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ARTIFICE AND THE NATURAL WORLD: MATHEMATICS, LOGIC, TECHNOLOGY

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If Tahiti suggested to theorists comfortably at home in Europe thoughts of noble savages without clothes, those who paid for and went on voyages there were in pursuit of a quite opposite human ideal. Cook's voyage to observe the transit of Venus in 1769 symbolises the eighteenth century's commitment to numbers and accuracy, and its willingness to spend a lot of public money on acquiring them. The state supported the organisation of quantitative researches, employing surveyors and collecting statistics to compute its power.[1] People volunteered to become more numerate;[2] even those who did not had the numerical rationality of the metric system imposed on them.[3] There was an increase of two orders of magnitude or so in the accuracy of measuring instruments and the known values of physical constants.[4] The graphical display of quantitative information made it more readily available and comprehensible.[5] On the research front, mathematics continued its advance, even if with notably less speed than in the two adjoining centuries. The methods of the calculus proved successful in more and more problems in mechanics, both celestial and terrestrial. Elasticity and fluid dynamics became mathematically tractable for the first time.[6] The central limit theorem brought many chance phenomena within the purview of reason.

These successes proved of interest for 'low philosophy', or philosophy-as-propaganda, as practised by the natural theologians and the *Encyclopédistes*. Both had their uses for scientific breakthroughs, though sometimes not much interest in the details. For 'high philosophy', as constituted by the great names, mathematics and science had a different importance. A feature common to the biographies of all the well-known philosophers of the eighteenth century is a mathematical youth. Wolff began as a professor of mathematics, and it was in that subject that he first made the contributions to the intellectual vocabulary and style of German for which he is so universally loathed. Kant taught mathematics, and his Prize Essay begins with an analysis of the mathematical method. D'Alembert, Condorcet, Lambert, even Diderot in a smaller way (and of course Leibniz earlier) made serious mathematical contributions. Reid also taught mathematics, and his first published work was on quantity. Paley was

Senior Wrangler in the Cambridge Mathematical Tripos. Berkeley's *Analyst* is one of the most successful interventions ever by a philosopher into mathematics. Hume and Vico, though no mathematicians, used mathematical examples as the first illustrations of their theories. Adam Smith's 'invisible hand' and Malthus's model of population growth both belong to what is now called dynamical systems theory.

Naturally, these philosophers did not all draw the same lessons from their mathematical experience. But philosophers have one thing in common in their attitude toward mathematics, in this last century before the surprise of non-Euclidean geometry undermined the pretensions of mathematics to infallibility. It is envy. What is envied, in particular, is the 'mathematical method', which apparently produced what philosophy wished it could but had been unable to: certain truths, agreed to by all, delivered by pure thought.

I. THE 'MATHEMATICAL METHOD' PRAISED

The eighteenth century was the last to accept, fundamentally without question, certain approved opinions of the ancients concerning the method and content of mathematics. The ideas were largely Aristotelian in origin but had survived the demise of scholasticism by being accepted almost in full by the Cartesians and Newton. The much admired 'mathematical method' is the deriving of truths by syllogisms from self-evident first principles; the method was believed to be instantiated by Euclid's *Elements*. As to the content, mathematics is the science of 'quantity', which is 'whatever is capable of increase, or diminution'.[7] Numbers arise from considering the ratio of quantities to an arbitrarily chosen unit. Quantity is of two kinds: discrete (studied by arithmetic) and continuous (studied by geometry). However, geometry is also the study of 'extension', or real space. Quantity in the abstract is studied by pure mathematics, while 'magnitude as subsisting in material bodies'[8] is the object of mixed or applied mathematics, which includes optics, astronomy, mechanics, navigation, and the like. Tendencies to regard mathematics as about some abstraction of reality did exist but were generally resisted. Euler, for example, says that in geometry one does not deal with an ideal or abstract triangle but with triangles in general, and that generality in mathematics is no different from generality elsewhere;[9] to the same purpose, d'Alembert defends an approximation theory, whereby the perfect circles of geometry allow us to 'approach' the truth, 'if not rigorously, at least to a degree sufficient for our use'.[10]

There are several philosophical problems with this complex of opinions, which are sufficiently obvious to keep surfacing in one form or another again and again:

- Why is the mathematical method found in *mathematics*?
- Why are the first principles in mathematics necessary, and how are they known?
- Is the reasoning in Euclid in fact all syllogisms? (in the strict sense, that is, of the form: All A are B, All B are C, so All A are C). If not, what kind of reasoning is it?

1. *Wolff*

Wolff at least had answers to these questions. The mathematical method, he thinks, is applicable everywhere; and there is no problem about the self-evidence of the first principles because there is only one of them, and it is the principle of non-contradiction. Geometrical demonstrations can all be resolved into formal syllogisms, and discoveries in mathematics are made exclusively by syllogistic means.[11] His central place in eighteenth-century philosophy results from his attempt to derive all philosophical truths from the principle of non-contradiction, by the 'mathematical method'.[12] A look at how he actually proposes to *prove* that everything has a sufficient reason, using only the principle of non-contradiction, reveals why Wolff's 'method' achieved less than universal agreement:

> Let us suppose A to be without a sufficient reason why it is rather than is not. Therefore nothing is supposed by which it can be understood why A is. Thus A is admitted to be, on the basis of an assumed nothing; but since this is absurd, nothing is without a sufficient reason.[13]

Wolff's ideal differs from those of others essentially in lacking anything like Plato's dialectic, or Aristotle's induction, or Kant's analysis: the roundabout discussion and sorting of experience which allows the intellect to come to an insight into first principles. It is unnecessary in Wolff's system because the principle of non-contradiction is the sole starting point. Any tendency to regard brute facts as contingent and outside the scope of explanation by necessary reasons is suppressed, in Wolff, by his acceptance of Leibniz's Best of All Possible Worlds theory. According to that theory, everything, however particular, has an explanation in principle.

2. *Mathematics as philosophical propaganda*

Mathematics, because of its immense prestige, is always destined to be used in support of various philosophical positions. It was a natural as a prop for the Enlightenment motif that there should be more Reason all round. The *Encyclopédie* says: 'M. Wolff . . . made it clear in theory, and especially in practice, and in the composition of all his works, that the mathematical method belongs to all the sciences, is natural to the human spirit, and leads to discoveries of truths of all kinds.'[14] Not that the French needed Wolff to tell them this, given the Cartesian ideals expressed by, for example, Fontenelle:

> The geometrical spirit is not so attached to geometry that it cannot be carried over to other knowledge as well. A work of ethics, of politics, of criticism, perhaps even of eloquence will be better, all things else being equal, if it is made by the hand of a mathematician. The order, clarity, precision and exactitude which have reigned in the better works recently, can well have had their first source in this geometrical spirit which extends itself more than ever and which in some fashion communicates itself even to those who have no knowledge of geometry.[15]

Of course, there were counter-currents. There were the complaints common in all centuries from self-proclaimed 'practical men' like Frederick the Great and Jefferson,[16] who regarded the higher abstractions of mathematics as useless, and from humanists like Vico and Gibbon, who abhorred the 'habit of rigid demonstration, so destructive of the finer feelings of moral evidence'.[17]

Mathematics was also called to the aid of more particular philosophical theses. On the one side, there was the support allegedly given to natural theology by the various 'principles of least action'. On the other, the success of prediction in astronomy could be a support to determinism. It was found that many phenomena in physics could be derived from 'methods of maxima and minima', or 'principles of least action', such as the one stating that the path of light from one point to another is the one which minimises the time of travel (even if the path is not straight, because of reflection or refraction). Maupertuis and Euler take this to be evidence of final causes, and for the existence of God.[18] Their idea owes something to the more general claim of Leibniz's *Theodicée* that everything is the necessary result of a maximum principle, namely, that the goodness of this world is the best possible. D'Alembert, to the contrary, warns of the danger of 'regarding as a primitive law of nature what is only a purely mathematical consequence of some formulae'.[19] He believes the best hope for the countries of Europe oppressed by superstition is to begin studying geometry, which will lead to sound philosophy.[20] Laplace also, in an image that haunts philosophy still, invites mathematics to assist an anti-religious worldview:

> Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it . . . it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom.[21]

History would thus be a subfield of the theory of differential equations.

