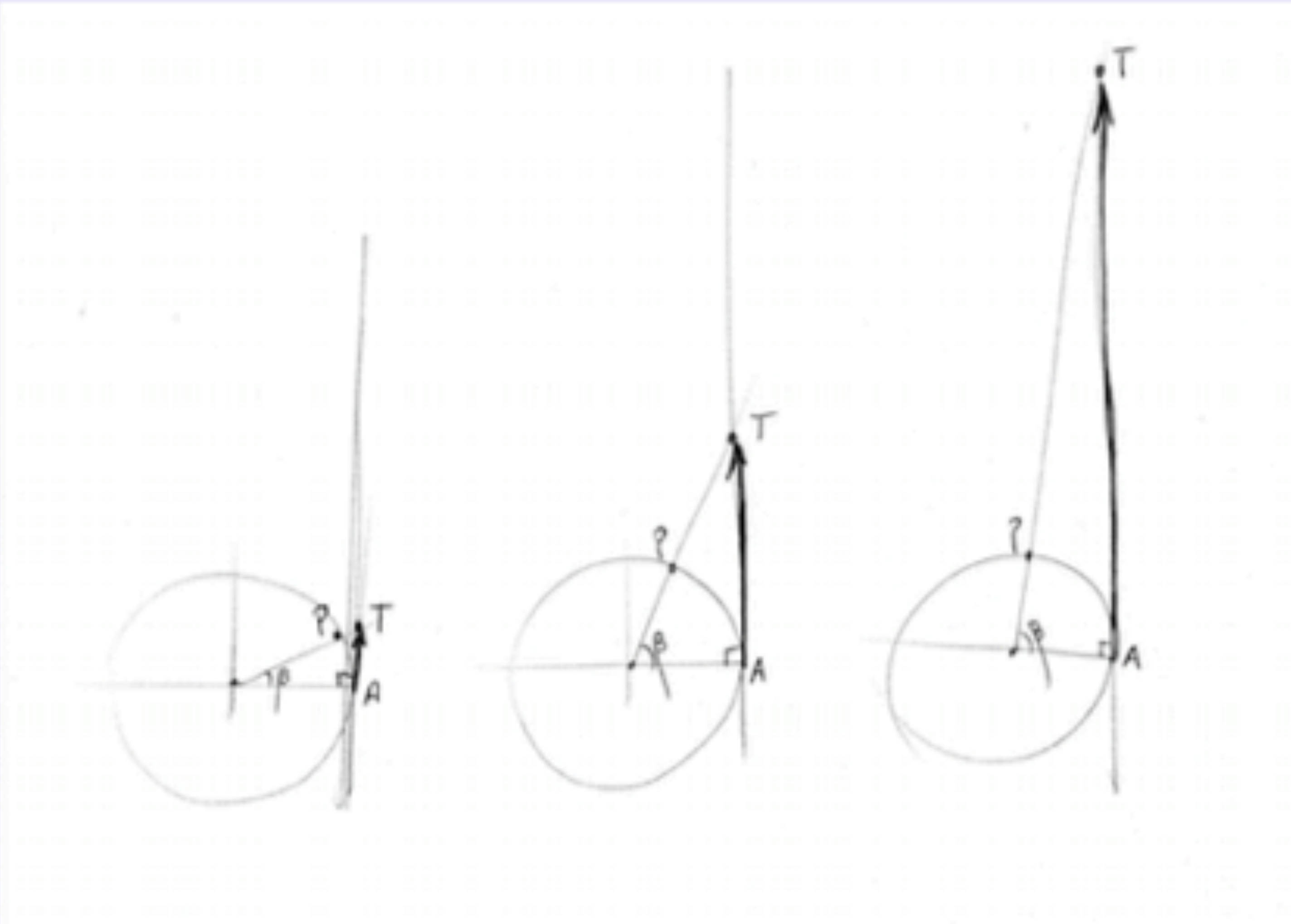
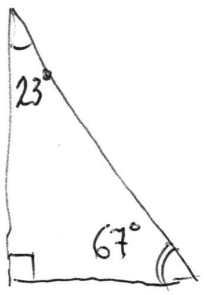
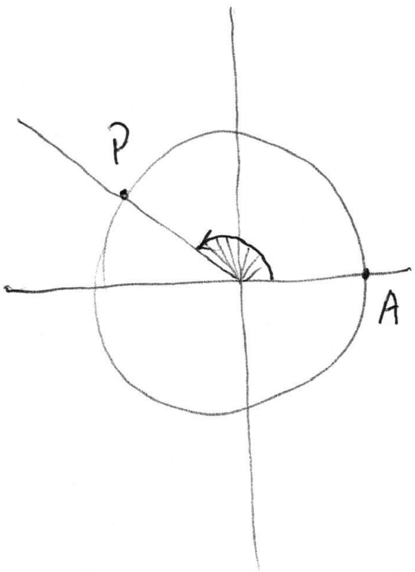
Some basic studies about trigonometry



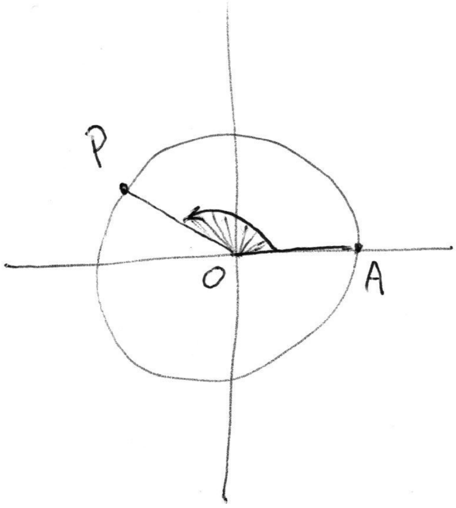
(Luiz Antonio Freire)



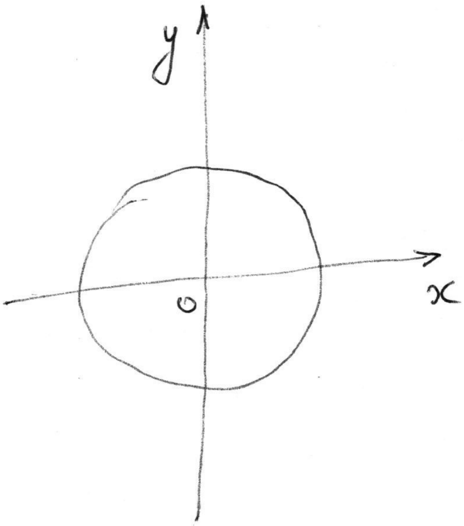
Usually, in
GEOMETRY
The concept of ANGLE
is something STATIC



However,
in TRIGONOMETRY
we will consider
an ANGLE
as the result of a
ROTATION



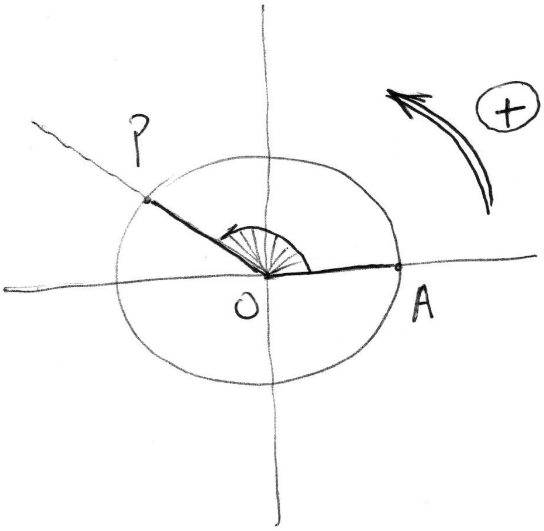
A rotation which will always begin on the segment $\overline{OA}$ and will finish somewhere in the circle, for instance, in this case on the segment $\overline{OP}$



This is the trigonometric circle.

The radius (in this context) will always be $\boxed{R=1}$ (by definition)

(we will soon see why this is so important)



Also,

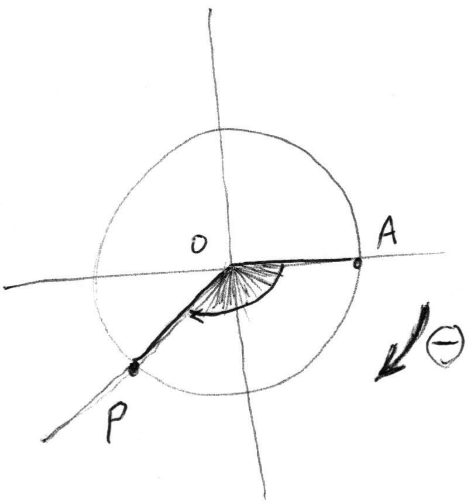
an angle α will be positive,

if it goes

in the

counter clock wise

direction

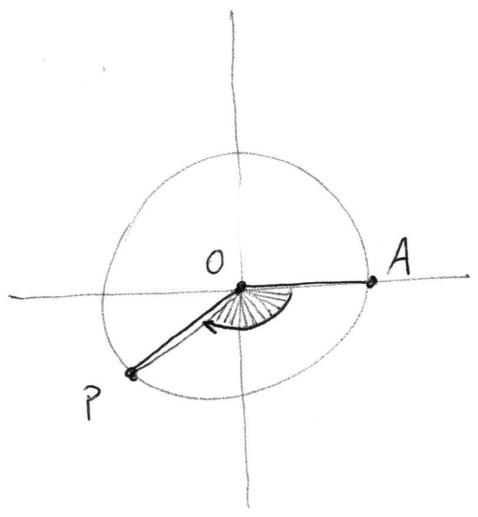


And it will be negative,

if it goes

in the

direction.



We've been observing

That The rotation (i.e.: The angle)

starts always on $\overline{OA}$

and ends at a certain point P

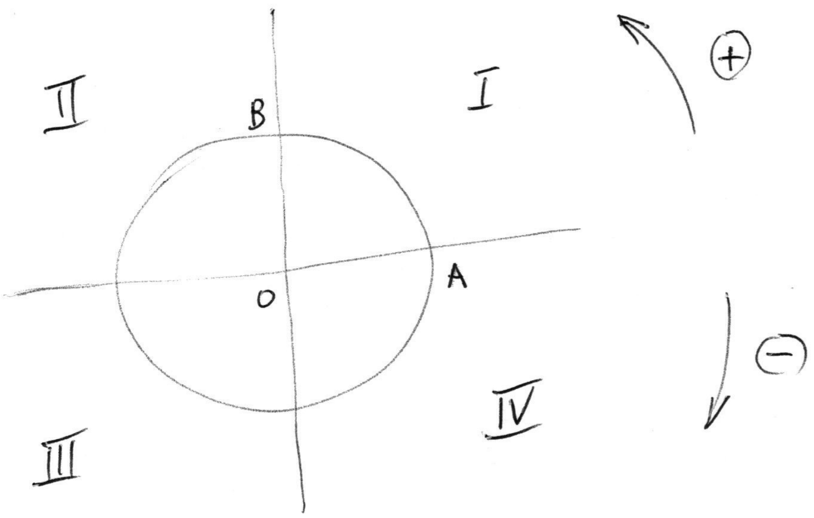
on The circle.

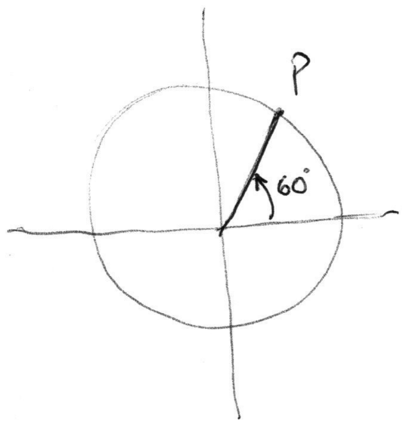
So, at This moment, we need to give names

to These four quadrants,

so That we can identify

in which quadrant P belongs to.





For instance, an angle $\angle = +60^\circ$

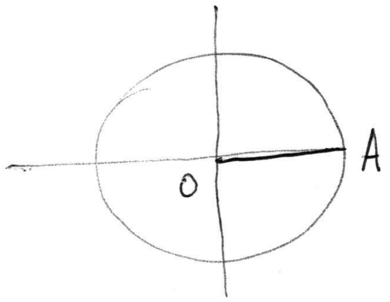
will start at A,

and will rotate 60 degrees

in a counterclockwise direction,

and will stop at point P

located in the $\boxed{\quad}$ quadrant.



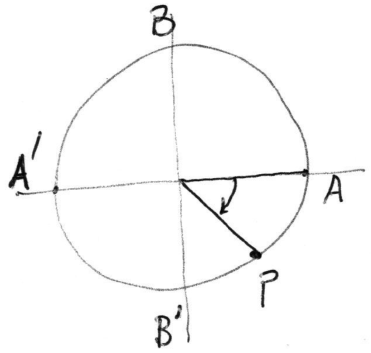
The angle

$\beta = -60°$

will be in

The

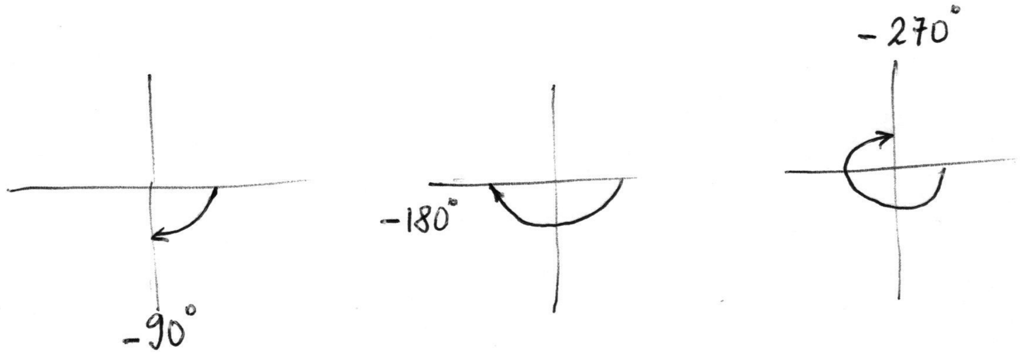
$\boxed{}$ quadrant.

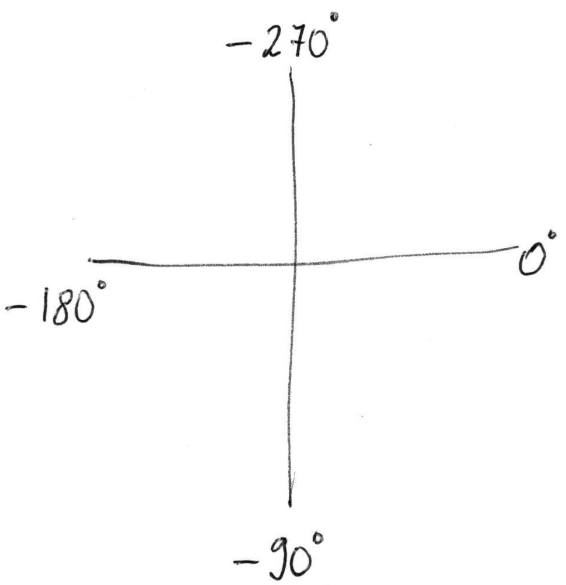


Why are we sure that the angle $\beta = -60°$ is in the $\underline{IV}$ quadrant?

OK; in order to be SURE that it is indeed in the $\underline{IV}$ quadrant, we need to have a certain control (awareness) of the specific angles that are exactly on each border separating each pair of neighboring quadrants.

Since we are, in this case,
dealing with a NEGATIVE rotation,
we need to have these kinds of pictures,
in our minds;





Suppose $\beta = -60^\circ$

$$0^\circ > \overset{-60^\circ}{\overset{\shortparallel}{\beta}} > -90^\circ$$

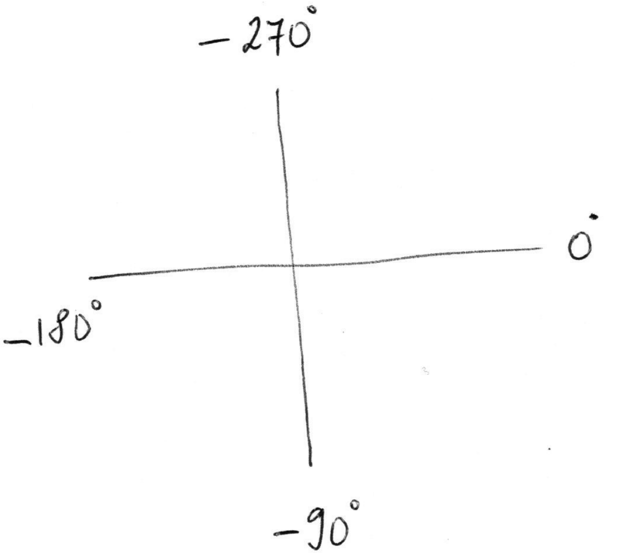
Then

$$-60^\circ \in \underline{\overline{\text{IV}}}$$

Sometimes we find ourselves reluctant to deal with this kind of mathematical notation, but... once we get used to it... we end up feeling comfortable with it.

All it says is that if we are considering an angle of negative 60 degrees, we can be sure that it belongs to the fourth quadrant because -60° is between 0° and -90°.

(That's all)



Again:

What we just did was:

in order to locate where

The angle $\beta = -60^\circ$ is:

we STARTED at $\overline{OA}$,

ROTATED -60°

and STOPPED at P

Moreover, we are sure it is

in the fourth quadrant,

BECAUSE

-60° is between 0° and -90°

Now, we can try to answer
The following question.

Where will The angle $\alpha = -300^\circ$ be ?

How about $\gamma = + 400^\circ$ (?)

(Be careful) ... Now we have a positive rotation.
We start at $\bar{OA}$ and rotate in The positive direction, so we need To be aware of The angles in The "borders", according To The perspective of someone who is "Travelling" in a counterclockwise (or positive) direcTion.

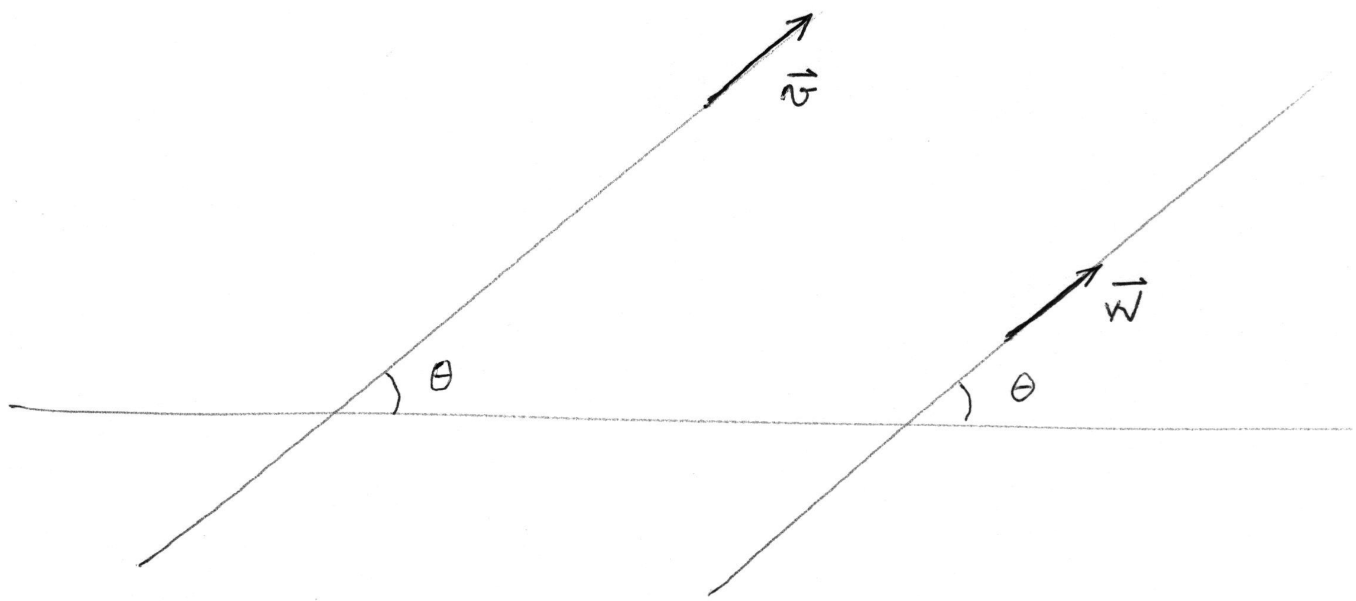
How abouT $\mu = + 1000^\circ$ (?)

How abouT $\omega = - 1000^\circ$ (?)

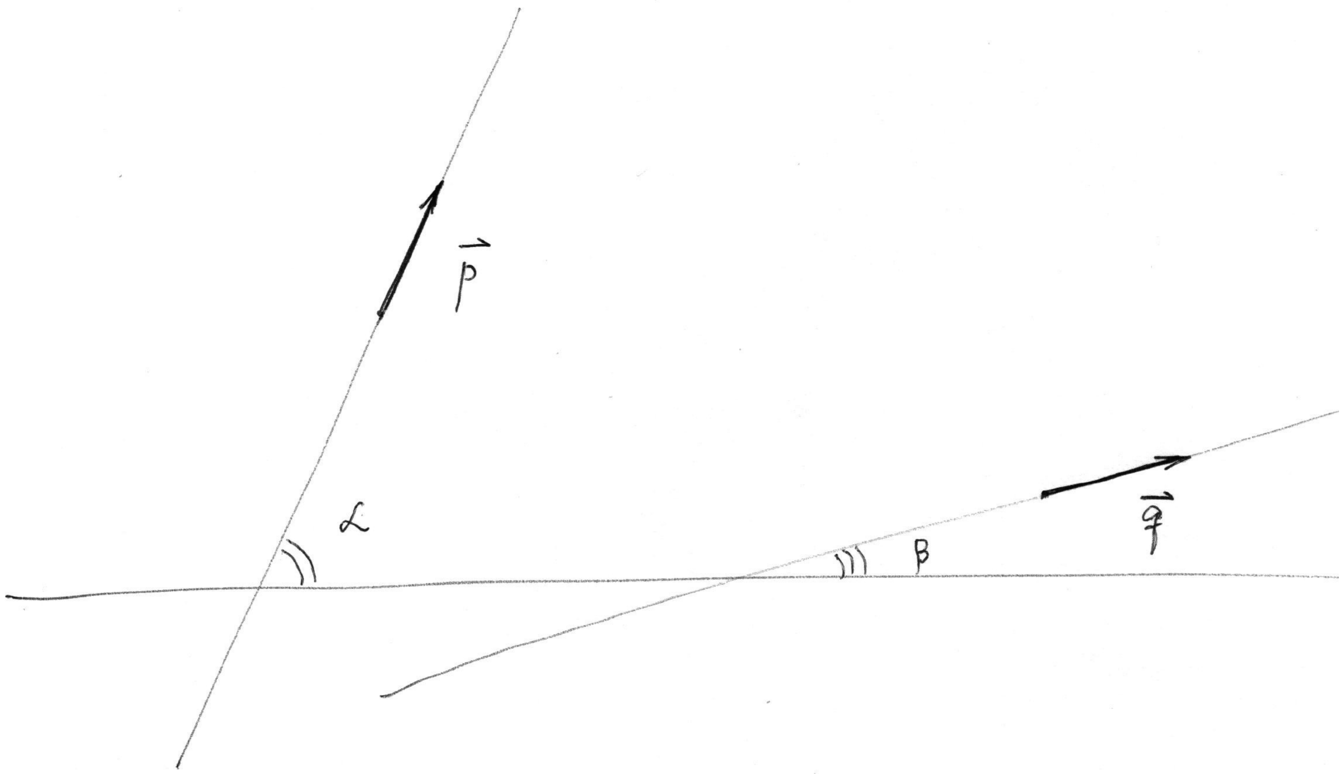
A vector is a concept extremelly useful in both: Mathematics and Physics.

At this moment, we can regard a "vector" simply as an "arrow" having two important characteristics: a length and a direction.

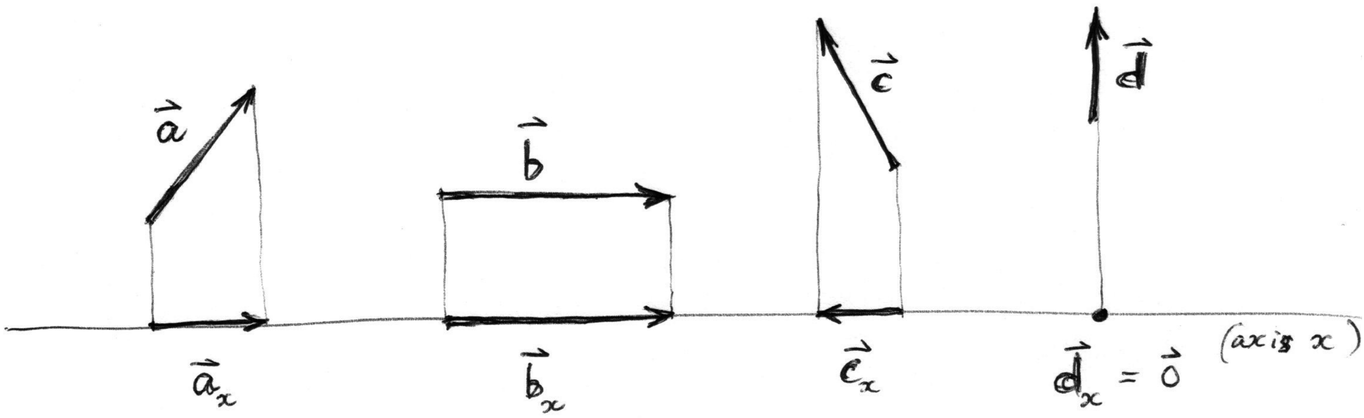
Vectors $\vec{v}$ and $\vec{w}$, below, are equal (i.e.: represent the same vector), because they both have the same length and the same direction.



Vectors $\vec{p}$ and $\vec{q}$ (below), represent two different vectors, because although they have the same length, their directions are different.



