A Note on Consistency and Platonism*

Alfredo Roque Freire ∧ V. Alexis Peluce

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Abstract

Is consistency the sort of thing that could provide a guide to mathematical ontology? If so, which notion of consistency suits this purpose? Mark Balaguer holds such a view in the context of platonism, the view that mathematical objects are non-causal, non-spatiotemporal, and non-mental. For the purposes of this paper, we will examine several notions of consistency with respect to how they can provide a platonist epistemology of mathematics. Only a Gödelian notion, we suggest, can provide a satisfactory guide to a platonist ontology.

Is consistency the sort of thing that could provide a guide to mathematical ontology? If so, *which* notion of consistency suits this purpose? Mark Balaguer holds such a view in the context of platonism, the view that mathematical objects are non-causal, non-spatiotemporal, and non-mental. Balaguer's version of Platonism, *Full-Blooded Platonism* (FBP), is the view that "there are as many abstract mathematical objects as there could be—i.e., there actually exist abstract mathematical objects of all possible kinds" (Balaguer, 2017, p. 381). He continues:

> Since FBP says that there are abstract mathematical objects of all possible kinds, it follows that if FBP is true, then every purely mathematical theory that *could* be true—i.e., that is internally consistent—accurately describes some collection of actually existing abstract objects. Thus it follows from FBP that in order to acquire knowledge of abstract objects, all we have to do is come up with an internally consistent purely mathematical theory (and know that it is internally consistent). (Balaguer, 2017, p. 381)

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Balaguer, here, wants to avoid the problem of explaining access to non-spatio-temporal reality. In the current philosophical-mathematical literature, this is known as Benacerraf's Problem (Benacerraf, 1973). His strategy is to recast the target of mathematical knowledge; instead of committing to the claim that mathematical knowledge is about this or that *particular* mathematical structure, Balaguer suggests that it is about portions of mathematical reality carved out by consistent theories. But even if consistency, understood in some suitable way, can provide a guide to mathematical ontology, the task still remains to articulate just which reading of the notion is to serve this function. In this paper we explore more traditional and more liberal ways of understanding *consistency*, with an eye toward whether or not they can provide a guide for a platonist ontology.

There are many ways to understand consistency, some more standard and others more liberal. For the purposes of this paper, we will examine the following, which include admittedly more liberal, understandings of the notion of consistency:

1. Semantic
2. Syntactic
3. Folk Intuitions
4. Expert Intuitions
5. Brouwerian Construction
6. Gödelian Perception

The first two are indeed our standard understandings of consistency. The second two have to do with the intuitions of reasoners. The last two instead have to do with the abilities of agents. Only the last, we suggest, can provide a satisfactory guide to a platonist ontology.

Let us begin with the first understanding of consistency. What is meant by *semantic consistency*? We use this in the usual sense as existence of **a** model. As has already been pointed out by Balaguer in *Platonism and Anti-Platonism in Mathematics*, knowledge of consistency then would thus amount to knowledge of a model, which is already an abstract object (Balaguer, 1998, p. 70). For this reason, the notion of semantic consistency cannot ground the possibility of knowledge of abstract objects in general. What kind of guide for ontology does semantic consistency grant us? Through knowledge of consistency, we obtain a collection of non-specified objects. That is, we know nothing about these objects except that they form a model of the theory in question. So, we can only use the existent structure we have arrived at to guide us to other structures, e.g. from the existence of a model of arithmetic, we may argue for a model of arithmetic that eliminates non-standard numbers. This view, as we will see, has the problem of justifying access to mathematical objects. We will return to this in our discussion of Gödelian Perception.

What about *syntactic consistency*? By syntactic consistency we mean the absence of a derivation of a contradiction. But perhaps we can say more. There are at least two types of syntactic consistency. The first being *external consistency*, by which we mean the inability of a logical system to derive a sentence and its negation. The second, call it *internal consistency*, is the sort of consistency we would express with, say, a Gödelian proof predicate. When we understand *formal knowledge* of F as the ability to prove F in a formal system, we already know that *formal knowledge* of internal consistency with a Gödelian proof predicate is impossible, by the second incompleteness theorem (Gödel, 1951, p. 308 – 310). Admittedly, this argument may rely too heavily on the assumption of the use of a *Gödelian* proof predicate, as opposed to a non-standard one. But even if we have good reasons to choose a particular proof predicate, it will be only a theoretical representation of consistency insofar as its connection to the represented concept is justified.

