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Discrete and Continuous: A Fundamental Dichotomy in Mathematics

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Synopsis

The distinction between the discrete and the continuous lies at the heart of mathematics. Discrete mathematics (arithmetic, algebra, combinatorics, graph theory, cryptography, logic) has a set of concepts, techniques, and application areas largely distinct from continuous mathematics (traditional geometry, calculus, most of functional analysis, differential equations, topology). The interaction between the two — for example in computer models of continuous systems such as fluid flow — is a central issue in the applicable mathematics of the last hundred years. This article explains the distinction and why it has proved to be one of the great organizing themes of mathematics.

1. Introduction

Global versus local, discrete versus continuous, simple versus complex, linear versus nonlinear, deterministic versus stochastic, ordered versus random, analytic versus numerical, constructive versus nonconstructive — these contrasts are among the great organizing themes of mathematics. They are forks in the road of mathematical technique — the concepts along one fork are very different from those along the other, even when they give complementary views on the same phenomena.

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It is hard to find a clear exposition of any one of those contrasts. (An account of the global/local distinction is in [14].) Perhaps the most deep-rooted contrast is that between discrete and continuous. It is so ubiquitous in mathematics that the lack of a straightforward overview of the whole topic and explanation of its significance is astonishing.

The basic distinction is clear enough, but hard to define in terms of anything simpler.

Discrete: "separate; detached from others; individually distinct";

Continuous: "extending without interruption of substance; having no interstices or breaks; having its parts in immediate connection" (Oxford English Dictionary).

The integers are discrete, the real line is a continuum. Matter may be continuous or discrete (atomic) — it cannot be determined a priori and scientific investigation is needed.

According to the view of mathematics standard up to modern times, mathematics is the "science of quantity". Quantity is a property that physical things have, and the way to find out about the quantity of something is to count (if the quantity is discrete) or measure (if it is continuous). Mathematics thus has two main branches, arithmetic (dealing with the discrete) and geometry (dealing with the continuous). That view certainly still makes good sense of elementary mathematics as taught in school, and indeed, of almost all the mathematics discovered up to the seventeenth century (by which time the calculus came to seem more the science of the continuous as such) [13, Chapter 3]. The origins of this bifurcation in mathematics lie, like so much else, with the Greeks.

2. The incommensurability of the diagonal

The significance of the discrete/continuous distinction, which established it as one of the great themes of mathematics, became clear with the ancient Greek discovery of the incommensurability of the diagonal. It is necessary to explain this discovery in its own terms, as the common modern reinterpretation of the result as the "irrationality of the square root of 2" obscures its original meaning.

In the most obvious continuous cases, like length, one chooses a unit arbitrarily and measures the ratio of all other lengths to the unit. ("By Number", Newton says in his magisterial prose, "we understand not so much a Multitude of Unities, as the abstracted Ratio of any Quantity, to another Quantity of the same kind, which we take for Unity" [27, page 2].) That can give the impression that continuous quantity is not fundamentally different from discrete: to convert any continuous problem to a discrete one, it is just necessary to find a small enough unit to measure all the continuous quantities involved. Given a ruler divided finely enough, it should be possible to find the ratio of any continuous quantities, say lengths, by counting exactly how many times the ruler's unit is needed to measure each quantity. One length might be 127 times the unit and another 41 times, showing that the ratio of the lengths is 127 to 41.

That natural and even compelling thought is incorrect, as the ancient Greeks discovered. Perhaps the first truly surprising result in mathematics was the one attributed (traditionally but without much evidence) to Pythagoras, the proof of the incommensurability of the side and diagonal of a square. There is no unit, however small, which measures diagonal and side, that is, of which both are whole number multiples.

The method by which the Pythagoreans discovered this is unknown. (It certainly did not resemble our modern algebraic proof of the irrationality of $\sqrt{2}$.) No method is entirely easy. It was most likely something like this (the relevant brief ancient texts of Theon of Smyrna and Proclus are given in [10, pages 58 and 101] and in [39]). Given any two lengths (not yet divided into units), it is possible to find the largest unit which "measures" them (if there is one) by a process of *anthyphairesis* or "reciprocal subtraction". It is the same process as the Euclidean algorithm for finding the greatest common divisor of two numbers, but applied to continuous magnitudes.

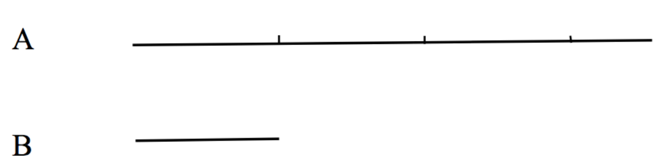


Figure 1: First stage in *anthyphairesis* of two lengths.