3. D'Alembert versus Diderot

Wolff's ant-like progress through a farrago of equivocations and circularities is merely dispiriting and only brings rationalism into disrepute. D'Alembert is not so easily dismissed when he argues for essentially the same conclusions. His immediate claim, is not, indeed, that *all* subjects are entirely amenable to the geometric method, only that mechanics is. But mechanics is very inclusive, on a typical eighteenth-century view. If La Mettrie and d'Holbach were right about the nature of man, for example, psychology would be a sub-branch of mechanics. D'Alembert argues, more convincingly than Descartes, that mechanics is a branch of mathematics, based like arithmetic on absolutely necessary first principles. In a kind of mathematical version of Hume's scepticism about causes, he regards forces as 'beings obscure and metaphysical': 'All we see distinctly in the movement of a body is that it crosses a certain space and that it employs a certain time to cross it.'[22] Hence collisions are to be explained in terms of impenetrability, and the density of a body is merely 'the ratio of its mass (that is, the space it would occupy if it were absolutely without pores) to its volume, that is, to the space it actually occupies'.[23] It might seem that there is no hope of demonstrating the conservation of momentum purely geometrically:

> However, if we consider the matter carefully, we shall see that there is one case in which equilibrium manifests itself clearly and distinctly; that is where the masses of the two bodies are equal, and their velocities equal and opposite.[24]

The *Encyclopédie* article 'Expérimentale', which one expects to be along Baconian lines, is in fact used by d'Alembert to propagate his extreme anti-experimental views. He regards collecting facts as a rather medieval exercise, superseded by Newton's introduction of geometry into physics. The laws of colliding bodies are demonstrable: nature could not be any other way. But there is an admission that how fast a body falls under gravity, and what the weight of a fluid is, must be measured; only after that do the relevant sciences become 'entirely or almost entirely mathematical':

> No theory could have allowed us to find the law that heavy bodies follow in their vertical fall, but once this law is found through experience, all that belongs to the movement of heavy bodies, whether rectilinear or curvilinear, whether inclined or vertical, is found entirely by theory.[25]

While these cases appear as unfortunate weakenings of his original wish for purely deductive science, d'Alembert's comments here are perhaps his most solid achievement. In the more mathematical sciences, experience does appear only in support of a few easily checkable symmetry principles and simple laws, while most of the weight of explanation rests on the difficult mathematical derivations of more subtle phenomena from these. And, as Leibniz points out,[26] a symmetry principle has a special logical status, being an application of the principle of (in)sufficient reason: in d'Alembert's example, if two bodies have equal and opposite velocities, their momenta must balance, as there is no reason why one should overcome the other. D'Alembert was widely thought to have succeeded in showing that the principles of mechanics had 'a necessity as rigorous as the first elementary truths of geometry'.[27] Kant concurred in d'Alembert's unlikely conclusion.[28] Lagrange's *Mécanique analytique* of 1788 confirmed further that mechanics could look like a deductive system, managing almost to conceal the existence of forces.[29]

Of one mind on the iniquity of priestcraft, the inevitable progress of mankind, and other such Enlightenment staples, the two prime movers of the *Encyclopédie* fell out over mathematics. Diderot believed mathematics had reached its highest point and was now in decline.[30] He preferred sciences full of life and ferment, like chemistry and biology, criticising mathematics as abstract and over-simple.[31] D'Alembert, on the other hand, held that an abundance of experiential 'principles' is 'an effect of our very poverty'.[32] Diderot's attack is not all invective; he has a philosophical argument to undermine rationalist pretensions about mathematics which is the same as the contention of twentieth-century empiricists and positivists that mathematics is essentially trivial. Geometrical truths are merely identities, saying the same thing in a thousand different ways without generating any new facts.[33] D'Alembert allowed this argument to appear in the *Discours préliminaire* to the *Encyclopédie*, but replied that it just showed how powerful mathematics was to be able to get so much from so little.[34]

II. THE 'MATHEMATICAL METHOD' DOUBTED

As in the twentieth century, the success of science and mathematics attracted from professional philosophers not praise, but complaints, to the effect that they, the philosophers, could not see how so much knowledge could possibly be achievable. While very few were prepared to go as far as Diderot, much argument was undertaken to show that the claims of mathematical proof were not all they seemed. The argument mostly centred on geometry. The problem at its simplest, as Gauss put it, is, 'if number is entirely a product of our own minds, space has a reality outside of our minds and we cannot prescribe its laws *a priori*'.[35] The century inherited what could be called the Euclid-Newton view of space and time. The essential features are these: Space and time are infinite in extent in all directions, homogeneous, flat, and infinitely divisible. Truths about space and time may be proved with absolute certainty, in the style of Euclid. After two thousand years of success, the facade seemed unbreakable. This prevented two developments which, at various times during the eighteenth century, seemed on the point of happening. The first is the discovery of non-Euclidean geometry. The second is the adoption of the philosophical opinion that knowledge of space, which is something real outside the mind, must be empirical and fallible. The problem with the Euclidean claims is that it is difficult to see how they could be known, if true. Without the scholastic magic of an intellect equipped with a natural aptitude for truth, and with epistemological worries becoming more central to philosophy, empiricism and rationalism were in equal but opposite quandaries. For the empiricist, the infinitely large and the infinitely small are not available for inspection, so where is knowledge of them to come from? Hume will pursue this thought to its limit. For the rationalist, the certainty of the deliverances of reason on space and time will suggest a dependence of those concepts on the mind. Kant will pursue this idea to, or beyond, its limits.

1. Bayle and Saccheri: doubts on the foundations of geometry

The problem as it appeared at the beginning of the century can be seen in two widely known semi-philosophical works: Bayle's *Dictionary* (1697) and Saccheri's *Euclid Cleansed from All Spot* (1733). Bayle remarks that the certainty of the mathematical method is not all it is claimed to be, since there are disputes even among mathematicians, for example, over infinitesimals.[36] He argues that space can consist neither of mathematical points nor of Epicurean extended atoms, nor can it be infinitely divisible. He takes this to exhaust all the possibilities, and concludes, in a remark that contains seeds of both Hume and Kant, that the attempted geometrical proofs that space is infinitely divisible 'serve no other use but to show that extension exists only in our understanding'.[37]

An essential claim of admirers of the 'mathematical method' was that Euclid's axioms were self-evident. But how true is this? Somewhere, Euclidean geometry must claim that space is infinite, which seems a claim beyond the capacity of experience to know. Euclid's Fifth Postulate, in particular, asserts something that seems to require an intuition about arbitrarily distant space:

> That if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Saccheri undertook to derive the fifth postulate from the others by showing that the first four postulates, plus the negation of the fifth, led to a contradiction. This is in fact impossible to show. If true, it would have removed all doubts about the self-evidence of Euclid's axioms. He proceeds well for some time, demonstrating what would later be called theorems in hyperbolic geometry, the non-Euclidean geometry in which the sum of the angles in any triangle is less than 180 degrees. Then he derives a 'contradiction', but it is unconvincing, as it involves common perpendiculars to two straight lines 'at infinity'. He makes another attempt and again claims success, but there is a mistake. He hints that the result is not as clear as it might be, and publication of his book was withheld in his lifetime (possibly entitling him to a footnote in the history of ethics).[38]

The problem became well-known: d'Alembert called it the 'scandal of the elements of geometry'. G. S. Klügel's dissertation of 1763 reviewed twenty-eight attempts to prove the fifth postulate, concluding that they were all deficient. He gave his opinion that the postulate was not provable, its truth thus resting on the judgment of the senses.[39] Kant's only serious attempt to do work of his own in mathematics was an attempt to prove the fifth postulate.

