Godel number of the element $\widehat{\zeta} \in \widehat{\mathbf{S}^{\#}}$ of the set $\widehat{\mathbf{S}^{\#}} = \cup_{n \in \mathbb{N}} \widehat{\mathbf{S}}^n$, $\widehat{\mathbf{S}} = \widehat{\Delta}_1 \cup \widehat{\Delta}_2 \cup \widehat{\Delta}_3 \cup \widehat{\Delta}_4$, correspondinging to the expression $\zeta \in \mathbf{S}^{\#}$, of ZFC_{Nst},where

$\mathbf{S}^{\#} = \cup_{n \in \mathbb{N}} \mathbf{S}^n, \mathbf{S} = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$,see Definition 2.1.6.

(vi) $\mathbf{Trm}^{ZFC_{Nst}}(x, \widehat{\tau})$: x is the Godel number of the element $\widehat{\tau} \in \widehat{\Upsilon}_1$ of the set $\widehat{\Upsilon} = \wp_{\tau}^{-1}(\Upsilon_1)$,correspondinging to the term $\tau = \wp_{\tau}(\widehat{\tau})$ of ZFC_{Nst}.

(vii) $\mathbf{Atfml}^{ZFC_{Nst}}(x, \widehat{\pi})$: x is the Godel number of the element $\widehat{\pi} \in \widehat{\Sigma}_1$ of the set $\widehat{\Sigma}_1 = \wp_{\pi}^{-1}(\Sigma_1)$,correspondinging to atomic wff $\pi = \wp_{\pi}(\widehat{\pi})$ of ZFC_{Nst}.

(viii) $\mathbf{Fml}^{ZFC_{Nst}}(y, \widehat{\varphi})$: y is the Godel number of the element $\widehat{\varphi} \in \widehat{\Xi}_1$ of the set $\widehat{\Xi}_1 = \wp_{\Xi_1}^{-1}(\Xi_1)$,correspondinging to the wff formula $\varphi = \wp_{\Xi_1}(\widehat{\varphi})$ of ZFC_{Nst}.

(ix) $\mathbf{Fr}^{ZFC_{Nst}}(y, v, \widehat{\varpi})$: y is the Godel number of the element $\widehat{\varpi} \in \widehat{\Xi}_{1,\nu}$ of the set $\widehat{\Xi}_{1,\nu} = \wp_{\Xi_{1,\nu}}^{-1}(\Xi_{1,\nu})$,correspondinging to the wff formula or term $\varpi = \wp_{\Xi_{1,\nu}}(\widehat{\varpi})$

of ZFC_{Nst} that contains free occurrences of the variable with Godel number v.

Thus finally we obtain:

$$\begin{gathered}
\mathbf{EVbl}^{ZFC_{Nst}}(x) \iff \exists\widehat{\alpha}\left(\widehat{\alpha}\in\widehat{\Delta}_1\right)\mathbf{EVbl}^{ZFC_{Nst}}(x,\widehat{\alpha}) \iff (\exists z\in M_{st}^{\overline{PA}})_{z<x}(1\leq z\wedge x=2^{13+8z}),\\
\mathbf{EIC}^{ZFC_{Nst}}(x) \iff \exists\widehat{\beta}\left(\widehat{\beta}\in\widehat{\Delta}_2\right)\mathbf{EIC}^{ZFC_{Nst}}(x,\widehat{\beta}) \iff (\exists y\in M_{st}^{\overline{PA}})_{y<x}(\mathbf{IC}^{ZFC_{Nst}}(y)\wedge x=2^y),\\
\mathbf{EFL}^{ZFC_{Nst}}(x) \iff \exists\widehat{\gamma}\left(\widehat{\gamma}\in\widehat{\Delta}_3\right)\mathbf{EFL}^{ZFC_{Nst}}(x,\widehat{\gamma}) \iff (\exists y\in M_{st}^{\overline{PA}})_{y<x}(\mathbf{FL}^{ZFC_{Nst}}(y)\wedge x=2^y),\\
\mathbf{EPL}^{ZFC_{Nst}}(x) \iff \exists\widehat{\delta}\left(\widehat{\delta}\in\widehat{\Delta}_4\right)\mathbf{EPL}^{ZFC_{Nst}}(x,\widehat{\delta}) \iff (\exists y\in M_{st}^{\overline{PA}})_{y<x}(\mathbf{PL}^{ZFC_{Nst}}(y)\wedge x=2^y),
\end{gathered}$$

$$\begin{gathered}
\mathbf{Gd}^{ZFC_{Nst}}(x) \iff \exists\widehat{\zeta}\exists n\left(n\in M_{st}^{\overline{PA}}\wedge\widehat{\zeta}\in\widehat{\mathbf{S}}^n\right)\mathbf{Gd}^{ZFC_{Nst}}(x,\widehat{\zeta}) \iff\\
\mathbf{EVbl}^{ZFC_{Nst}}(x)\vee\mathbf{EIC}^{ZFC_{Nst}}(x)\vee\mathbf{EFL}^{ZFC_{Nst}}(x)\vee\mathbf{EPL}^{ZFC_{Nst}}(x)\vee\\
x=2^7\vee x=2^9\vee x=2^{11}\vee x=2^{13}\vee\\
(\exists u\in M_{st}^{\overline{PA}})_{u<x}(\exists v\in M_{st}^{\overline{PA}})_{v<x}(x=u*v\wedge\mathbf{Gd}^{ZFC_{Nst}}(u)\wedge\mathbf{Gd}^{ZFC_{Nst}}(v)).
\end{gathered}$$

$$\begin{gathered}
\mathbf{Trm}^{ZFC_{Nst}}(x) \iff \exists\widehat{\tau}\left(\widehat{\tau}\in\widehat{\Upsilon}\right)\mathbf{Trm}^{ZFC_{Nst}}(x,\widehat{\tau}) \iff\\
\mathbf{EVbl}^{ZFC_{Nst}}(x)\vee\mathbf{EIC}^{ZFC_{Nst}}(x)\vee(\exists y\in M_{st}^{\overline{PA}})_{y<(p_x!)^x}[x=(y)_{lh(y)\dot{-}1}\wedge\\
lh(y)=\mathbf{Arg}_{\mathbf{T}}((x)_0)+1\wedge\mathbf{FL}^{ZFC_{Nst}}(((y)_0)_0)\wedge((y)_0)_1=3\wedge\\
lh((y)_0)=2\wedge(\forall u)_{u<lh(y)\dot{-}1}(\exists v)_{v<x}((y)_{u+1}=(y)_u*v*2^7\wedge\mathbf{Trm}^{ZFC_{Nst}}(v))\wedge\\
(\exists v)_{v<x}((y)_{lh(y)\dot{-}1}=(y)_{lh(y)\dot{-}2}*v*2^5\wedge\mathbf{Trm}^{ZFC_{Nst}}(v))].
\end{gathered}\tag{2.3.6}$$

$$\begin{gathered}
\mathbf{Atfml}^{ZFC_{Nst}}(x) \iff \exists\widehat{\pi}\left(\widehat{\pi}\in\widehat{\Sigma}\right)\mathbf{Atfml}^{ZFC_{Nst}}(x,\widehat{\pi}) \iff\\
(\exists y)_{y<(p_x!)^x}[x=(y)_{lh(y)\dot{-}1}\wedge lh(y)=\mathbf{Arg}_P((x)_0)+1\wedge\\
\mathbf{PL}^{ZFC_{Nst}}(((y)_0)_0)\wedge((y)_0)_1=3\wedge lh((y)_0)=2\wedge\\
(\forall u)_{u<lh(y)\dot{-}2}(\exists v)_{v<x}((y)_{u+1}=(y)_u*v*2^7\wedge\mathbf{Trm}^{ZFC_{Nst}}(v))\wedge\\
(\exists v)_{v<x}((y)_{lh(y)\dot{-}1}=(y)_{u<lh(y)\dot{-}2}*v*2^5\wedge\mathbf{Trm}^{ZFC_{Nst}}(v))].
\end{gathered}$$

$$\begin{gathered}
\mathbf{Fml}^{ZFC_{Nst}}(y) \iff \exists\widehat{\varphi}\left(\widehat{\varphi}\in\widehat{\Xi}\right)\mathbf{Fml}^{ZFC_{Nst}}(y,\widehat{\varphi}) \iff\\
\mathbf{Atfml}^{ZFC_{Nst}}(y)\vee(\exists z\in M_{st}^{\overline{PA}})_{z<y}[(\mathbf{Fml}^{ZFC_{Nst}}(z)\wedge=2^3*2^9*z*2^5)\vee\\
(\mathbf{Fml}^{ZFC_{Nst}}((z)_0)\wedge\mathbf{Fml}^{ZFC_{Nst}}((z)_1\wedge y=2^3*(z)_0*2^{11}*(z)_1*2^5)\vee\\
(\mathbf{Fml}^{ZFC_{Nst}}((z)_0)\wedge\\
\wedge\mathbf{EVbl}^{ZFC_{Nst}}((z)_1)\wedge y=2^3*2^3*2^{13}*((z)_1*2^5*(z)_0*2^5)].
\end{gathered}$$

$$\begin{gathered}
\mathbf{Fr}^{ZFC_{Nst}}(y,v) \iff \exists\varpi\left[(\varpi\in\Xi_{1,\nu})\vee\left(\varpi\in\widehat{\Upsilon}_1\right)\right]\mathbf{Fr}^{ZFC_{Nst}}(y,v,\varpi) \iff\\
(\mathbf{Fml}^{ZFC_{Nst}}(y)\vee\mathbf{Trm}^{ZFC_{Nst}}(y))\wedge\mathbf{EVbl}^{ZFC_{Nst}}(2^v)\wedge\neg\mathbf{Subst}^{ZFC_{Nst}}(y,y,2^{13+8v},v).
\end{gathered}$$