Given two lengths A and B, we see how many times the smaller one (say B) fits into the larger one A (in the example, 3 times). If it does not fit exactly a whole number of times, there is a remainder R that is smaller than both A and B. (If B does fit exactly, then of course B itself is the unit that measures both A and B.) Any unit that measures both A and B must also measure R (since R is just A minus a whole number of B's). So we can repeat the process with R and B, either finding that R measures B (and hence A as well), or that there is a smaller remainder R', which must also be measured by any unit that measures A and B. And so on. Since we always get smaller remainders at each step, we work our way down until the last remainder is the unit that measures all previous remainders and hence also measures A and B.

Now, what happens if we apply *anthyphairesis* to those two very naturally occurring lengths, the side and diagonal of a square? The side fits once into the diagonal, with a remainder left over, which we can lay off against the side, and ...

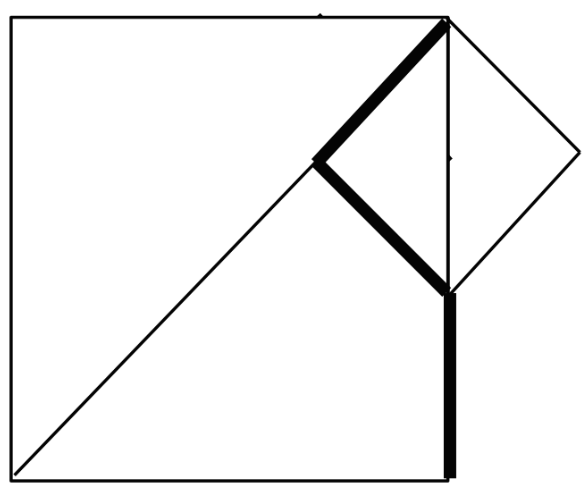


Figure 2: First stage in *anthyphairesis* of diagonal and sides of a square.

The first remainder (diagonal minus side) is the length drawn in thick lines. It appears three times in the diagram. It fits twice into the (original) side, and when we take the (small) side length out of the (small) diagonal, we are in the same position as we were originally with the larger square: taking a side out of a diagonal. Thus the small square, with its diagonal, is a repeat of — the same shape as — the large square with its diagonal, so *anthyphairesis* goes into a loop and keeps repeating: at each stage, one side-length is taken out of one diagonal. Therefore the remainders just keep getting smaller and smaller and the process never ends. There is thus no unit that measures the original diagonal and side. The diagonal and side of a square are "incommensurable".

So the ratios of continuous quantities are more varied than the relations of discrete quantities. The Greeks drew the conclusion, surely rightly, that geometry, and continuous quantity in general, is in some fundamental sense richer than arithmetic and not reducible to it via choice of units. While much about the continuous can be captured through discrete approximations, it always has secrets in reserve.

Later work showed that the continuum could, in some sense, be reconstructed in the discrete. But not easily, naturally or neatly. Infinite decimals proved usable, but they take a lot of symbols and it is awkward that 0.9999... is equal to 1.0000... although the symbol strings are completely different. (For a history of this gradual development, see [26].) Infinitesimals, which look discrete but are supposed to be tiny enough to merge into the continuous, are very messy [2, 43]. Cantor's standard discrete "construction" of the continuum involves equivalence classes of Cauchy sequences of rationals (themselves equivalence classes of pairs of integers), of cardinality an infinity higher than that of the whole numbers. Even then, the result requires that some infinite sets of points add up to a finite length and some do not, so one has to start again to define measure. If that is what it takes to imitate the continuous in a discrete structure, the Greek insight that the two are fundamentally distinct is arguably vindicated.

3. Discrete Structures from Continuous

The second major discovery of the Greeks was that the continuous sometimes naturally gives rise to discrete structures, structures not immediately visible but which give a deep insight into the original continuous phenomenon.

The ancient classification of mathematics included not only pure geometry and pure arithmetic, but three "subalternate" or applied mathematical sciences, astronomy and optics (subalternate to geometry) and music (subalternate to arithmetic). The reason music was considered allied to arithmetic was the discovery, also attributed to Pythagoras, of the connection between the harmony of notes and the whole-number ratios of the lengths of the strings producing them. Although the lengths of strings and the pitch of notes are both quantities whose nature is to vary continuously, a discrete pattern emerges from them: for strings of a fixed tension, the easily perceived harmonies of octave, fifth and so on are produced by pairs of strings whose lengths are in small whole-number ratios. The simplest non-trivial ratio, 2 : 1, produces the most prominent harmony, the octave. Later science has explained the mystery and at the same time found one mechanism for how continuous variation can produce discreteness: pitch is caused by the frequency of soundwaves, and there is something special about the frequencies of waves in small integer ratios. They interact to produce regular large peaks that affect the eardrums, which randomly-attuned frequencies do not.