Lambert came closest to thinking in terms of an actual alternative geometry, writing, 'I should almost conclude that the third hypothesis [of angle sum less than 180 degrees] holds on an imaginary sphere'.[40] Nevertheless, like Saccheri, he incorrectly claims to derive a contradiction, and the genuine possibility of a non-Euclidean geometry was not recognised until well after 1800. The philosophical commitment to the self-evidence of Euclid certainly stimulated important mathematical work but at the same time delayed the discovery of the correct answer, which was not that desired by philosophy.

2. *Berkeley's* Analyst: *calculus and infinitesimals*

Berkeley's general philosophy of mathematics shows intellectual independence, to say the least. Rejecting completely views that mathematics is about either quantity or abstractions, he is the first formalist philosopher of arithmetic, maintaining that there is only the manipulation of symbols according to rules.[41] Geometry, he believes, can only be about perceived extension. He is thus led to reject the infinite divisibility of space; like Hume after him (and this is where Hume's and Berkeley's philosophies come closest) he denies the meaningfulness of any talk about lengths less than the *minimum visibile or minimum tangibile*.[42]

Prepared by these non-standard speculations, Berkeley, in his *Analyst* of 1734, attacked the mathematicians' understanding of the foundations of the calculus as hopelessly confused and contradictory. The episode has a special place in the history of philosophy, as one of the very few cases where a technical field eventually admitted that philosophy strictly so-called had won a victory over the technical practitioners. Berkeley intended the argument to serve a purpose in the philosophy of religion, by showing that there were mysteries as incomprehensible as those of religion even in the paradigm of reason, mathematics.[43] The argument itself, however, is quite independent of its purpose.

At issue is the meaning of a derivative, or rate of change of a variable quantity – the 'fluxion' of a 'fluent', in Newton's terminology. If we wish to measure the speed of a moving object, that is, the rate of change of distance, we use a unit like miles per hour. To find the numerical value of an object's speed, therefore, we divide the distance it travels in any time interval by the length of the interval. If the speed is constant, no problems arise: the answer is the same whatever interval is taken: 12 miles divided by 3 hours gives the same answer as 8 miles divided by 2 hours, namely 4 miles per hour. But if the speed is itself variable, conceptual problems arise in trying to explain what the instantaneous speed is, at any given instant. For the speed calculated from dividing any finite distance traversed by the finite time taken to do so is not an instantaneous speed but the average speed over the interval. It is natural to approximate the speed at an instant more closely by taking smaller and smaller intervals including that instant, but the problem remains that an instantaneous speed and an average speed are different, both conceptually and numerically. Newton used such doubtfully intelligible language as

> Fluxions are very nearly as the augments of the fluents generated in equal, but very small, particles of time; and, to speak accurately, they are in the *first ratio* of the nascent augments . . .[44]

In calculating the speed if the distance travelled in time x is x^n, he first finds the distance travelled in the time between x and $x + o$, divides it by the 'augment' of time o, and finally claims that when the augment o vanishes, their 'ultimate ratio' is as nx^{n-1} to 1. Berkeley's criticism is perfectly correct:

> For when it is said, let the increments vanish, or let there be no increments, the former supposition that the increments were something, or that there were increments, is destroyed, and yet a consequence of that supposition, *i.e.*, an expression got by virtue thereof, is retained.[45]

Indeed, the division by o to find the average speed requires that o not be zero, while later o is taken to be zero. It is no use maintaining that o is small, since as Berkeley again says, 'the minutest errors are not to be neglected in mathematics'. Newton's attempts to speak of the augments as 'nascent' and 'evanescent', and the ratios as 'first' and 'ultimate' attracts Berkeley's most famous piece of ridicule: 'And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?'(§35, 4: 89). Berkeley also attacks with justice other parts of Newton's calculus, notably the higher derivatives. A speed is itself a variable quantity, so it has a fluxion, or rate at which it is changing, the acceleration. If it is hard to explain a first derivative in terms of the ratios of 'evanescent' quantities, it is doubly so to explain in such terms what a second or third derivative is:

> The incipient celerity of an incipient celerity, the nascent augment of a nascent augment, *i.e.*, of a thing which hath no magnitude, – take it in what light you please, the clear conception of it will, if I mistake not, be found impossible. (§4, 4: 67)

Bad answers to Berkeley began with his own,[46] and a flood of them appeared from as far away as America.[47]

There are other possible ways of expressing what it is of which ratios are being taken. On the continent, it was common to speak in terms of 'infinitesimals'. These were conceived of as quantities smaller than any finite quantities, yet not zero. An instantaneous speed might then be regarded as exactly the ratio of infinitesimal augments, though only approximately the ratio of any finite augments.[48] 'The clear conception of them' proved no easier to achieve than that of fluxions.

One of the more serious attempts to resolve the problems was Maclaurin's *Treatise of Fluxions* of 1742. It attempts to show that fluxions are a generalisation of the 'geometry of the antients', by using Archimedes' method of exhaustion to replace infinitesimals, of which he complains, 'From geometry the infinities and infinitesimals passed into philosophy, carrying with them the obscurity and perplexity which cannot fail to accompany them'.[49] His idea is in principle the same as the modern treatment using limits, but Maclaurin retains kinematic notions which would later be regarded as inadequate. In particular, he defines a fluxion obscurely in terms of a counterfactual: 'the increment or decrement that would be generated in a given time by this notion, if it was continued uniformly.'[50] D'Alembert and L'Huilier tried to base calculus on limits, in a way that was essentially correct, but still lacked the precision achieved in the next century by the use of multiple quantifiers.[51] Lazare Carnot's *Réflexions sur la métaphysique du calcul infinitésimal*, of 1797, achieved much greater popular success, by repeating all the worst excesses of infinitesimals.[52] The debates over whether infinitesimals are zero or not, whether they can be conceived, and whether a limit is or is not actually attained often read more like a Kantian antinomy than the real thing.