Remark 2.3.3. Let $g_{ZFC_{Nst}}(u)$ be a Gödel number of given an expression $u\in\Omega$ of the language of the set theory $ZFC_{Nst}\triangleq ZFC+\exists M_{Nst}^{ZFC}$. Recall that $\wp_{\Omega_1}^{-1}(\Omega_1)=\widehat{\Omega}_1$ see Definition 2.1.6. We set now

$$\widehat{g}_{ZFC_{Nst}}(u)=g_{ZFC_{Nst}}(u) \tag{2.3.7}$$

(ii) Let $\mathbf{Fr}^{ZFC_{Nst}}(y,v)$ be the relation : y is the Gödel number of a wff of the set theoryZFC_{Nst} that contains free occurrences of the variable X with Gödel number v,see Eq.(2.2.3)

(iii) Note that the relation $\mathbf{Fr}^{ZFC_{Nst}}(y,v)$ is expressible in ZFC_{Nst} by a wff $\widehat{\mathbf{Fr}^{ZFC_{Nst}}}(y,v)$

(iv) Note that for any $y,v\in\mathbb{N}$ by definition of the relation $\mathbf{Fr}^{ZFC_{Nst}}(y,v)$ follows that

$$\begin{gathered}
\widehat{\mathbf{Fr}^{ZFC_{Nst}}}(y,v) \iff \exists!\Psi_X\left[(g_{ZFC_{st}}(\Psi_X)=y)\wedge(g_{ZFC_{Nst}}(X)=\nu)\right] \iff\\
\exists!\widehat{\Psi}_X\left[\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{\Psi}_X\right)=y\right)\wedge\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{X}\right)=\nu\right)\right],
\end{gathered}\tag{2.3.8}$$

where $\Psi_X=\Psi(X)$ is a unique wff of ZFC_{Nst} which contains free occurrences of the variable X with Gödel number v.We denote such unique wff $\Psi(X)$ defined by equivalence (2.3.8) by symbol $\Psi_{y,\nu}(X)$,i.e.

$$\begin{gathered}
\widehat{\mathbf{Fr}^{ZFC_{Nst}}}(y,v) \iff \exists!\Psi_{y,\nu}(X)\left[(g_{ZFC_{Nst}}(\Psi_{y,\nu}(X))=y)\wedge(g_{ZFC_{Nst}}(X)=\nu)\right] \iff\\
\exists!\widehat{\Psi_{y,\nu}(X)}\left[\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{\Psi_{y,\nu}(X)}\right)=y\right)\wedge\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{X}\right)=\nu\right)\right].
\end{gathered}\tag{2.3.9}$$

where $\Psi_X=\Psi(X)$ is a unique wff of ZFC_{Nst} which contains free occurrences of the variable X with Gödel number v.We denote such unique wff $\Psi(X)$ defined by equivalence (2.3.9) by symbol

$\Psi_{y,\nu}(X)$,i.e.

$$\begin{gathered}\mathbf{Fr}^{\widehat{ZFC}_{Nst}}(y,v) \iff \exists!\Psi_{y,\nu}(X)\left[\left(g_{ZFC_{Nst}}(\Psi_{y,\nu}(X))=y\right)\wedge\left(g_{ZFC_{Nst}}(X)=\nu\right)\right] \iff \\ \exists!\Psi_{\widehat{y,\nu}(X)}\left[\left(\widehat{g}_{ZFC_{Nst}}\left(\Psi_{\widehat{y,\nu}(X)}\right)=y\right)\wedge\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{X}\right)=\nu\right)\right].\end{gathered} \tag{2.3.10}$$

Remark 2.3.4. Note that a function $g_{ZFC_{Nst}}(\Psi_{y,\nu}(X)) = y$ is expressible in set theory ZFC_{Nst} by a wff of the set theory ZFC_{Nst} that contains free occurrences of the variable $y \in \mathbb{N}$.

Note that any formula $\Psi_{y,\nu}(X)$ is given by an expression $u_0u_1..u_j...u_r$,i.e. $\Psi_{y,\nu}(X) =: u_0u_1..u_j...u_r$,
where each u_j is a symbol of ZFC_{Nst}. We introduce now a functions $[\Psi_{y,\nu}(X);j] : \Psi_{y,\nu}(X) \to u_j, j = 0,1,...$,i.e. $[\Psi_{y,\nu}(X);j] =: u_j$ and revrite expression $u_0u_1..u_j...u_r$ in the following equivalent form

$$[\Psi_{y,\nu}(X);0]\,[\Psi_{y,\nu}(X);1]\,...\,[\Psi_{y,\nu}(X);j]\,...\,[\Psi_{y,\nu}(X);r]. \tag{2.3.11}$$

By definitions we obtain that

$$\begin{gathered} g_{ZFC_{Nst}}(\Psi_{y,\nu}(X)) = y \\ \iff y = 2^{g\left([\Psi_{y,\nu}(X);0]\right)}\cdot 3^{g\left([\Psi_{y,\nu}(X);1]\right)}\cdot...\cdot p_j^{g\left([\Psi_{y,\nu}(X);j]\right)}\cdot...\cdot p_r^{g\left([\Psi_{y,\nu}(X);r]\right)}. \end{gathered} \tag{2.3.12}$$

and

$$\begin{gathered} \widehat{g}_{ZFC_{Nst}}\left(\widehat{\Psi_{y,\nu}(X)}\right) = y \\ \iff y = 2^{\widehat{g}\left(\left[\widehat{\Psi_{y,\nu}(X)};0\right]\right)}\cdot 3^{\widehat{g}\left(\left[\widehat{\Psi_{y,\nu}(X)};1\right]\right)}\cdot...\cdot p_j^{\widehat{g}\left(\left[\widehat{\Psi_{y,\nu}(X)};j\right]\right)}\cdot...\cdot p_r^{\widehat{g}\left(\left[\widehat{\Psi_{y,\nu}(X)};r\right]\right)}. \end{gathered} \tag{2.3.13}$$

correspondingly.Let us denote by $(y)_j$ the exponent $g([\Psi_{y,\nu}(X);j])$ in this factorization

$$y = 2^{g\left([\Psi_{y,\nu}(X);0]\right)}\cdot 3^{g\left([\Psi_{y,\nu}(X);1]\right)}\cdot...\cdot p_j^{g\left([\Psi_{y,\nu}(X);j]\right)}\cdot...\cdot p_r^{g\left([\Psi_{y,\nu}(X);r]\right)}. \tag{2.3.14}$$

If $y = 1, (y)_j = 1$ for all j. If $x = 0$, we arbitrarily let $(y)_j = 0$ for all j. Then the functions $(y)_j, j = 0,1,...$ is primitive recursive, since $(y)_j = \mu_{z<y}\left(p_j^z|y \wedge \neg p_j^{z+1}|y\right)$,is primitive recursive.

Thus the function $(y)_j$ is expressible in set theory ZFC_{Nst} by formula denoted below by $\lambda_j(y, g([\Psi_{y,\nu}(X);j]))$.

For $y > 0$, let $lh(y)$ be the number of non-zero exponents in the factorization of y into powers of primes, or, equivalently, the number of distinct primes that divide y. Let $lh(0) = 0$, then $lh(y)$ is primitive recursive.