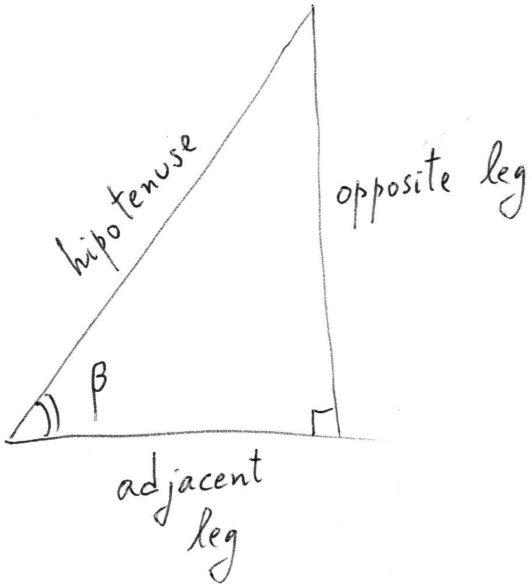
Given a set of vectors, and an axis x, we can observe how their projections (over the axis) are formed.

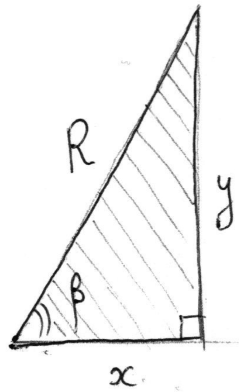


In general, the length of the projection is smaller than the original vector, except when it is parallel to the axis.

Another important observation is that the projection will be reduced to a point (whose length is zero) whenever the original vector is perpendicular to the axis.

Basic definitions in Trigonometry :





$$\sin \beta = \frac{y}{R}$$

$$\cos \beta = \frac{x}{R}$$

$$\tan \beta = \frac{y}{x}$$

$$\cot \beta = \frac{x}{y}$$

$$\sec \beta = \frac{R}{x}$$

$$\csc \beta = \frac{R}{y}$$

$$\text{sine } \beta = \frac{\text{opp.}}{\text{hip.}}$$

$$\text{cosine } \beta = \frac{\text{adj.}}{\text{hip.}}$$

$$\text{tangent } \beta = \frac{\text{opp.}}{\text{adj.}}$$

$$\text{cotangent } \beta = \frac{\text{adj.}}{\text{opp.}}$$

$$\text{secant } \beta = \frac{\text{hip.}}{\text{adj.}}$$

$$\text{cosecant } \beta = \frac{\text{hip.}}{\text{opp.}}$$

Observe That

sine and cosecant are reciprocal of each other.

cosine and secant are also reciprocal of each other.

tangent and cotangent are also reciprocal of each other.

sin

cos

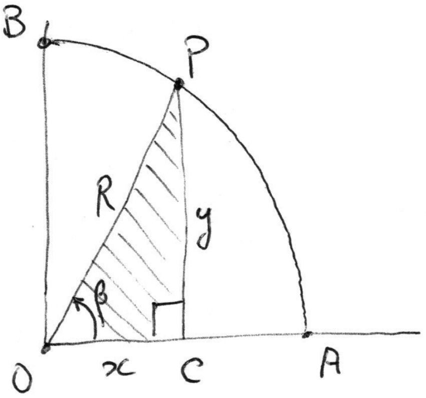
tan

cot

sec

csc

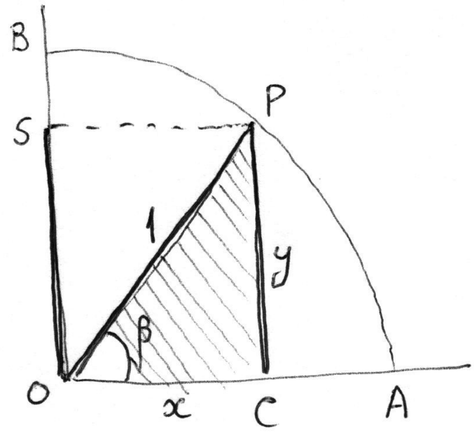
Now, it's time to focus our attention on the sine and cosine of a certain angle β on the first quadrant



Recalling that, in the trigonometric circle, the radius R is always equal to 1, we have:

$$\sin\beta = \frac{opp}{hip} = \frac{y}{R} = \frac{y}{1} = y$$

$$\cos\beta = \frac{adj}{hip} = \frac{x}{R} = \frac{x}{1} = x$$



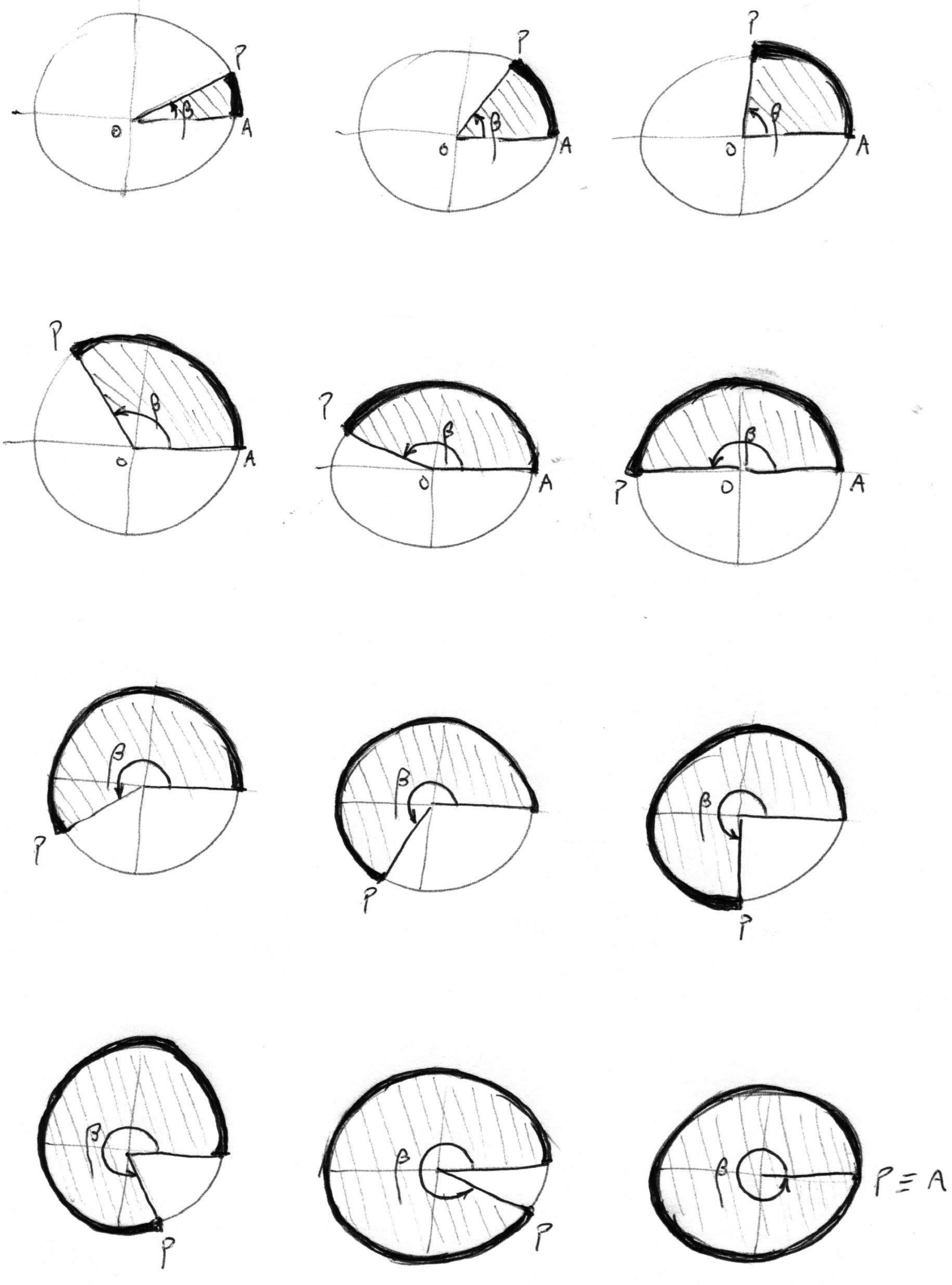
In other words:

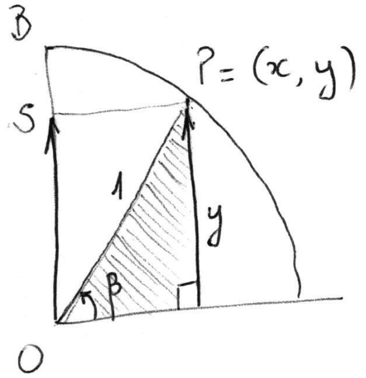
$\sin\beta$ can be regarded as the y-coordinate of point P,

whereas

$\cos\beta$ can be regarded as the x-coordinate of point P

We can imagine the point P moving counterclockwise through the circle, generating a sequence of "screen-shots" each one displaying their respective points P's.





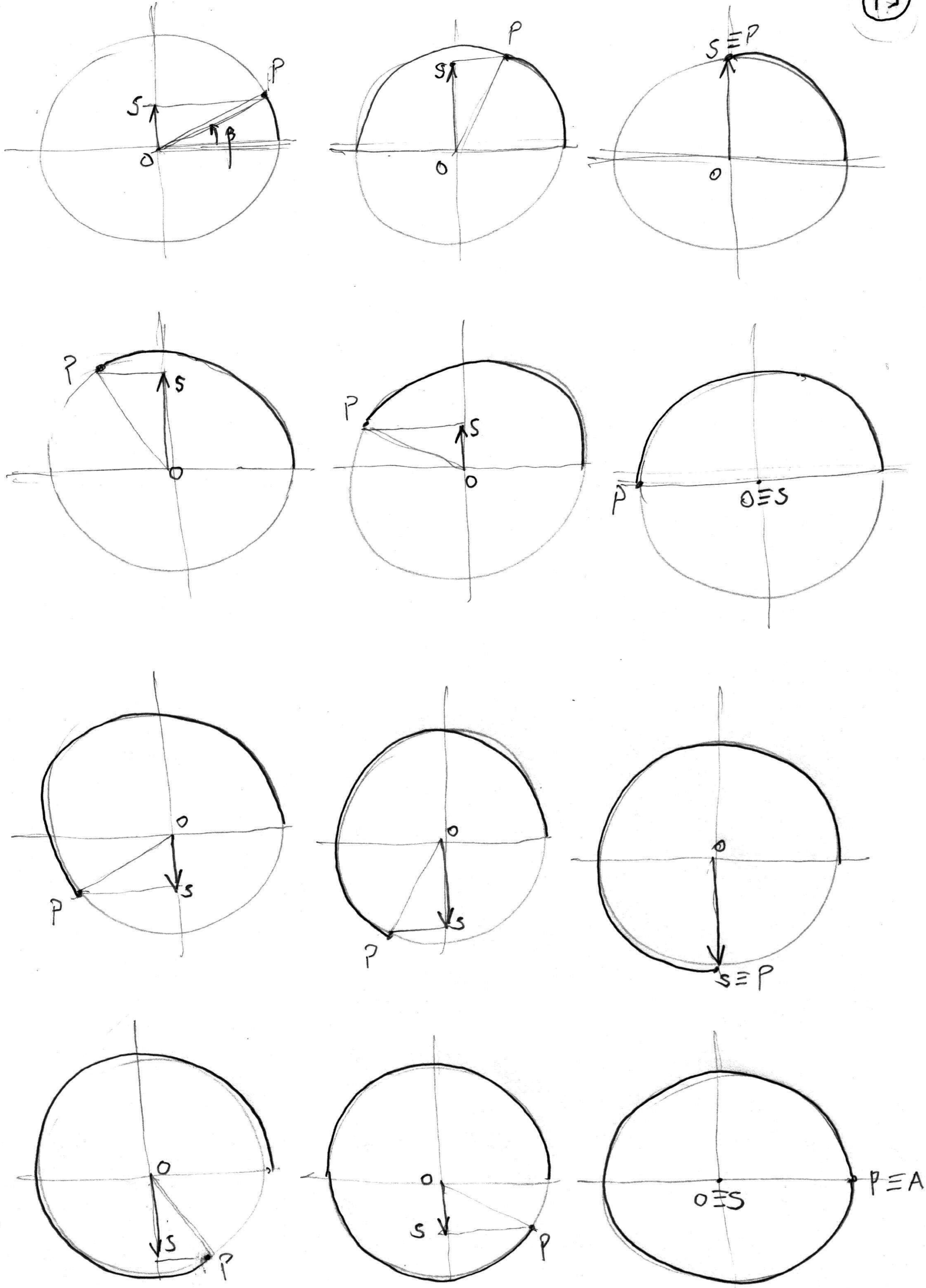
Now, recalling that $\sin\beta$ is the y-coordinate of point P

$$\sin\beta = \frac{\text{opp}}{\text{hip}} = \frac{\text{opp}}{1} = y$$

... we can extend this same principle to all four quadrants ...

Although $\sin\beta$ is not a vector, it would be useful if we allow ourselves to represent $\sin\beta$ by the vector $\overrightarrow{OS}$

As P travels through the circle, the vector $\overrightarrow{OS}$ will give us an interesting idea of what is going on with $\sin\beta$.


S
P
O
β
P
S
O
S ≡ P
O
P
S
O
P
S
O
P
O ≡ S
O
S
P
O
S
P
O
S ≡ P
O
S
P
O
S
P
P ≡ A
O ≡ S

In the first quadrant,

as β goes from $0°$ to $90°$

$\sin\beta$ goes from zero to 1

In the second quadrant,

as β goes from $90°$ to $180°$,

$\sin\beta$ goes from 1 to zero

In the third quadrant,

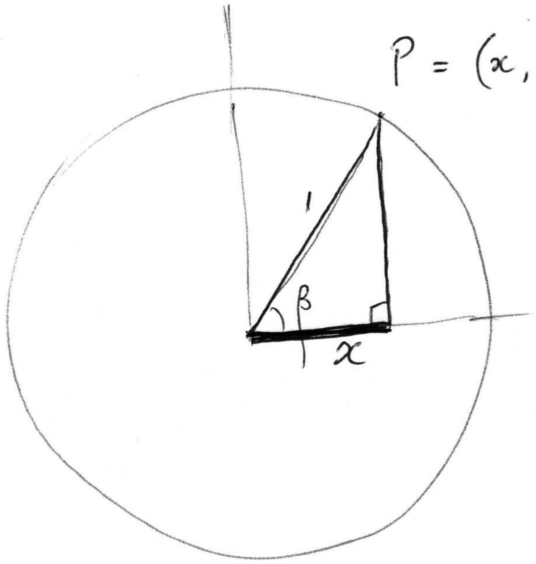
as β goes from 180 to $270°$,

$\sin\beta$ goes from zero to -1

In the fourth quadrant,

as β goes from $270°$ to $360°$,

$\sin\beta$ goes from -1 to zero



Now, let's take a look at the COSINE:

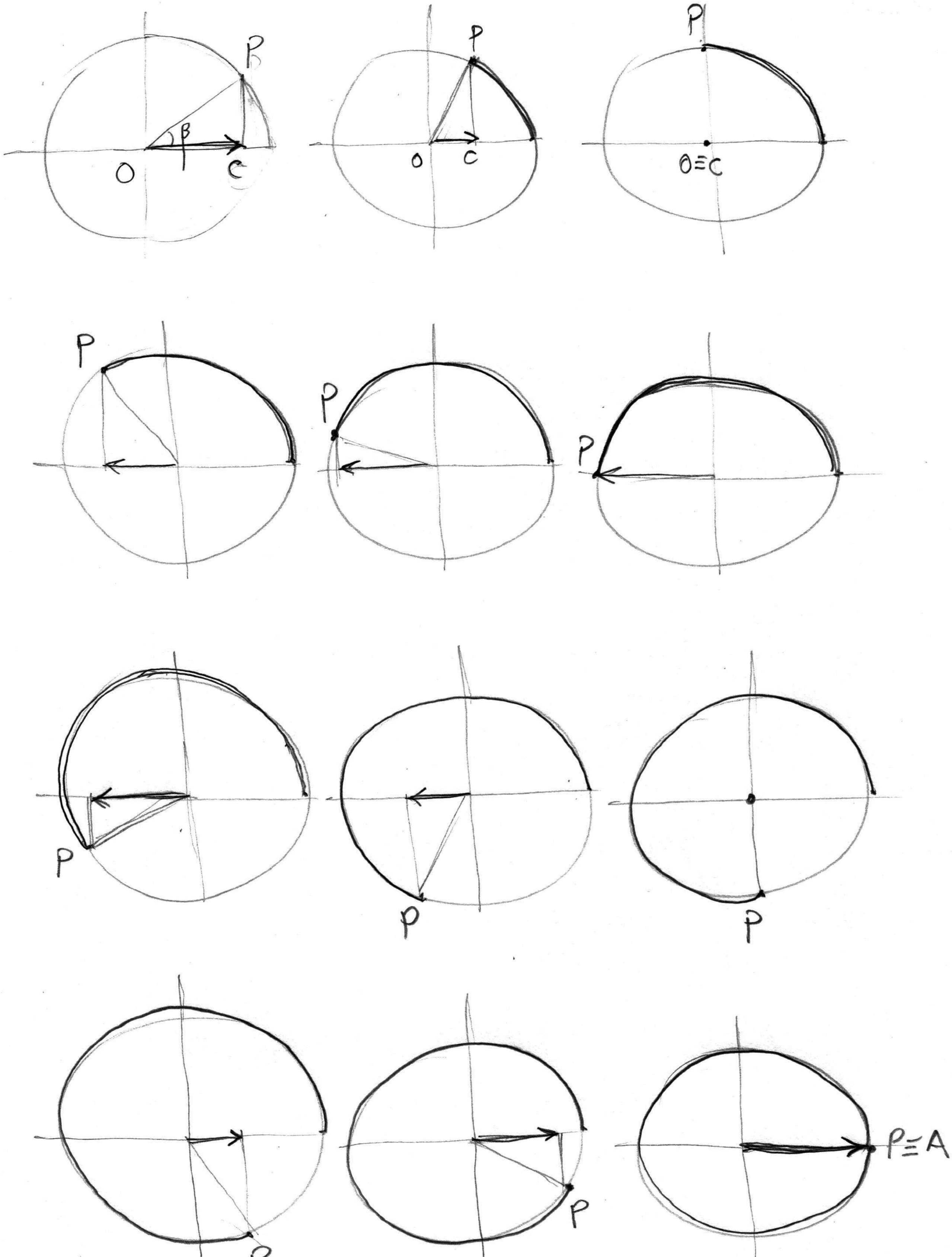
$$\cos \beta = \frac{adj}{hip} = \frac{x}{R} = \frac{x}{1} = x$$

That is:

$\cos \beta$ is the x-coordinate of point P

A similar behavior happens now, as we look at the COSINE.

The difference is that, while in the previous case (SINE) the radius was projecting itself onto the vertical axis, now, in this (COSINE) case, the radius will project itself onto the horizontal axis.


P
β
O
C
P
o
C
P
O≡C
P
P
P
P
P
P
P
P
P≡A

Observing the behavior of the COSINE,

in the first quadrant,

as β goes from $0°$ to $90°$,

$\cos\beta$ goes from 1 to zero

In the second quadrant,

as β goes from $90°$ to $180°$,

$\cos\beta$ goes from $\boxed{\text{zero}}$ to $\boxed{-1}$

In the third quadrant,

as β goes from $\boxed{\quad}$ to $\boxed{\quad}$,

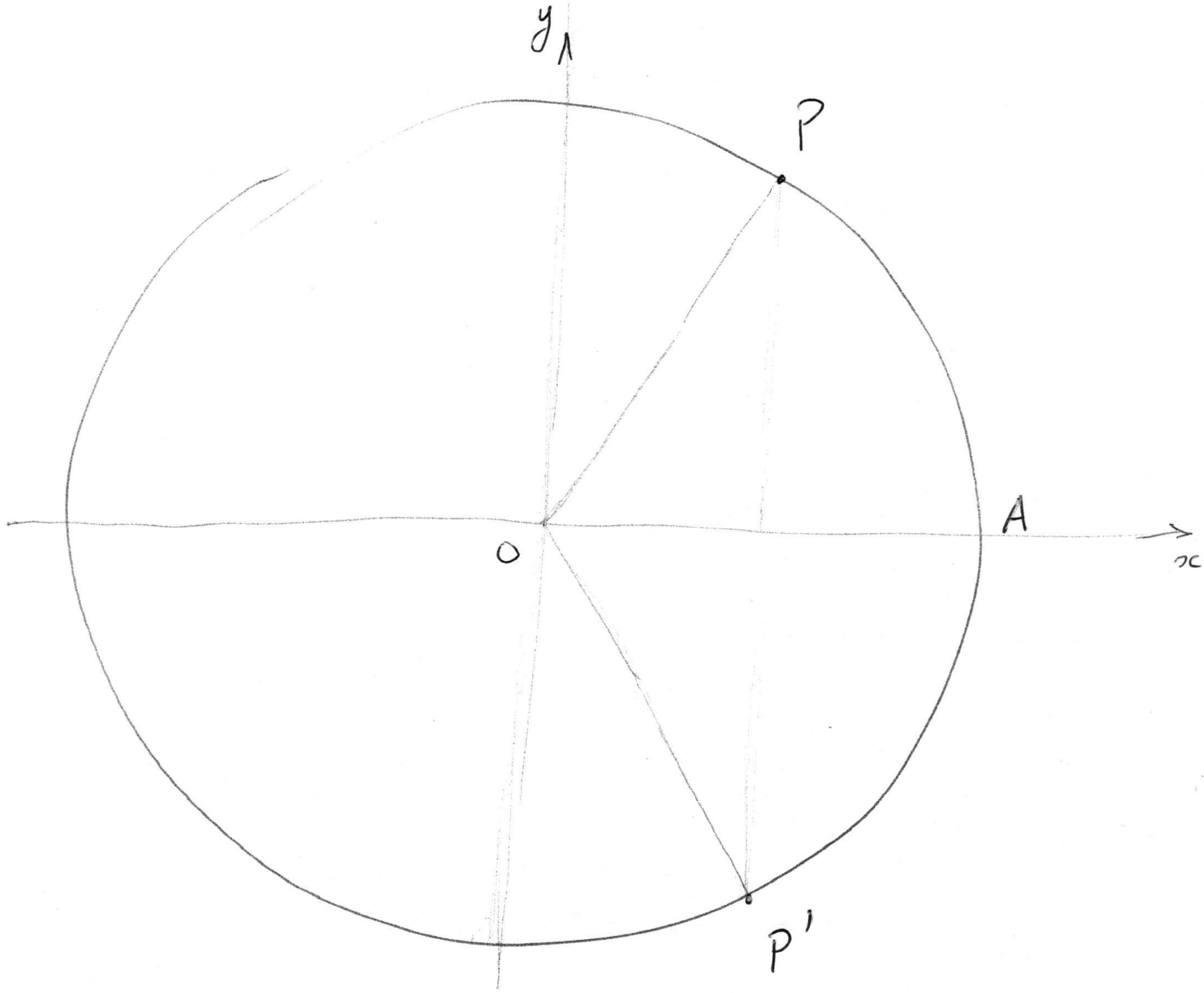
$\cos\beta$ goes from $\boxed{\quad}$ to $\boxed{\quad}$

In the $\boxed{\quad}$ quadrant

as β goes from $\boxed{\quad}$ to $\boxed{\quad}$

$\boxed{\quad}$ goes from $\boxed{\quad}$ to $\boxed{\quad}$

Now, let's put aside our (dear) trigonometry, and we are going to "navigate" a little bit through the world of basic GEOMETRY.



Let angle $A\hat{O}P = 60°$

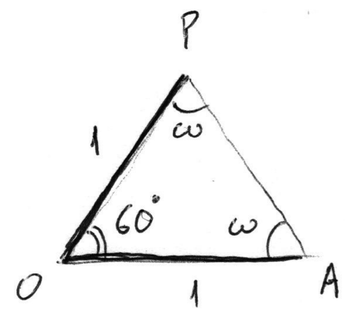
Let P' be the symmetric of P, with respect to the horizontal axis.

Let the radius $R = 1$

Observe that :

(1) Triangle OAP is isosceles, since $\overline{OA} = \overline{OP} = 1 = \text{radius}$

(2) Triangle OAP is more than isosceles,
it is actually an EQUILATERAL triangle,
because

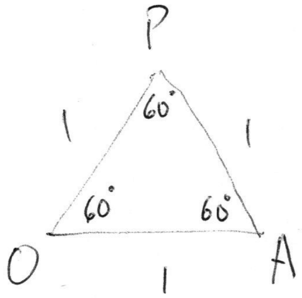


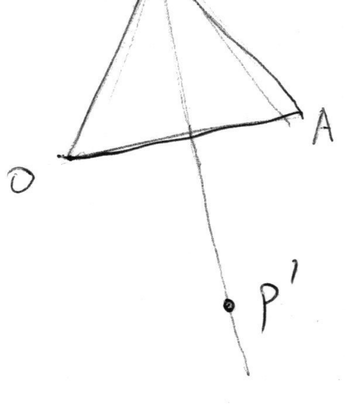
$$60^\circ + \omega + \omega = 180^\circ$$

$$60^\circ + 2\omega = 180^\circ$$

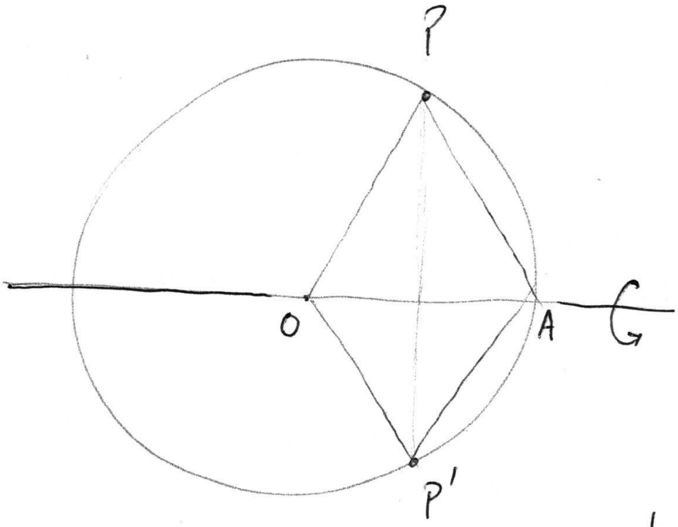
$$2\omega = 120^\circ$$

$$\omega = 60^\circ$$



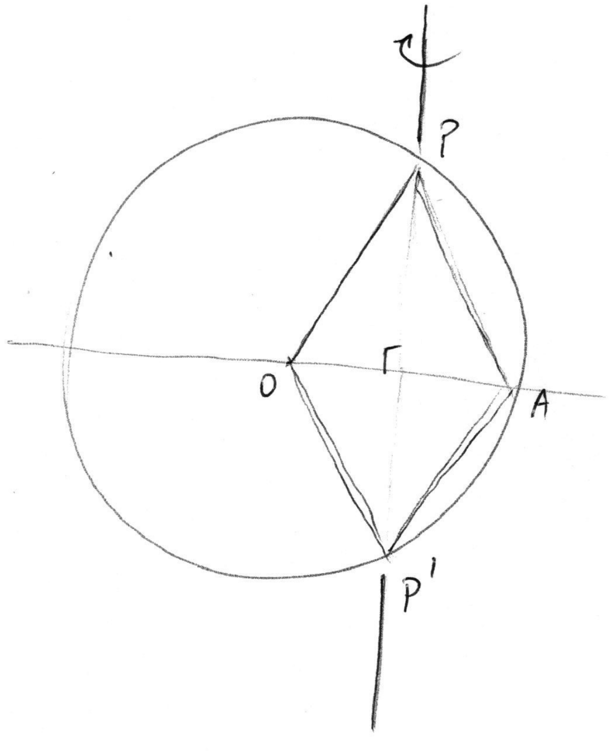


Recalling that
P' is the symmetric of P
with respect to the horizontal axis,
it follows that
$\overline{PP'}$ and $\overline{OA}$
are perpendicular

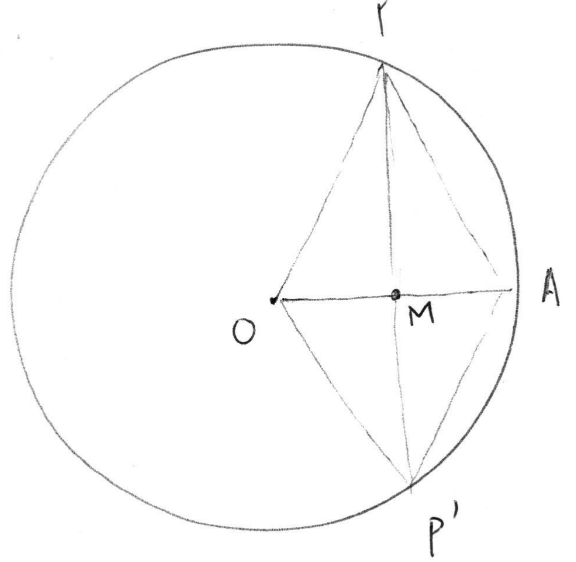


Triangles OAP
and OAP'
are both symmetric
to each other
with respect to
the horizontal axis.