Since then, the interplay between the discrete and the continuous has been at the heart of some of the most profound advances in science and mathematics.

4. Eigenvalues, resonance, algebraic topology, and quantum mechanics

Consider the eigenvectors of a linear map of a vector space to itself. A linear map is a continuous function, but its eigenvectors (with their eigenvalues) form a small discrete structure which gives, so to speak, the essence or skeleton or "big picture" of the map. If we calculate that the eigenvectors of the matrix

$$\begin{pmatrix} 13 & 4 \\ 4 & 7 \end{pmatrix}$$

are

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ (with eigenvalue 15) and } \begin{pmatrix} -1 \\ 2 \end{pmatrix} \text{ (with eigenvalue 5),}$$

then we know in full what the map "looks like": it expands the plane 15 times in the direction of the first eigenvector and 5 times in the direction of the second eigenvector, as in Figure 3 on the next page.

The phenomenon can be literally seen in the motion of coupled oscillators, as in Figure 4. Equal weights A and B moved freely on a frictionless surface impelled by three equal springs, the outer two of which are fixed to walls.

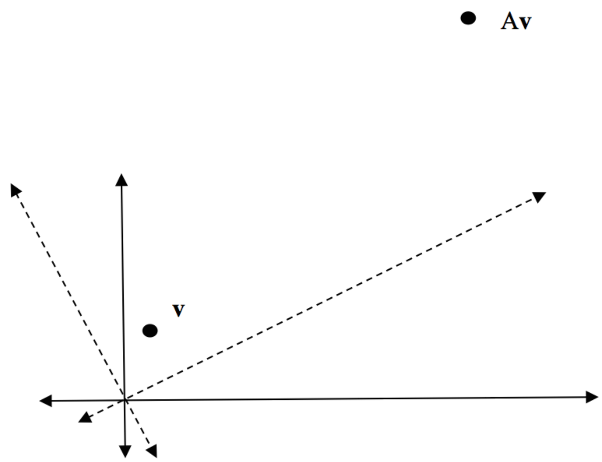


Figure 3: Linear map with eigenvalues 15 and 5 in perpendicular directions.

There are two "normal modes", eigenfunctions of the motion, which can easily be seen by choosing the initial conditions carefully. If we pull A left and B right an equal distance and release them, they move in and out in equal and opposite harmonic motions. That is the first normal mode. The second arises if we move both an equal distance in the same direction. On release, they move back and forth in tandem. These simulations are easily run in the mental visualization facility. It is harder to imagine what happens if we release the weights from more complicated initial positions, but the mathematics shows that general motions are superpositions (that is, linear combinations) of the two normal modes. (Animations are easily available, see for example http://lectureonline.cl.msu.edu/~mmp/applist/coupled/osc2.htm, accessed on June 17, 2017.) They are the eigenfunctions, acting as a natural basis, or discrete "skeleton", of the continuous space of all the possible motions.

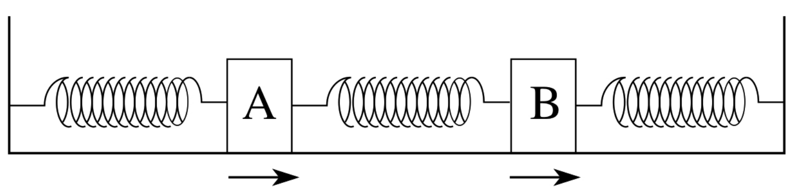


Figure 4: Coupled oscillators. Image created by Gerald O. Nolan.

The most central and deepest interplay between discrete and continuous, however, lies in algebraic topology. Jean Dieudonné predicted that "the 20th century will come to be known in the history of mathematics as the century of topology, and more precisely of what used to be called combinatorial topology, and which has developed in recent times into algebraic topology and differential topology" [5, page 7]. The word "topology" refers to the continuous spaces that are the original objects of study, while "algebraic" refers to the discrete structures such as homology groups, Betti numbers and the like which form the deep structure which distinguishes those spaces which are "really" — topologically — different.

To take a simple illustrative example, the fundamental group of a (path-connected) space consists of the equivalence classes of loops in the space, loops being identified if one can be deformed (in the space) into the other. Thus the fact that the fundamental group of the torus is $\mathbb{Z} \times \mathbb{Z}$ corresponds to the fact that a loop cross-sectionally across the torus cannot be deformed to one around the middle, as in Figure 5:

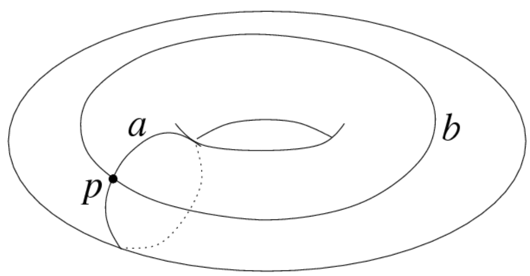


Figure 5: Loops a and b on the torus are not deformable to each other. Image created by `Yonatan-commonswiki`, available at https://commons.wikimedia.org/wiki/File:Fundamental_group_torus2.png with a CC-SA 3.0 license.