Thus (i) function $g_{ZFC_{Nst}}(\Psi_{y,\nu}(X)) = y$ is expressible in set theory ZFC_{Nst} by the following formula $\widetilde{\Xi}(\Psi_{y,\nu}(X), y)$

$$\widetilde{\Xi}(\Psi_{y,\nu}(X), y) \iff {}_{j\leq lh(y)}\lambda_j(y, g([\Psi_{y,\nu}(X);j])). \tag{2.3.15}$$

(ii) function $g_{ZFC_{Nst}}(\Psi_{y,\nu}(X)) = y$ is expressible in set theory ZFC_{Nst} by the following formula $\widetilde{\Xi}(\Psi_{y,\nu}(X), y)$

$$\widetilde{\Xi}\left(\widehat{\Psi_{y,\nu}(X)}, y\right) \iff {}_{j\leq lh(y)}\lambda_j\left(y, \widehat{g}\left(\left[\widehat{\Psi_{y,\nu}(X)};j\right]\right)\right). \tag{2.3.16}$$

Definition 2.3.5. Let $g_{ZFC_{Nst}}(X) = \nu$.Let Γ_ν^{Nst} be a set of the all Gödel numbers of the 1-place open wff's of the set theory ZFC_{Nst} that contains free occurrences of the variable X with Gödel number v,i.e.

$$\Gamma_\nu^{Nst} = \left\{y \in \mathbb{N} | \langle y, \nu\rangle \in \mathbf{Fr}^{ZFC_{Nst}}(y,v)\right\}, \tag{2.3.17}$$

or in the following equivalent form:

$$\forall y\,(y\in\mathbb{N})\left[y\in\Gamma_\nu^{Nst}\iff(y\in\mathbb{N})\wedge\mathbf{Fr}^{\widehat{ZFC_{Nst}}}(y,v)\right].\tag{2.3.18}$$

Remark 2.3.5. Note that from the axiom of separation it follows directly that Γ_ν^{Nst} is a set in the sense of the set theory ZFC_{Nst}.

We define now a set $\Gamma_\nu^{Nst}\subsetneqq_\nu^{Nst}$ by the following first order formula

$$\begin{gathered}\forall y\,(y\in\mathbb{N})\left\{y\in\Gamma_\nu^{Nst}\iff\left(y\in_\nu^{Nst}\right)\right.\\ \left.\wedge\exists\widehat{\Psi_X}\left[\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{\Psi_X}\right)=y\right)\wedge\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{X}\right)=\nu\right)\left(\exists!X\left(X\in M_{Nst}^{ZFC}\right)\Psi_X\right)\right]\right\}\end{gathered}\tag{2.3.19}$$

where $\Psi_X=\Psi(X)$ is a unique wff of ZFC_{Nst} which contains free occurrences of the variable X with Gödel number v. or in the following equivalent form

$$\begin{gathered}\forall y\,(y\in\mathbb{N})\left[y\in\Gamma_\nu^{Nst}\iff\left(y\in_\nu^{Nst}\right)\wedge\mathbf{Fr}^{\widehat{ZFC_{Nst}}}(y,v)\searrow\right.\\ \left.\wedge\exists\widehat{\Psi_X}\left[\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{\Psi_X}\right)=y\right)\wedge\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{X}\right)=\nu\right)\right]\wedge\left(\exists!X\left(X\in M_{Nst}^{ZFC}\right)\Psi_X\right)\right],\end{gathered}\tag{2.3.20}$$

Remark 2.3.6. Note that from the axiom of separation it follows directly that Γ_ν^{Nst} is a set in the sense of the set theory ZFC_{Nst}.

Definition 2.3.6. Let $\Psi_1=\Psi_1(X)$ and $\Psi_2=\Psi_2(X)$ be 1-place open wff's of the set theory ZFC_{Nst}.

(i) We define now the equivalence relation $\left(\cdot\sim_{\widehat{X}}\cdot\right)\subset\widehat{\Gamma}_X^{Nst}\times\widehat{\Gamma}_X^{Nst}$ by

$$\begin{gathered}\widehat{\Psi_1(X)}\sim_{\widehat{X}}\widehat{\Psi_2(X)}\iff\\ \Psi_1(X)\sim_X\Psi_2(X)\iff\left(\forall X\left(X\in M_{Nst}^{ZFC}\right)\left[\Psi_1(X)\iff\Psi_2(X)\right]\right)\end{gathered}\tag{2.3.21}$$

or more precisely

$$\begin{gathered}\forall\widehat{\Psi}_1\forall\widehat{\Psi}_2\left(\widehat{\Psi}_1\sim_{\widehat{X}}\widehat{\Psi}_2\right)\iff\forall\widehat{\Psi_1(X)}\forall\widehat{\Psi_2(X)}\left\{\left[\widehat{\Psi_1(X)}\sim_{\widehat{X}}\widehat{\Psi_2(X)}\right]\right.\\ \left.\iff\forall\widehat{\Psi_1(X)}\forall\widehat{\Psi_2(X)}\left[\forall X\left(X\in M_{Nst}^{ZFC}\right)\left[\Psi_1(X)\iff\Psi_2(X)\right]\right]\right\}\iff\\ \forall\widehat{\Psi_1(X)}\forall\widehat{\Psi_2(X)}\left\{\left[\widehat{\Psi_1(X)}\sim_{\widehat{X}}\widehat{\Psi_2(X)}\right]\iff\right.\\ \left.\forall\widehat{\Psi_1(X)}\forall\widehat{\Psi_2(X)}\left[\forall X\left(X\in M_{Nst}^{ZFC}\right)\Psi_1(X)\iff\forall X\left(X\in M_{Nst}^{ZFC}\right)\Psi_2(X)\right]\right\}.\end{gathered}\tag{2.3.22}$$

or in the following equivalent form

$$\begin{gathered}\forall\widehat{\Psi}_1\forall\widehat{\Psi}_2\left(\widehat{\Psi}_1\sim_{\widehat{X}}\widehat{\Psi}_2\right)\iff\forall\widehat{\Psi_1(X)}\forall\widehat{\Psi_2(X)}\left[\widehat{\Psi_1(X)}\sim_{\widehat{X}}\widehat{\Psi_2(X)}\right]\iff\\ \forall\widehat{\Psi_1(X)}\forall\widehat{\Psi_2(X)}\left\{\left[\widehat{\Psi_1(X)}\sim_{\widehat{X}}\widehat{\Psi_2(X)}\right]\iff\right.\\ \left(\exists y_1\in M_{st}^{PA}\right)\mathbf{Fr}^{\widehat{ZFC_{Nst}}}(y_1,v)\left(\exists y_2\in M_{st}^{PA}\right)\mathbf{Fr}^{\widehat{ZFC_{Nst}}}(y_2,v)\searrow\\ \left[\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{\Psi_1(X)}\right)=y_1\right)\wedge\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{\Psi_2(X)}\right)=y_2\right)\wedge\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{X}\right)=\nu\right)\right]\wedge\\ \left.\left[\forall X\left(X\in M_{Nst}^{ZFC}\right)\Psi_1(X)\iff\forall X\left(X\in M_{Nst}^{ZFC}\right)\Psi_2(X)\right]\right\}.\end{gathered}\tag{2.3.23}$$

or in the following equivalent form

$$\begin{gathered}\forall\widehat{\Psi}_1\forall\widehat{\Psi}_2\left(\widehat{\Psi}_1\sim_{\widehat{X}}\widehat{\Psi}_2\right)\iff\forall\widehat{\Psi_1(X)}\forall\widehat{\Psi_2(X)}\left[\widehat{\Psi_1(X)}\sim_{\widehat{X}}\widehat{\Psi_2(X)}\right]\iff\\ \forall\widehat{\Psi_1(X)}\forall\widehat{\Psi_2(X)}\left\{\left[\widehat{\Psi_1(X)}\sim_{\widehat{X}}\widehat{\Psi_2(X)}\right]\iff\right.\\ \exists y_1\mathbf{Fr}^{\widehat{ZFC_{Nst}}}(y_1,v)\exists y_2\mathbf{Fr}^{\widehat{ZFC_{Nst}}}(y_2,v)\searrow\\ \left[\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{\Psi_1(X)}\right)=y_1\right)\wedge\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{\Psi_2(X)}\right)=y_2\right)\wedge\left(\widehat{g}_{ZFC_{Nst}}\left(\widehat{X}\right)=\nu\right)\right]\wedge\\ \left.\left[\forall X\left(X\in M_{Nst}^{ZFC}\right)\Psi_1(X)\iff\forall X\left(X\in M_{Nst}^{ZFC}\right)\Psi_2(X)\right]\right\}.\end{gathered}\tag{2.3.24}$$

(ii) A subset $\widehat{\Lambda}_X^{Nst}$ of $\widehat{\Gamma}_X^{Nst}$ such that $\widehat{\Psi_1(X)}\sim_{\widehat{X}}\widehat{\Psi_2(X)}$ holds for all $\widehat{\Psi_1(X)}$ and $\widehat{\Psi_2(X)}$ in $\widehat{\Lambda}_X^{Nst}$, and never for $\widehat{\Psi_1(X)}$ in $\widehat{\Lambda}_X^{Nst}$ and $\widehat{\Psi_2(X)}$ outside $\widehat{\Lambda}_X^{Nst}$, is an equivalence class of $\widehat{\Gamma}_X^{Nst}$.