They are both equilateral triangles
and $\overline{PP'}$ is perpendicular to $\overline{OA}$.



So, there will also
be another symmetry
going on here,
as we can see that
triangles PAP' and POP'
are symmetric to each other
with respect to the
vertical line PP'

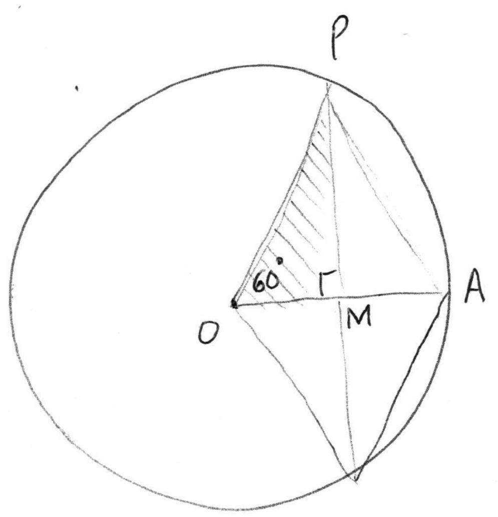


Therefore,

The line $\overline{PP'}$

intercepts the line $\overline{OA}$

at its midpoint M



Recalling that $\overline{OA}$

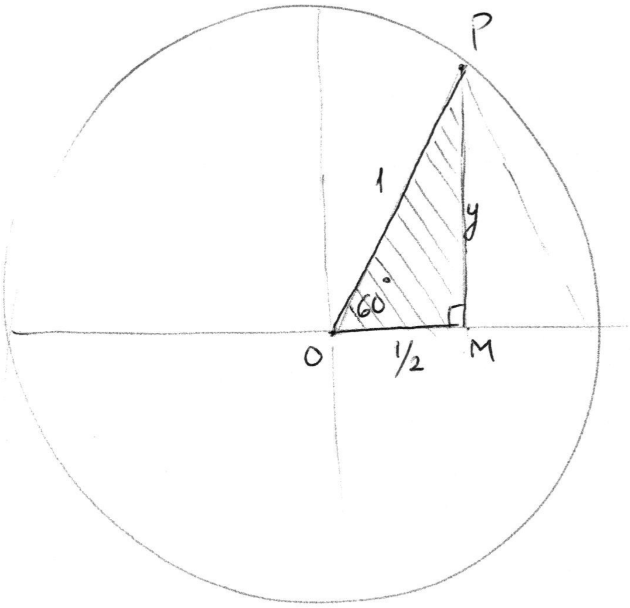
is the radius (equal to 1),

we can now conclude

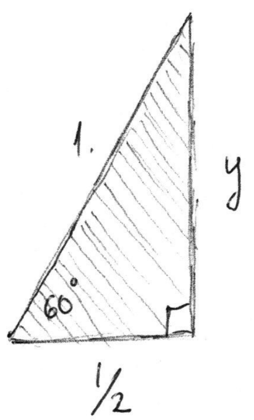
that $\overline{OM} = \frac{1}{2}$

$$\cos 60^\circ = \frac{adj}{hip} = \frac{\overline{OM}}{1} = \overline{OM}$$

$$\cos 60^\circ = \frac{1}{2}$$



Once we know that the small leg $\overline{OM}$ is 1/2 we can find the other leg "y" using the Pythagorean Theorem



$$1^2 = y^2 + \left(\frac{1}{2}\right)^2$$

$$1 = y^2 + \frac{1}{4}$$

$$1 - \frac{1}{4} = y^2$$

$$\frac{3}{4} = y^2$$

$$y = \pm\sqrt{\frac{3}{4}}$$

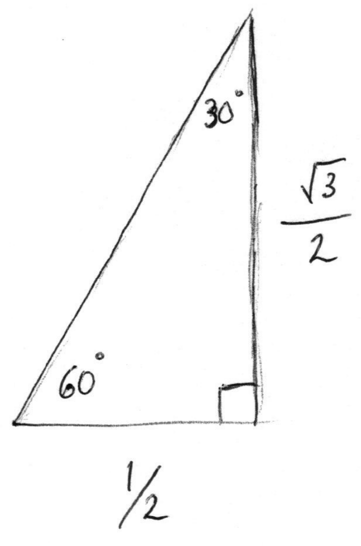
We will select only the positive case $\sqrt{\frac{3}{4}}$, because the other option $-\sqrt{\frac{3}{4}}$ wouldn't make sense, here, since we are dealing with the length of the side of a triangle, which has to be, necessarily, a positive number.

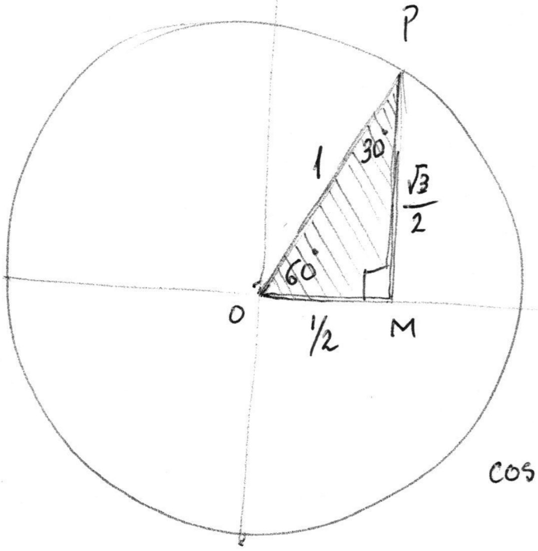
So,

$$y = \sqrt{\frac{3}{4}}$$

$$y = \frac{\sqrt{3}}{\sqrt{4}}$$

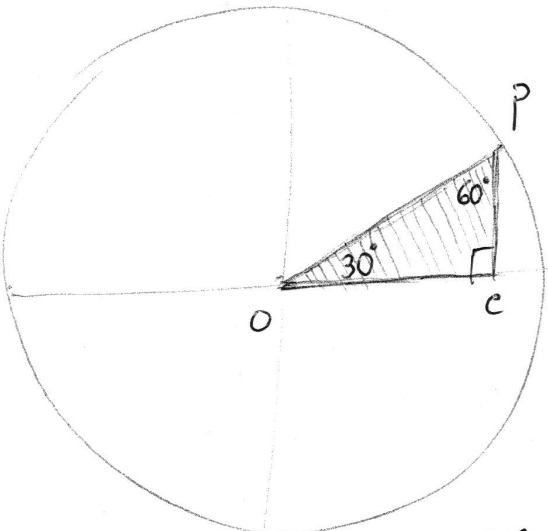
$$y = \frac{\sqrt{3}}{2}$$





$$\cos 60^\circ = \frac{adj}{hip} = \frac{\overline{OM}}{1} = \overline{OM} = \frac{1}{2}$$

$$\sin 60^\circ = \frac{opp}{hip} = \frac{\overline{MP}}{1} = \overline{MP} = \frac{\sqrt{3}}{2}$$



Triangls OMP (described above)

and OCP (described here)

are congruent,

since They have all Three angles equal to 30°, 60°, 90°.

Moreover,

Their hipotenuses are both equal to 1.

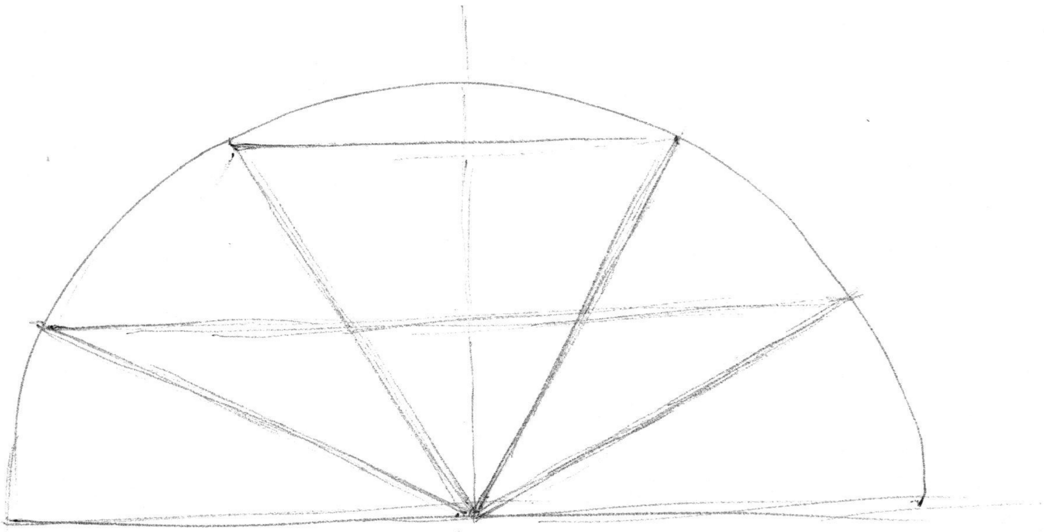
If they had all three angles coincident (in this case ; 30°, 60°, 90°) that would tell us that they are similar.

But, actually, they are more than similar: they are congruent, because: the fact that they have hypotenuses with the same length (in this case, equal to 1), means that the ratio of similarity is 1 to 1.

In other words: they are congruent.

Therefore, their big legs will both measure $\frac{\sqrt{3}}{2}$

while their small legs will both measure $\frac{1}{2}$



Based on what we've been observing,

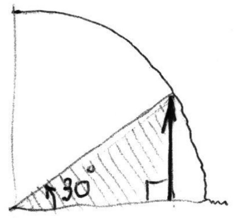
what is

$\sin 30^\circ$

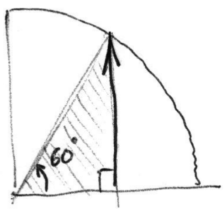
$\sin 60^\circ$

$\sin 120^\circ$

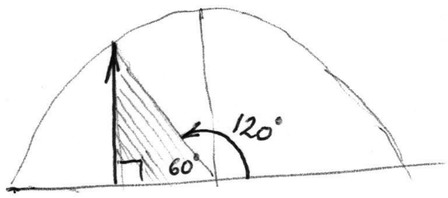
$\sin 150^\circ$ (?)



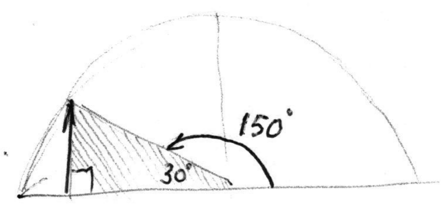
$\sin 30^\circ =$



$\sin 60^\circ =$



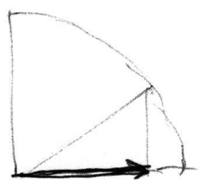
$\sin 120^\circ =$



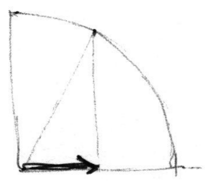
$\sin 150^\circ =$

In all (four) examples above
$\sin\beta$ will be positive because the "arrow"
is pointing upwords.

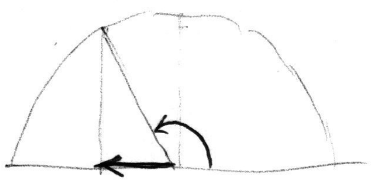
Applying the same (four) examples to the cosine, we should observe that now the cosine will be positive in the first quadrant, and negative in the second quadrant, since the "arrow" which represents each case is pointing to the left (in the second quadrant).



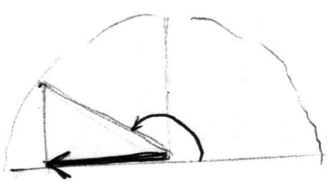
cos 30° =



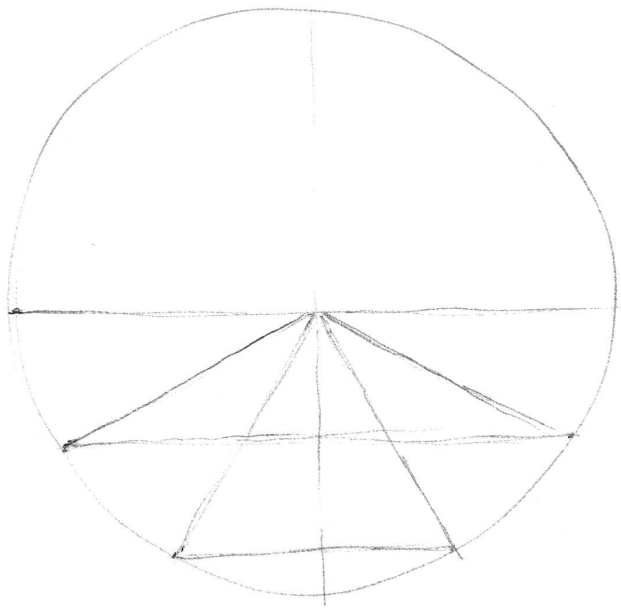
cos 60° =



cos 120° =



cos 150° =



$\sin 210^\circ =$

$\cos 210^\circ =$

$\sin 240^\circ =$

$\cos 240^\circ =$

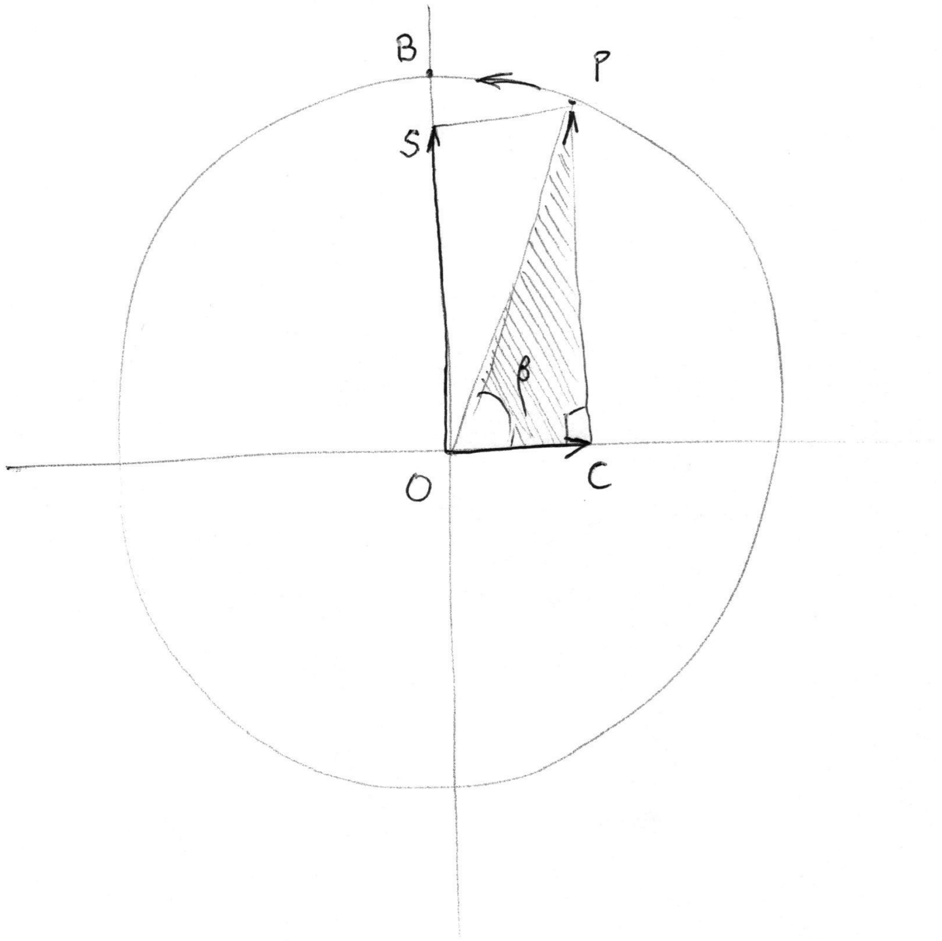
$\sin 300^\circ =$

$\cos 300^\circ =$

$\sin 330^\circ =$

$\cos 330^\circ =$

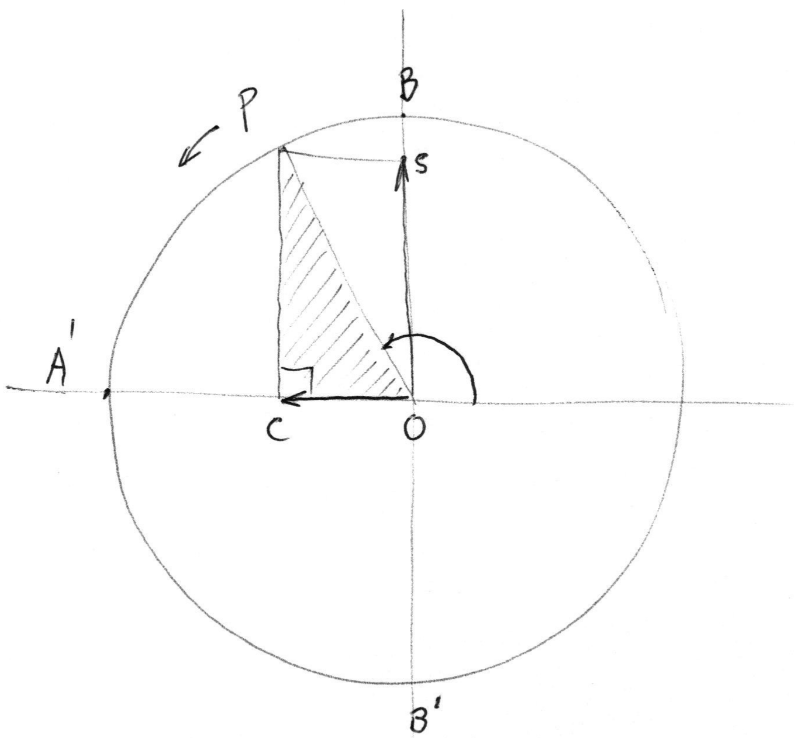
Now, we need to go back a few pages, and review those sequences of "screen-shots", in order to observe again what is happening with both the SINE and COSINE, when the angle β "crosses" the "borders" of each pair of neighboring quadrants.



As P approaches B, $\vec{OC}$ becomes smaller and smaller, meaning that $\cos\beta$ is also becoming smaller and smaller ...

... until it reaches the value zero,

when β reaches exactly 90°.



As soon as P continues its (counterclockwise) journey,

(now, in the second quadrant), $\cos\beta$ becomes negative.

— // —

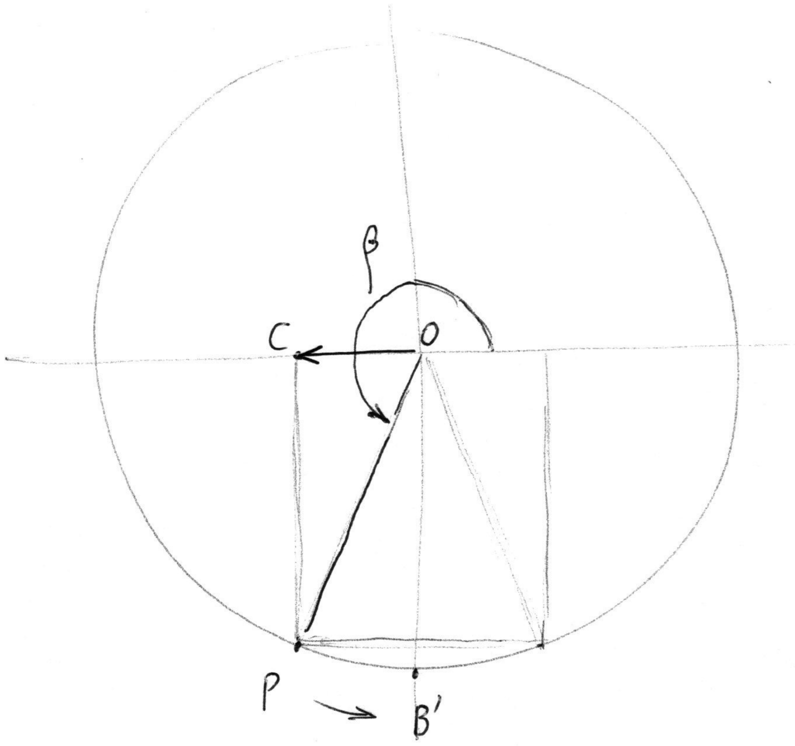
Continuing our journey ... as P approaches A',

β approaches 180°,

while $\overrightarrow{OC}$ becomes more and more

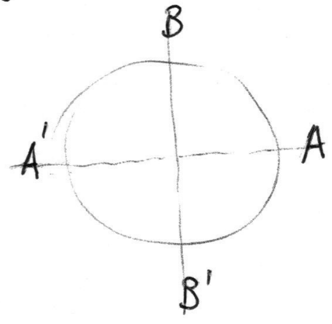
negative, until it reaches $\overline{OA'}$,

meaning that $\cos 180^\circ = -1$



Near the "border" of the third and fourth quadrants, as P approaches B', $\overrightarrow{OC}$ becomes less and less negative, approaching zero, which is very understandable since $\overrightarrow{OC}$ will become positive (i.e.: pointing to the right) as soon as P crosses the "border".

Similar analysis should be done,
for both the SINE and COSINE's behaviors
as P crosses each border
at points B, A', B' and A.



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Inspired on the philosophy that the best way to learn is to teach, I would recommend as an excellent exercise, the following project:

Prepare (either individually, or in group), a similar exposition about this topic, explaining each detail of the behavior of the SINE, COSINE, and, in the near future, as we shall see, the behavior of the tangent, cotangent, secant and cosecant.

Such exposition can be done,
using traditional tools, like the ones used
here: (plain paper, pencil and eraser)...

... or you can use a software dealing
with animation, such as GEOGEBRA.

—— // ——

What we've done so far,
was just the very first steps
into an environment of thoughts,
logic, and reasoning,
typical of fields like Mathematics and Physics,

where a lot of Geometry (with its proofs) is necessary if we want to build a good level of understanding about these subjects without having to memorize too many formulas, since, as we adopt such approach we would have the "keys" of the proofs allowing us to deduce the formulas through a coherent line-of-reasoning based on a small number of important and fundamental principles (as opposed to someone who uses the formulas, without knowing where they came from).

— // —

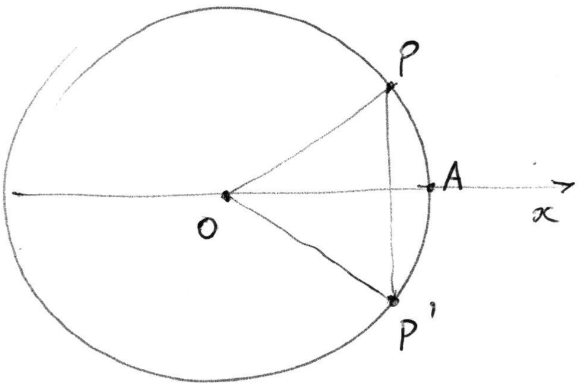
Now, we are (almost) ready to move on, and begin to work on the tangent function.

But, before we do so, I would like to share with you a certain detail that revealed itself while I was re-reading what we've done so far.

We had proved that $\cos 60° = \frac{1}{2}$ and from there, we deduced that $\sin 60° = \frac{\sqrt{3}}{2}$ through the Pythagorean Theorem.

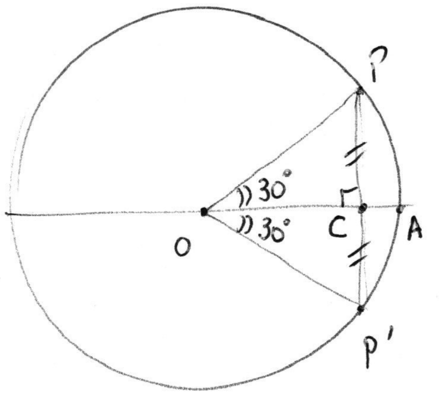
But, we could also have adopted another route: we could have proved first that $\sin 30° = \frac{1}{2}$ and from there, arrive at $\cos 30°$ (through Pythagoras).

This (new) route is also interesting, so we will take a closer look at the sequence of arguments that explains why $\sin 30° = \frac{1}{2}$



Let angle $A\hat{O}P = 30°$

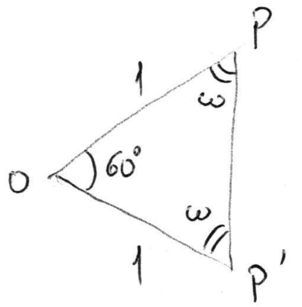
Let P' be symmetric to P with respect to the horizontal axis x.