The loops themselves are continuous objects, but their identification under deformability creates a discrete object, the fundamental group, which characterizes the global topological structure of the space.

The ways in which the Schrödinger Equation (a partial differential equation in continuous space and time) gives rise to discrete entities like electron orbitals and the collapse of the wave function is of course a major theme of quantum mechanics. That is what makes it quantum mechanics. The story is too large to go into here, but parallels the way in which eigenvectors arise from linear maps.

5. Dispatches from the long war between the continuous and the discrete

The problem is that the continuous is easier to imagine and prove results about, but the discrete is unavoidable when it comes to calculation. Since Hipparchus and Ptolemy predicted the smooth orbits of the planets by calculating with up to five places of sexagesimals [36, pages 57-58], it has been an exceedingly painful process to make the continuous and the discrete match approximately. Mathematicians sometimes say, especially when advertising themselves to outsiders, that they are guided by mathematical beauty, and seek only beautiful mathematics. “Beauty is the first test,” said G.H. Hardy, “There is no permanent place in this world for ugly mathematics” [19, page 85]. Surely no-one can say that with a straight face after struggling with how to prevent a fourth-order Runge-Kutta method from degrading the sharpness of a wave front. Numerical analysis of schemes to approximate ordinary and partial differential equations is essential mathematics, but excruciating.

The most familiar discrete approximation of the continuous is probably the definition of the Riemann integral. The integral of a continuous function is a continuous entity, while the lower and upper sums are discrete entities — finite sums — whose values approximate it. (See Figure 6 on the next page.) There are plenty of theorems on the goodness of the approximation, in terms of the variation of the function and the fineness of the partition. That means that the way the discrete approximation works is tightly controlled and well understood.

What is misleading about the Riemann integral as a case study of discrete approximations to the continuous is simply that integration is more numerically stable than differentiation. The definition of a derivative involves division by a difference of nearly equal numbers, leading to any errors in the denominator blowing up. Nothing like that occurs with numerical integration, where small errors in the abscissa or ordinate make little difference to the computed integral (unless there is something very strange about the integrand). That is one source of the problems that are endemic in numerical approximations to differential equations, both ODEs and PDEs. (Other sources include the chaotic nature of some of the differential equations.) Trade-offs are inevitable. A finer discretization of space and time is more desirable — for the same reason as a finer partition to approximate a Riemann integral

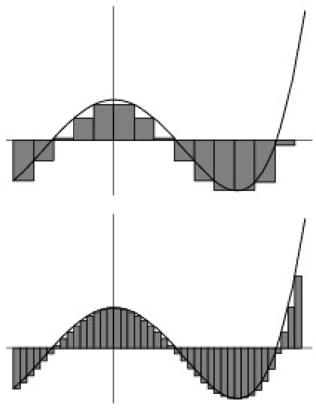


Figure 6: Lower sums approximating the Riemann integral of a continuous function. Image from http://mathworld.wolfram.com/LowerIntegral.html, accessed on June 17, 2017.

— but at the same time it gives more opportunity for errors to accumulate across the many computational steps, so that the discretized system loses track of the continuous reality being modeled.