(iii) For any $\widehat{\Psi(X)} \in \widehat{\Gamma}_X^{Nst}$ let $\left[\widehat{\Psi(X)}\right]_{Nst} \triangleq \left\{\widehat{\Phi(X)} \in \widehat{\Gamma}_X^{Nst} \middle| \widehat{\Psi(X)} \sim_{\widehat{X}} \widehat{\Phi(X)}\right\}$ denote the equivalence class to which $\widehat{\Psi(X)}$ belongs. All elements of $\widehat{\Gamma}_X^{Nst}$ equivalent to each other are also elements of the same equivalence class.

(iv) The set of all possible equivalence classes of $\widehat{\Gamma}_X^{Nst}$ by $\tilde{}_{\widehat{X}}$, denoted by $\widehat{\Gamma}_X^{Nst}/\sim_{\widehat{X}}$:

$$\widehat{\Gamma}_X^{Nst}/\sim_{\widehat{X}} \triangleq \left\{\left[\widehat{\Psi(X)}\right]_{Nst} \middle| \widehat{\Psi(X)} \in \widehat{\Gamma}_X^{Nst}\right\}. \tag{2.3.25}$$

Definition 2.3.7.(i)We define now the equivalence relation $(\cdot \sim_\nu \cdot) \subset \widehat{\Gamma}_\nu^{Nst} \times \widehat{\Gamma}_\nu^{Nst}$ in the sense of the set theory ZFC_{Nst} by

$$y_1 \sim_\nu y_2 \iff \left[\widehat{\Psi_{y_1,\nu}(X)} \sim_{\widehat{X}} \widehat{\Psi_{y_2,\nu}(X)}\right] \tag{2.3.26}$$

Note that from the axiom of separation it follows directly that the equivalence relation $(\cdot \sim_\nu \cdot)$ is a relation in the sense of the set theory ZFC_{Nst}.

(ii) A subset $\widehat{\Lambda}_\nu^{Nst}$ of $\widehat{\Gamma}_\nu^{Nst}$ such that $y_1 \sim_\nu y_2$ holds for all y_1 and y_1 in $\widehat{\Lambda}_\nu^{Nst}$, and never for y_1 in $\widehat{\Lambda}_\nu^{Nst}$ and y_2 outside $\widehat{\Lambda}_\nu^{Nst}$, is an equivalence class of $\widehat{\Gamma}_\nu^{Nst}$.

(iii) For any $y \in \widehat{\Gamma}_\nu^{Nst}$ let $[y]_{Nst} \triangleq \left\{z \in \widehat{\Gamma}_\nu^{Nst} | y \sim_\nu z\right\}$ denote the equivalence class to which y belongs. All elements of $\widehat{\Gamma}_\nu^{Nst}$ equivalent to each other are also elements of the same equivalence class.

(iv)The set of all possible equivalence classes of $\widehat{\Gamma}_\nu^{Nst}$ by $\tilde{}_\nu$, denoted by $\widehat{\Gamma}_\nu^{Nst}/\sim_\nu$:

$$\widehat{\Gamma}_\nu^{Nst}/\sim_\nu \triangleq \left\{[y]_{Nst} | y \in \widehat{\Gamma}_\nu^{Nst}\right\}. \tag{2.3.27}$$

Remark 2.3.7. Note that from the axiom of separation it follows directly that $\widehat{\Gamma}_\nu^{Nst}/\sim_\nu$ is a set in the sense of the set theory ZFC_{Nst}.

Definition 2.3.8. Let $\Im_{Nst}$ be a countable set of the all sets definable by 1-place open wff of the set theory ZFC_{Nst}, i.e. by using second order lenguage correspondng definition reads

$$\begin{gathered}\forall Y \{Y \in \Im_{Nst} \iff \\ \exists \Psi(X) \left[\left([\Psi(X)]_{Nst} \in \Gamma_X^{st}/\sim_X\right) \wedge \left[\exists! X \left(X \in M_{Nst}^{ZFC}\right)[\Psi(X) \wedge Y = X]\right]\right]\}.\end{gathered} \tag{2.3.28}$$

We rewrite now (2.3.28) by using first order lenguage of the set theory ZFC_{Nst} in the following equvalent form

$$\begin{gathered}\forall \{Y \in \Im_{Nst} \iff \\ \exists \widehat{\Psi(X)} \left[\left(\left[\widehat{\Psi(X)}\right]_{Nst} \in \widehat{\Gamma}_X^{Nst}/\sim_{\widehat{X}}\right) \wedge \left[\exists! X \left(X \in M_{Nst}^{ZFC}\right)[\Psi(X) \wedge Y = X]\right]\right]\}.\end{gathered} \tag{2.3.29}$$

Remark 2.3.8. Note that from the axiom of replacement it follows directly that $\Gamma_\nu^{st}/\sim_\nu$ is a set in the sense of the set theory ZFC_{Nst}.

Definition 2.3.9. We rewrite now (2.3.29) in the following equivalent form

$$\forall Y \left\{Y \in \Im_{st} \iff \exists \widehat{\Psi(X)} \left[\left(\left[\widehat{\Psi(X)}\right]_{Nst} \in \widehat{\Gamma}_X^{*Nst}/\sim_{\widehat{X}}\right) \wedge (Y = X)\right]\right\}, \tag{2.3.30}$$

where the countable set $\Gamma_X^{*Nst}/\sim_{\widehat{X}}$ is defined by

$$\forall\widehat{\Psi(X)}\left\{\left[\widehat{\Psi(X)}\right]_{Nst} \in \widehat{\Gamma}_X^{*Nst}/\sim_{\widehat{X}} \iff \left[\left(\left[\widehat{\Psi(X)}\right]_{Nst} \in \widehat{\Gamma}_X^{st}/\sim_X\right) \wedge \exists! X\left(X \in M_{Nst}^{ZFC}\right)\Psi(X)\right]\right\}. \quad (2.3.31)$$

Definition 2.3.10. Let $\Re_{Nst}$ be the countable set of the all sets such that

$$\forall X\left(X \in \Im_{Nst}\right)\left[X \in \Re_{Nst} \iff X \notin X\right]. \quad (2.3.32)$$

Remark 2.3.9. Note that $\Re_{st} \in \Im_{Nst}$ since $\Re_{Nst}$ is a set definable by 1-place open wff

$$\Psi\left(Z, \Im_{Nst}\right) \triangleq \forall X\left(X \in \Im_{Nst}\right)\left[X \in Z \iff X \notin X\right]. \quad (2.3.33)$$

From (2.3.32) and Remark 2.3.9 one obtains directly

$$\Re_{Nst} \in \Re_{Nst} \iff \Re_{Nst} \notin \Re_{Nst}. \quad (2.3.34)$$

But (2.3.34) immediately gives a contradiction

$$\left(\Re_{Nst} \in \Re_{Nst}\right) \wedge \left(\Re_{Nst} \notin \Re_{Nst}\right). \quad (2.3.35)$$

The contradiction (2.3.35) it is a true contradiction inside ZFC_{Nst} for the reason that the countable set $\Im_{Nst}$ is a set in the sense of the set theory ZFC_{Nst}.

Definition 2.3.11. Let $\Xi_{1,X}$ be a ZFC-set of the all first order 1-open wiff's of the set theory ZFC_{Nst},then we abbreviate $\Xi_{1,X} \triangleq \mathbf{Wff}_{1,X}\left[ZFC_{Nst}\right]$.

Theorem 2.3.1. Let ZFC_{Nst}^* be a theory $ZFC_{Nst}^* \triangleq ZFC + \exists M_{Nst}^{ZFC}$ and $\mathbf{Wff}_{1,X}\left[ZFC_{Nst}\right] \in M_{Nst}^{ZFC}$.

Then set theory ZFC_{Nst}^* is inconsistent.

Proof. Immediately from (2.3.33).

Remark 2.3.10. In order to obtain a contradiction inside ZFC_{Nst} in more general case,i.e., without any refference to Assumption 2.3.1 we introduce the following definitions.