Such symmetry of P' and P guarantees us that $\overline{CP} = \overline{CP'}$ and also, that lines $\overline{OA}$ and $\overline{PP'}$ are perpendicular.

If we can prove that triangle $OP'P$ is equilateral, then we are done, because $\overline{PP'}$ would be equal to the radius (thus, equal to 1).

The proof that triangle $OP'P$ is equilateral is very easy and similar to the one we did before:



$\overline{OP} = \overline{OP'} = 1$ (both $\overline{OP}$ and $\overline{OP'}$ represent the radius)

$P\hat{O}P' = 60°$ (by hypothesis)

$60° + 2\hat{\omega} = 180°$

So... $\hat{\omega} = 60°$

Once we showed to ourselves that triangle OPP' is indeed equilateral,

Then $\overline{PP'} = 1$

and $\overline{CP} = \overline{CP'} = \frac{1}{2}$

That is: $\sin 30^\circ = \frac{1}{2}$

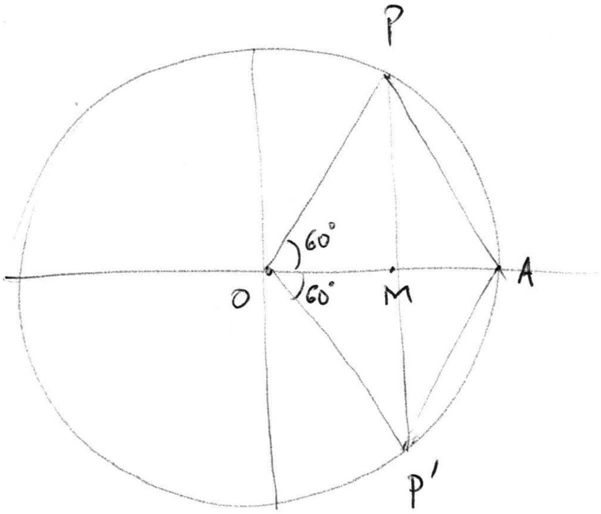
— // —

The main difference between these two approaches (i.e.: first route: find $\cos 60^\circ$, then $\sin 60^\circ$; second route: find $\sin 30^\circ$, then $\cos 30^\circ$),

was that, (in the first route),

we proved that $\cos 60^\circ = \frac{1}{2}$

using the picture below:



Then, we developed a sequence of arguments concluding that

$$\overline{OM} = \overline{MA}$$

(i.e.: M is the midpoint of $\overline{OA}$).

Well...

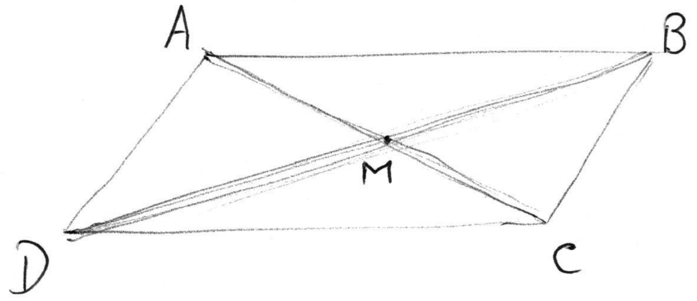
since we are reviewing this proof,

I should add, now, the observation that the quadrilateral OPAP' is a rhombus because the equilateral triangles OAP and OAP' are symmetric to each other with respect to the horizontal axis.

(A rhombus is a parallelogram whose four sides have the same length)

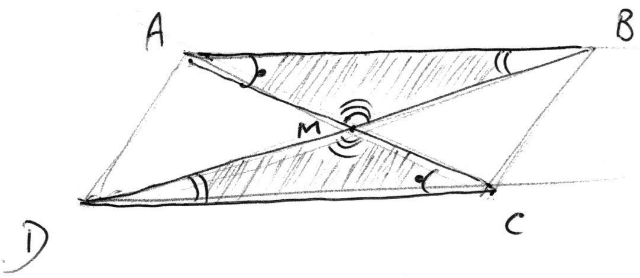
Quadrilateral OPAP' being a rhombus, conducts our thoughts to a theorem that says:

" In every parallelogram,
their diagonals intercept each other
at their midpoints. "



Given: ABCD is a parallelogram

Prove: $\left\{ \begin{array}{l} M \text{ is the midpoint of } \overline{DB} \\ \text{(and also, the midpoint of } \overline{AC}\text{).} \end{array} \right.$



The proof is based on the observation that triangles AMB and DMC are congruents:

Let's observe that:

(1) angles $\measuredangle = \hat{ABD} = \hat{BDC}$ are congruents
(because they are alternate interior in a configuration where $\overline{AB}$ is parallel to $\overline{DC}$ and $\overline{DB}$ is the transversal).

(2) angles $\measuredangle = \hat{BAC} = \hat{ACD}$ are congruents
(They are alternate interior in a configuration where $\overline{AB}$ is parallel to $\overline{DC}$ and $\overline{AC}$ is the transversal).

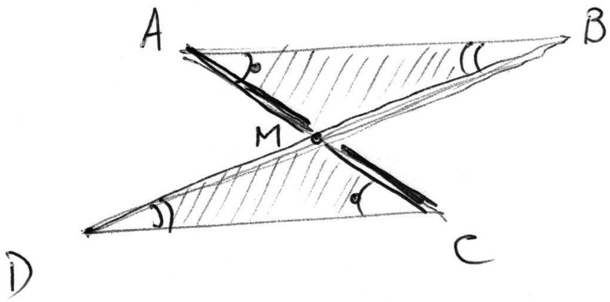
(3) angles $\measuredangle = \hat{AMB} = \hat{DMC}$ are congruents
(because they are vertical angles)

(4) Moreover, the corresponding sides $\overline{AB}$ and $\overline{DC}$ have the same length,
(since we are dealing with a parallelogram).

(The four observations above, (simultaneously),
imply that triangles AMB and CMD
are congruents).

Now, we need to be, a little bit, careful,
and realize that $\overline{AM}$ is the corresponding side of $\overline{MC}$,
because they are both, opposite sides of the angles
represented by $\measuredangle$.

Likewise,
$\overline{DM}$ and $\overline{MB}$ will be another pair
of corresponding sides,
because both are opposite sides
of the angles
represented by $\measuredangle$.



The fact that triangles AMB and CMD are congruents, allows us to conclude that

$$\overline{AM} = \overline{MC} \qquad \text{(both opposite to } \measuredangle \text{)}$$

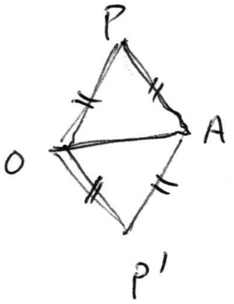
and

$$\overline{DM} = \overline{MB} \qquad \text{(both opposite to } \measuredangle \text{)}$$

In other words:

In any parallelogram, the two diagonals intercept each other at their MIDPOINT.

If this is true for any parallelogram,
then it will also be true for any rhombus,
(since every rhombus is a parallelogram)



—//—

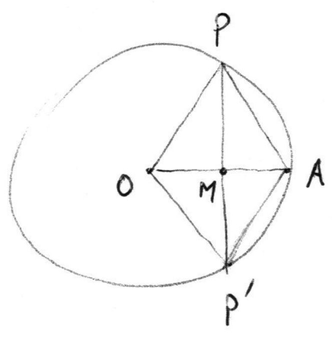
Returning to our proof (whose goal was to prove that $\cos 60^\circ = \frac{1}{2}$),

we had previously observed that
Triangles OAP and OAP' are equilateral.

Therefore OPAP' is a rhombus,
and hence, a parallelogram.

Being a parallelogram, the Theorem (above) dealing with diagonals in a parallelogram, guarantees us that

the point M
is indeed the
MIDPOINT of $\overline{OA}$



and so, $\cos 60^\circ = \frac{1}{2}$

—//—

As I was re-reading our very first "proof" about $\cos 60^\circ = \frac{1}{2}$, something was telling me that the passage that moved from the symmetry around the horizontal axis x, to the vertical "axis" $\overline{PP'}$, was, in a way, not very clear.

So, I decided to go over it again, this time using some additional analysis, so that we can keep alive this (healthy) environment of exercising these kinds of observations and thoughts typical of Euclidean Geometry which, (as we had mentioned before), is an excellent way to prepare ourselves for disciplines such as Mathematics and Physics, (among others), because, as we do this kind of activity, we are developing the ability of explaining (to ourselves) each detail of the line-of-reasoning.

In other words; we are learning how to think ... in a logical way.

—//—

Perhaps, now, it's a good time to close this chapter, and, (after the "break"), continue our work, focusing on some other graphical representations associated to the tangent, cotangent, secant and cosecant.

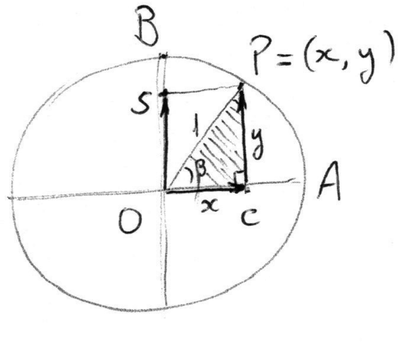
The Tangent function.

When we were studying the sine and the cosine, the fact that the radius of the trigonometric circle was defined to be always equal to 1, helped us a lot, in the sense of eliminating the denominator of:

$$\sin\beta = \frac{opp}{hip} = \frac{\overline{CP}}{\overline{OP}} = \frac{\overline{CP}}{1} = \overline{CP} = y$$

and:

$$\cos\beta = \frac{adj}{hip} = \frac{\overline{OC}}{\overline{OP}} = \frac{\overline{OC}}{1} = \overline{OC} = x$$



Now, in the case of the tangent,

we are not as "lucky" as before,

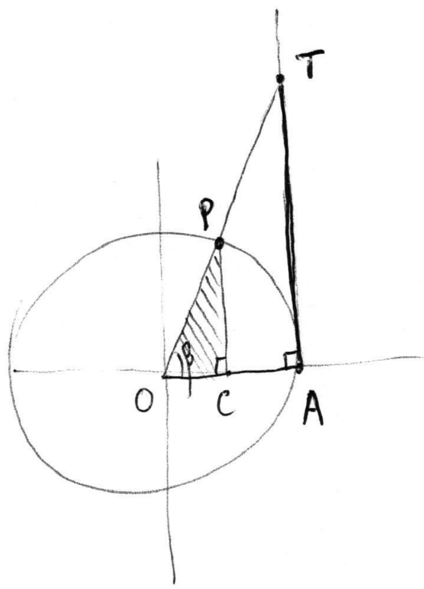
because the denominator of

$$\tan\beta = \frac{opp}{adj}$$ is not equal to 1.

In order to solve this (little) difficulty,

we will construct a triangle OAT

which is similar to triangle OCP since they have all their three sides respectively parallel to each other.



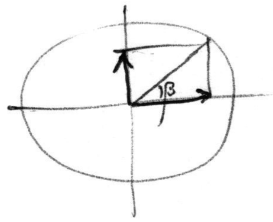
(lines $\overline{OP}$ and $\overline{OT}$ are coincident,
lines $\overline{OC}$ and $\overline{OA}$ are coincident,
and lines $\overline{CP}$ and $\overline{AT}$ are parallel)

The fact that triangles OCP and OAT are similar, means that

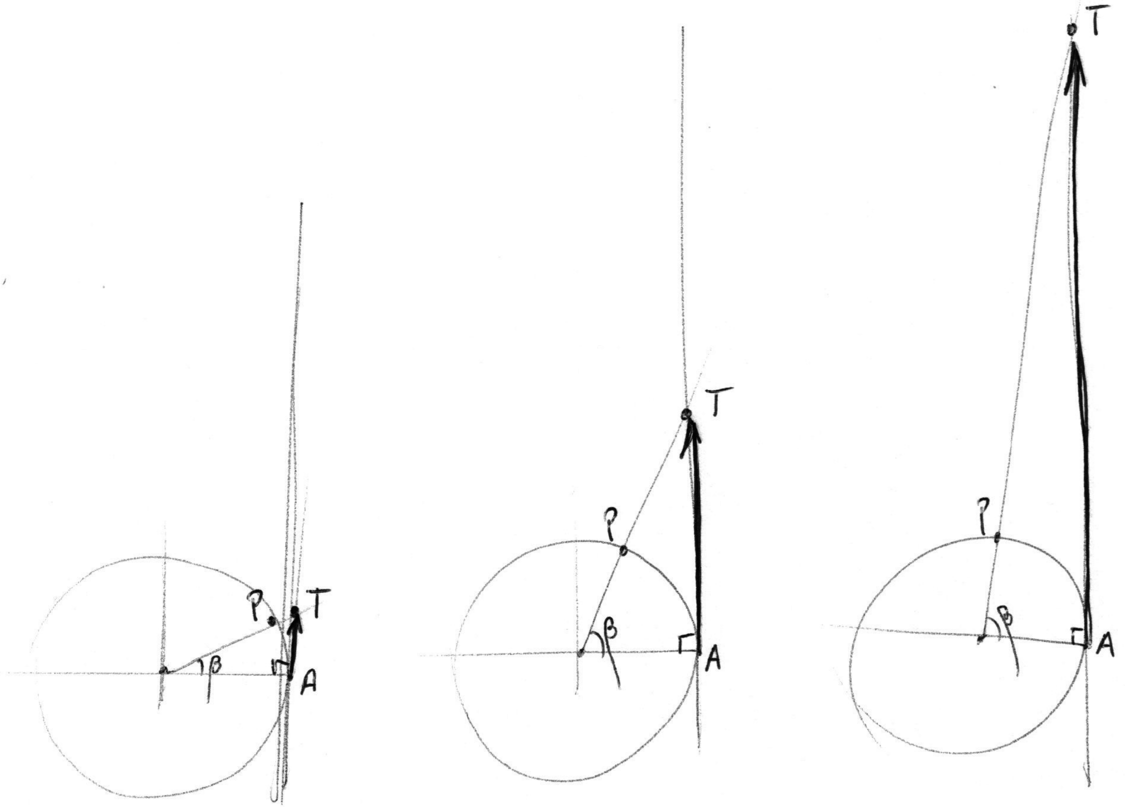
$$\tan\beta = \frac{opp}{adj} = \frac{\overline{CP}}{\overline{OC}} = \frac{\overline{AT}}{\overline{OA}} = \frac{\overline{AT}}{radius} = \frac{\overline{AT}}{1} = \overline{AT}$$

This result was a pretty interesting one, because now, we can allow ourselves to REPRESENT $\tan\beta$ by the "vector" $\overrightarrow{AT}$, (although we know that $\tan\beta$ is not a vector).

So, as we did before, when we associated the sine and cosine to two "vectors"...



... now, we can regard $\tan\beta$ as being associated to the "vector" $\overrightarrow{AT}$, where the point T is the intersection of the extended line $\overline{OP}$ with the vertical line $\overline{AT}$.



In the first quadrant,

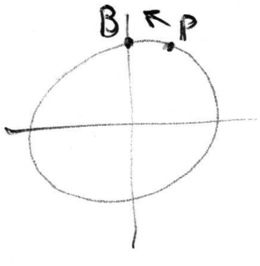
when β goes from 0° to 90°,

the "vector" $\overrightarrow{AT}$ initially has length zero (when $A \equiv T$),

and, as the angle β increases from 0° to 90°,

$\overrightarrow{AT}$ increases in length, (initially slowly),

but when P approaches B,



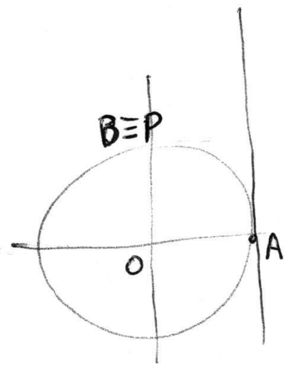
$\overrightarrow{AT}$ increases very rapidly,

until P reaches B

at the angle $\beta = 90^\circ$.

At This angle $(\beta = 90°)$,

There is no point "T" anymore, because The extension of



The straight line $\overline{OP}$ would be PARALLEL To what have been our (previous) line $\overline{AT}$.

So ... tan 90° does not exist.

—//—

Another way To see This, is To observe That

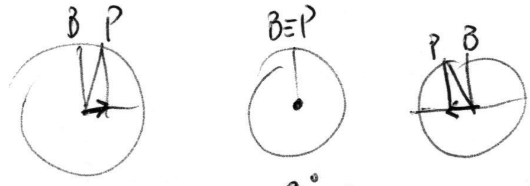
$\tan\beta$ can be expressed as $\dfrac{\sin\beta}{\cos\beta}$

because $\tan\beta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\frac{\text{opp}}{\text{Radius}}}{\frac{\text{adj}}{\text{Radius}}} = \dfrac{\sin\beta}{\cos\beta}$

... and, in the previous chapter,

we had observed that

$$\cos 90^\circ = 0$$



$\cos 90^\circ = 0$

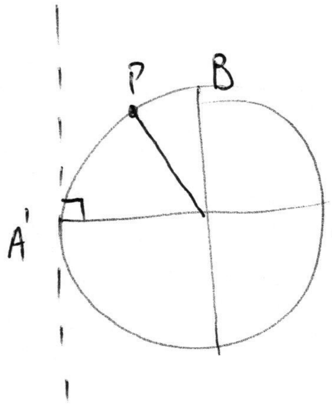
So ...

$$\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{\sin 90^\circ}{0}$$

, and we know that a fraction cannot have zero as its denominator.

(This would be another reason why Tan 90° does not exist)

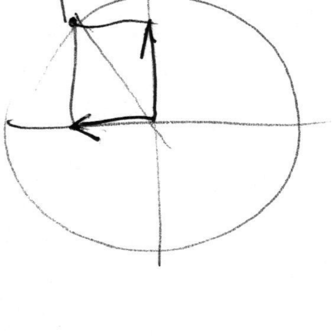
Now, as P enters into the second quadrant, our initial intuition would suggest us to construct from point A' (on the left), a parallel line to the y-axis.



But we won't follow this idea, because, in the second quadrant,

$$\tan\beta = \frac{\sin\beta}{\cos\beta}$$ is a negative number,

(i.e.: the numerator ($\sin\beta$) is positive, whereas ($\cos\beta$) is negative).

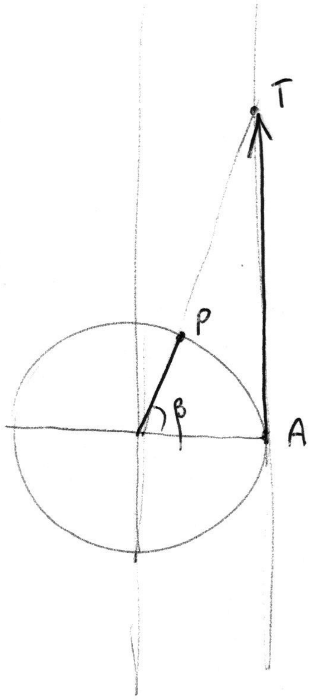


$$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{\oplus}{\ominus} = \ominus$$

(Tan β would be negative, in the second quadrant)

So, we will decide to keep our previous vertical line constructed from point A (on the right), and that's where our ("dear") vector $\overrightarrow{AT}$ will be all the time, regardless of the quadrant where point P is located, during its journey through the circle.

Such approach is convenient, because the "vector" $\overrightarrow{AT}$ would be representing a positive number, whenever it is pointing upwards; and a negative number, whenever it is pointing downwards.

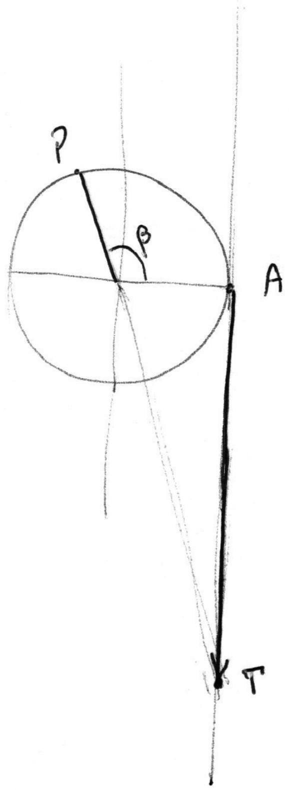


In the first quadrant

Tan β is positive

$$\left(\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{+}{+} = + \right)$$

(Also, $\overrightarrow{AT}$ is pointing upwards)

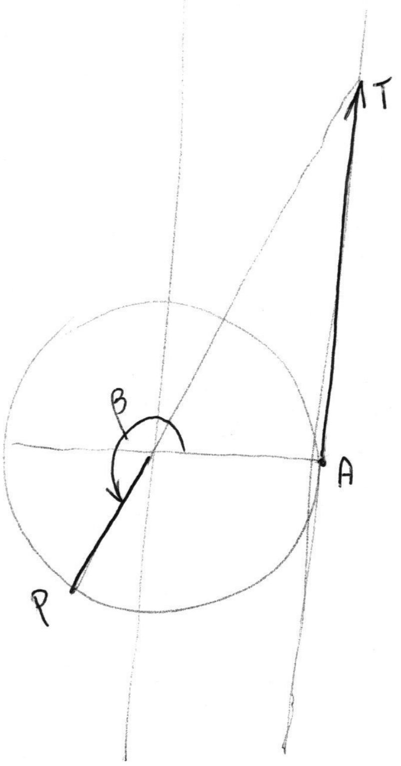


In the second quadrant

Tan β is negative

$$\left(\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{+}{-} = - \right)$$

(Also, $\overrightarrow{AT}$ is pointing downwards)

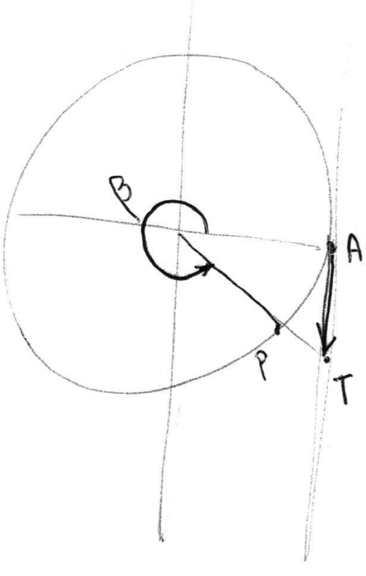


In the third quadrant,

$\tan\beta$ is positive

$\left(\tan\beta = \dfrac{\sin\beta}{\cos\beta} = \dfrac{\ominus}{\ominus} = \oplus \right)$

(Also, $\overrightarrow{AT}$ points upwards)



In the fourth quadrant,

$\tan\beta$ is $\boxed{}$

$\left(\tan\beta = \dfrac{\boxed{}}{\boxed{}} = \dfrac{\square}{\square} = \square \right)$

(Also, $\overrightarrow{AT}$ points $\boxed{}$)

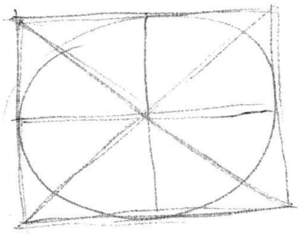
Now, we will take a look at what would be happening at:

tan 45°

tan 135°

tan 225°

tan 315°

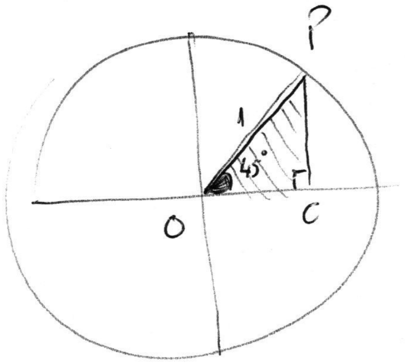


But, before we do so, let's go back to the previous chapter (about sine and cosine), and do a little exercise that we should have done at that time, but we didn't, (probably because we had some other priorities).

The missing exercise was:

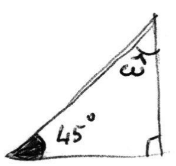
What is $\sin 45^\circ$ (?)

(also ... what is $\cos 45^\circ$ (?) ...)



Triangle OPC is an interesting triangle, because, besides being a right triangle it is also an isosceles triangle.

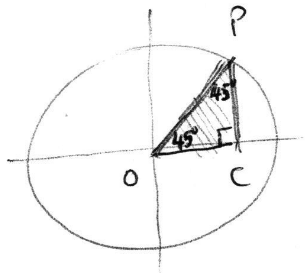
Angle $\hat{\omega}$ would have to be 45° ... because...