3. Hume on mathematics

Hume's philosophy of mathematics is a natural outgrowth of his combining the usual 'science of quantity and extension' view with his requirement that all concepts be explained in terms of impressions and ideas. In the division of truths between relations of ideas and matters of fact, mathematics falls entirely on the side of ideas. But whereas relations of ideas like resemblance and contrariety are 'discoverable at first sight', this is not so with '*proportions of quantity or number*'. Though not different in kind from resemblance, because of their complexity 'their relations become intricate and involved', so that coming to know them may need some 'abstract reasoning and reflexion', or demonstration (unless 'the difference is very great and remarkable'). The relations treated in arithmetic and algebra are the best known, because of their 'perfect precision and exactness'. For example, two numbers may be infallibly pronounced equal when 'the one always has an unite answering to every unite of the other', since this is something directly checkable.[53] Such complicated mathematical facts as that a number is divisible by nine if the sum of its digits is also divisible by nine may at first appear due to chance or design, but reasoning shows they result from 'the nature of these numbers'.[54] Hume thus does not agree that mathematics is syllogistic, or in any other way 'analytic' in any trivial or vacuous sense. But he does hold that mathematical truths are known by subjecting ideas (of quantity) to some kind of purely conceptual 'analysis' (not Hume's word).[55]

Even demonstrated mathematical knowledge is in practice fallible, however. For the certainty that results from discovering the relations is only an 'in principle' one, since an actual reasoner can make mistakes. 'The rules are certain and infallible; but when we apply them, our fallible and uncertain faculties are very apt to depart from them.'[56]

Geometry has less certainty than algebra and arithmetic, because it deals with continuous quantities, which cannot be measured exactly. The result is that Hume becomes, with Berkeley, one of the few philosophers in history to reject the infinite divisibility of space. The topic belongs more properly to his philosophy of space than to his philosophy of mathematics – even granted that the distinction is anachronistic. But his replies to the alleged *mathematical* demonstrations of the infinite divisibility of space, approved by such good authorities as the Port-Royal *Logic* and Isaac Barrow,[57] are of some worth. The mathematical arguments simply consist in extracting the assumption of infinite divisibility that is contained implicitly in Euclid, and cannot determine whether actual space is infinitely divisible. Hume goes some way toward exposing this flaw when he doubts the exact correspondence between the axioms of geometry and our ideas of space: 'none of these demonstrations can have sufficient weight to establish such a principle, as this of infinite divisibility; and that because with regard to such minute objects, they are not properly demonstrations, being built on ideas, which are not exact.'[58] There is no way to be sure, for example, that two straight lines with a small angle between them meet in only one point (1.3.1.4, SBN 71).

But the errors of geometry 'are never considerable'. It would seem that Hume has been substantially vindicated by subsequent developments, which have revealed that deciding whether space is exactly Euclidean is an empirical question, although it is obviously approximately Euclidean in our region.

An aspect of Hume and Berkeley's writing on geometry that is central but that some commentators have found odd is their talk of 'parts of ideas'. They discuss, for example, into how many indivisible parts an idea of extension 'as conceived by the imagination' can be divided.[59] If one takes this seriously, it would appear that ideas or imagination themselves have a quasi-spatial quality. Hume does not develop this notion further, but Reid does, and goes so far as to say what exactly is the spatial structure of the 'geometry of visibles'. It is the geometry of the surface of a sphere.[60]

4. Kant

The key to Kant's views on mathematics, and much else, is the notion of *construction* in geometry. In Euclid, there are postulates, such as 'To draw a straight line from any point to any point', which assert that certain things exist, or may be constructed. The first thing Euclid proves is that an equilateral triangle may be *constructed* on any line. In learning how to prove in geometry, as of course all educated eighteenth-century persons did, one must spend a good deal of time deciding which lines to prolong, when to draw a new circle, and so on. From the point of view of modern formal logic, this can be regarded as a defect in Euclid's treatment of geometry, but from an earlier point of view it reinforces two convictions: that Euclidean geometry is not purely syllogistic, and that it is about real space.

A fascination with construction had already been evident in Vico. The first statement of his *verum-factum* theory is in the context of geometry: 'We demonstrate geometrical things because we make them.'[61] Even when his 'New Science' of human things is fully developed and its contrast with the natural sciences emphasised, its links with geometry are retained. Both construct the world they study.[62] It was realised that constructions did not fit well into the Wolffian view of sciences as demonstrating truths about universals from their definitions. Wolff was prepared to assert that the drawing of a straight line between two points flowed from the definition of a line, but this is not plausible. Andreas Rüdiger alleges that the Wolffian 'mathematical method' is a travesty of real mathematics, by recalling the inspection of particulars in geometrical constructions and in counting.[63] Johann Heinrich Lambert describes the experience of reading Euclid after Wolff and recognising that Euclid is nothing like Wolff says he should be. He notes that Euclid does not derive things from the definition of space, but starts with lines as simples, and exhibits first the possibility of an equilateral triangle.[64]

So, when the Berlin Academy posed the question, whether metaphysical truth could be equated with mathematical truth, Kant in his 'Prize Essay' replied: mathematics has construction, or synthesis, while metaphysics does not.[65] In Kant, construction in geometry is used to fill out the vaguer notions of the previous one hundred and fifty years along the lines that the possible is what can be clearly and distinctly conceived (in the 'imagination', conceived as a mental visualisation facility). Kantian 'intuition' is, like the scholastics' 'intelligible matter', a medium in which can be drawn not just a few simple ideas to be compared with one another, in the style of Locke, but whole geometrical diagrams. What can be so drawn is more restricted than what merely does not contain a logical contradiction. For example, there is no contradiction in the concept of two straight lines meeting in two points and enclosing a figure; nevertheless, no such figure is possible, since it cannot be constructed: 'That between two points there is only one straight line . . . can[not] be derived from some universal concept of space; [it] can only be *apprehended* concretely, so to speak, in space itself.'[66] Similarly, that there is a plane passing through any three given points is evident because the intuition constructs the figure 'immediately'.[67] These necessities and possibilities are 'synthetic', in the sense that they do not follow simply from formal logical principles, and also in the sense that they involve 'synthesis', or construction. These truths are also a priori, since Kant is not prepared to compromise the absolute certainty of mathematics. So Leibniz, he thinks, cannot be right about space arising out of relations between real objects because that would make geometry empirical, and there might be a non-Euclidean space, which Kant takes to be impossible.[68] It is his own theory, that space is imposed by the mind, that is needed to ensure the certainty of geometry: 'Assuredly, had not the concept of space been given originally by the nature of the mind . . . then the use of geometry in natural philosophy would be far from safe' (§15E, Ak 2: 404–5).

It is clear then how Kant's synthetic a priori, on which so much in his philosophy depends, is the result of combining three pre-existing ideas: Euclidean construction, the reduction of concepts to ideas in 'imagination' or 'intuition', and the certainty of geometry.

Kant finds construction also in arithmetic, in the thinking of how many times a unit is contained in a quantity: 'this how-many-times is grounded on successive repetition, thus on time and the synthesis (of the homogeneous) in it' (*Kritik* B 300). The concept of number is one which 'in itself, indeed, belongs to the understanding but of which the actualisation in the concrete requires the auxiliary notions of time and space (by successively adding a number of things and setting them simultaneously side by side)' (*De mundi*, §12, Ak 2: 397). In the famous passage of the *Kritik der reinen Vernunft* explaining why the proposition '$7 + 5 = 12$' is synthetic, Kant writes:

> The concept of twelve is by no means already thought merely by my thinking of that unification of seven and five, and no matter how long I analyze my concept of such a possible sum I will still not find twelve in it. . . . For I take first the number 7, and, as I take the fingers of my hand as an intuition for assistance with the concept of 5, to that image of mine I now add the units that I have previously taken together in order to constitute the number 5 one after another to the number 7, and thus see the number 12 arise. (B 15–16, see also B 205 and B 299)

The construction here is with real fingers, not 'in the imagination', but Kant means exactly to assimilate the mind's structuring of experience while perceiving fingers to construction in the imagination: 'this very same formative synthesis, by means of which we construct a figure in imagination is entirely identical with that which we exercise in the apprehension of an appearance, in order to make a concept of experience of it' (B 271). Kant emphasises that he does not just mean reading off results from a picture; there is an intellectual operation involved, which is responsible for the necessity of the truth. Large numbers, for example, obviously cannot be counted by an immediate glance; what is important is the 'schema' of successive addition of units that allows the aggregate to be synthesised, that is, counted (B 16, B 179–81). The essentials of this discovery, that the necessity in mathematical knowledge comes from assimilating an image or experience to construction or synthesis according to some rule, Kant attributes to the earliest Greek geometers (B xii).