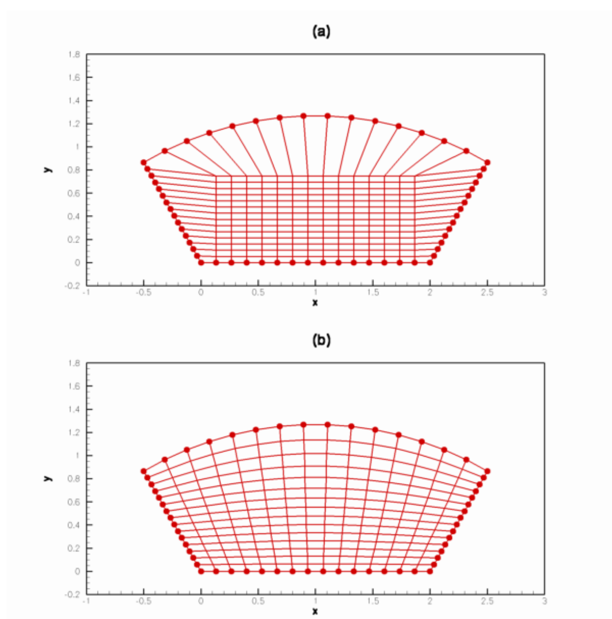
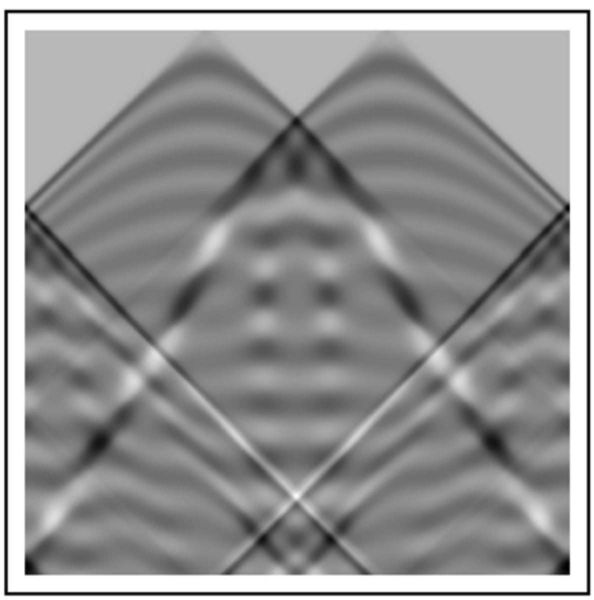


Figure 7: Adapting the grid for numerical solution of a PDE to the region of integration. This image is licensed under the GNU Free Documentation License. It uses material from the CFD Wiki article 'Mesh generation' at CFD Online.

It is not surprising then that several fronts have opened up in the war between continuous and discrete perspectives, in areas where the exact relation is harder to establish. One concerns the value of discrete "agent-based" or cellular automata models of self-organizing phenomena in economics, ecology, combat, traffic and so on, versus continuous PDE models. Agent-based models propose individual agents with a few basic properties, which interact with nearby agents according to a few update rules. It is often observed that quite simple rules will generate complex emergent behavior in the mass, and it is easy to attribute a kind of "swarm intelligence" to the collection of agents. PDE models, by contrast, appear to admit of only simple local influences and would be expected to give rise to less complex and more predictable global behavior.

That natural thought appears to be wrong. The remarkable pictures on pages 165–166 of Wolfram's *A New Kind of Science* [42] show that the phenomenon central to Wolfram's book, the generation of complex behavior from simple rules, is found in PDEs as much as in cellular automata.



sine-Gordon equation

Figure 8: A partial differential equation imitates the "simple rule, complex behavior" of cellular automata. Stephen Wolfram, *A New Kind of Science*, Wolfram Media, Champaign IL, 2002, 923, Copyright: Stephen Wolfram, LLC, http://www.wolfram.com.

Similarly in a field like combat modelling, the tendency to anthropomorphize simple agents which exhibit complex behavior in the mass will be lessened when we realize that PDE models can do the same. ([20, 6]; in the reverse direction [34])

Another front concerns the modelling of physical reality, such as fluid flow. Real fluids are discrete ensembles of molecules undergoing flow that is partly smooth and deterministic and partly stochastic (when molecules collide). Computation of fluid flow is also discrete, but one very rarely uses a model at the molecular level. Usually one makes do with a continuous model of flow which is hoped to approximate the real situation, then approximates that with a discretized PDE as above. Both stages of the approximation involve major mathematical difficulties, the first from the awkwardness of approximating a stochastic process by a continuous flow [33], and the second from (in addition to numerical difficulties) the well-known and obvious problems of the chaotic nature of turbulent flow.

What should be emphasized is that the success (or not) of these approximations is not to be explained by philosophical hand-waving about "idealizations". It is a provable mathematical fact (or not, as the case may be) that the continuous model of fluid flow approximates well the real mathematical structure of bouncing molecules on the one hand, and on the other hand the system of difference equations used in the computation.

Other fronts in the discrete-continuous war — too extensive to be surveyed here — include the poor attempt of Artificial Intelligence to ground discrete symbols in the continuous flow of perception (e.g. to transform continuous visual input into a list of objects in a scene), and the endemic inability of discrete logic to cope with arguments involving continuous variation, such as fuzzy logic and extrapolation arguments [12, 29].

As the radical Left used to say, "The struggle continues."

6. The discretized continuous versus the truly discrete

The examples of the Riemann integral and numerical simulations of differential equations raise a complex question, hard to focus on, concerning whether such fine-grained discretizations of continuous phenomena are "really" discrete mathematics. The question has been discussed in philosophy, following David Lewis, under the rubric: Is the analog/digital distinction the same as the continuous/discrete distinction?