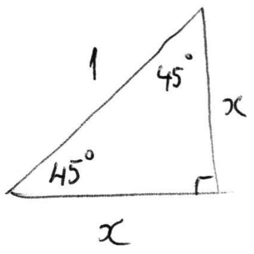


$$45^\circ + \hat{\omega} + 90^\circ = 180^\circ$$

So ... $\hat{\omega} = 45^\circ$

(actually, in any right triangle, the other two angles are complementary angles, that is: their sum is equal to 90°)



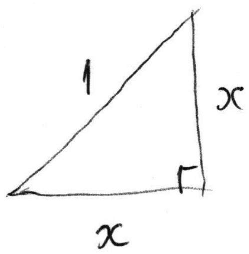


Being an isosceles (right) triangle,

The two legs are equal (i.e.: congruents),

and we can replace our (old) "y"

by "x" (The horizontal projection of the radius)



By Pythagoras,

$$1^2 = x^2 + x^2$$

$$1 = 2x^2$$

$$\frac{1}{2} = x^2$$

$$x = \pm\sqrt{\frac{1}{2}}$$

Again, we discard the negative option, (because we are dealing with length, which is always positive) and so, we have:

$$x = \sqrt{\frac{1}{2}}$$

$$x = \frac{\sqrt{1}}{\sqrt{2}}$$

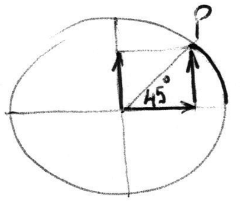
$$x = \frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}}$$

$$x = \frac{\sqrt{2}}{2}$$

(Very good)... That means that

$$\cos 45^\circ = \frac{\sqrt{2}}{2}$$

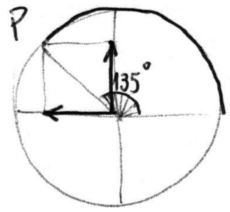
and also ... $\sin 45^\circ = \frac{\sqrt{2}}{2}$



In the first quadrant,

$$\sin 45^\circ = \frac{\sqrt{2}}{2}$$

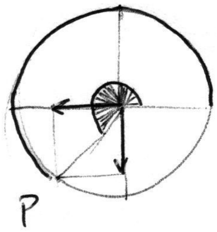
$$\cos 45^\circ = \frac{\sqrt{2}}{2}$$



In the second quadrant,

$$\sin 135^\circ = \boxed{}$$

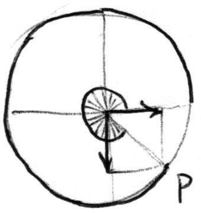
$$\cos 135^\circ = \boxed{}$$



In the third quadrant,

$$\sin 225^\circ = \boxed{}$$

$$\cos 225^\circ = \boxed{}$$



In the fourth quadrant,

$$\sin 315^\circ = \boxed{}$$

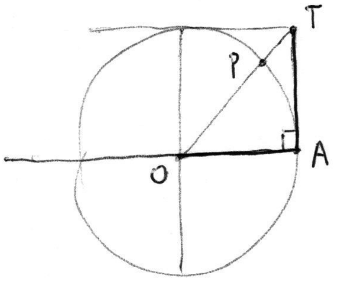
$$\cos 315^\circ = \boxed{}$$

Now, let's go back to the point where we were earlier:

We were involved with the question:

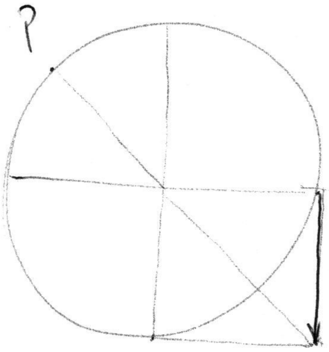
What is $\tan 45^\circ$
$\tan 135^\circ$
$\tan 225^\circ$
$\tan 315^\circ$ (?)

Recalling that $\tan \beta$ can be represented by the "vector" $\overrightarrow{AT}$, in the particular case when $\beta = 45^\circ$, triangle OAT, besides being a right triangle, it will also be an isosceles triangle where $\overline{OA} = \overline{AT}$

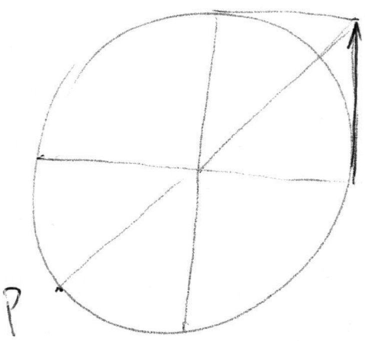


But $\overline{OA}$ is the radius, so $\overline{AT}$ will also be equal to 1.

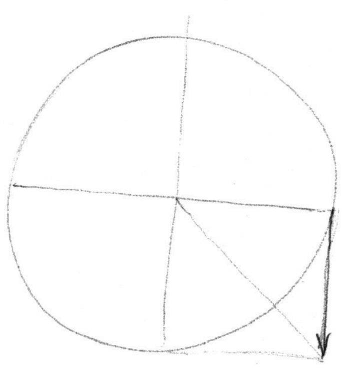
Therefore $\boxed{\tan 45^\circ = 1}$



$\tan 135^\circ =$



$\tan 225^\circ =$

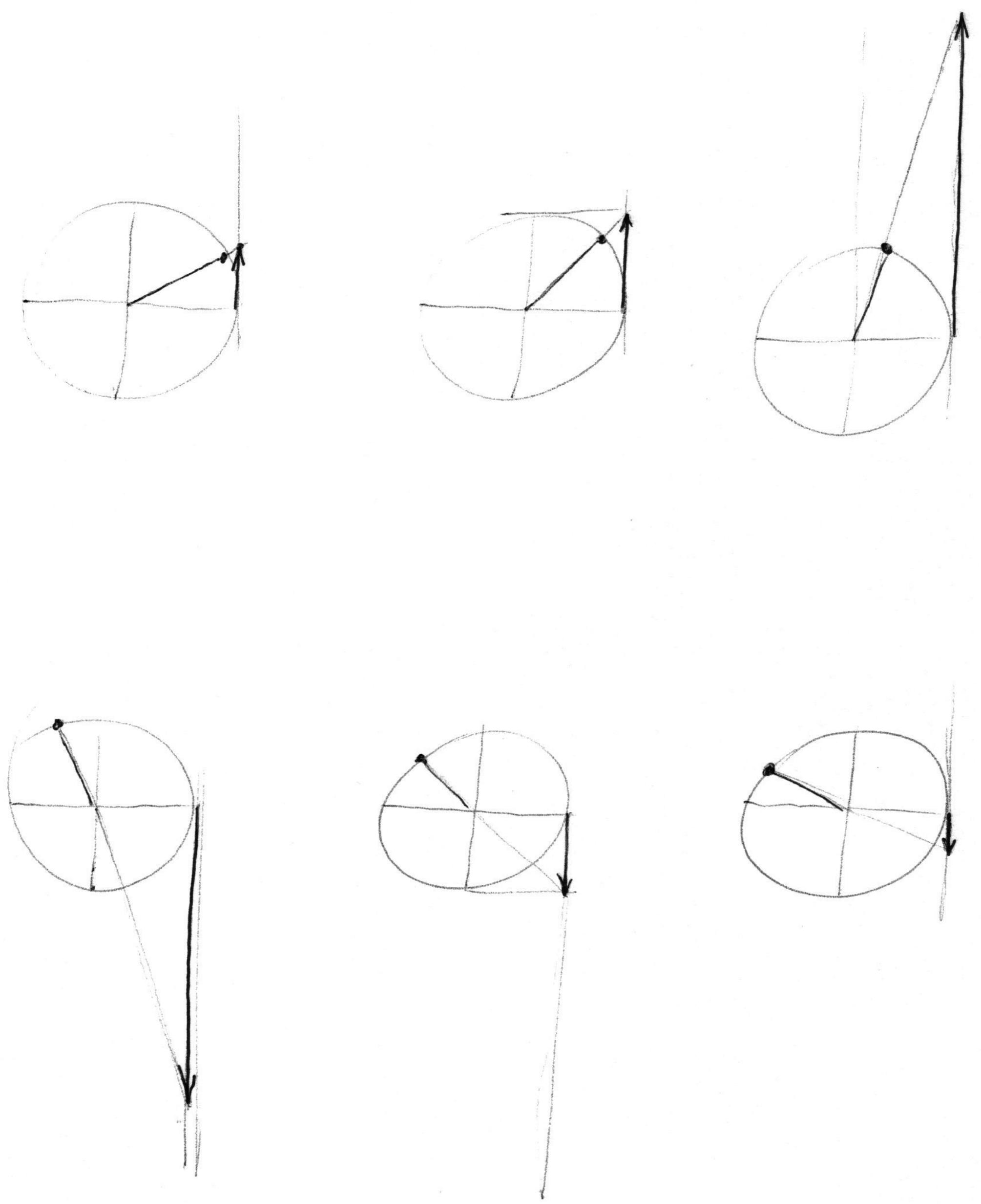


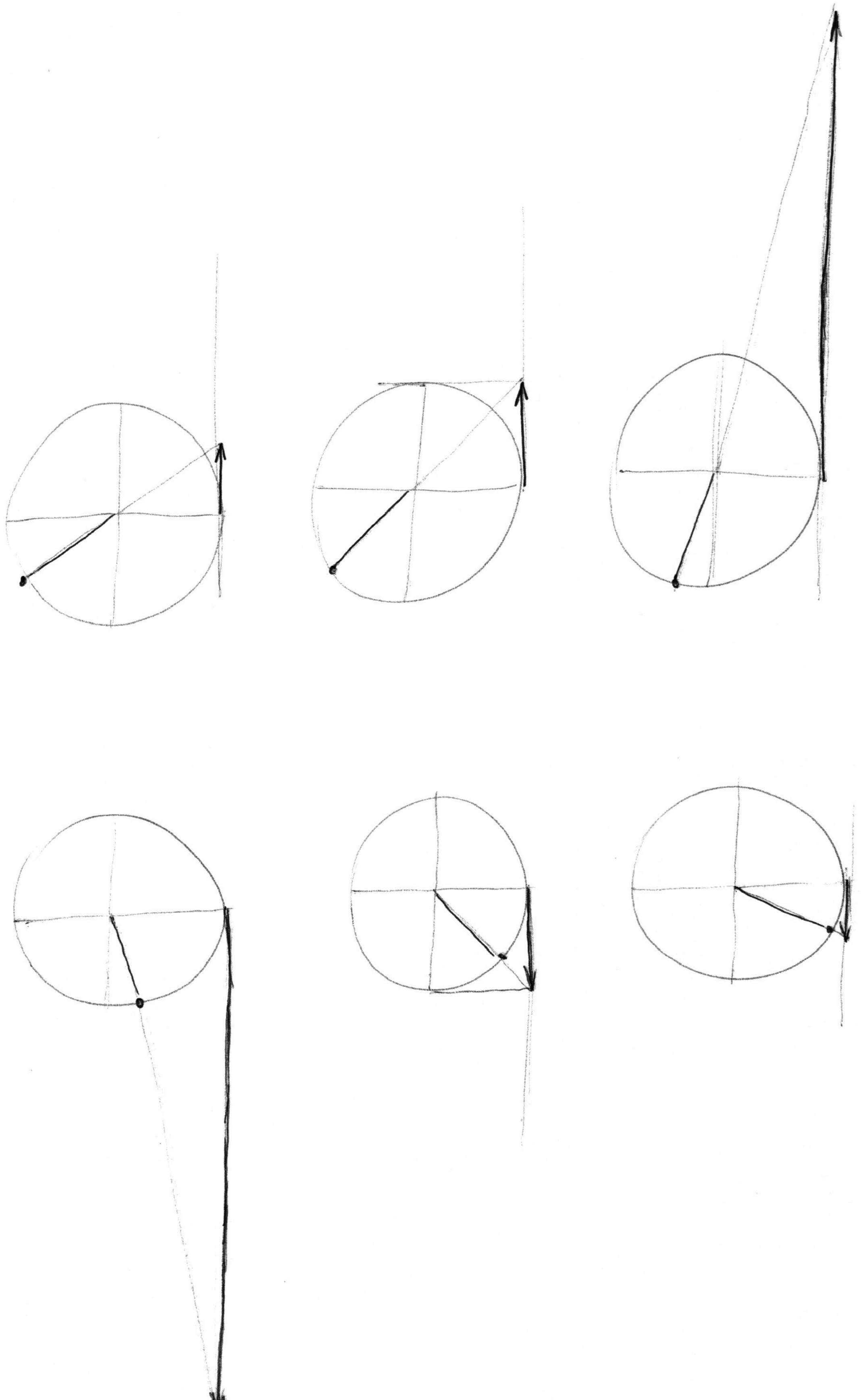
$\tan 315^\circ =$

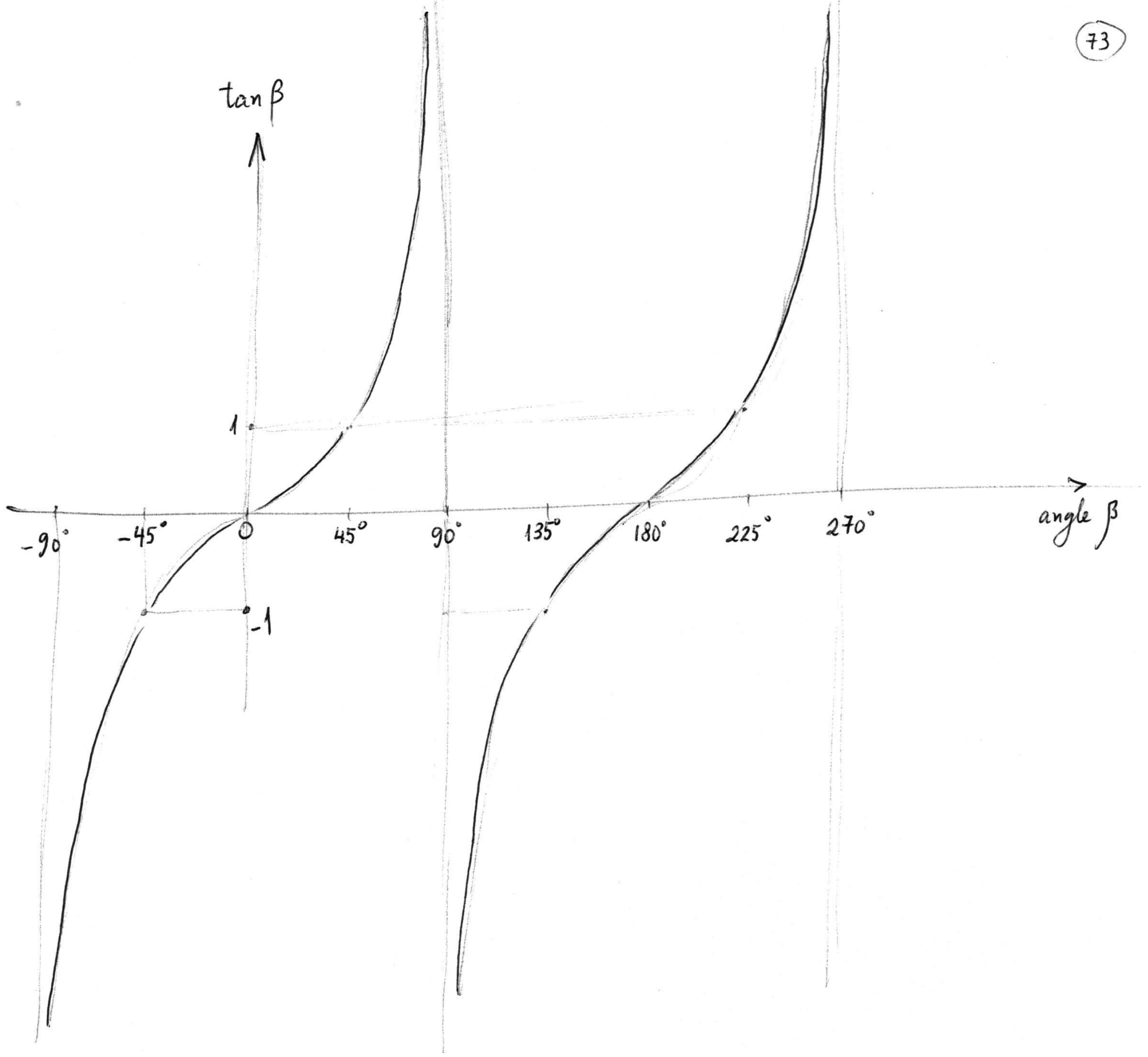
Now, we are ready to construct (again)

The sequence of "screen-shots"

for the TANGENT.







This is the graph of $f(\beta) = \tan\beta$, where

the horizontal axis is the angle β

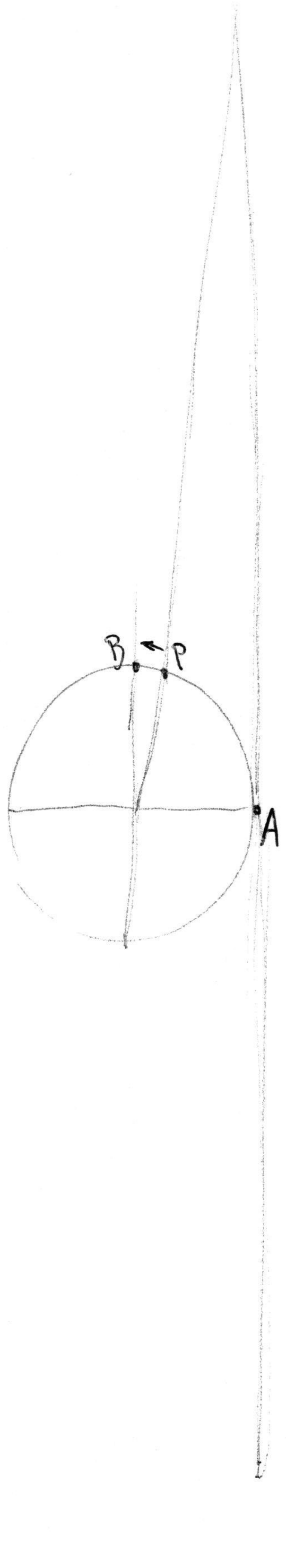
and the vertical axis is the value of $\tan\beta$.

As we can see, $\tan(-45°) = -1$

$\tan(45°) = 1$

$\tan(135°) = -1$

$\tan(225°) = 1$

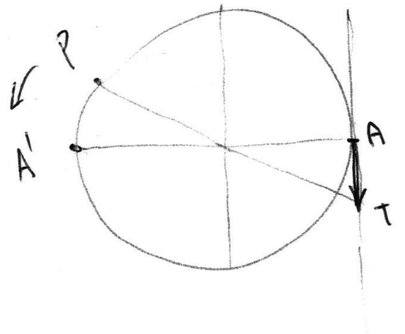


... which confirms what we had observed earlier.

Also, as P approaches B, $\overrightarrow{AT}$ becomes infinitely big and positive.

When P reaches B (β is exactly 90°), $\tan 90^\circ$ does not exist.

As soon as P enters into the second quadrant, $\overrightarrow{AT}$ points downwards becoming infinitely negative.



Near the border A'

(between the second and third quadrants)

as P approaches A',

$\overrightarrow{AT}$ is negative,

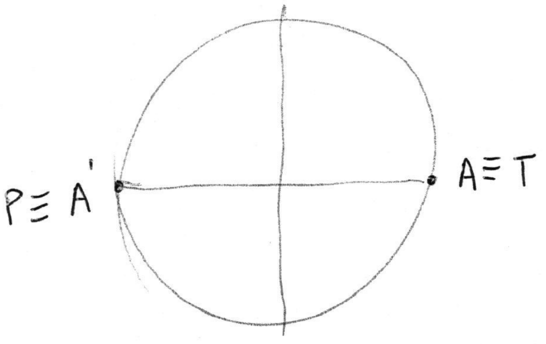
and it becomes less and less

negative until it vanishes

while T becomes coincident with A,

meaning that $\overline{AT} = 0$

(i.e.: $\tan 180° = 0$)

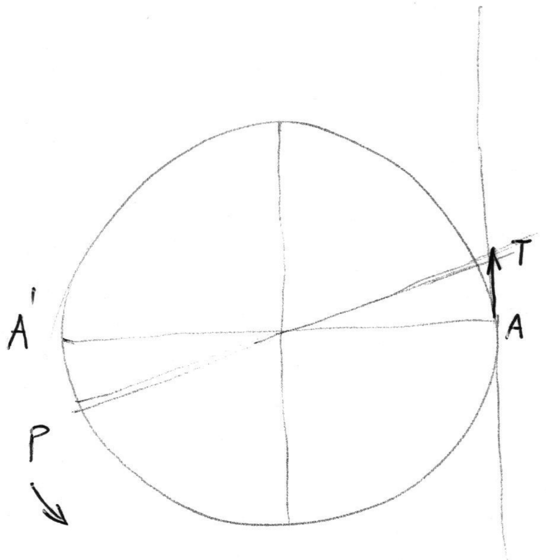


matching (once more)

the informations displayed on the graph,

with what we've been observing here,

in the trigonometric circle.

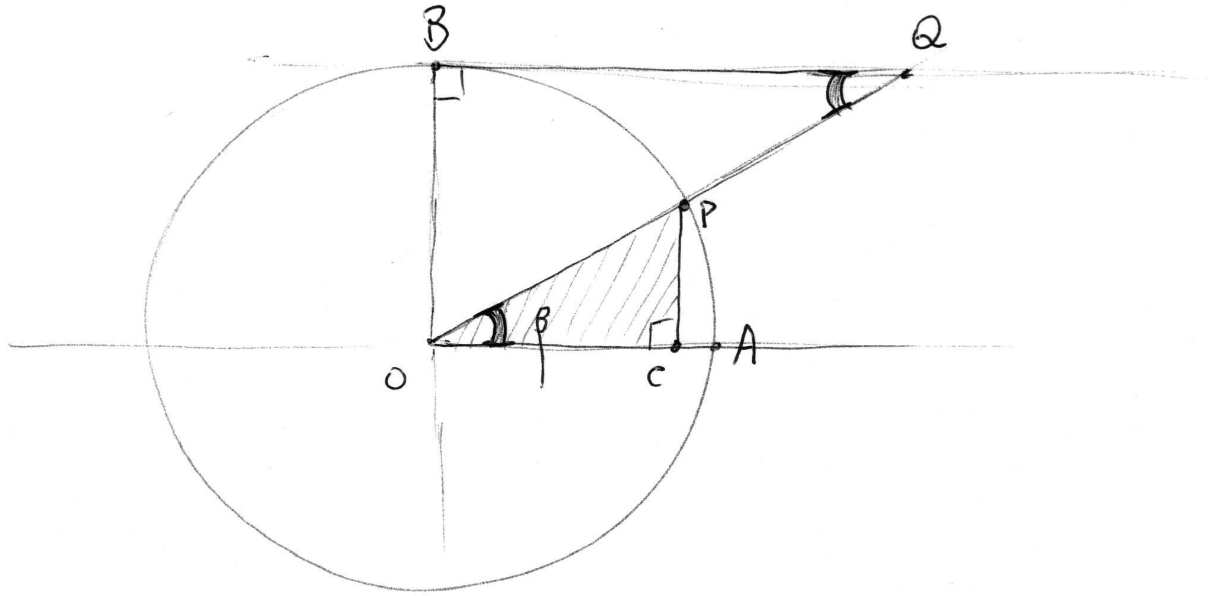


In the third quadrant, $\overrightarrow{AT}$ becomes positive (again), repeating the same behavior as in the first quadrant.

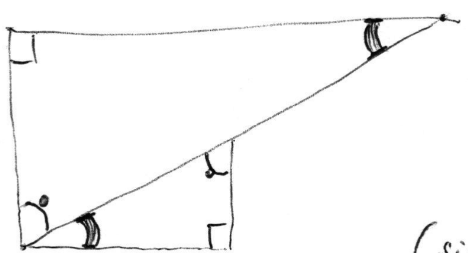
As we can see,

the fact that we are using $\overrightarrow{AT}$ as a representative of $\tan\beta$, greatly helped us to better understand what is going on with the graph of the function $f(\beta) = \tan\beta$.

The cotangent function



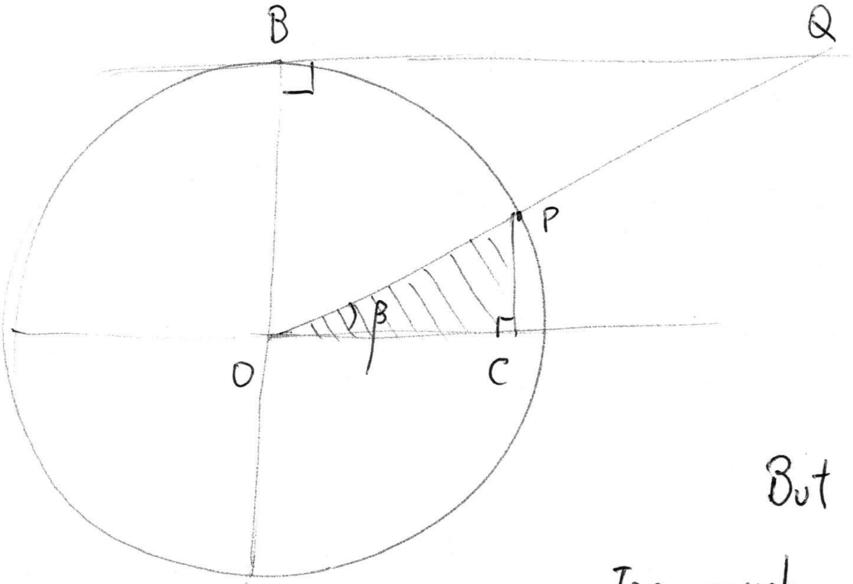
Angles $\measuredangle = C\hat{O}P = O\hat{Q}B$ are congruents because they are alternate interior.



Therefore the two right triangles OBQ and OCP will be similar, (since they have all three angles respectively congruent).

$$\sin\beta = \frac{opp}{hip}$$
$$\cos\beta = \frac{adj}{hip}$$
$$\tan\beta = \frac{opp.}{adj}$$
$$\cot\beta = \frac{adj}{opp}$$
$$\sec\beta = \frac{hip}{adj.}$$
$$\csc\beta = \frac{hip}{opp}$$

$\cot\beta$ is by definition, equal to $\frac{adj}{opp}$



$$\cot\beta = \frac{\overline{OC}}{\overline{CP}}$$

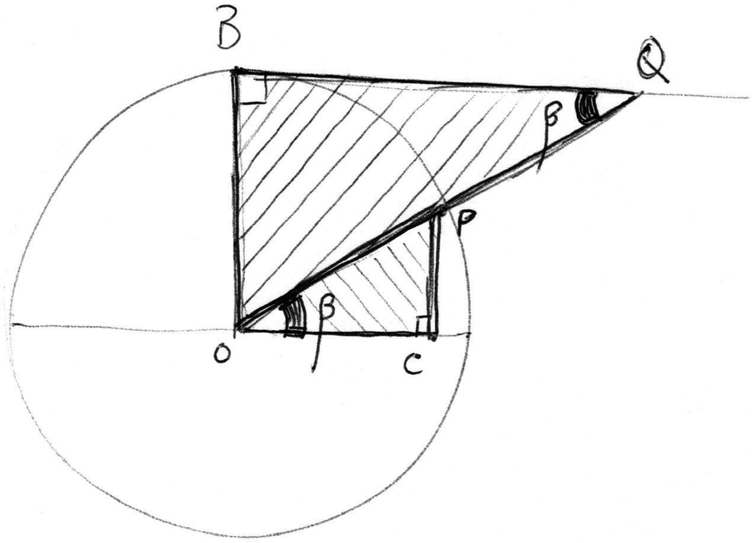
(according to triangle OCP)

But This doesn't help too much, because we don't have The denominator equal to 1 (in order to try to use The same "strategy" we have done, before).

So, you might be (naturally) guessing,

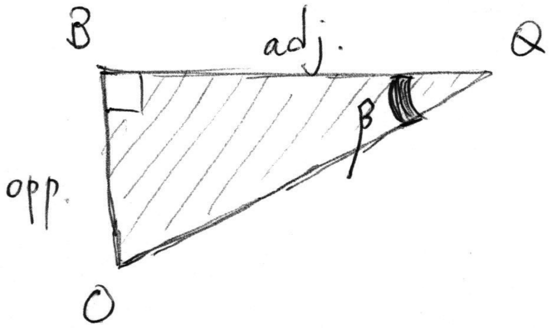
That we are going to use triangle OQB

since it is similar to our (previous) triangle OCP.