Kant brings the same ideas to the problems of the infinite. How those problems appeared to mathematicians in Kant's time is apparent from the terms of the prize set by the Berlin Academy of Sciences (Mathematical Section) for 1786:

> *There is needed a clear and precise theory of what is called Infinite in Mathematics*. . . certain eminent modern analysts admit that the phrase *infinite magnitude* is a contradiction in terms. The Academy, therefore, desires an explanation of how it is that so many correct theorems have been deduced from a contradictory supposition, together with enunciation of a sure, a clear, in short a truly mathematical principle that may be substituted for that of the *infinite*.[69]

Kant sees this problem too in terms of construction or synthesis: 'Since *unrepresentable* and *impossible* are commonly treated as having the same meaning, the concepts both of the *continuous* and of the *infinite* are frequently rejected' (*De mundi* §1, Ak 2: 388). So the notion of a completed infinity contains no contradiction, but since it 'can never be completed through a successive synthesis' (*Kritik*, B 454), it is not the object of any possible experience, intuition, or construction. But, on the other hand, there seems no limit to space or time either; for example, 'the beginning always presupposes a preceding time' (B 515). So to the question, 'But what is the magnitude of the world we live in, finite or infinite?', Kant replies: neither; to demand an answer is to assume 'that the world (the whole series of appearances) is a thing in itself. For the world remains, even though I may rule out the infinite or the finite regress in the series of its appearances'.[70]

Many of the same considerations apply to the infinitely small. To see the problems about the continuity of space in terms of 'infinite divisibility' already plays into Kant's hands. Division is a human act, suggesting to Kant that the act of constructing a line by a continuous flowing motion comes first, followed by the construction of its parts by a further act of division (rather than the parts coming first and together forming space).[71]

The demand for constructibility is, it appears, at the bottom of such central Kantian themes as the ideality of the world. Also of the noumenon, unreachable by experience, of which the infinite is, so to speak, the first example. Lest it seem that the problem of construction is an artifact of the eighteenth century's primitive view of geometry, it may be noted that the problem recurs in the modern foundations of mathematics. There, one normally proves the consistency of a concept by constructing it out of sets, but to do so requires an 'axiom of infinity', which ensures that sufficiently many sets 'exist', in particular, that a completed infinity of them exists.

III. NEW OBJECTS OF MATHEMATICS

1. *Algebra*

Algebra tended to take over more and more of mathematics in the eighteenth century. Where Newton had recast his reasoning in geometrical form for public consumption, Joseph-Louis de Lagrange's *Mécanique analytique* of 1788 says:

> No drawings are to be found in this work. The methods which I present require neither constructions nor geometrical or mechanical arguments, but only algebraic operations, subject to a regular and uniform progression.[72]

Condorcet says that Euler

> sensed that algebraic analysis was the most extensive and certain instrument one can employ in all sciences, and he sought to render its usage universal. This revolution . . . earned him the honour, unique so far, of having as many disciples as Europe has mathematicians.[73]

And in a rare moment of agreement with Euler, d'Alembert says algebra

> is the foundation of all possible discoveries concerning quantity. . . . This science is the farthest outpost to which the contemplation of the properties of matter can lead us, and we would not be able to go further without leaving the material universe altogether. (*Oeuvres* 1: 26, see also 30–31; transl. 20 and 26)

But, having established that algebra is a good thing, what exactly is it? Originally, it was a method of solving problems by making letters stand for unknown quantities, and manipulating the letters as if they were numbers. Even on this narrow view, algebra had philosophical significance, as it was a method for discovering answers, and thus seemed on the side of 'analysis', as opposed to the 'synthetic' deriving of known truths from axioms in the style of Euclid.[74] But by 1700 it seemed more than that. Noting that the letters could stand for geometrical quantities as easily as for numerical ones, various thinkers proclaimed algebra to be the science of quantity in general, that is, virtually the whole of mathematics.[75] Even if that were agreed, many things remained unclear. For example, what could the letters represent – complex numbers? Infinities? Infinitesimals? And if algebra was a general mathematics, where were its axioms?[76] Another view of algebra was that of Wolff, who saw it as part of Leibniz's universal characteristic, that is, as a general method of reasoning symbolically.[77] In the same vein, Condillac's idea of the mathematical method that ought to be imposed on philosophy was not so much Euclid as the solving of equations, manipulating known and unknown quantities until the knowns appeared by themselves.

> *Equations*, *propositions* and *judgements* are basically the same thing, and . . . consequently one reasons in the same manner in all the sciences . . . we have seen that, just as the equations $x - 1 = y + 1$, and $x + 1 = 2y - 2$, pass through different transformations to become $y = 5$ and $x = 7$, sensation passes equally through different transformations to become the understanding.[78]

Regarding French as a language lacking taste and precision, Condillac proposed to reform it on the basis of the grammar of algebra.[79] Kant says that algebra proceeds by manipulating uninterpreted symbols 'until eventually, when the conclusion is drawn, the meaning of the symbolic conclusion is deciphered'. The simple pushing around of symbols is what gives 'the degree of assurance characteristic of seeing something with one's own eyes' (whereas with philosophy one must keep the meanings in mind all the time).[80] These are the same claims made around 1900 for formal logic. The possibility of manipulating symbols without attending to their meaning is not without problems. One may end up with conclusions that do not mean anything either. Euler is famous for his lack of rigour in calculating with infinite series without worrying about their convergence; it is typical not only of him but of the century to conduct long 'philosophical' debates about the true sum of the series:

$$1 - 1 + 1 - 1 + 1 - \ldots$$ [81]

The case is even worse when manipulating symbols that are explicitly stated to have no meaning, such as those denoting the square roots of negative numbers. Though necessary for calculations, they do not satisfy the definition of quantities as being 'capable of increase and decrease', as they cannot be less than or greater than one another. Euler describes them as 'impossible', but proceeds to calculate extensively with them.[82]

Euler played a crucial role in emphasising the centrality of the notion of *function* in mathematics (its significance is indicated by the fact that about half of modern pure mathematics is 'functional analysis'). His aim was to replace vague geometrical notions and dynamical metaphors of 'fluents' with something more precise and amenable to calculation. He initially defined a function in algebraic terms as an *expression* involving variables: 'A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.' For example, $az + \sqrt{a^2 - z^2}$ is a function of z (where a is a constant).[83] But later, his debate with d'Alembert over the vibrating string convinced him this notion was too narrow, because of the need to consider more irregular functions, which might not be expressible by an algebraic formula. He had little success in explaining what this notion should be.[84] Lagrange also attempted a purely algebraic notion of function and tried to use it as a foundation for the calculus 'independent of all metaphysics'. He claimed to prove that every differentiable function could be expressed as a power series, that is, represented algebraically (except perhaps at isolated points).[85] This is false, as Cauchy soon showed. When the best mathematicians in the world begin claiming to have proved what is false – a rare event, much to the credit of mathematics – it is time to conclude that rigour is not a luxury. The nineteenth century drew the correct conclusion, leading to the correct foundations of calculus, and to set theory.

Just visible in the work of Lagrange are the beginnings of modern abstract algebra. This is the subject that perhaps most obviously deals not with quantity but with certain kinds of abstract structure. Lagrange, inquiring why it had not been possible to find formulas for solving equations of degree 5 or higher, considers functions of the roots of the equation which do not change if the roots are permuted, or interchanged. He understands that some permutations may be 'independent' of others, thus thinking of the permutations as themselves entities with interrelationships. These permutations form the first of a new kind of subject matter of mathematics, later the object of modern group theory.[86] Paolo Ruffini's work of 1799 goes further, considering the totality of permutations (a group, in modern terms) and their composition.[87]

Any or all of these developments in mathematics might have provided the philosophers of the eighteenth century with perfect examples of the advancement of knowledge through the analysis of ideas, had they informed themselves about them.