What about *external consistency*? External consistency has to do with the sorts of symbols that can be arrived at by manipulating axioms with certain rules. We can reason very generally that either the axioms of a formal system are taken to be true or not. This is just Gödel's distinction between hypothetico-deductive and proper mathematics (Gödel, 1951, 305). If axioms of proper mathematics, as opposed to algebraic-like axioms, are taken to be true, then it would seem question begging to infer existence of platonic objects from their consistency, provided the platonist accept that the correctness of axioms has to do with the mathematical ontology itself. If axioms are not taken to be true, then there is no reason to think they would be relevant to mathematical ontology as opposed to physical ontology, or anything else. But how can syntactic consistency guide a platonist ontology? A platonist who thinks that an axiom is true already assumes it describes properties of mathematical structures, so the knowledge of syntactic consistency of a set of axioms provides nothing not already assumed in the truth of the axioms. On the other hand, a platonist that does not think axioms are true will have no reason to establish a connection of consistent sets of axioms with mathematical structures.

Perhaps we might think of consistency in a different way. Balaguer suggests the Full-Blooded Platonist might make use of an alternative notion, with "[t]he main idea here [being] that 'consistent' is simply a *primitive* term. More precisely, the claim is that in addition to the syntactic and semantic notions of consistency, there is also a primitive or intuitive notion of consistency that is not defined in any platonistic way" (Balaguer, 1998, p. 70). Balaguer suggests the Kreisel-Field view of intuitive consistency, that "the intuitive notion is related to the two formal notions [semantic and syntactic consistency] in analogous was: neither of the formal notions provides us with a *definition* of the primitive notion, but they both provide us with information about the *extension* of the primitive notion" (Balaguer, 1998, p. 70). Nonetheless, even if there is a privileged intuitive notion of consistency that relates somehow to formal consistency, we need to say something more about *which* notion of intuitive consistency we have in mind.

One suggestion might be that the intuitive notion of consistency is the folk notion of consistency. The thought here might be that humans already seem to have a decent understanding of consistency. If we show students a sentence that they are able to parse, the view might be, they can reliably tell whether or not it is consistent (Balaguer, 1998, p. 72). But again, it is not obvious that the folk notion of consistency should tell us anything about mathematical ontology. Why should it, after all? Should folk ideas of decent chess moves accurately tell us something about what actually is a winning move in the game? By analogy, we should not expect that a folk idea of consistency of mathematical statements should provide a guide to mathematical ontology.

A next suggestion would be to appeal to an expert's notion of consistency. The thought would be that intuitive consistency of the sort of thing that experts have in mind could perhaps provide a guide to platonistic ontology. But it would seem we value the expert understanding of consistency *because* they have antecedent understanding of the relevant objects, as opposed to a privileged access to consistency simpliciter. After all, to become a mathematical expert it is not sufficient to just acquire this special understanding of *consistency*. Instead, one must cultivate a special understanding of *mathematical objects*. It thus seems that what is doing the philosophical work here is the expert understanding of *mathematical objects*, as opposed to that of *consistency*. But then, we musk ask: just what is this expert understanding of mathematical objects?

There is a Brouwerian notion of construction that is worth examining. Thinking of consistency more liberally, we see that this can be thought of as the intuitionistic correlate of consistency. And this, as we will see, was exactly the view of Brouwer. In his dissertation *On the Foundations of Mathematics* of 1907, L.E.J. Brouwer writes a reply to his interlocutor, the logician, who wants to emphasize the role of logical laws:

> The words of your mathematical demonstration merely accompany a mathematical *construction* that is affected without words. At the point where you announce the contradiction, I simply perceive that the construction no longer *goes*, that the required structure cannot be embedded in the given basic structure. And when I make this observation, I do not think of a principium contradictionis." (Brouwer, 1907, p. 73)

The thought here is that the primary phenomenon is that of construction; what can and cannot be arrived at in this way by an agent. Formal consistency and contradiction are just what the logician would tie to the possibility for a construction to proceed or be forced to stop. While the Brouwerian thought is fruitful, that (real) consistency is *constructability*, it is not in the context of platonism in the philosophy of mathematics. Constructed objects, from the platonist perspective, are but a particular non-exhaustible kind within the platonist ontology. More importantly, their construction is not thought of as their existence criteria even if a given platonist ends up concluding that every object is in fact constructable.