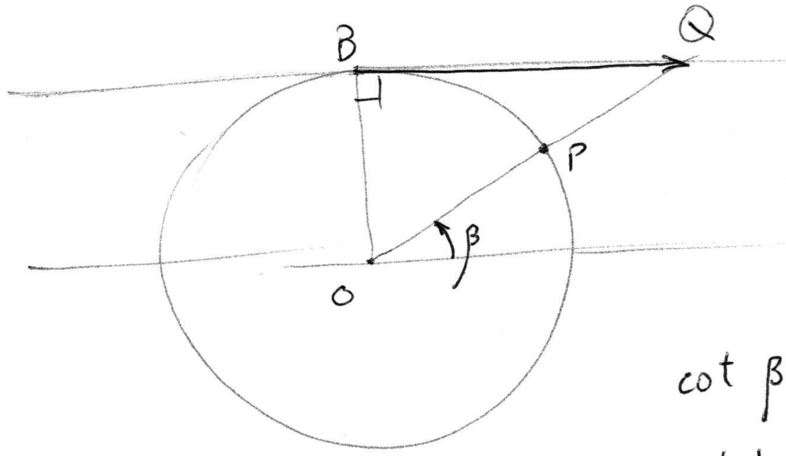
And so, according to triangle OQB,

$$\cot \beta = \frac{adj}{opp} = \frac{\overline{BQ}}{\overline{BO}} = \frac{\overline{BQ}}{radius} = \frac{\overline{BQ}}{1} = \overline{BQ}$$



(Wonderful...)

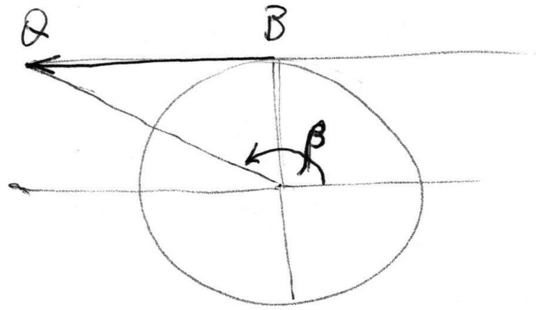
Now, we have a representative for $\cot\beta$.



$\cot\beta$ will be represented by the "vector" $\overrightarrow{BQ}$

(although we know (as always) that $\cot\beta$ is not a vector).

—//—　　　　　　—//—



Once more, we will "resist the temptation" of constructing another horizontal line passing through B' (on the bottom),

... whenever we are in the third or fourth quadrants, because, if we do so, we would end up "disturbing" the orientation of the "vector" $\overrightarrow{BQ}$.

(we want to "see" $\overrightarrow{BQ}$ as positive,

whenever it points to the right,

and negative whenever it points to the left).

So, we will keep $\overrightarrow{BQ}$ always

on the top, passing through B,

thus discarding the idea of constructing

another auxiliary line passing through B',

horizontally.

In the chapter of the TANGENT we had observed

That $\tan \beta = \dfrac{\sin \beta}{\cos \beta}$

$$\left(\text{because} \quad \tan \beta = \frac{\text{opp}}{\text{adj.}} = \frac{\frac{\text{opp}}{\text{Radius}}}{\frac{\text{adj.}}{\text{Radius}}} = \frac{\sin \beta}{\cos \beta} \right)$$

Likewise,

$$\cot \beta = \frac{\text{adj}}{\text{opp}} = \frac{\frac{\text{adj.}}{\text{Radius}}}{\frac{\text{opp}}{\text{Radius}}} = \frac{\cos \beta}{\sin \beta}$$

So, in the first quadrant, $\cot \beta = \dfrac{\cos \beta}{\sin \beta} = \dfrac{+}{+} = +$

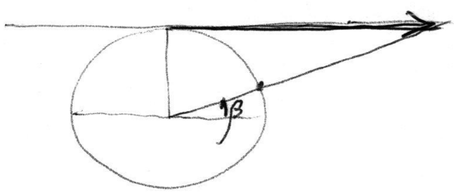
In the second quadrant, $\cot \beta = \dfrac{\square}{\square} = \dfrac{\square}{\square} = \square$

In the $\square$ quadrant, $\cot \beta = \dfrac{\square}{\square} = \dfrac{\square}{\square} = \square$

In the $\square$ quadrant,

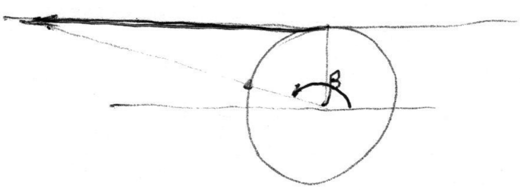
$\cot \beta = \dfrac{\square}{\square} = \dfrac{\square}{\square} = \square$

The above observations should confirm what the pictures (below) are going to tell us, since we are assuming that, (in this context), a vector pointing to the right means that it is representing a positive number, whereas if it points to the left, it represents a negative number.



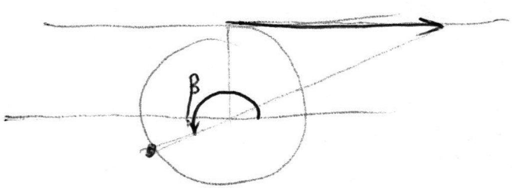
In the first quadrant,

cot β is $\boxed{\text{positive}}$



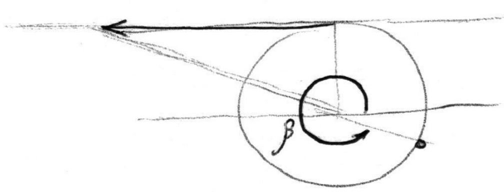
In the second quadrant,

cot β is $\boxed{\quad\quad}$.



In the third quadrant,

cot β is $\boxed{\quad\quad}$.



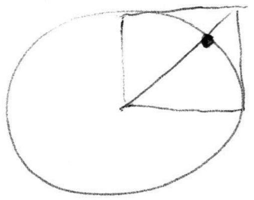
In the fourth quadrant,

cot β is $\boxed{\quad\quad}$.

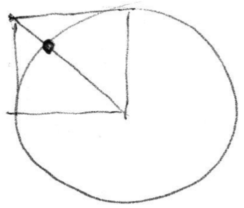
Now, focusing on $\beta = 45°$,

(and also, on angles such as $135°$; $225°$ and $315°$),

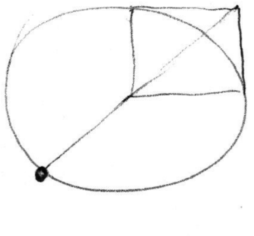
we have :



$\cot 45° =$

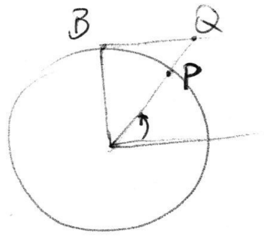


$\cot 135° =$



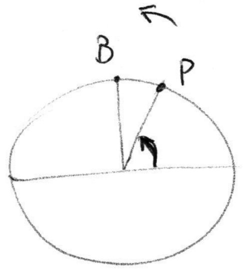
$\cot 225° =$

$\cot 315° =$



Near B as P approaches B

$\overrightarrow{BQ}$ becomes smaller and smaller,



until it reaches zero at $\beta = 90°$

So ... $\cot 90° = 0$

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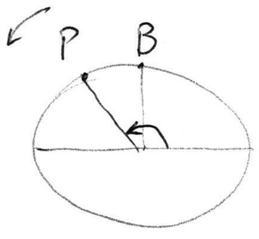
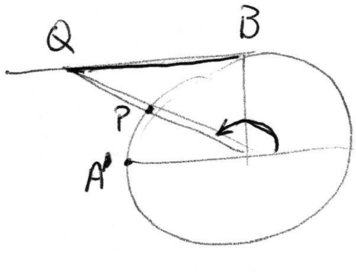


As P moves away from B,

$\overrightarrow{BQ}$ becomes more and more

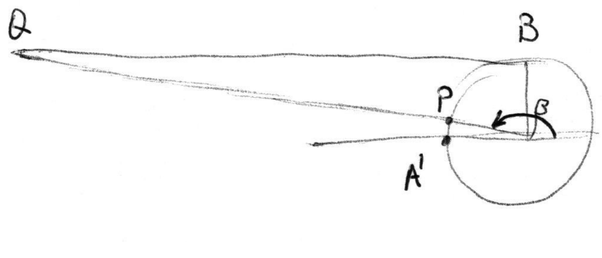


negative, and, as P gets

near A', $\overrightarrow{BQ}$ is so negative,

(being represented by a "huge"

arrow pointing to the left)



That, when P reaches A',

(at $\beta = 180°$)

$\cot \beta$ does not exist

(since the lines become parallel).

As we can see,
similar things happen often,
while we observe each one of these
six trigonometric functions.

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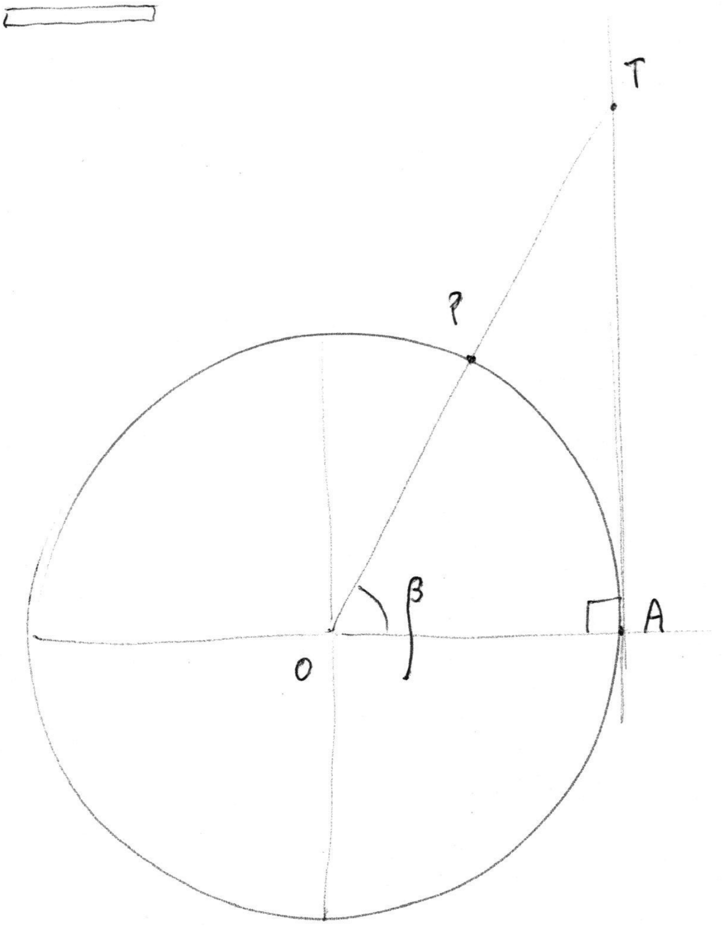
In order to conclude this type of
analysis, we should take a quick look
at the remaining two functions:
the secant and the cosecant.

The secant



$$\sec\beta = \frac{1}{\cos\beta} = \frac{1}{\frac{adj}{hip}} = \frac{hip}{adj} = \frac{\overline{OT}}{\overline{OA}} =$$

$$= \frac{\overline{OT}}{radius} = \frac{\overline{OT}}{1} = \overline{OT}$$

sin
cos
tan
cot
sec
csc

In other words,

$\sec\beta$ can be represented by

the length of the segment $\overline{OT}$

In the case of the SECANT,
we will not regard $\overrightarrow{OT}$ as a vector,
because, in general, $\overrightarrow{OT}$ is neither horizontal
nor vertical. So, we can't tell,
just by looking at it, whether the
SECANT will be positive or negative.

But, fortunately, since the SECANT is the
reciprocal of the COSINE, we will be able to tell
whether the SECANT is positive or not,
by looking at the COSINE:
if the COSINE is positive, then the SECANT
will also be positive;
if the COSINE is negative, then the SECANT
will also be negative.

Anyway, the length of the vector $\overrightarrow{OT}$ will tell us the value of the absolute value of the secant,

That is: $|\overrightarrow{OT}| = |\sec\beta|$

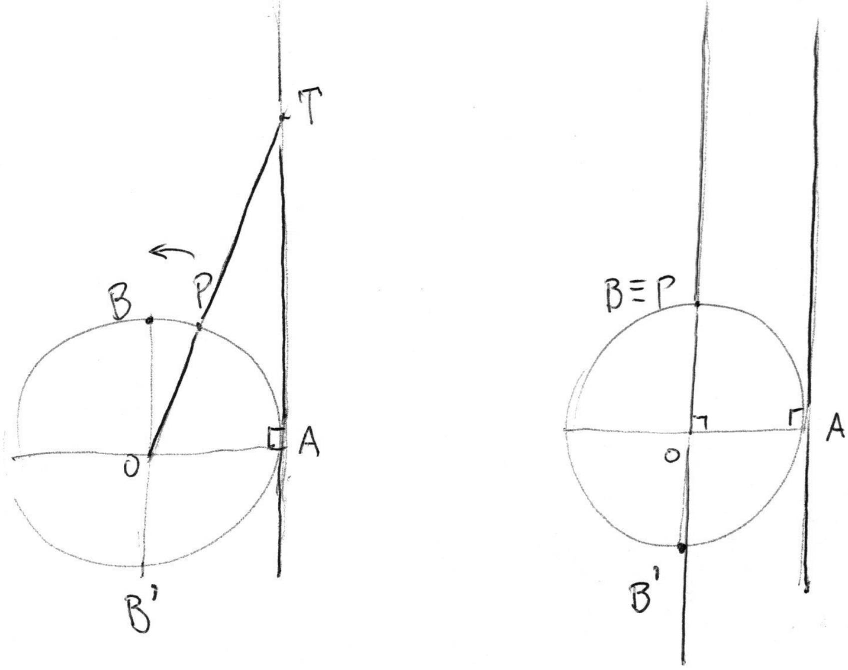
—//—

We must also observe that

$\sec 90^\circ$ does not exist

because $\sec 90^\circ = \dfrac{1}{\cos 90^\circ} = \dfrac{1}{0}$ which wouldn't make sense.

Likewise, $\sec 270^\circ$ does not exist.



Whenever point P is at B or B',

the point T cannot exist,

due to the parallelism of the lines

whose intersection would define it.

Such observation is coincident with the fact that none of the expressions (below) exist:

sec 90°

tan 90°

sec 270°

tan 270°

Another way to look at this "phenomenon" is to observe that ...

$$\sec 90^\circ = \frac{1}{\cos 90^\circ} = \frac{1}{0}$$

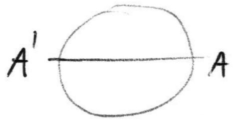
$$\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{\sin 90^\circ}{0}$$

$$\sec 270^\circ = \frac{1}{\cos 270^\circ} = \frac{1}{0}$$

$$\tan 270^\circ = \frac{\sin 270^\circ}{\cos 270^\circ} = \frac{\sin 270^\circ}{0}$$

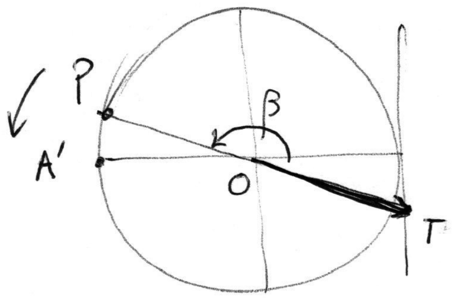
... generating a situation of impossibility, since fractions do not admit zero as their denominators.

Now, it would be worthwhile to take a look at what happens when the SECANT is near points



(That is : when $\beta = 180^\circ$ or 360°)

Near point A', The SECANT is either in The second quadrant or in The Third quadrant, so it will be negative in both quadrants, (since The COSINE is negative in These quadrants).



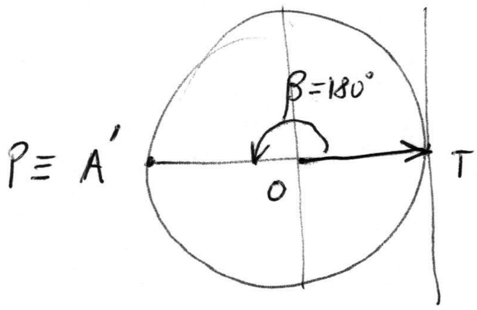
As β approaches 180°

P approaches A'

and $|\overrightarrow{OT}|$ approaches 1

($|\overrightarrow{OT}|$ is The length of $\overrightarrow{OT}$)

—//—



At exactly 180°, $(\beta = 180^\circ)$

$P \equiv A'$

and $|\overrightarrow{OT}| = 1$

Although $|\vec{OT}| = 1$, we should be careful

and not "jump to the conclusion" that $\sec 180^\circ = 1$,

because here, in this situation,

we have already observed that $\sec \beta$ is negative

(near 180°).

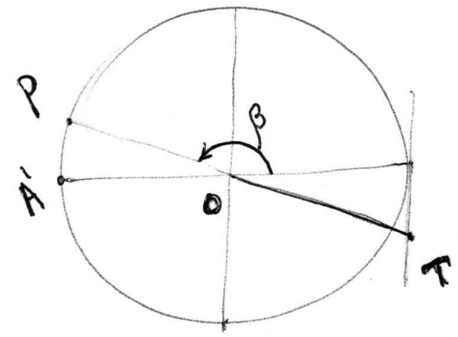
So ... since $\begin{cases} |\vec{OT}| = 1 \\ \text{and } \sec\beta \text{ must be negative} \end{cases}$

Then, $\sec 180^\circ = -1$

(Another way to look at it, would be:

$$\sec 180^\circ = \frac{1}{\cos 180^\circ} = \frac{1}{-1} = -1$$).

Returning to the picture around A',



When P is near A', (and still in the second quadrant),

$|\overrightarrow{OT}|$ tells us that $\sec\beta$ is a number

whose ABSOLUTE VALUE is a little bit bigger than 1,

(for instance 1.2)

And since we know that, in the second quadrant

$\sec\beta$ is negative, we would have, (in this case),

$$\sec\beta = -1.2$$

whose ABSOLUTE VALUE is $|\sec\beta| = 1.2$

Such remarks are important,

to emphasize that the role of the vector $\overrightarrow{OT}$ in the context of the SECANT is simply to give us the information about the value of $|\sec\beta|$.

In other words,

$$\left|\overrightarrow{OT}\right| = \left|\sec\beta\right|$$

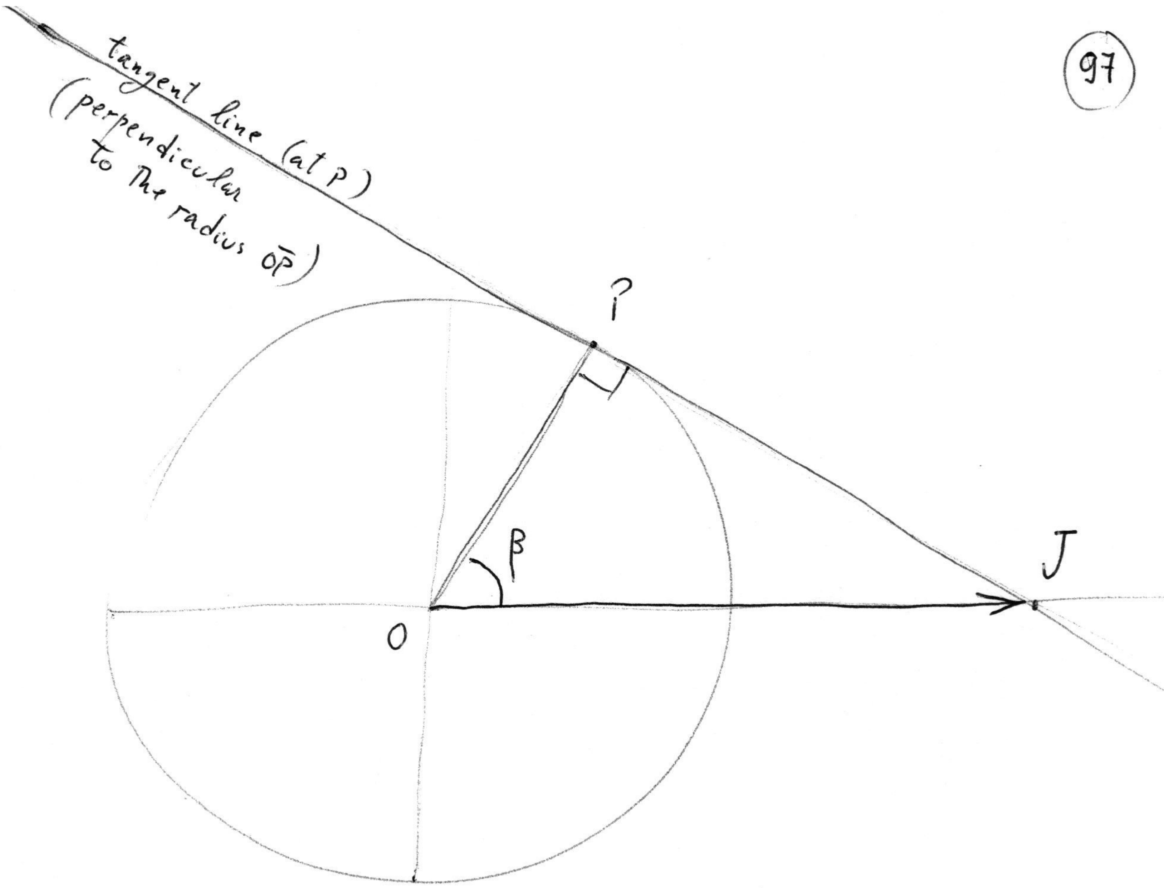
Once we have the information about its absolute value,

then we should check whether it is positive or negative,

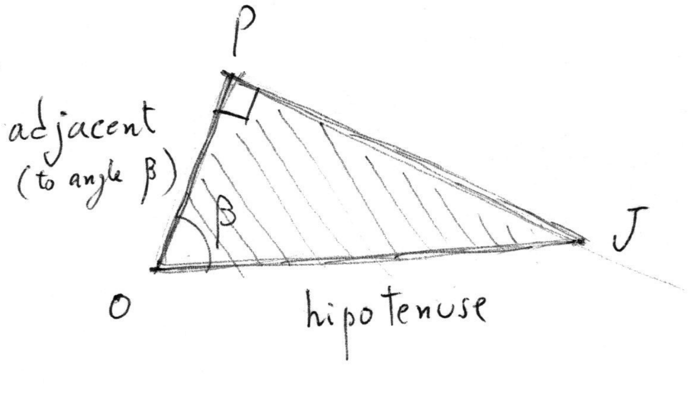
(depending on which quadrant we are).

Well, it seems like, our choice of the vector $\overrightarrow{OT}$, as a visual representation for $\sec\beta$, is not helping us too much, since it doesn't give us a complete information about it.

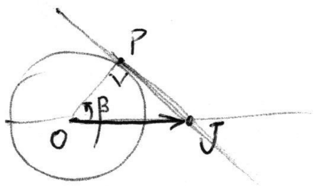
But, fortunately, there is another approach, that will introduce to us another "candidate vector" as a (new) representative for $\sec\beta$ which will fulfill, in a very nice way, our needs.



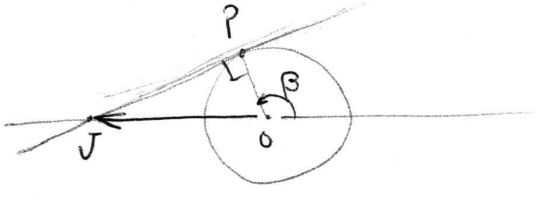
$$\sec\beta = \frac{1}{\cos\beta} = \frac{1}{\frac{adj}{hip}} = \frac{hip}{adj} = \frac{\overline{OJ}}{\overline{OP}} = \frac{\overline{OJ}}{1} = \overline{OJ}$$



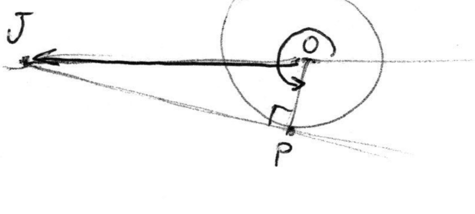
The vector $\overrightarrow{OJ}$ would be an excellent representative for $\sec\beta$, because, in all four quadrants it points to the right or to the left, in accordance with the sign of the secant, i.e.: positive in the first and fourth quadrants, and negative in the second and third quadrants, similarly to the way the cosine behaves, in terms of signs.



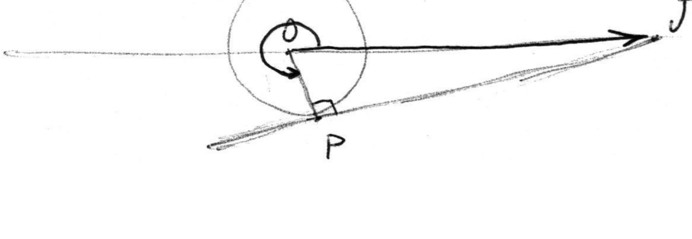
first quadrant



second quadrant

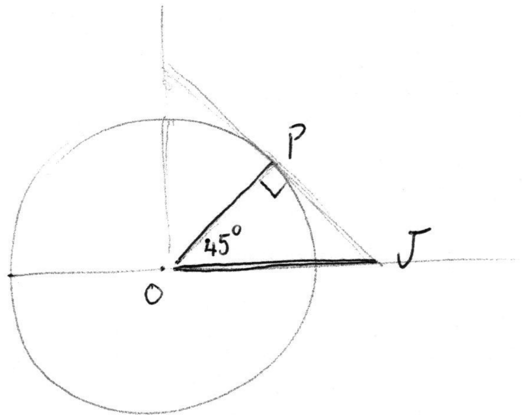


third quadrant

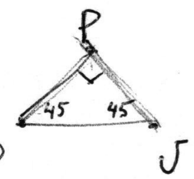


fourth quadrant

Now, that we know that sec β can be represented by the vector $\overrightarrow{OJ}$, it would be a good exercise to check what would be sec 45° ; sec 135° ; sec 225° ; sec 315°.