2. *Experimental evidence in mathematics*

While the eighteenth century admired the rigour of Euclid, its own mathematics is famous for a lack of rigour. It may be that the philosophical emphasis on ideas as against formal logic contributed to a disregard of formal rigour.[88] In any case, if mathematical conclusions are to be supported by anything less than complete formal demonstration, there is a need to consider how there can be a less than deductive logical support. Euler was the first, among either philosophers or mathematicians, to argue explicitly for the use of experimental, or probable, reasoning in mathematics.

> It will seem not a little paradoxical to ascribe a great importance to observations in that part of the mathematical sciences which is usually called Pure Mathematics, since the current opinion is that observations are restricted to physical objects that make impression on the senses. As we must refer the numbers to the pure intellect alone, we can hardly understand how observations and quasi-experiments can be of use in investigating the nature of the numbers. Yet, in fact, as I shall show here with very good reasons, the properties of the numbers known today have been mostly discovered by observation, and discovered long before their truth has been confirmed by rigid demonstrations. There are even many properties of the numbers with which we are well acquainted, but which we are not yet able to prove; only observations have led us to their knowledge.[89]

Euler's works contain a number of examples of how to reason probabilistically in mathematics. He used, for example, some daring and obviously far from rigorous methods to conclude that the infinite sum $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$ (where the numbers on the bottom of the fractions are the successive squares of whole numbers) is equal to the prima facie unlikely value $\pi^2/6$. Finding that the two expressions agreed to seven decimal places, and that a similar argument led to the already proved result $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots = \frac{\pi}{4}$, Euler concluded, 'For our method, which may appear to some as not reliable enough, a great confirmation comes here to light. Therefore we shall not doubt at all of the other things which are revealed by the same method.'[90]

Laplace and Gauss, who were in a position to know, agreed casually that such reasoning was central to mathematics.[91] Even Wolff writes that 'examples of hypotheses are also found in arithmetic, which first influenced me to look upon philosophical hypotheses more favourably'. What he has in mind is the calculation of answers by successive approximation, the initial guess being the hypothesis.[92] Yet philosophers pronouncing on mathematics since have rarely given it a place.

A different connection between probability and pure mathematics was discovered by Lambert. He understands that a series of digits produced by a random process, like throwing a die, will be disordered or patternless, but that the same can be said of the digits of π or of $\sqrt{2}$, which are completely determined. He is prepared to say that the probability of the hundredth digit of $\sqrt{2}$ being five is $1/10$.[93] Whether a notion of probability can be applied in such a deterministic case is still a crucial issue in the philosophy of probability.

3. Topology

Topology provided the clearest example of an object of mathematics that would not fit under the old rubric, 'the science of quantity'. The citizens of Königsberg noticed that it seemed to be impossible to walk over all seven of the bridges connecting the two banks of the River Pregel and its islands, without walking over at least one of them twice. Euler proved they were right. This is a problem in the area now called the topology of networks. There is no quantity involved in the problem, only the *arrangement* of the system of bridges and land areas. Euler writes:

> The branch of geometry that deals with magnitudes has been zealously studied throughout the past, but there is another branch that has been almost unknown up to now; Leibniz spoke of it first, calling it the 'geometry of position'. This branch of geometry deals with relations dependent on position alone, and investigates the properties of position; it does not take magnitudes into consideration, nor does it involve calculation with quantities. But as yet no satisfactory definition has been given of the problems that belong to this geometry of position.[94]

What Leibniz said about the 'geometry of position' was both short and extremely vague,[95] but Euler was not the only one to find it suggestive. Buffon relates it briefly to the folding of seeds and to symmetry in plaiting, and remarks that 'the art of knowing the relations that result from the position of things would be as useful as and perhaps more necessary than that which has the magnitude of things only for its object'.[96] Kant sees a connection between it and his ideas on incongruent counterparts.[97] The subject was given some more definite content by Vandermonde, who first drew a graph, in the modern sense of a system of nodes connected by lines. He used it to solve the problem of the knight's tour in chess, 'using numbers which do not represent quantities at all, but regions in space'.[98]

4. Social mathematics and moral algebra

If ideas have parts, perhaps one should count them, or maybe weigh them, if they differ in their force. There would result a 'moral arithmetic', or 'moral physics'. Naturally, there are measurement problems: how are psychological units to be measured and compared, or even identified? Hume suggests that the size of the smallest impression can be found from measuring the least visible dot (*Treatise*, 1.2.1.4, SBN 27–8); such measurements of the threshold of visibility were carried out in Hume's lifetime.[99] Buffon suggests regarding the probability of sudden death in the next twenty-four hours, for one in the prime of life, as a standard unit of 'moral impossibility', to which the reasonable man gives no serious thought.[100] Maupertuis reduces morality to prudence, and prudence to a hedonistic calculus: 'The estimation of happy and unhappy moments is the product of the intensity of the pleasure or pain by the duration.' Measurement of intensities may be difficult, but Maupertuis invites introspection on the inevitability of comparing, for example, the pain of an operation for the stone with the longer but lesser pain of forgoing the operation.[101] The problem is urgent for economics, which can hardly avoid being quantitative, when explaining prices, but seems to rely on a subjective 'utility' whose measurement is as dubious as that of pleasure and pain. Adam Smith achieves the trick, so useful in these matters, of claiming the right to speak quantitatively, while avoiding the responsibility of commitment to any actual quantities or formulas. He writes that the value of any wealth to its owners 'is precisely equal to the quantity of labour which it can enable them to purchase or command. . . . Equal quantities of labour, at all times and places, may be said to be of equal value to the labourer', but he undercuts the apparent accuracy of his measure by adding:

> It is often difficult to ascertain the proportion between two different quantities of labour. The time spent in two different sorts of work will not always alone determine this proportion. The different degrees of hardship endured, and of ingenuity exercised, must likewise be taken into account. There may be more labour in an hour's hard work than in two hours easy business.[102]

He appeals to the market to coordinate different people's measures, but 'not by any accurate measure'.

Benjamin Franklin advises, in cases of perplexity about a decision, the listing of the reasons for and against in two columns:

> When I have thus got them all together in one View, I endeavour to estimate their respective Weights; and where I find two, one on each side, that seem equal, I strike

them both out: If I find a Reason *pro* equal to some two Reasons *con*, I strike out the three . . . and thus proceeding I find at length where the Ballance lies. . . . And tho' the Weight of Reasons cannot be taken with the Precision of Algebraic Quantities . . . in fact I have found great Advantage from this kind of Equation, in what may be called *Moral or Prudential Algebra*.[103]

The special difficulties of measurement in the social and mental realm suggested to Condorcet that social mathematics should rely chiefly on the theory of probability.[104] There, the equality of the beliefs one should have that a die will fall on any side is inferred from symmetry, or 'insufficient reason': there is no reason to prefer any side to any other. He believed he had proved, using probability, that decisions taken by majority vote were perfect for achieving the truth.[105] Before being hounded to death by a regime that exalted Equality over Liberty and Fraternity, Condorcet had the opportunity to reconsider the assumptions of his proof, and wonder if perhaps he did not mean that those voting had to reach some standard of Reason.[106]

As long as there has been 'social mathematics', there have been explanations of why it fails to work, or at least lacks anything like the success of mathematics as applied to physics. The suggestion that lack of exact measurement is the problem was anticipated, and argued against, as the quotations above indicate. Another idea was that of Reid, who thought the problem lay in the definition of quantity as 'whatever has increase or diminution'. This is too wide, he says, as it allows in pleasure and pain, which admit of degrees but cannot be measured in units.[107]