Definition 2.3.12. We define now countable set $\widehat{\Gamma}_\nu^{*Nst}/\sim_\nu$by the following formula

$$\forall y\left\{[y]_{st} \in \widehat{\Gamma}_\nu^{*Nst}/\sim_\nu \iff \left([y]_{Nst} \in \widehat{\Gamma}_\nu^{Nst}/\sim_\nu\right) \wedge \widehat{\mathbf{Fr}^{ZFC_{Nst}}}(y, v) \wedge \left[\exists! X\left(X \in M_{Nst}^{ZFC}\right)\Psi_{y,\nu}(X)\right]\right\}. \quad (2.3.36)$$

Remark 2.3.11. Note that from the axiom of separation it follows directly that $\widehat{\Gamma}_\nu^{*st}/\sim_\nu$is a set in the sense of the set theory ZFC_{Nst}.

Definition 2.3.13. We define now the countable set $\Im_{st}^*$ by the following formula

$$\forall Y\left\{Y \in \Im_{Nst}^* \iff \exists y\left[\left([y]_{Nst} \in \widehat{\Gamma}_\nu^{*Nst}/\sim_\nu\right) \wedge \left(\widehat{g}_{ZFC_{Nst}}(X) = \nu\right) \wedge Y = X\right]\right\}. \quad (2.3.37)$$

Note that from the axiom schema of replacement it follows directly that $\Im_{Nst}^*$ is a set in the sense of the set theory ZFC_{Nst}.

Definition 2.3.14. We define now the countable set $\Re_{Nst}^*$ by the following formula

$$\forall X\left(X \in \Im_{Nst}^*\right)\left[X \in \Re_{Nst}^* \iff X \notin X\right]. \quad (2.3.38)$$

Note that from the axiom schema of separation it follows directly that $\Re_{Nst}^*$ is a set in the sense of the set theory ZFC_{Nst}.

Remark 2.3.12. Note that $\Re_{Nst}^* \in \Im_{Nst}^*$ since $\Re_{Nst}^*$ is a definable by the following formula

$$\Psi^*(Z) \triangleq \forall X\,(X \in \Im^*_{Nst})\,[X \in Z \iff X \notin X]\,. \quad (2.3.39)$$

Theorem 2.3.2. Set theory ZFC_{Nst} is inconsistent.

Proof. From (2.3.39) and Remark 2.3.12 we obtain

$$\Re^*_{Nst} \in \Re^*_{Nst} \iff \Re^*_{Nst} \notin \Re^*_{Nst}\,. \quad (2.3.40)$$

From (2.3.40) immediately one obtains a contradiction $(\Re^*_{Nst} \in \Re^*_{Nst}) \wedge (\Re^*_{Nst} \notin \Re^*_{Nst})$.

3 AVOIDING THE CONTRADICTIONS FROM SET THEORY ZFC_2^{HS} AND SET THEORY ZFC_{ST} USING QUINEAN APPROACH

In order to avoid difficulties mentioned above we use well known Quinean approach [19].

3.1 Quinean Set Theory NF

Remind that the primitive predicates of Russellian unramified typed set theory (TST), a streamlined version of the theory of types, are equality = and membership $\in$. TST has a linear hierarchy of types: type 0 consists of individuals otherwise undescribed. For each (meta-) natural number n, type $n+1$ objects are sets of type n objects; sets of type n have members of type $n-1$. Objects connected by identity must have the same type. The following two atomic formulas succinctly describe the typing rules: $x^n = y^n$ and $x^n \in y^{n+1}$.

The axioms of TST are:

Extensionality: sets of the same (positive) type with the same members are equal.

Axiom schema of comprehension:

If $\Phi(x^n)$ is a formula, then the set $\{x^n \mid \Phi(x^n)\}^{n+1}$ exists i.e., given any formula $\Phi(x^n)$, the formula

$$\exists A^{n+1}\forall x^n[x^n \in A^{n+1} \leftrightarrow \Phi(x^n)] \quad (3.1.1)$$

is an axiom where A^{n+1} represents the set $\{x^n \mid \Phi(x^n)\}^{n+1}$ and is not free in $\Phi(x^n)$.

Quinean set theory [19] (New Foundations) seeks to eliminate the need for such superscripts.

New Foundations has a universal set, so it is a non-well founded set theory. That is to say, it is a logical theory that allows infinite descending chains of membership such as $\ldots \in x_n \in x_{n-1} \in \ldots \in x_3 \in x_2 \in x_1$. It avoids Russell's paradox by only allowing stratifiable formulae in the axiom of comprehension. For instance $x \in y$ is a stratifiable formula, but $x \in x$ is not (for details of how this works see below).

Definition 3.1.1. In New Foundations (NF) and related set theories, a formula Φ in the language of first-order logic with equality and membership is said to be stratified if and only if there is a function f which sends each variable appearing in Φ [considered as an item of syntax] to a natural number (this works equally well if all integers are used) in such a way that any atomic formula $x \in y$ appearing in Φ satisfies $f(y) = f(x) + 1$ and any atomic formula $x = y$ appearing in Φ satisfies $f(x) = f(y)$

Quinean set theory.

Axioms and stratification are:

the well-formed formulas of New Foundations (NF) are the same as the well-formed formulas of TST, but with the type annotations erased. The axioms of NF are.

Extensionality: two objects with the same elements are the same object.

A comprehension schema: all instances of TST Comprehension but with type indices dropped (and without introducing new identifications between variables).

By convention, NF's Comprehension schema is stated using the concept of stratified formula and making no direct reference to types. Comprehension then becomes.

Axiom schema of comprehension:

$\{x \mid \Phi^s\}$ exists for each stratified formula Φ^s.

Even the indirect reference to types implicit in the notion of stratification can be eliminated. Theodore Hailperin showed in 1944 that Comprehension is equivalent to a finite conjunction of its instances, [20] so that NF can be finitely axiomatized without any reference to the notion of type. Comprehension may seem to run afoul of problems similar to those in naive set theory, but this is not the case. For example, the existence of the impossible Russell class $\{x \mid x \notin x\}$ is not an axiom of NF, because $x \notin x$ cannot be stratified.

3.2 Set Theory $\overline{ZFC}_2^{Hs}, ZFC_{st}$ and Set Theory ZFC_{Nst} with Stratified Axiom Schema of Replacement

The stratified axiom schema of replacement asserts that the image of a set under any function definable by stratified formula of the theory ZFC_{st} will also fall inside a set.

Stratified Axiom schema of replacement.

Let $\Phi^s(x, y, w_1, w_2, \ldots, w_n)$ be any stratified formula in the language of ZFC_{st} whose free variables are among $x, y, A, w_1, w_2, \ldots, w_n$, so that in particular B is not free in Φ^s. Then

$$\begin{aligned} &\forall A \forall w_1 \forall w_2 ... \forall w_n \left[\forall x \left(x \in A \implies \exists! y \Phi^s(x, y, w_1, w_2, \ldots, w_n)\right) \implies \right. \\ &\left. \implies \exists B \forall x \left(x \in A \implies \exists y \left(y \in B \wedge \Phi^s(x, y, w_1, w_2, \ldots, w_n)\right)\right)\right], \end{aligned} \quad (3.2.1)$$

$i.e., iftherelation \Phi^s(x, y, ...)$ represents a definable function f, A represents its domain, and $f(x)$ is a set for every $x \in A$, then the range of f is a subset of some set B.

Stratified Axiom schema of separation.

Let $\Phi^s(x, w_1, w_2, \ldots, w_n)$ be any stratified formula in the language of ZFC_{st} whose free variables are among $x, A, w_1, w_2, \ldots, w_n$, so that in particular B is not free in Φ^s. Then

$$\forall w_1 \forall w_2 ... \forall w_n \forall A \exists B \forall x \left[x \in B \iff \left(x \in A \wedge \Phi^s(x, w_1, w_2, \ldots, w_n)\right)\right], \quad (3.2.2)$$

Remark 3.2.1. Notice that the stratified axiom schema of separation follows from the stratified axiom schema of replacement together with the axiom of empty set.

Remark 3.2.2. Notice that the stratified axiom schema of replacement (separation) obviously violeted any contradictions (2.1.20), (2.2.18) and (2.3.18) mentioned above. The existence of the countable Russell sets $\Re_2^{*Hs}, \Re_{st}^*$ and $\Re_{Nst}^*$ impossible, because $x \notin x$ cannot be stratified.

4 SECOND-ORDER SET THEORY ZFC_2 WITH THE FULL SECOND-ORDER SEMANTICS

4.1 Second-order Set Theory ZFC_2 with Urlogic

Remind that urlogic has the following characteristics [13].