The issue is easily appreciated in the example of a clock. Mechanical clocks represent time continuously, while a clock with a digital display represents it discretely. But what about a watch with a normal face which has a second hand that moves forward each second? It is in one way discrete, as it represents no intervals smaller than one second. But it is in one sense "analog", as in analogous, because it represents time by a quantity (angle) that varies linearly with time. It, so to speak, pictures time directly [1, 22, 24]. How finely it does so is irrelevant to the mode in which it represents, and is quite different from the way a digital display does so. A discrete analog representation of time thus has much in common with a continuous representation, and is quite unlike a digital display. So it is arguable that the distinction analog/digital cuts across the distinction continuous/discrete.

That debate is about representation, and is intended to cast light on how the mind or brain represents reality: is that fundamentally analog or digital? But the issues are not confined to that context. A discretized PDE is like the PDE it approximates — that is the point of it. Discrete analysis, as in the analysis of finite approximations to a Riemann integral, is obviously close to real analysis, and discrete but finely divided functions behave in many ways like continuous functions. (One can pursue discrete analysis by itself, pretending that its continuum approximation does not exist, but it is not easy [44]. Smooth functions, like smooth chocolate, are our preference, but we can cope with the gritty variety if need be.)

Perhaps there is no definite right or wrong answer to the question of whether the discrete and the digital are really the same. But since the original aim of this article was to discuss the fork in the road between discrete and continuous mathematics, it seems fair to note that there is a major difference in style of thinking between the discretized continuous, as in simulations, and mathematics that is inherently discrete from the start, like number theory, combinatorics and cryptography.

7. The signature of the discrete: simple integer ratios in Dalton and Mendel

We now turn to some more applied fields and ask what evidence is relevant to deciding whether a reality being studied is discrete or continuous. If something is continuous as far as perception can tell, it can be very hard to tell if it is discrete at a smaller scale.

What is the evidence that atoms exist? The ancient Epicureans and some seventeenth-century philosophers proposed them as neat explanations of various phenomena, but they could not point to any convincing cases where an explanation in terms of atoms was superior to one using the alternative and more natural hypothesis that matter is continuous. The first convincing argument for atoms was that of Dalton, who discovered instances of what is now called the law of multiple proportions — if two elements form more than one compound between them, then the ratios of the masses of the second element which combine with a fixed mass of the first element will be ratios of small whole numbers. An atomic structure of matter gives a good explanation while a continuous one has little prospect. Dalton wrote:

> The element of oxygen may combine with a certain portion of nitrous gas, or with twice that portion, but with no intermediate quantity. In the former case nitric acid is the result; in the latter nitrous acid ... Nitrous oxide is composed of two particles of azote and one of oxygen. This was one of my earliest atoms. I determined it in 1803, after long and patient consideration and reasoning. [4, 35]

Another easily appreciated example is Mendel's work on genetics. One of the many problems with Darwin's original theory of evolution concerned "blending inheritance". Natural selection, according to Darwin, acts on small mutations that occur rarely and at random in populations. But since the possessor of such a mutation has to breed with the "normals" of the population, surely a favorable mutation should blend back into the population over a few generations before natural selection has a chance to act? Like many other problems with evolutionary theory, it was only admitted to be serious once it had been solved. The solution came with Gregor Mendel's discovery that the genetic material was, often at least, discrete. And the discovery of discreteness, through Mendel's experiments with peas in his monastery garden, came, as it had with the Pythagoreans and Dalton, with the noticing of small whole-number ratios in material that was in its nature continuous.

Mendel took two lines of peas, short and long, each bred over several generations to be "pure" or "true". That is, the short ones produced only short ones and the tall ones only tall ones. Then he crossed them, that is, pollinated the flowers of the short ones with pollen from the tall ones. With a genius for "population thinking" exceedingly rare in his age, he thought to perform this experiment many times and count the results. He found, not very excitingly, that all of the crosses were tall (not somewhere between short and tall, but clearly tall). But if this generation of tall ones were crossed with one another, the results were not all tall but consistently about 3/4 tall and 1/4 short. The simple phrase "consistently about 3/4..." is deceptive. It makes something hard to find look easy. The discreteness of the ratio 3 : 1 does not reveal itself in any single experiment, especially a small one with one or two plants. One must take a count in a large number of experiments, and be content that the ratio is not exactly 3 : 1.

Mendel correctly posited a theory of discrete factors or genes that would explain the simple ratio. He supposed that length of peas depends on each individual's having two genes for height, each either "tall" or "short", with the combination "tall" and "short" resulting in the same appearance as two "talls", and with random assortment of genes between generations. When first generation plants (all appearing tall but in fact all having one "tall" and one "short" gene) are crossed with one another, the next generation has about one quarter "tall-tall", half "tall-short" and a quarter "short-short"; only the last quarter appear short.

With one bound, Darwin's theory was free of the blending inheritance problem. Assortment of discrete genetic material is capable of giving a mutation several chances to express itself fully.