Mathematical modelling of social, as opposed to introspective, phenomena was attempted qualitatively in Hume's and Adam Smith's conception of the economy as a self-regulating system,[108] but the most successful quantitative project was that of Malthus, whose conclusions about the poor laws are intended to follow from a purely mathematical fact:

Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in arithmetical ratio. A slight acquaintance with numbers will shew the immensity of the first power in comparison with the second.[109]

He takes these ratios to be evident and feels no need to support them with empirical evidence. This kind of a priori fitting of formulas has particularly afflicted economics, so it is interesting to see Condorcet criticising Verri's mathematical economics on just this ground. It is true, Condorcet says, that more buyers mean a higher price, but what justification is there for Verri's assumption of a *direct* proportion between the two, if no empirical data are considered?[110] A remedy is to fit formulas to actual social statistical data. This is not a strong point of eighteenth-century mathematics, but Lambert had some understanding of how to do it.[111]

IV. LOGIC

1. *Textbook logic*

Eighteenth-century writing on 'logic' is extensive,[112] but neither the school of traditional Aristotelian logic nor their opponents, the 'men of ideas', produced much that has commanded respect since.

As systems of thought go, Aristotelian logic was one of the great survivors. For several centuries, it was attacked vigorously, in almost identical terms, by almost all the thinkers remembered by history. It was defended by nonentities. In each generation, when the dust settled, it was found to be still in control of the field (that is, of the undergraduate syllabus). This was true in 1700, just after Locke had renewed the attacks of Bacon and Descartes. It was equally true in 1800, when Aristotelian logic was about to undergo a revival in Britain. Its contents are the traditional logic of terms, judgments, and inference by syllogism, largely unchanged since the logic textbooks of the thirteenth century.

The English representative of the old school was Henry Aldrich's *Artis Logicae Compendium*, the standard Oxford textbook for the whole century. First published in 1691, it appeared in many editions, epitomes, and expansions until the mid-nineteenth century, including an English translation by John Wesley.[113] On the continent, Wolff found traditional logic satisfactory – so much so that he in effect tried, as we saw, to incorporate the rest of philosophy into it. It is a little more surprising to find Kant largely on the side of the syllogism. He has some minor criticisms of traditional arrangements of the four figures, which he thinks over-elaborate,[114] but he accepts that the syllogism is not intended to be a method of discovery, and on the whole his logic teaching agrees with tradition. He is clear about, and opposed to, psychologism in logic.[115]

2. *'Facultative' logic and Hume's 'psychologism'*

The opposing school produced voluminous works of 'logic', but they are full of what would now be called cognitive psychology, epistemology, semiotics, philosophy of logic, and introspection. They are full also of invective, against 'scholastic headpieces', full indeed of everything except logic, in the modern sense of formal logic. It is not that logic was confused with (not yet existent) disciplines like psychology. On the contrary, the scholastics had been clear about formal logic, and it was in deliberate opposition to them that the followers of the 'way of ideas' identified logic with 'psychologistic' notions instead.

Bacon, Descartes, and Locke, between them, had convinced most that the syllogism, and formal logic generally, was of no use or interest.[116] The essence of Locke's attack was that the syllogism was not useful for the *discovery* of truths, and that it concealed the fact that inference consisted in the 'agreement or disagreement' or 'connexion' of ideas.[117] Traditional logic had certainly opened the way to such criticisms by holding that logic is about 'thought' or 'judgment', and by concentrating on a single argument form, the syllogism, which has an air of being analytic and trivial. Whatever the justice of the Lockean criticisms, they failed to issue in anything better, either in new logical ideas or in textbooks. If adherence to the syllogism restricts logic, the 'agreement and disagreement of ideas' is if anything an even worse straitjacket. It does nothing to encourage the discovery of logical structure, and instead diverts logic into vapidity. It is all very well to offer advice like, 'Enlarge your general Acquaintance with Things daily, in order to attain a rich Furniture of Topics or middle Terms'[118], but how do you examine that? The logics of Crousaz, Duncan, and Watts followed the Port-Royal *Logic* in including enough of the traditional classifications, distinctions, and so on to provide some content, and in simply adding critical observations in the style of Bacon and Descartes.[119] At the end of the century, however, Reid and Campbell revert to a purely negative approach, speaking as if they have just discovered that the syllogism is not a logic of discovery.[120] But neither they nor any of their school have a replacement to offer. Campbell ventures the opinion that mathematical demonstrations are not syllogisms but does not suggest what their form is, if not syllogistic. Reid is closer to the truth in holding that mathematical reasoning cannot usually be syllogistic, as it deals with *relations* of quantities, and the syllogism is not applicable to relations.

Hume takes to its extreme the 'psychologising' of logic, and so exhibits most dramatically the problems in doing so. Plainly, there are tensions in 'naturalising' logic by reducing it to manipulations the mind happens to perform on ideas, while relying on logic as normative for argument. Hume applies general principles, such as, '*like objects, plac'd in like circumstances, will always produce the same effects*', to particular cases, without apparently noticing that he is using a formal logical principle of instantiation. Much the same could be said of his use of 'not'.[121] Hume exacerbates these difficulties by adding a sceptical project to his naturalising one. What he is sceptical about is the *logical* force of common inferences: causal inferences, inferences from 'is' to 'ought', and so on.[122] It is odd, certainly, to say that causal inference is not logically cogent but only an unavoidable habit, at the same time maintaining that all logical inference is only an unavoidable habit.

Hume's views on *inference* are seen to better advantage if they are thought of not in terms of formal logic, or even introspection, but as a research proposal to be implemented in, say, silicon chips. Modern Artificial Intelligence, like most eighteenth-century writing, is concerned with the *implementation* of a system of inference, not just the formal structure of the system itself. From that point of view, it is necessary to answer questions that do not arise in formal logic, such as how the symbols become attached to the things they mean. One must consider, in short, the 'natural history of the understanding'. It is then a matter for debate whether the syllogism needs to be explicitly represented internally, and whether one can replace an explicit generality with exemplars linked by 'custom', so that when one individual is activated, the linked ones 'immediately crowd in upon us' (*Treatise*, 1.1.7.7–8, SBN 20–1). The links between the exemplars are to be induced by the resemblance, constant conjunction, and like relations that hold between them. Hume's claim for his rules about causes that 'Here is all the LOGIC I think proper to employ in my reasoning' (1.3.15.11, SBN 175) is then a claim that can be investigated empirically: will a mechanism equipped with only the principles of association Hume names be able to reason adequately?

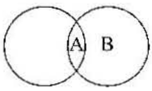
Logic's place at the centre of the curriculum makes certain wider effects of eighteenth-century logic more interesting than the subject itself. One student of logic who took the natural undergraduate reaction against the subject to an extreme was Swift, whose inversion of the stock logical examples, 'Man is a rational animal; a horse is a whinnying animal' led to the satire of the Yahoos and Houyhnhnms.[123] Another who used its rhetoric to good effect was Thomas Jefferson.[124] The claim of the Declaration of Independence, 'We hold these truths to be self-evident, that all men are created equal', combines the logical theme of self-evidence with the mathematical one of deriving such self-evidence from a symmetry principle. A true logician will ask why, if a principle is indeed self-evident, it is necessary to 'hold' it to be so. The French were clearer that Equality is not a given but a goal.