1. Sentences of urlogic are finite strings of symbols. That a string of symbols is a sentence of urlogic, is a non-mathematical judgement.

2. Some sentences are accepted as axioms. That a sentence is an axiom, is a non-mathematical judgement.

3. Derivations are made from axioms. The derivations obey certain rules of proof. That a derivation obeys the rules of proof, is a non-mathematical judgement.

4. Derived sentences can be asserted as facts.

Remark 4.1.1. Let ZFC_2^{Ul} be second order set theory ZFC_2 with Ur logic. Note that in ZFC_2^{Ul} by using the rules of $\mathbf{DED}_2$ we dealing without any reference to semantics, i.e. satisfiability in some standard model, validity etc.

Definition 4.1.1. Let Γ_X^{Ul} be the countable set of the all first order 1-place open wff's of the set theory ZFC_2^{Ul} that contains free occurrences of the variable X.

Let $\Psi_1(X), \Psi_2(X)$ be a first order 1-place open wff's of the set theory ZFC_2^{Ul}. We define now the equivalence relation $(\cdot \sim_X \cdot) \subset \Gamma_X^{Ul} \times \Gamma_X^{Ul}$ by

$$\Psi_1(X) \sim_X \Psi_2(X) \iff \forall X [\Psi_1(X) \iff \Psi_2(X)] \quad (4.1.1)$$

For any $\Psi(X) \in \Gamma_X^{Ul}$ let $[\Psi(X)]_{Ul} \triangleq \{\Phi(X) \in \Gamma_X^{Ul} | \Psi(X) \sim \Phi(X)\}$ denote the equivalence class to which $\Psi(X)$ belongs. All elements of Γ_X^{Ul} equivalent to each other are also elements of the same equivalence class. The set of the all possible equivalence classes of Γ_X^{Ul} by $\tilde{}_X$, denoted by $\Gamma_X^{Ul}/\sim_X$

$$\Gamma_X^{Ul}/\sim_X \triangleq \{[\Psi(X)]_{Ul} | \Psi(X) \in \Gamma_X^{Ul}\}. \quad (4.1.2)$$

Let $\mathbf{Fr}_1^{Ul}(y, v)$ be the relation : y is the Gödel number of a first order 1-open wff of the set theoryZFC_2^{Ul} that contains free occurrences of the variable X with Gödel number v [15].

Note that the relation $\mathbf{Fr}_1^{Ul}(y, v)$ is expressible in ZFC_2^{Ul} by a wff $\widehat{\mathbf{Fr}}_1^{Ul}(y, v)$.

Note that for any $y, v \in \mathbb{N}$ by definition of the relation $\mathbf{Fr}_1^{Ul}(y, v)$ follows that

$$\widehat{\mathbf{Fr}}_1^{Ul}(y, v) \iff \exists! \Psi(X) \left[\left(g_{ZFC_2^{Ul}}(\Psi(X)) = y\right) \wedge \left(g_{ZFC_2^{Ul}}(X) = \nu\right)\right], \quad (4.1.3)$$

where $\Psi(X)$ is a unique wff of ZFC_2^{Ul} which contains free occurrences of the first order variable X with Gödel number v. We denote a unique wff $\Psi(X)$ defined by using equivalence (4.1.3) by symbol $\Psi_{y,\nu}^{Ul}(X)$, i.e.

$$\widehat{\mathbf{Fr}}_1^{Ul}(y, v) \iff \exists! \Psi_{y,\nu}^{Ul}(X) \left[\left(g_{ZFC_2^{Ul}}(\Psi_{y,\nu}^{Ul}(X)) = y\right) \wedge \left(g_{ZFC_2^{Ul}}(X) = \nu\right)\right]. \quad (4.1.4)$$

Definition 4.1.2. Let $g_{ZFC_2^{Ul}}(X) = \nu$. Let Γ_ν^{Ul} be a set of the all Gödel numbers of the first order 1-place open wff's of the set theory ZFC_2^{Ul} that contains free occurrences of the variable X with Gödel number v, i.e.

$$\Gamma_\nu^{Ul} = \{y \in \mathbb{N} | \langle y, \nu \rangle \in \mathbf{Fr}_1^{Ul}(y, v)\}, \quad (4.1.5)$$

or in the following equivalent form:

$$\forall y\,(y \in \mathbb{N})\left[y \in \Gamma_{\nu}^{Ul} \iff (y \in \mathbb{N}) \wedge \widehat{\mathbf{Fr}}_1^{Ul}(y,v)\right]. \qquad (4.1.6)$$

Remark 4.1.2. Note that from the axiom of separation it follows directly that Γ_{ν}^{Ul} is a set in the sense of the set theory ZFC_2^{Ul}.

Definition 4.1.3. (i) We define now the equivalence relation

$$(\cdot \sim_{\nu} \cdot) \subset \Gamma_{\nu}^{Ul} \times \Gamma_{\nu}^{Ul} \qquad (4.1.7)$$

inthesenseofthesettheoryZFC_2^{Ul} by

$$y_1 \sim_{\nu} y_2 \iff \left(\forall X\left[\Psi_{y_1,\nu}^{Ul}(X) \iff \Psi_{y_2,\nu}^{Ul}(X)\right]\right). \qquad (4.1.8)$$

For any $y_1 \in \Gamma_{v}^{Ul}$ let $[y_1]_{Ul} \triangleq \left\{y \in \Gamma_X^{Ul} | y_1 \sim_{\nu} y_2\right\}$ denote the equivalence class to which y_1 belongs. The set of the all possible equivalence classes of Γ_{ν}^{Ul} by $\tilde{}_{\nu}$, denoted $\Gamma_{\nu}^{Ul}/\sim_{\nu}$

$$\Gamma_{\nu}^{Ul}/\sim_{\nu} \triangleq \left\{[y]_{Ul} | y \in \Gamma_{\nu}^{Ul}\right\}. \qquad (4.1.9)$$

Remark 4.1.3. Note that from the axiom of separation it follows directly that $\Gamma_{\nu}^{Hs}/\sim_{\nu}$ is a set in the sense of the set theory ZFC_2^{Ul}.

Definition 4.1.4. Let $\Im_2^{Ul}$ be the countable set of the all sets definable by a first order 1-place open wff of the set theory ZFC_2^{Ul}, i.e.

$$\forall Y\left\{Y \in \Im_2^{Ul} \iff \exists \Psi(X)\left[\left([\Psi(X)]_{Ul} \in \Gamma_X^{Ul}/\sim_X\right) \wedge [\exists! X[\Psi(X) \wedge Y = X]]\right]\right\}. \qquad (4.1.10)$$

Definition 4.1.5. We rewrite now (4.1.10) in the following equivalent form

$$\forall Y\left\{Y \in \Im_2^{Ul} \iff \exists \Psi(X)\left[\left([\Psi(X)]_{Ul} \in \Gamma_X^{*Ul}/\sim_X\right) \wedge (Y = X)\right]\right\}, \qquad (4.1.11)$$

where the countable set $\Gamma_X^{*Ul}/\sim_X$ is defined by the following formula

$$\forall \Psi(X)\left\{[\Psi(X)]_{Ul} \in \Gamma_X^{*Ul}/\sim_X \iff \left[\left([\Psi(X)]_{Ul} \in \Gamma_X^{Ul}/\sim_X\right) \wedge \exists! X \Psi(X)\right]\right\}. \qquad (4.1.12)$$

Definition 4.1.6. Let $\Re_2^{Ul}$ be the countable set of all sets such that

$$\forall X\left(X \in \Im_2^{Ul}\right)\left[X \in \Re_2^{Ul} \iff X \notin X\right]. \qquad (4.1.13)$$

Remark 4.1.4. Note that $\Re_2^{Ul} \in \Im_2^{Ul}$ since $\Re_2^{Ul}$ is a set definable by first order 1-place open wff

$$\Psi\left(Z, \Im_2^{Ul}\right) \triangleq \forall X\left(X \in \Im_2^{Ul}\right)[X \in Z \iff X \notin X]. \qquad (4.1.14)$$

From (4.1.13) one obtains

$$\Re_2^{Ul} \in \Re_2^{Ul} \iff \Re_2^{Ul} \notin \Re_2^{Ul}. \qquad (4.1.15)$$

But (4.1.15) gives a contradiction

$$\left(\Re_2^{Ul} \in \Re_2^{Ul}\right) \wedge \left(\Re_2^{Ul} \notin \Re_2^{Ul}\right). \qquad (4.1.16)$$

421ontradiction (4.1.16) is a contradiction inside ZFC_2^{Ul} for the reason that the countable set $\Im_2^{Ul}$ is a set in the sense of the set theory ZFC_2^{Ul}.