There have been widespread doubts as to whether Mendel's published results were too close to the 3 : 1 ratio to be convincing. R. A. Fisher loudly accused him of fraud. As his personal papers do not survive, it is hard to know for sure. The results he claimed for the tall-short experiment were 1787 tall to 227 short, or 2.84 : 1. That is not very close to 3 : 1, and indeed would only be recognized as a 3 : 1 ratio in the context of several other similar results. When looking for the discrete in the continuous, one needs to know what one is looking for.

Defenses of Mendel have appeared, but to little effect [7, 28]. "Hero's feet found not of clay after all" — what kind of a headline is that?

8. Space and time: discrete or continuous?

Is space, on the small scale, continuous or discrete? That is not known at present, nor it is known whether it could be known in principle.

We have become used to thinking that matter is discrete but space and time are continuous. That is inherently rather unlikely — why the difference? The ancients took it for granted that either matter, space and time were all continuous (as Aristotle thought) or all of them were discrete (as the Epicureans thought) [41]. Euclid's geometry incorporates the assumption of the continuity of space (for example, in two concentric circles, all the radii of the larger circle intersect the smaller circle in distinct points — it cannot happen that there are insufficient points in the inner circle to keep the inner intersections distinct). As Euclidean geometry was thought to be necessarily true of real physical space, that put the immense authority of mathematics behind the assumption that space is continuous. Pascal, for example, in his correspondence with Fermat ([30], discussed in [11, page 305]) which founded the mathematical theory of probability in response to an alleged paradox from the Chevalier de Méré, wrote that de Méré's incompetence in mathematics was obvious from the fact that he thought an interval of space had only a finite number of points.

But de Méré was right, at least to the extent that the continuity of space cannot be proved. The small-scale shape of space, just as much as its curvature or its large-scale topology, is an empirical matter, to be decided on the basis of experimental results [9]. The possibilities for the microstructure of space are quite diverse: not just $\mathbb{R}^3$ and $\mathbb{Z}^3$, but various discrete lattices regular and irregular, non-locally-simply-connected spaces that are so to speak foamy all the way down, and homogeneous but imprecise spaces where, below a certain scale, there is no fact of the matter as to where anything is.

The verdict of modern physics on the question is so far ambiguous. If space is discrete — at the Planck length or below, around the scale of 10^{-35}m — it has left no clear signature on the observable world. It might have, since some microstructures of space, no matter how small, can make a difference at the macroscopic level. For example, if space at the small scale consists of atoms arrange in a cubic lattice, then the axial directions are distinguished, and if, as one would naturally suppose, the distance from A to B is the number of the shortest path through adjacent atoms, then the diagonal of a 1×1 square with sides in the axial directions has length 2, not $\sqrt{2}$, see Figure 9.

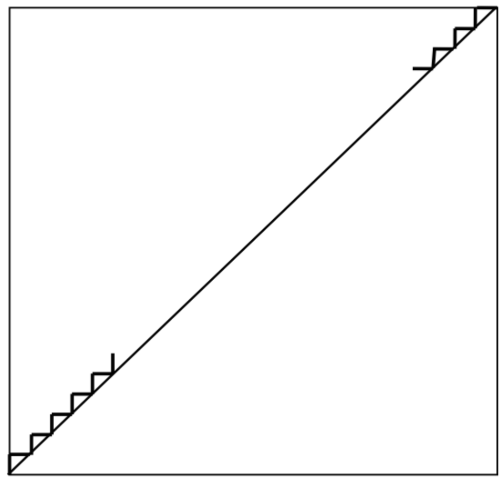


Figure 9: Length of shortest diagonal path in $\mathbb{Z}^2$.

It would seem to follow that motion in some directions should be observably much slower than in others. However, no such macroscopic anisotropy is observed in real space [40, page 43].

The standard modern physical theories, relativity and quantum mechanics, are expressed in terms of continuous space and time but there are no observations — perhaps no possible observations — to confirm directly that that is so. Erwin Schrödinger, like many physicists dealing with the very small, was impressed with how elaborate the structure of the mathematical continuum is and how little observational support there is for supposing it is instantiated in its entirety in real space. The observations on which quantum mechanics are based are discrete, and Schrödinger wrote that the "facts of observation are irreconcilable with a continuous description in space and time" [31, page 40]. Richard Feynman said:

> It always bothers me that, according to the laws as we understand them today, it takes a computing machine an infinite number of logical operations to figure out what goes on in no matter how tiny a region of space, and no matter how tiny a region of time. How can all that be going on in that tiny space? Why should it take an infinite amount of logic to figure out what one tiny piece of space/time is going to do? [8, page 57]