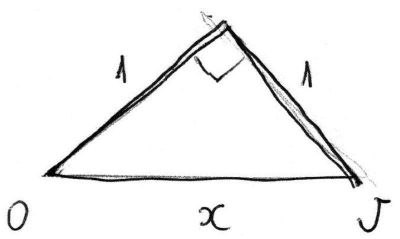


In this case, ($\beta = 45^\circ$), triangle OPJ is a right triangle isosceles ...



... where $\overline{OP}$ = radius = $\overline{PJ}$

(That is:) $\overline{OP} = 1 = \overline{PJ}$



Let $\overline{OJ} = x$

By Pythagoras,

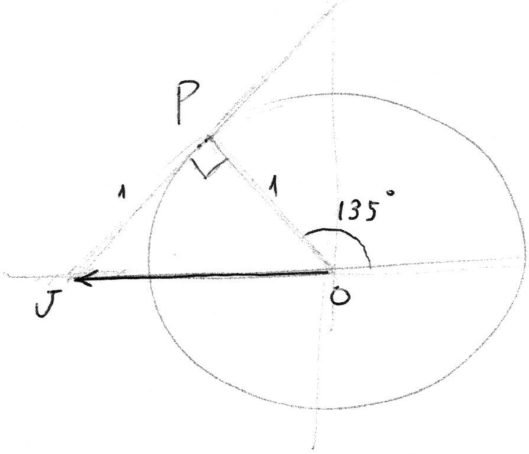
$x^2 = 1^2 + 1^2$

$x^2 = 2$

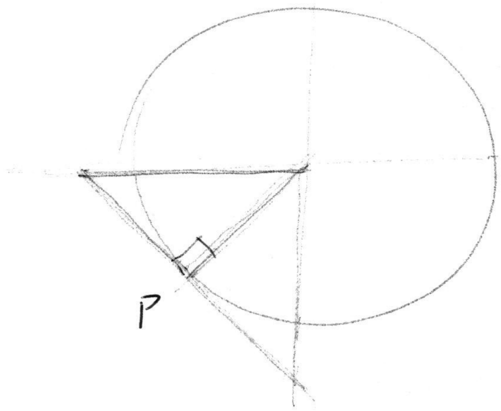
$x = \pm\sqrt{2}$ (Let's discard the negative option)

$x = \sqrt{2}$

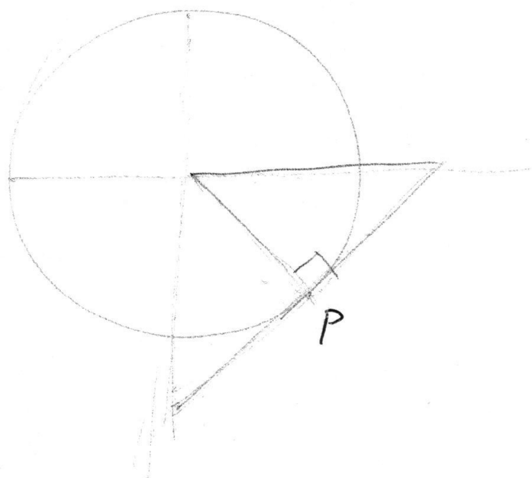
So... $\sec 45^\circ = \sqrt{2}$



$\sec 135^\circ = \boxed{-\sqrt{2}}$

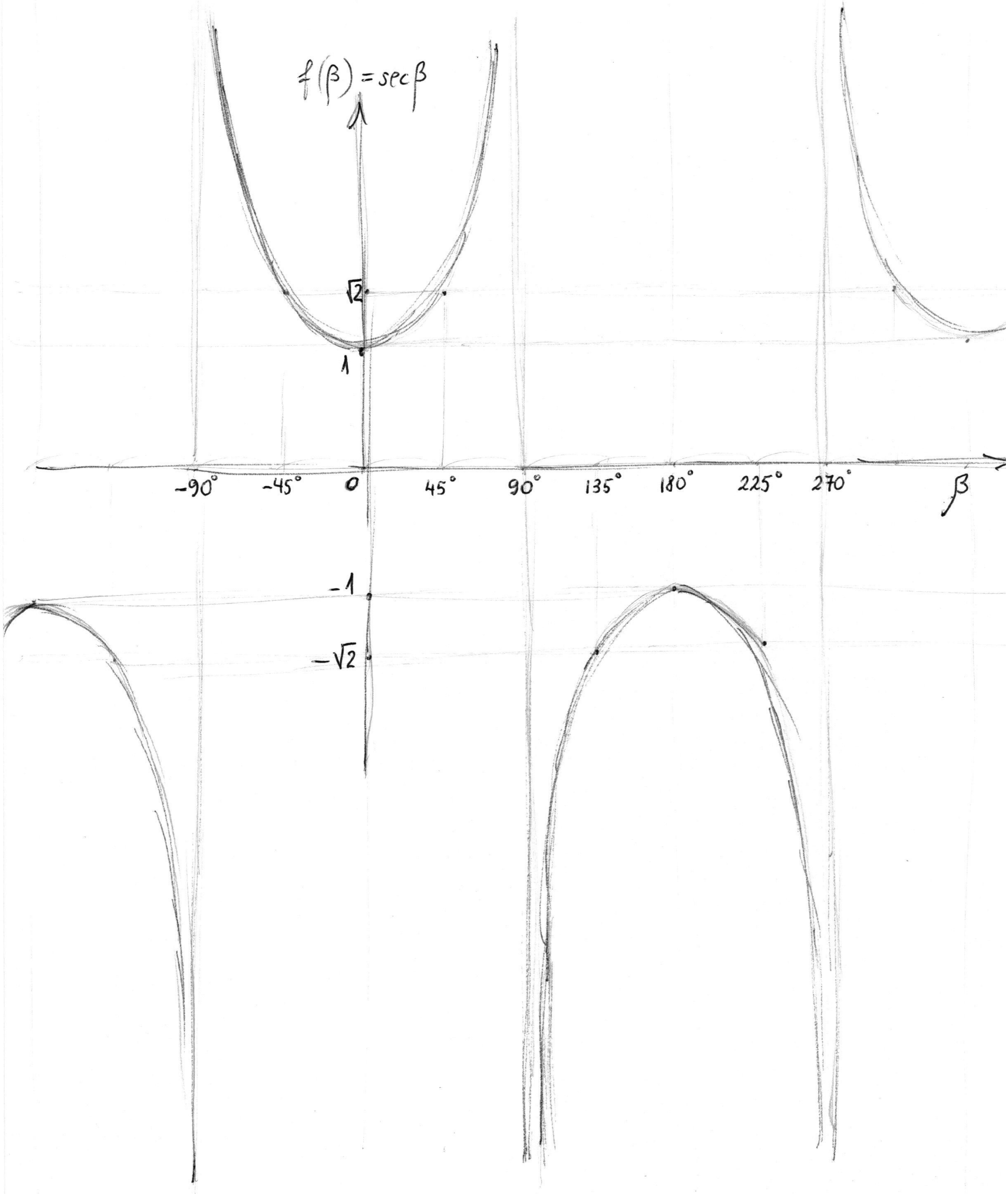


$\sec 225^\circ = \boxed{\phantom{-\sqrt{2}}}$



$\sec 315^\circ = \boxed{\phantom{-\sqrt{2}}}$

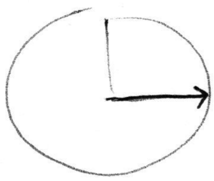
This is the graph of the function $f(\beta) = \sec\beta$



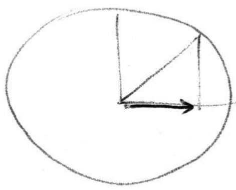
Recalling that the secant and cosine are reciprocal of each other, (i.e.: $\sec\beta = \frac{1}{\cos\beta}$),

it would be interesting if we take a look at the graph of $\cos\beta$.

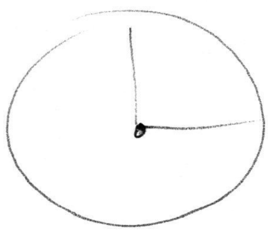
Before constructing it, we need to get some scratch paper and do a few annotations, while we review and think about them, (one more time)



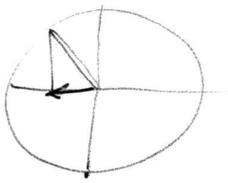
$\cos 0^\circ = 1$



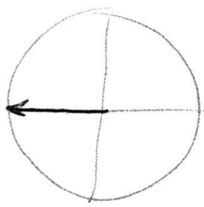
$\cos 45^\circ = \frac{\sqrt{2}}{2}$



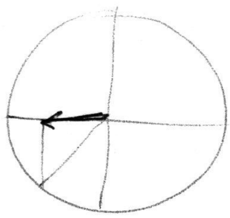
$\cos 90^\circ = 0$



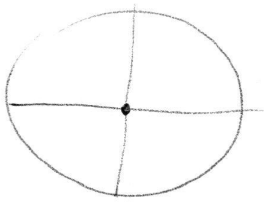
$\cos 135^\circ = -\frac{\sqrt{2}}{2}$



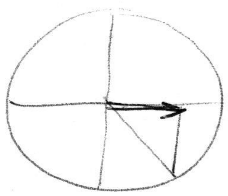
$\cos 180^\circ = -1$



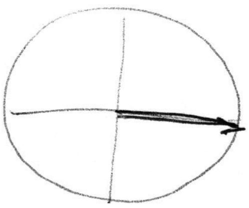
$\cos 225^\circ = -\frac{\sqrt{2}}{2}$



$\cos 270^\circ = 0$

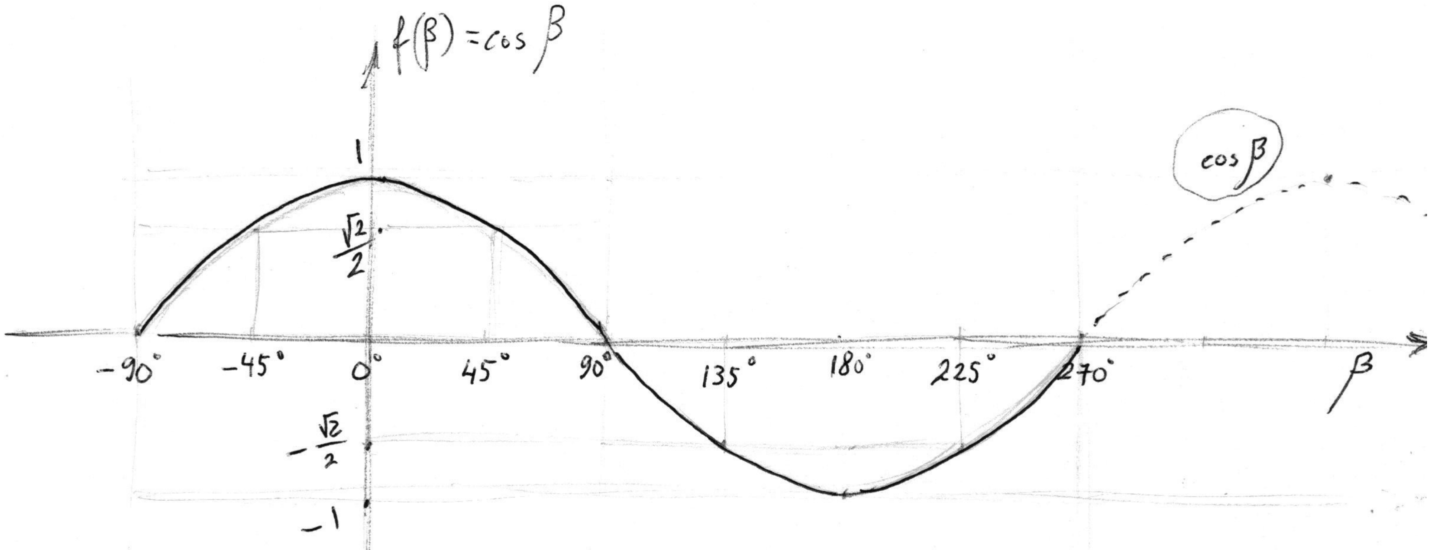


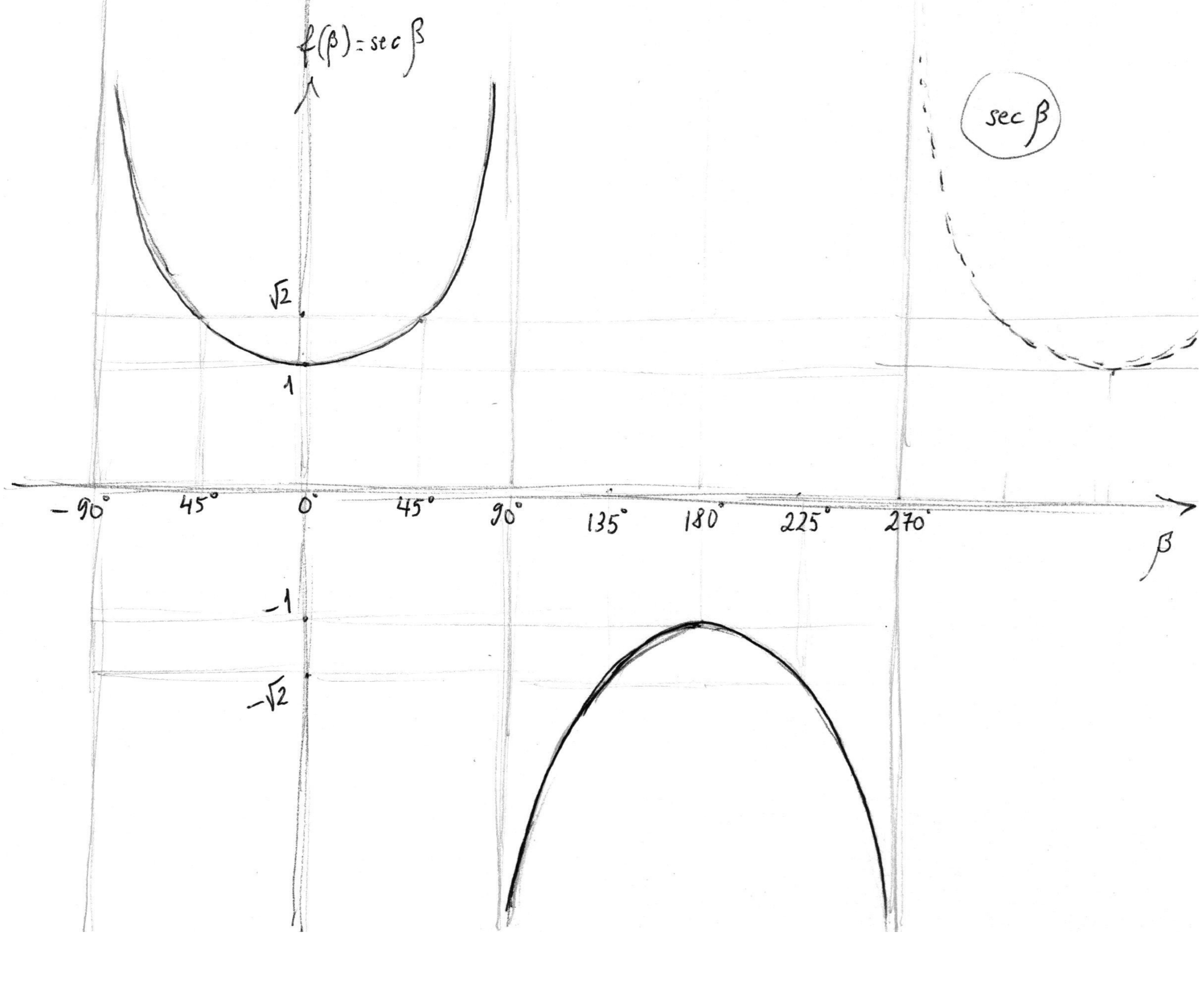
$\cos 315^\circ = \frac{\sqrt{2}}{2}$



$\cos 360^\circ = 1$

Now, we are ready to construct the graph of the cosine, and we would like to compare it to the graph of the secant.





At $\beta = 0^\circ$

$$\cos 0^\circ = 1$$

and $$\sec 0^\circ = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1$$

At $\beta = 45^\circ$

$$\cos 45^\circ = \frac{\sqrt{2}}{2} \quad \left(\text{or } \frac{1}{\sqrt{2}}\right)$$

and $$\sec 45 = \frac{1}{\cos 45^\circ} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

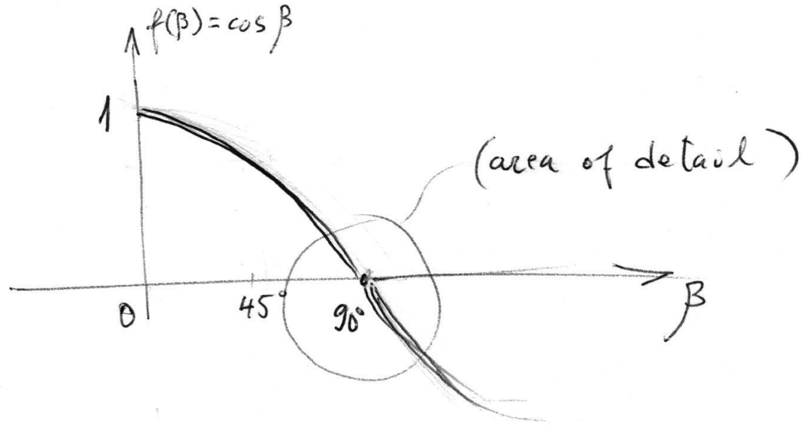
At $\beta = 90^\circ$

$$\cos 90^\circ = 0$$

and $$\sec 90^\circ = \frac{1}{\cos 90^\circ} = \frac{1}{0} \Rightarrow$$ $\sec 90^\circ$ does not exist

However, if β approaches $90°$, (from the left),

Then $\cos\beta$ approaches zero, assuming positive values, before reaching zero.



So, in this case (i.e.: as β approaches $90°$ (from the left),

$\sec\beta$ "explodes" to $+\infty$ ...

(see the graph of $\sec\beta$)

... because

The reciprocal of

a positive number infinitely small,

would be a positive number

infinitely big.

As a last remark,

notice that sec β never reaches the interval $(-1, 1)$

(i.e.: the interval of real numbers between (-1) and (1)).

One explanation for this, could be given by

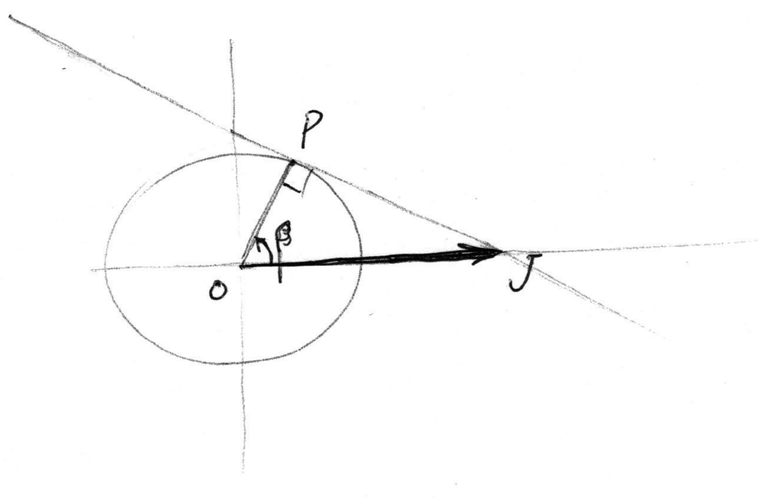
the fact that: from the way it was constructed,

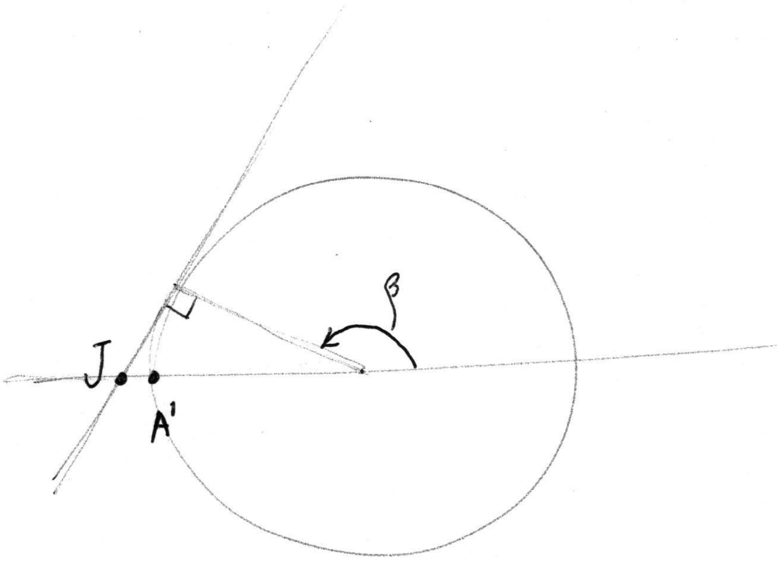
it is impossible for the vector $\vec{OJ}$,

to assume a length less than 1,

regardless of the location of point P

on its journey through the circle.



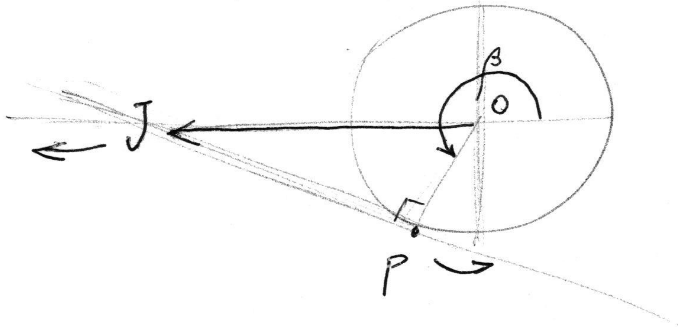


For instance, as β gets very near 180°, points J and A' get very close to each other, until β reaches 180°, at the precise moment when J and A' become coincident.

So, at $\beta = 180^\circ$

$$|\overrightarrow{OJ}| = 1$$

and $\overrightarrow{OJ}$ points to the left.



As β continues to rotate through the third quadrant, the length of the vector $\overline{OJ}$ continues to increase,

although the vector itself continues representing numbers which, in turn, become more and more negative,

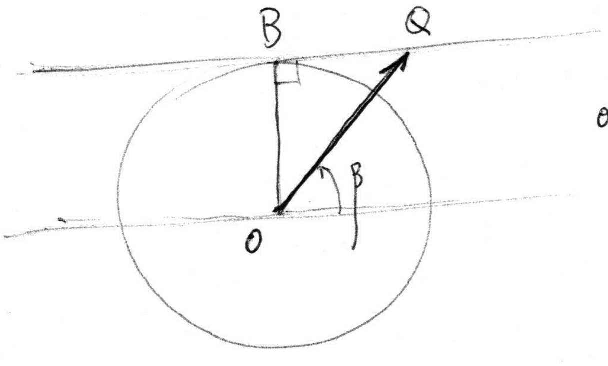
as β rotates counterclockwise through the third quadrant.

Well ... It seems like it's a good time to "move on" to other subjects ...

And, before we do so, let us say a few words about the cosecant.

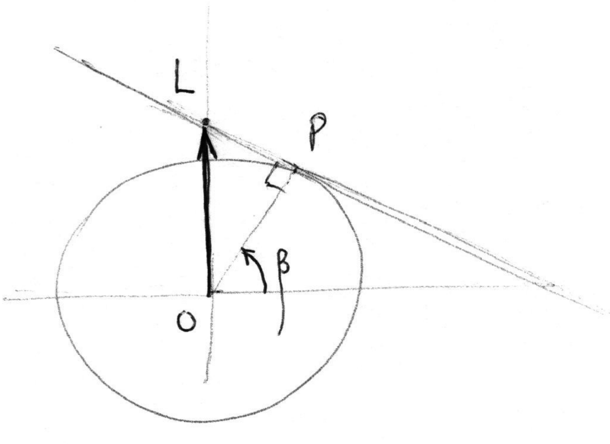
—//—

The cosecant function can be represented by any one of these two pictures below:



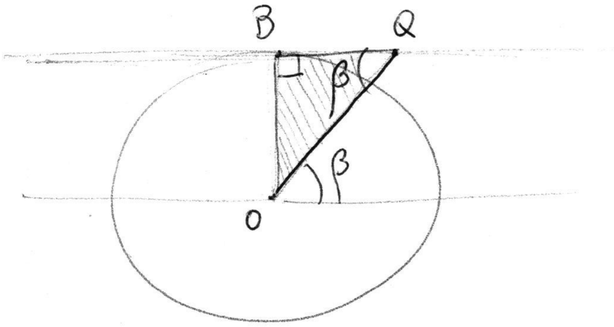
$\overrightarrow{OQ}$

or



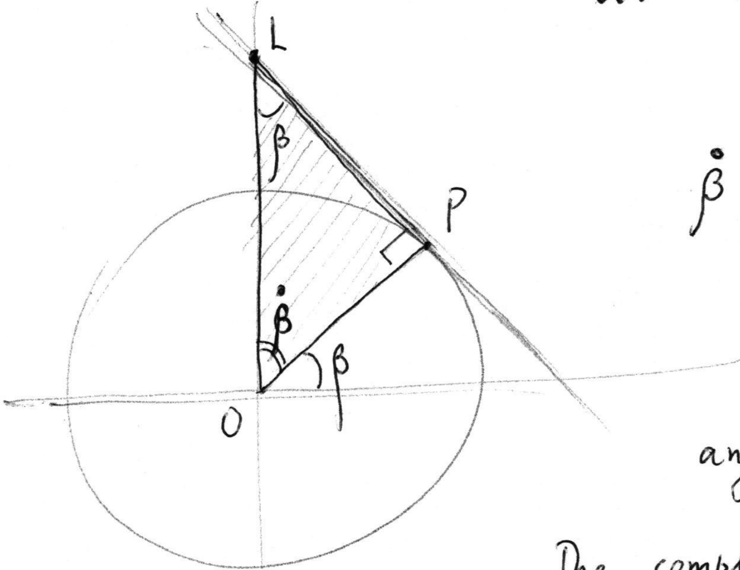
$\overrightarrow{OL}$

(Yes, this is the same Q we used in the study of the cotangent) because, in the first case ...



$$\csc\beta = \frac{1}{\sin\beta} = \frac{1}{\frac{opp}{hip}} = \frac{hip}{opp} = \frac{\overline{OQ}}{\overline{OB}} = \frac{\overline{OQ}}{radius} = \overline{OQ}$$

... and, in the second case,



$\dot{\beta}$ (β dot) is the $\boxed{\text{complement of } \beta}$

so...

angle $O\hat{L}P$ would be the complement of the complement of β

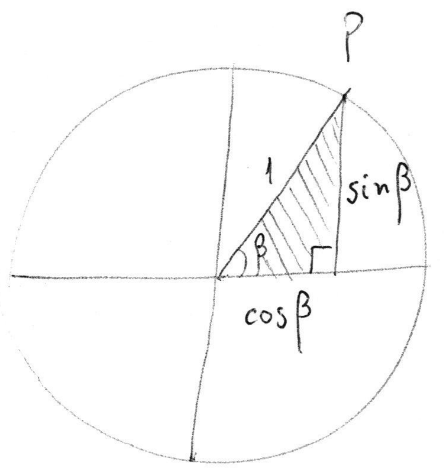
(That is: angle $O\hat{L}P = \beta$)

Therefore,

$$\csc\beta = \frac{1}{\sin\beta} = \frac{1}{\frac{opp}{hip}} = \frac{hip}{opp} = \frac{\overline{OL}}{\overline{OP}} = \frac{\overline{OL}}{radius} = \overline{OL}$$

Now, that we are familiar with the representatives of each trigonometric function, we will be able to deduce some important formulas, just by looking at certain pictures.