In order to obtain a contradiction inside ZFC_2^{Ul} in more general case we introduce the following definitions.

Definition 4.1.7. We define now the countable set $\Gamma_{\nu}^{*Ul}/\sim_{\nu}$ by the following formula

$$\forall y\left\{[y]_{Ul} \in \Gamma_{\nu}^{*Ul}/\sim_{\nu} \iff \left([y]_{Ul} \in \Gamma_{\nu}^{Ul}/\sim_{\nu}\right) \wedge \widehat{\mathbf{Fr}}_1^{Ul}(y,v) \wedge \left[\exists! X \Psi_{y,\nu}^{Ul}(X)\right]\right\}. \qquad (4.1.17)$$

Remark 4.1.5. Note that from the axiom of separation it follows directly that $\Gamma_{\nu}^{*Ul}/\sim_{\nu}$ is a set in the sense of the set theory ZFC_2^{Ul}.

Definition 4.1.8. We define now the countable set $\Im_2^{*Ul}$ by the following formula

$$\forall Y \left\{ Y \in \Im_2^{*Ul} \iff \exists y \left[\left([y]_{Ul} \in \Gamma_\nu^{*Ul} / \sim_\nu \right) \wedge \left(g_{ZFC_2^{Ul}}(X) = \nu \right) \wedge Y = X \right] \right\}. \quad (4.1.18)$$

Note that from the axiom schema of replacement (1.1.1) it follows directly that $\Im_2^{*Hs}$ is a set in the sense of the set theory ZFC_2^{Ul}.

Definition 4.1.9. We define now the countable set $\Re_2^{*Ul}$ by formula

$$\forall X \left(X \in \Im_2^{*Ul} \right) \left[X \in \Re_2^{*Ul} \iff X \notin X \right]. \quad (4.1.19)$$

Note that from the axiom schema of separation it follows directly that $\Re_2^{*Ul}$ is a set in the sense of the set theory ZFC_2^{Ul}.

Remark 4.1.6. Note that $\Re_2^{*Ul} \in \Im_2^{*Ul}$ since $\Re_2^{*Ul}$ is definable by the following formula

$$\Psi^*(Z) \triangleq \forall X \left(X \in \Im_2^{*Ul} \right) \left[X \in Z \iff X \notin X \right]. \quad (4.1.20)$$

Theorem 4.1.1. Set theory ZFC_2^{Ul} is inconsistent.

Proof. From (4.1.19) and Remark 4.1.6 we obtain $\Re_2^{*Ul} \in \Re_2^{*Ul} \iff \Re_2^{*Ul} \notin \Re_2^{*Ul}$ from which immediately one obtains a contradiction

$$\left(\Re_2^{*Ul} \in \Re_2^{*Ul} \right) \wedge \left(\Re_2^{*Ul} \notin \Re_2^{*Ul} \right). \quad (4.1.21)$$

4.2 Second-order Set Theory ZFC_2 with the Full Se-condorder Semantics

Remind that the canonical approach of second order logic with full second-order semantics to the foundations of mathematics is that mathematical propositions have the form

$$\mathbf{U} \models \Phi \quad (4.2.1)$$

where **U** is a mathematical structure, such as integers, reals etc., and is a mathematical statement written in second order logic. If A is one of the structures, such as $(\mathbb{N}, +, \times, <)$ or $(\mathbb{R}, +, \times, <)$, for which there is a second order sentence $\Xi_{\mathbf{U}}$ such that

$$\forall \mathbf{W} \left(\mathbf{W} \models \Xi_{\mathbf{U}} \iff \mathbf{W} \cong \mathbf{U} \right), \quad (4.2.2)$$

then (4.2.2) can be expressed as a second order semantic logical truth

$$\models \Xi_{\mathbf{U}} \implies \Phi. \quad (4.2.3)$$

Remark 4.2.1. Let ZFC_2^{fss} be second order set theory ZFC_2 with the full second-order semantics.

(1) There is no completeness theorem for second-order logic.

(2) Nor do the axioms of second-order ZFC_2^{fss} imply a reflection principle which ensures that if a sentence of second-order set theory is true, then it is true in some standard model.

Remark 4.2.2. Thus there may be sentences of the language of second-order set theory ZFC_2^{fss} :

(i) that are true but unsatisfiable, or

(ii) sentences that are valid, but false.

Remark 4.2.3. For example let Z be the conjunction of all the axioms of second-order ZFC_2^{fss}. Z is surely true. But the existence of a model for Z requires the existence of strongly inaccessible cardinals. The axioms of ZFC_2^{fss} don't entail the existence of strongly inaccessible cardinals, and hence the satisfiability of Z is independent of ZFC_2^{fss}. Thus, Z is true but its unsatisfiability is consistent with ZFC_2^{fss}.

Definition 4.2.1. We will say that Ψ is a well formed first order formula Ψ of ZFC_2^{fss} (wff$_1$) if Ψ contain only first-order variables and first-order quantifiers.

Let $\Gamma_X^{\sharp fss}$ be the countable set of the all first order 1-place open wff$_1$'s of the set theory ZFC_2^{fss} that contains free occurrences of the first-order variable X.

Let $\Psi_1(X), \Psi_2(X)$ be 1-place open wff$_1$'s of the set theory ZFC_2^{fss}. We define now the equivalence relation $(\cdot \sim_X \cdot) \subset \Gamma_X^{\sharp fss} \times \Gamma_X^{\sharp fss}$ by

$$\Psi_1(X) \sim_X \Psi_2(X) \iff \forall X [\Psi_1(X) \iff \Psi_2(X)] \quad (4.2.4)$$

For any $\Psi(X) \in \Gamma_X^{\sharp fss}$ let $[\Psi(X)]_{\sharp fss} \triangleq \left\{\Phi(X) \in \Gamma_X^{\sharp fss} | \Psi(X) \sim \Phi(X)\right\}$ be the equivalence class to which $\Psi(X)$ belongs. All elements of $\Gamma_X^{\sharp fss}$ equivalent to each other are also elements of the same equivalence class. The collection of all possible equivalence classes of $\Gamma_X^{\sharp fss}$ by $\tilde{}_X$, denoted $\Gamma_X^{\sharp fss}/\sim_X$

$$\Gamma_X^{\sharp fss}/\sim_X \triangleq \left\{[\Psi(X)]_{\sharp fss} | \Psi(X) \in \Gamma_X^{\sharp fss}\right\}. \quad (4.2.5)$$

Let $\mathbf{Fr}_2^{\sharp fss}(y, v)$ be the relation : y is the Gödel number of a wff of the set theory $ZFC_2^{\sharp fss}$ that contains free occurrences of the first-order variable X with Gödel number v [17].

Note that the relation $\mathbf{Fr}_1^{\sharp fss}(y, v)$ is expressible in ZFC_2^{fss} by a wff $\widehat{\mathbf{Fr}}_1^{\sharp fss}(y, v)$.

Note that for any $y, v \in \mathbb{N}$ by definition of the relation $\mathbf{Fr}_1^{\sharp fss}(y, v)$ follows that

$$\widehat{\mathbf{Fr}}_1^{\sharp fss}(y, v) \iff \exists! \Psi(X) \left[\left(g_{ZFC_2^{fss}}(\Psi(X)) = y\right) \wedge \left(g_{ZFC_2^{fss}}(X) = \nu\right)\right], \quad (4.2.6)$$

where $\Psi(X)$ is a unique wff$_1$ of ZFC_2^{fss} which contains free occurrences of the variable X with Gödel number v. We denote a unique wff$_1$ $\Psi(X)$ defined by using equivalence (4.2.6) by symbol $\Psi_{y,\nu}^{\sharp fss}(X)$, i.e.

$$\widehat{\mathbf{Fr}}_1^{\sharp fss}(y, v) \iff \exists! \Psi_{y,\nu}^{\sharp fss}(X) \left[\left(g_{ZFC_2^{fss}}\left(\Psi_{y,\nu}^{\sharp}(X)\right) = y\right) \wedge \left(g_{ZFC_2^{fss}}(X) = \nu\right)\right]. \quad (4.2.7)$$

Remark 4.2.4. In order to avoid difficulties mentioned above,see Remark 4.2.1-Remark 4.2.3 we dealing with the countable set $\Gamma_X^{\sharp fss}$ of the all first order 1-place open wff$_1$'s of the set theory ZFC_2^{fss}.