However orthodoxy in quantum mechanics has taken discreteness to be a fact about observation, and has founded the theory on the (unobservable) wave function, which gives the description of a system as a function of the usual continuous space and time. Discreteness then reenters in the "collapse of the wave function", which produces discrete observations but does not cast doubt on the continuity of space or time [17, Chapter 1]. Some less standard later physical theories have raised many proposals for deriving our apparently continuous space and time from something more basic, possibly discrete, but no such theories have become firmly established; nor on the other hand has that approach been ruled out (e.g. [25, section 4]; [32, 38]; the search in quantum gravity in [18])

In those circumstances, it is possible to suggest that the universe is digital in its entirety and to offer a discrete mathematics to do all of physics, as sketched in Wolfram's *A New Kind of Science* [42, Chapter 9]. But that is as much pure speculation as Epicurus's atomism was. At the moment, it is simply unknown whether space and time really are continuous.

9. Should we abolish the continuous?

If the continuous were abolished, how much serious mathematics would be left? Would we cope?

We know the answer to that question, because we have been teaching mathematics to computers for decades. When computers came to do mathematics, the continuous was abolished, since digital computers are finite objects. They deal in whole numbers with a fixed maximum size. Instead of real numbers, they are restricted to "floating point numbers" of limited precision, whose mathematics is awkward but in quite a different way from the continuum [15, 37]. Computer graphics packages do geometry on a large but finite grid of points. Symbolic manipulation packages such as Mathematica, Matlab and Maple manipulate finite formulas and solve differential equations, draw graphs and can pass mathematics exams more reliably than most mathematics students. The search for theorem-proving and especially theorem-discovering software has been much less successful, but there are some worthwhile advances [3, 23]. The end result is that finite machines with finite resources can output a product that reads to humans like mathematics, in greater quantity and quality than any individual human.

The success of computers in doing mathematics (or imitating it, if one insists that mathematics must be a product of a mental process) depends on two facts. The first is that so much of the mathematics we need to do is finite. The second is a purely mathematical fact about the abilities of discrete and continuous mathematical structures to imitate one another.

First, much of what we really need to do in mathematics is finite and requires no reference even remotely to infinity. The truth that $2 \times 3 = 3 \times 2$ is purely finite. It stands on its own irrespective of any generalizations of which it may be an instance. One could choose to derive it from Peano's axioms of arithmetic, which do refer to the infinite system of numbers, but that does not make the truth itself have any reference to infinity — the derivability is just a consequence of the obvious fact that a finite structure can be embedded in an infinite one. All the (finitely many) arithmetical facts that can be output by a standard electronic calculator are finite, and such facts are generally sufficient for applications of mathematics in the fields that are its bread-and-butter, accounting and data analysis. Since J.G. Kemeny's classic 1957 textbook on finite mathematics [21], there have been very many books and courses on "finite" and "discrete" mathematics, including topics such as logic, combinatorics, matrices and networks with their applications and alleged applications to business and the social sciences.

Even beyond that, it is not clear that the standard mathematics of the continuum is needed for work in applied mathematics. While humans most naturally think of space, time, mass and other such quantities as continuous, perceptions and measurements have finite precision, so it could reasonably be hoped that any practical mathematical tasks in physics and engineering could be accomplished with finite-precision arithmetic. But it is not obvious whether that hope is realistic: although direct measurement might well need only numbers as precise as the limits of the measuring device, it is not clear whether, for example, computing the advance of a wave might prove impossible if space and time in the computation are restricted to a discrete approximation.

The development of digital computers spurred the effort to see whether continuous processes could be calculated via discrete approximations. Indeed, that is what computers were invented for. Long before word-processing, spreadsheets and databases were thought of, computers were built to compute ballistic tables and simulate the weather, that is, to compute discrete approximations to continuous dynamics [16, part II]. The verdict was as described above: discrete simulations work well across the board, but they are very painful to program and there are many pitfalls in making sure the discretized version correctly tracks the continuous process it is simulating. Chaos theory and the theorems of numerical analysis show there are fundamental limitations in certain cases on how far ahead in time the discrete simulation and the continuous process will stay close. Nevertheless, in principle computation with finite objects can imitate a continuous process to any required degree of precision.

We could just about get by if we abolished the continuous. But it would be unwise, as the continuous limit of the discrete is the right way to understand the discrete itself.

To the beginner in mathematics, arithmetic and geometry look very different, as do algebra and calculus. But at higher levels, the interplay of the continuous and the discrete has been a driving force in the creation of modern mathematics. That is true of applied mathematics, with its efforts to approximate continuous processes by computable and hence discrete systems. It is equally true of pure mathematics, where naturally discrete structures arising out of continuous ones, as in algebraic topology, determine which continuous objects are truly the same, and lead to many of the deepest theorems in all of mathematics.

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