The discrediting of logic in England had consequences in education that are still felt. While tradition-bound, High Church Oxford took no notice of the problem and continued to teach logic, in Latin from Aldrich's text, and examined by disputation,[125] Whig Cambridge did the opposite. It replaced logic by the only credible alternative, mathematics, and produced the ancestor of the modern written examination system, the Mathematical Tripos. By a happy feedback effect, mathematics permitted an ever finer objective grading of candidates, leading to ever more concentration on mathematics. Since mathematics was a substitute logic, however, the matter examined was confined largely to geometry, continental innovations like algebra being considered unpatriotic.[126] Geometry is also more amenable to being considered in terms of 'ideas' than the formal manipulations of algebra.[127]

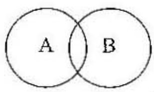
3. Symbolic logic and logic diagrams

Leibniz's vision of a universal characteristic, allowing logical inference by calculation, inspired some, but resulted in little of significance. The logical symbolism of Segner, Ploucquet, Holland, Maimon, and Castillon does not need much reinterpretation to yield various theorems in propositional and predicate calculus, but only the simplest ones.[128]

Euler developed the traditional theory of the syllogism in a popular work, illustrating it with diagrams similar to the later Venn diagrams. Particular (existential) propositions have always posed problems for such diagrams, as there needs to be some way of indicating which of the regions are non-empty. Euler distinguishes between 'Some A is B' and 'Some A is not B' as follows:



Some A is B



Some A is not B[129]

If one believes that it is a good thing for logic to become extensionalist, then logic diagrams will appear one of the century's few advances in logic. Euler himself does not mean to be taken this way. He uses only intentional vocabulary, such as 'If the notion C is entirely contained in the notion A . . . ' He takes no notice whatsoever of two centuries of criticism of the syllogism, and goes so far as to maintain that all truth arises from it.

A similar idea, but using lines instead of circles, appears in Lambert.[130] But this is only a small part of a larger project for the mathematising of logic. He proposes to give some precision to the analysis of concepts into simple ideas, thus doing for quality what geometers had done for quantity, and deducing everything from a firm basis.[131] An aspect of the project was a symbolic logic of concepts; one analyses a concept $a\gamma$ into its genus a and differentia $a\delta$; the equation

$$a\gamma = a - a\delta$$

then means that the genus of a is the result of abstracting the differentia from a. Lambert is sometimes misled by false analogies to ordinary algebra, to the extent of considering the square root of a relation.[132] In another attempt to bring logic and mathematics together, he considers the valid argument:[133]

$$\frac{3}{4} \textit{ of A are B, } \frac{2}{3} \textit{ of A are C, so some B are C.}$$

Few logicians since have cared to follow him into such numerical territory.

V. TECHNOLOGY

The subtitle of the *Encyclopédie* is *Dictionnaire raisonné des sciences, des arts, et des métiers*. The prominence given to the 'arts and trades' is due to Diderot, who writes:

> Let some man go out from the academies and down into the workshops, and gather material on the arts to explain them in a work which will persuade artisans to read, philosophers to think usefully, and the great to make at last some worthwhile use of their authority and wealth.[134]

'Useful science', as an idea, is Baconian, but science as an accepted route to profit, military superiority, and progress is really an eighteenth-century development. While the French government and the Royal Navy were among the largest investors in research, the practical orientation of research was especially evident in peripheral regions, where abstract thought, including philosophy, survived at all only on the promise of its practicality. Boundless confidence in the usefulness of science was as characteristic of the America of Franklin and Jefferson[135] as it was of Russia, where Euler and Lomonosov worked assiduously on 'improvements'. In England, the Industrial Revolution was associated less with London than with the provincial cities that were the homes of 'Philosophical Societies' devoted to practical science.[136]

Diderot found a significant fact about practical knowledge: it could not be written down adequately in text. Asking the practitioners to clarify it produced simply a garbled mass of unintelligibilities and inconsistencies. It proved essential to ask the tradesmen to *show* what they were doing and present the result in pictures. Hence the *Encyclopédie* has eleven volumes of plates (compared to seventeen of text).[137] Though there were no large-scale encyclopedic projects in England, their place was to some extent taken by public lectures on science, especially useful science. The famous London lectures of John Desaguliers taught by showing working machines. Science thus became accessible to those lacking mathematics; the Newtonian philosophy, Desaguliers says, 'tho' its truth is supported by Mathematicks, yet its Physical Discourses may be communicated without. The great Mr Locke was the first who became a Newtonian Philosopher without the help of Geometry'.[138] Inventions like the steam engine, the lightning rod and balloons were certainly spectacular and capable of conveying a message without the need for supporting captions.

But what message? The relation of machines to abstract thought was a vexed one. The formula for gravity is not much use, while the textile and steam engines were mostly invented by practical engineers, not scientists. Still, inventions are 'efforts of the mind and understanding which are calculated to produce new effects from the varied applications of the same cause, and the endless changes producible by different combinations and proportions',[139] that is, intellectual products. Adam Smith, who recognises the importance of machine inventions in improving productivity (though he tends to subordinate it to his *idée fixe* of division of labour), speaks of 'philosophers or men of speculation, whose trade it is, not to do anything, but to observe every thing; and who, upon that account, are often capable of combining together the powers of the most distant and dissimilar objects'.[140] The description is exactly true of James Watt, mathematical instrument maker to the University of Glasgow, who analysed the heat losses in Newcomen's steam engine and realised that the condensation was a separable process that could be better situated somewhere else.[141] The skills involved are cognitive, but they are not so much the formal geometry of Euclid as the draughtsmanship or design of the engineer – Diderot's 'experiential and manipulative mathematics', or the 'practical geometry' which Swift's Laputans 'despise as vulgar and mechanic'. And it was the eighteenth century's advances in cast iron and steel making that meant any shape could be made cheaply and durably. The availability of arbitrary rigid shapes, cheap, long-lasting, and reliably resistant to high pressures, stimulated imaginations to fashion intricate geometries of interacting parts. The iron machines are concrete realisations, so to speak, of several philosophical projects at once: Bacon's useful science, Kant's constructions, Vico's 'maker's knowledge', and Descartes's dream of explaining the world as the effect of interactions of rigid bodies.

There was some opposition to the idea of the beneficence of 'useful' science. Swift's satire attacked scientific research as either divorced from reality or productive of inventions that did not actually work.[142] But in general, technology had a positive glow, like mathematics, sufficient to tempt philosophers of most persuasions to claim it as on their side. Derham's *Physico-theology*, for example, saw the advances in mechanical inventions as evidence for God's providence.[143] *L'homme machine* may have seemed an idea of obviously atheist consequence in Paris, but Paley knew a good deal more about machines than La Mettrie, and convinced most, at least in the short term, that the teleological aspect of machines supported a theist interpretation of the man-machine analogy. 'Watches, telescopes, stocking-mills, steam-engines, &c.' are not the kind of things that can arise by chance – not even chance followed by selection.[144]

The consequences for political philosophy of Diderot's praise of artisans are generally left implicit in the *Encyclopédie*, but Hume's essay *Of Refinement in the Arts* supplies the gap:

We cannot reasonably expect, that a piece of woollen cloth will be wrought to perfection in a nation, which is ignorant of astronomy, or where ethics are neglected.... Can we expect, that a government will be well modelled by a people, who know not how to make a spinning-wheel, or to employ a loom to advantage?... a progress in the arts is rather favourable to liberty, and has a natural tendency to preserve, if not produce a free government.[145]

The idea that machines create progress autonomously has remained an attractive one for the Enlightened. Citizen Gateau, administrator of military provisions, writes of the machine that has come to be most associated with Liberty:

Saint Guillotine is most wonderfully active, and the beneficent terror accomplishes in our midst, as though by a miracle, what a century or more of philosophy and reason could not hope to produce.[146]

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