Definition 4.2.2. Let $g_{ZFC_2^{fss}}(X) = \nu$.Let $\Gamma_\nu^{\sharp fss}$ be a set of all Gödel numbers of the all first order 1-place open wff$_1$'s of the set theory ZFC_2^{fss} that contains free occurrences of the first-order variable X with Gödel number v, i.e.

$$\Gamma_\nu^{\sharp fss} = \left\{y \in \mathbb{N} | \langle y, \nu \rangle \in \mathbf{Fr}_1^{\sharp fss}(y, v)\right\}, \quad (4.2.8)$$

or in the following equivalent form

$$\forall y (y \in \mathbb{N}) \left[y \in \Gamma_\nu^{\sharp fss} \iff (y \in \mathbb{N}) \wedge \widehat{\mathbf{Fr}}_1^{\sharp fss}(y, v)\right]. \quad (4.2.9)$$

Remark 4.2.5. Note that from the axiom of separation it follows directly that $\Gamma_\nu^{\sharp fss}$ is a set in the sense of the set theory ZFC_2^{fss}.

Definition 4.2.3. (i) We define now the equivalence relation

$$(\cdot \sim_\nu \cdot) \subset \Gamma_\nu^{\sharp fss} \times \Gamma_\nu^{\sharp fss} \quad (4.2.10)$$

in the sense of the set theory ZFC_2^{fss} by

$$y_1 \sim_\nu y_2 \iff \left(\forall X \left[\Psi_{y_1,\nu}^{\sharp fss}(X) \iff \Psi_{y_2,\nu}^{\sharp fss}(X)\right]\right). \quad (4.2.11)$$

The collection of all possible equivalence classes of $\Gamma_\nu^{\sharp fss}$ by $\tilde{}_\nu$, denoted $\Gamma_\nu^{\sharp fss}/\sim_\nu$

$$\Gamma_v^{\sharp fss}/\sim_\nu \triangleq \left\{[y]_{\sharp fss} \,|\, y \in \Gamma_\nu^{\sharp fss}\right\}. \quad (4.2.12)$$

Remark 4.2.6. Note that from the axiom of separation it follows directly that $\Gamma_\nu^{\sharp fss}/\sim_\nu$ is a set in the sense of the set theory ZFC_2^{fss}.

Definition 4.2.4. Let $\Im_2^{\sharp fss}$ be the countable collection of the all sets definable by a first order 1-place open wff$_1$ of the set theory ZFC_2^{fss}, i.e.

$$\forall Y \left\{Y \in \Im_2^{\sharp fss} \iff \exists \Psi(X)\left[\left([\Psi(X)]_{\sharp fss} \in \Gamma_X^{\sharp fss}/\sim_X\right) \wedge [\exists! X [\Psi(X) \wedge Y = X]]\right]\right\}. \quad (4.2.13)$$

Definition 4.2.5. We rewrite now (4.2.13) in the following equivalent form

$$\forall Y \left\{Y \in \Im_2^{*\sharp fss} \iff \exists \Psi(X)\left[\left([\Psi(X)]_{\sharp fss} \in \Gamma_X^{*\sharp fss}/\sim_X\right) \wedge (Y = X)\right]\right\}, \quad (4.2.14)$$

where the countable collection $\Gamma_X^{*\sharp fss}/\sim_X$ is defined by the following formula

$$\forall \Psi(X) \left\{[\Psi(X)]_{\sharp fss} \in \Gamma_X^{*\sharp fss}/\sim_X \iff \left[\left([\Psi(X)]_{\sharp fss} \in \Gamma_X^{\sharp fss}/\sim_X\right) \wedge \exists! X \Psi(X)\right]\right\}. \quad (4.2.15)$$

Definition 4.2.6. Let $\Re_2^{*\sharp fss}$ be the countable collection of all sets such that

$$\forall X \left(X \in \Im_2^{\sharp fss}\right)\left[X \in \Re_2^{*\sharp fss} \iff X \notin X\right]. \quad (4.2.16)$$

Remark 4.2.7. Note that $\Re_2^{*\sharp fss} \in \Im_2^{*\sharp fss}$ since $\Re_2^{*\sharp fss}$ is a collection definable by 1-place open wff$_1$

$$\Psi\left(Z, \Im_2^{*\sharp fss}\right) \triangleq \forall X \left(X \in \Im_2^{*\sharp fss}\right)[X \in Z \iff X \notin X]. \quad (4.2.17)$$

From (4.2.16) and Remark 4.2.7 one obtains

$$\Re_2^{*\sharp fss} \in \Re_2^{*\sharp fss} \iff \Re_2^{*\sharp fss} \notin \Re_2^{*\sharp fss}. \quad (4.2.18)$$

But (4.2.18) gives a contradiction

$$\left(\Re_2^{*\sharp fss} \in \Re_2^{*\sharp fss}\right) \wedge \left(\Re_2^{*\sharp fss} \notin \Re_2^{*\sharp fss}\right). \quad (4.2.19)$$

The contradiction (4.2.19) it a contradiction inside ZFC_2^{fss} for the reason that the countable collection $\Im_2^{*\sharp fss}$ is a set in the sense of the set theory ZFC_2^{fss}.

In order to obtain a contradiction inside ZFC_2^{fss} in more general case we introduce the following definitions.

Definition 4.2.7. We define now the countable set $\Gamma_\nu^{*\sharp fss}/\sim_\nu$ by the following formula

$$\forall y \left\{[y]_{Ul} \in \Gamma_\nu^{*\sharp fss}/\sim_\nu \iff \left([y]_{\sharp fss} \in \Gamma_\nu^{*\sharp fss}/\sim_\nu\right) \wedge \widehat{\mathbf{Fr}}_2^{*\sharp fss}(y, v) \wedge \left[\exists! X \Psi_{y,\nu}^{\sharp fss}(X)\right]\right\}. \quad (4.2.20)$$

Remark 4.2.8. Note that from the axiom of separation it follows directly that $\Gamma_\nu^{*Ul}/\sim_\nu$ is a set in the sense of the set theory ZFC_2^{fss}.

Definition 4.2.8. We define now the countable set $\Im_2^{*\sharp fss}$ by formula

$$\forall Y \left\{ Y \in \Im_2^{*\sharp fss} \iff \exists y \left[\left([y]_{\sharp fss} \in \Gamma_\nu^{*\sharp fss} / \sim_\nu \right) \wedge \left(g_{ZFC_2^{fss}} (X) = \nu \right) \wedge Y = X \right] \right\}. \quad (4.2.21)$$

Note that from the axiom schema of replacement (1.1.1) it follows directly that $\Im_2^{*\sharp fss}$ is a set in the sense of the set theory ZFC_2^{fss}.

Definition 4.2.9. We define now the countable set $\Re_2^{*\sharp fss}$ by the following formula

$$\forall X \left(X \in \Im_2^{*\sharp fss} \right) \left[X \in \Re_2^{*\sharp fss} \iff X \notin X \right]. \quad (4.2.22)$$

Note that from the axiom schema of separation it follows directly that $\Re_2^{*\sharp fss}$ is a set in the sense of the set theory ZFC_2^{fss}.

Remark 4.2.9. Note that $\Re_2^{*\sharp fss} \in \Im_2^{*Ul}$ since $\Re_2^{*Ul}$ is definable by the following formula

$$\Psi^* (Z) \triangleq \forall X \left(X \in \Im_2^{*\sharp fss} \right) [X \in Z \iff X \notin X]. \quad (4.2.23)$$

Theorem 4.2.1. Set theory ZFC_2^{fss} is inconsistent.

Proof. From (4.2.22) and Remark 4.1.6 we obtain $\Re_2^{*\sharp fss} \in \Re_2^{*\sharp fss} \iff \Re_2^{*\sharp fss} \notin \Re_2^{*Ul}$ from which immediately one obtains a contradiction

$$\left(\Re_2^{*\sharp fss} \in \Re_2^{*\sharp fss} \right) \wedge \left(\Re_2^{*\sharp fss} \notin \Re_2^{*\sharp fss} \right). \quad (4.2.24)$$

5 CONCLUSIONS

a In this Chapter we have proved that set theory $ZFC + \exists M_{st}^{ZFC}$ is inconsistent in particular $\neg Con(ZF + V = L)$..

b This result originally was obtained in [2], [4]. [5] by using essentially another approach.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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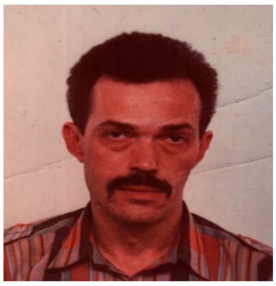

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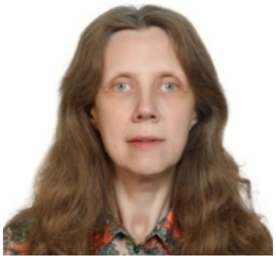

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