There is lastly a Gödelian notion of consistency, in the liberal sense that we have been making use of. It is like Brouwerian construction in that it has to do with the workings of the mind. It is unlike the Brouwerian notion in that it is the sort of thing suited for "concepts form[ing] an objective reality of their own, which we cannot create or change, but only perceive and describe" (Gödel, 1951, p. 320). Kurt Gödel writes:

> [By platonism] I mean the view that mathematics describes a non-sensual reality, which exists independently both of the acts and [of] the dispositions of the human mind and is only perceived, and probably perceived very incompletely, by the human mind. (Gödel, 1951, p. 323)

He has in mind the sort of platonism that we have been discussing, as a second world of non-sensual objects. He suggests that there is some faculty of the human mind that perceives this second world, albeit fallibly. He includes a quote of Charles Hermite (translated by Solomon Feferman and Marguerite Frank):

> There exists, unless I am mistaken, an entire world consisting of the totality of mathematical truths, which is accessible to us only through our intelligence, just as there exists the world of physical realities; each one is independent of us, both of them divinely created. (Gödel, 1951, p. 323)

The passage of Hermite that Gödel invokes emphasizes the sort of intellectual perception mentioned earlier. Gödel articulates his view of mathematical perception or intuition further in "Is mathematics syntax of language? V" as follows:

> The similarity between mathematical intuition and a physical sense is very striking. It is arbitrary to consider "This is red" an immediate datum, but not so to consider the proposition expressing modus ponens or complete induction (or perhaps some simpler propositions from which the latter follows). For the difference, as far as it is relevant here, consists solely in the fact that in the first case a relationship between a concept and a particular object is perceived, while in the second case it is a relationship between concepts. (Gödel, 1953, p. 359)

While obviously this is not *consistency* in the formal sense, it is closer to something like the informal understanding of consistency invoked by Balaguer. Such a view seems to have the advantages of the Brouwerian sort of consistency we put forth earlier, while being amenable to platonism. The thought is that if an object is perceived in Gödel's sense, *then* we have justification for inferring the existence of a platonic object.

A natural objection to the Gödelian view is that it is epistemologically unattractive. It posits that humans have a faculty that somehow connects them to abstract objects. In the context of a Gödel-style response to Benacerraf's epistemological argument, Balaguer writes:

> [If one pursues this strategy,] then the claim will presumably be that humans are capable of somehow "leaving" the physical, spatiotemporal world and "accessing" the platonic realm and gathering information about what abstract objects are like. Most people who work in this area would say that this view is pretty implausible. Indeed, if you endorse a naturalistic, scientific view of the world (and of human beings), then the view probably seems extremely implausible. (Balaguer, 2016, p. 723)

One might even think, as Balaguer argues, that *even if* we posit non-physical parts of human beings, we still ought to be cautious in positing a Gödel-style link between humans and abstract objects (Balaguer, 2016, p. 723).

If the question about whether or not we want to take mathematical objects to be non-spatiotemporal, non-causal, and non-mental is open, then perhaps the need to posit a Gödelian epistemology provides cause for finding platonism unattractive. After all, if the only way we could reach abstract objects is with the above sort of epistemology, then perhaps we should just not posit these objects in the first place. To avoid reaching outside physical reality, Balaguer suggests an inert kind of platonism, where all that is possible indeed exists and every consistent mathematical theory is true of some portion of this inaccessible reality. In fact, this platonism has no explanatory advantage since all we can know about those structures is what is already given by their formal theories. He allow us to "save" platonism at the expense of platonism having any meaningful explanatory power.

While in the above context the counterintuitiveness of Gödelian epistemology works as a consideration against platonism, things are of course different if we antecedently assume that platonism is true and the world may be richer than can be described with a formal theory. If we begin from the belief in the existence of non-spatiotemporal, non-causal, and non-mental mathematical objects, it is not obvious that it is so implausible to posit a connection between those objects and reason. From the belief that this just is the sort of thing that mathematical objects are, we should ask what serves as a guide for our ontology. Here the Gödelian view seems much more attractive; if we are to posit abstract objects in the first place, what better way to access them than through a corresponding faculty of Gödelian perception?

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