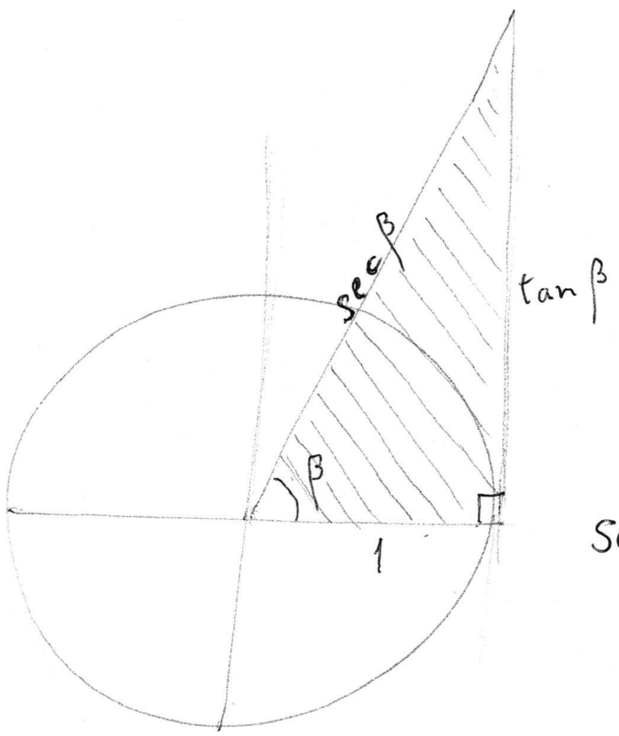
Some important formulas



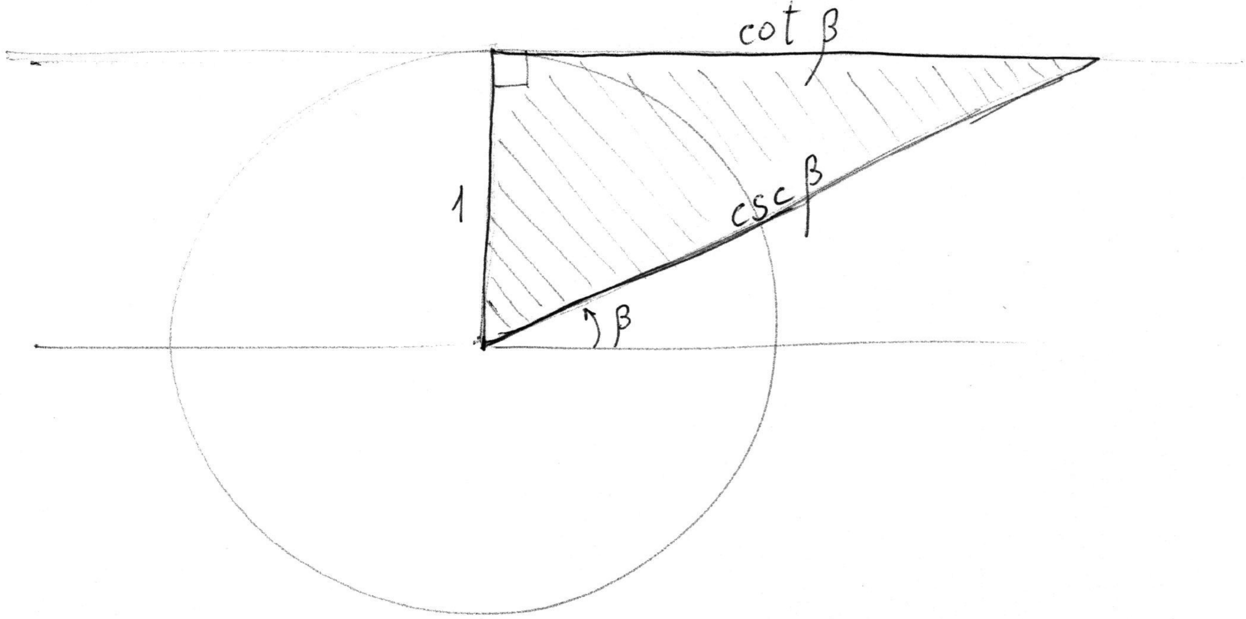
from Pythagoras:

$$1^2 = (\sin\beta)^2 + (\cos\beta)^2$$

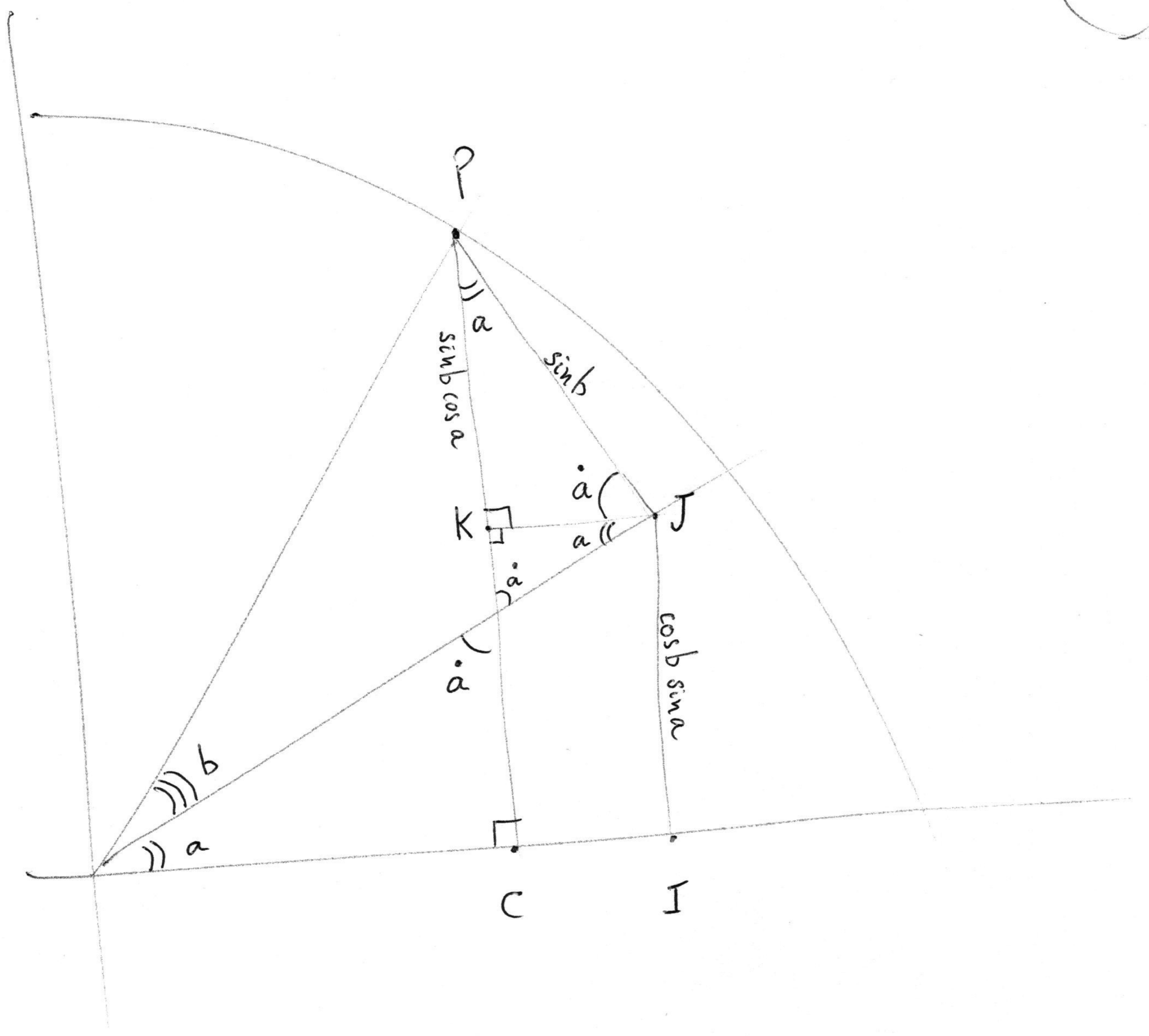
$$1 = \sin^2\beta + \cos^2\beta$$



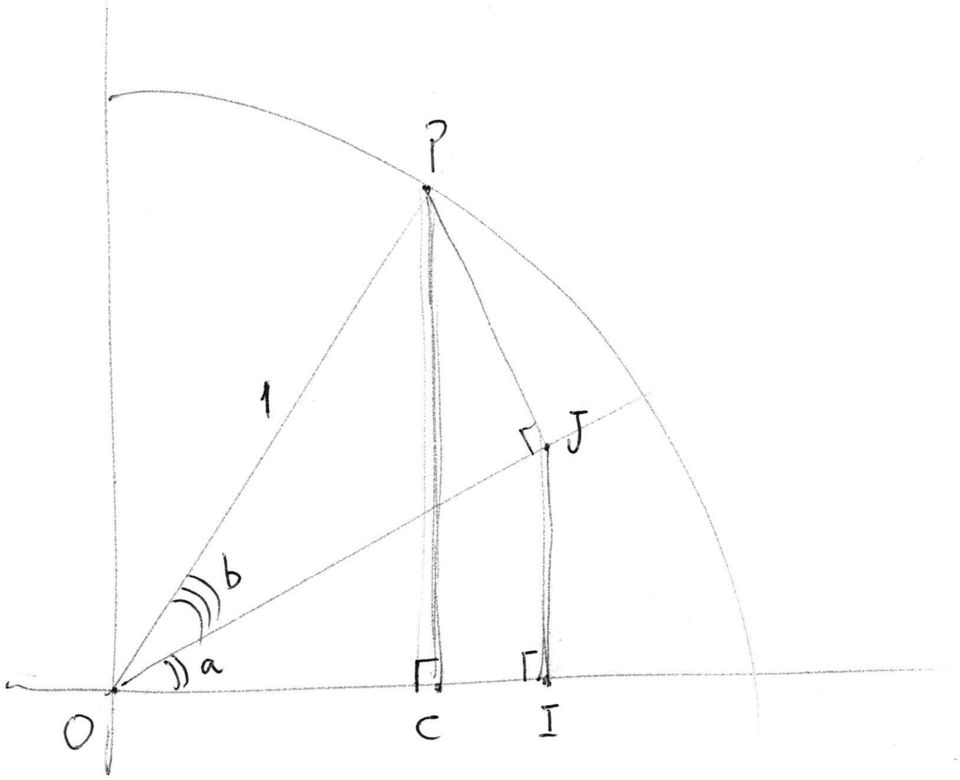
$$\sec^2\beta = 1 + \tan^2\beta$$



$$\csc^2\beta = 1 + \cot^2\beta$$



P
a
sin b cos a
sin b
K
J
a
a
a
a
cos b sin a
b
a
C
I



Our goal, now, would be

to deduce a formula for the

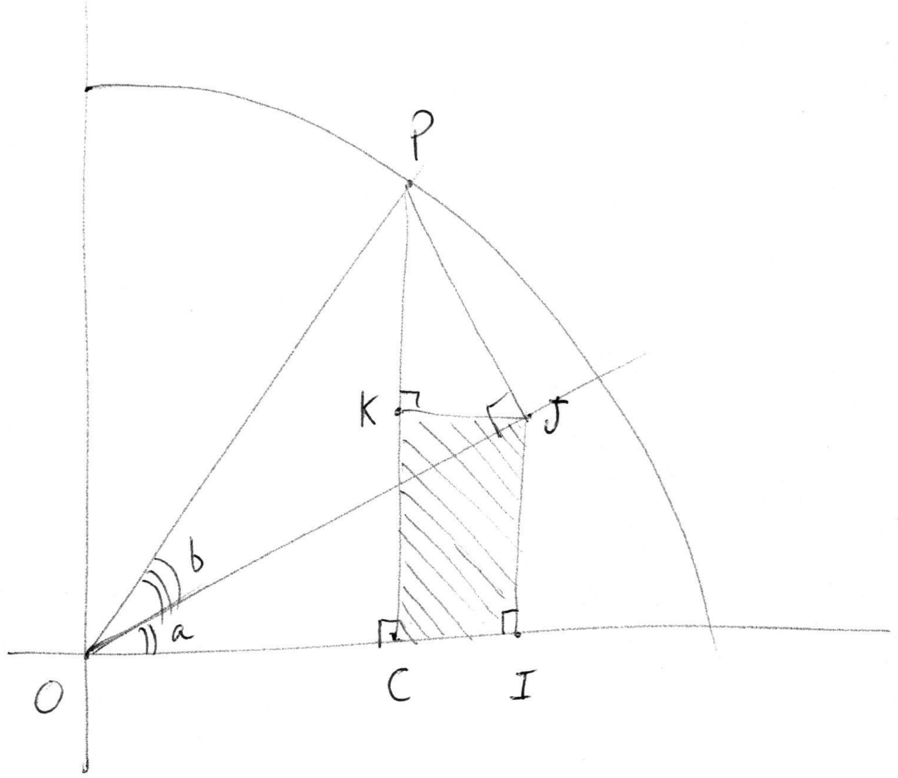
sine of a SUM of TWO angles:

(angle "a", and

angle "b")

$\sin(a+b) = ?$

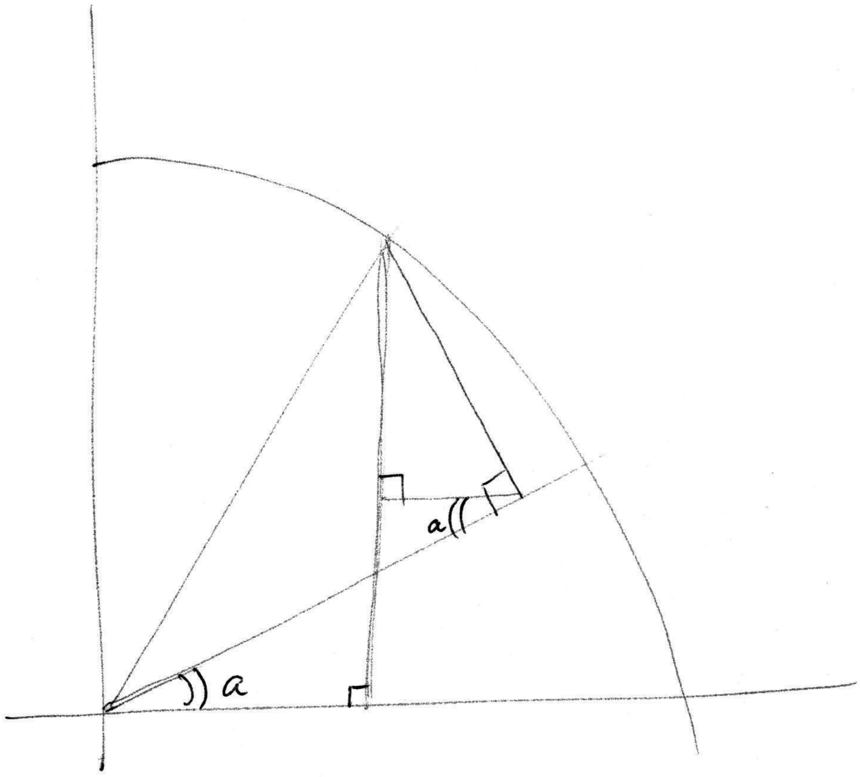
$\sin(a+b) = \overline{CP}$ (from triangle OCP)



$$\overline{CP} = \overline{CK} + \overline{KP}$$

$$\overline{CP} = \overline{IJ} + \overline{KP}$$

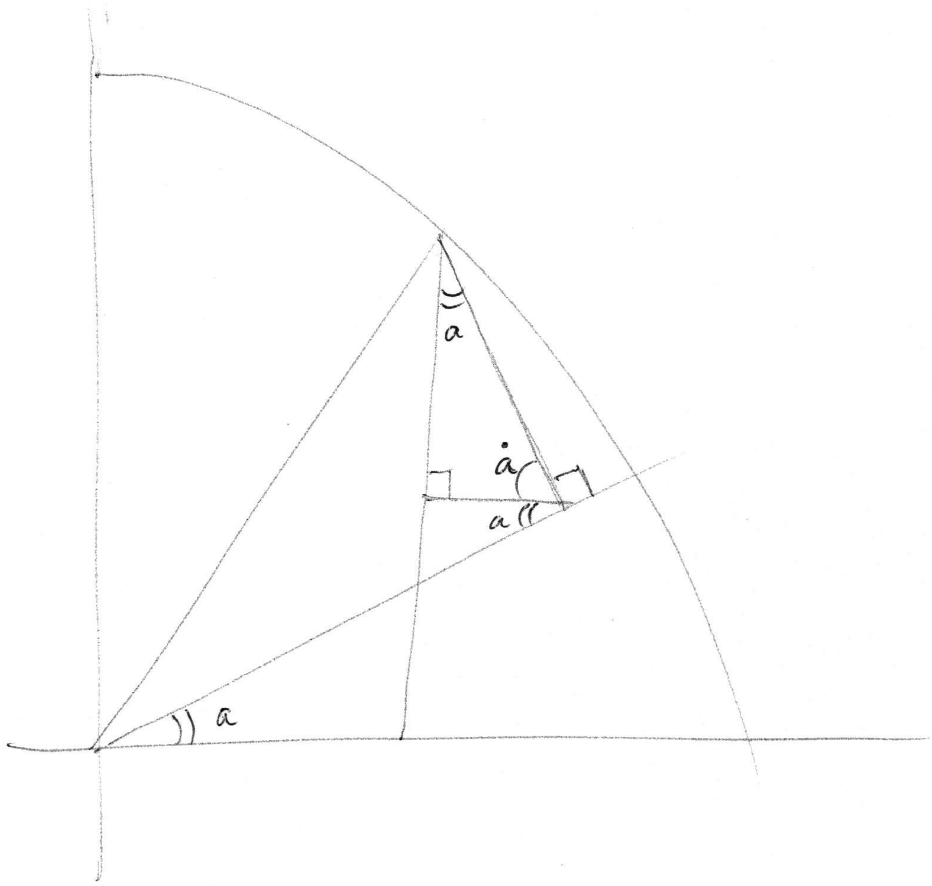
$$\sin(a+b) = \overline{IJ} + \overline{KP}$$

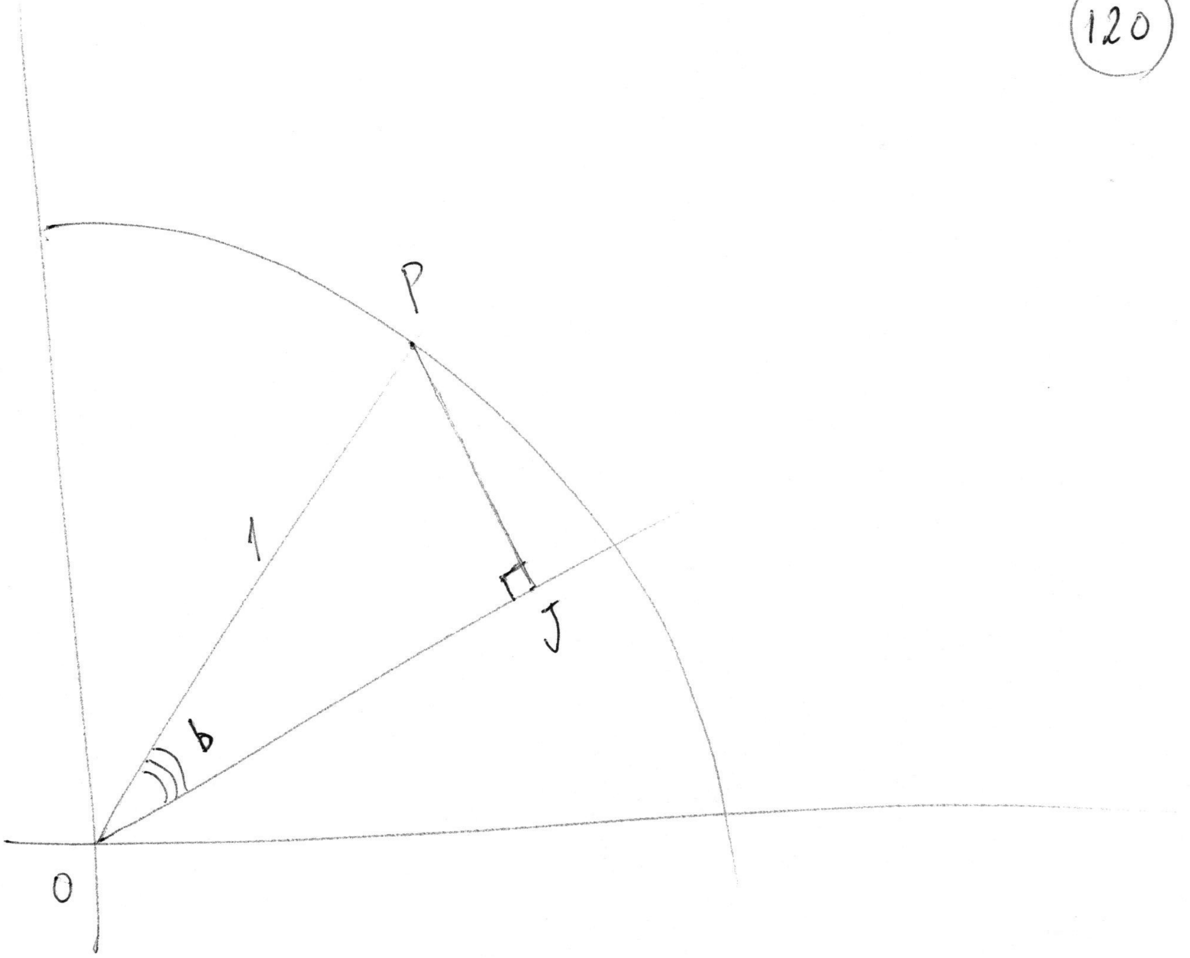


$a = a$ (alternate interior angles)

if $\dot{a}$ (a dot) is the complement of a,

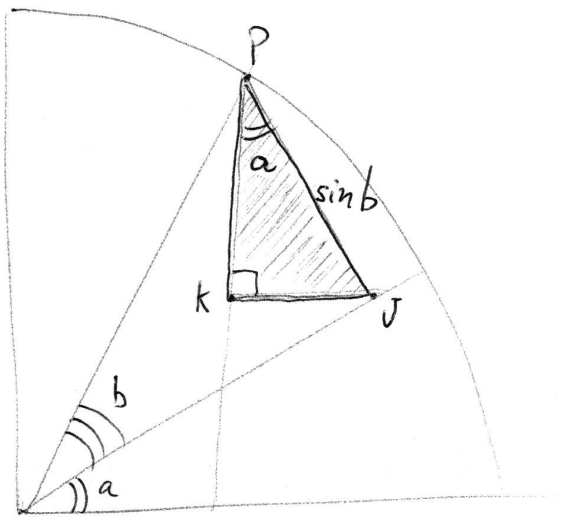
Then

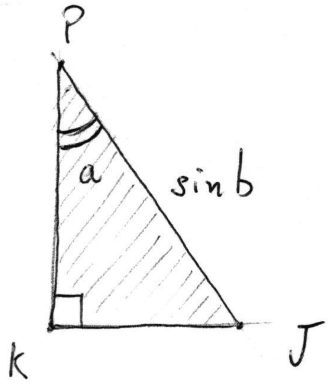




$$\sin b = \frac{opp}{hip} = \frac{\overline{JP}}{1}$$

So...
$$\overline{JP} = \sin b$$

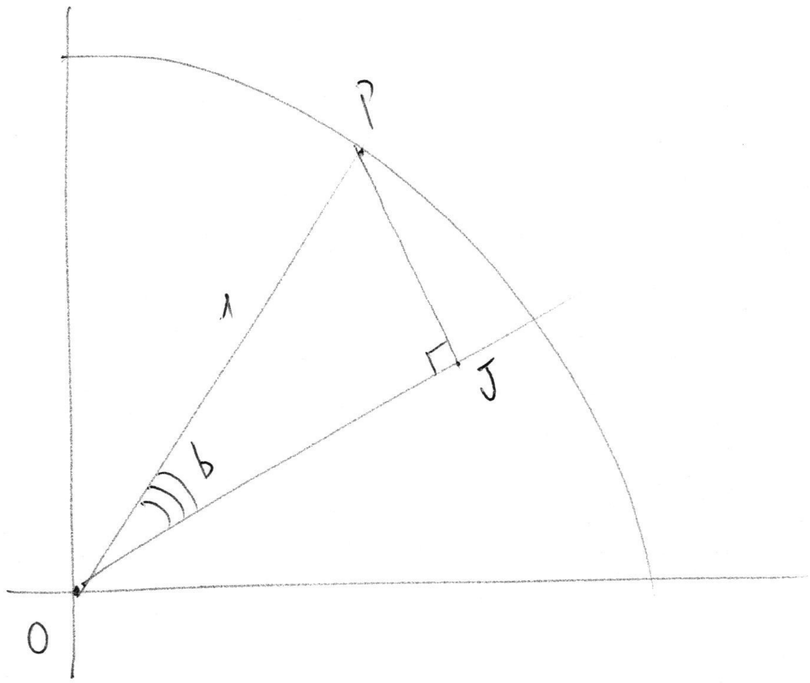




$$\cos a = \frac{\text{adj}}{\text{hip}} = \frac{\overline{KP}}{\sin b}$$

So ...

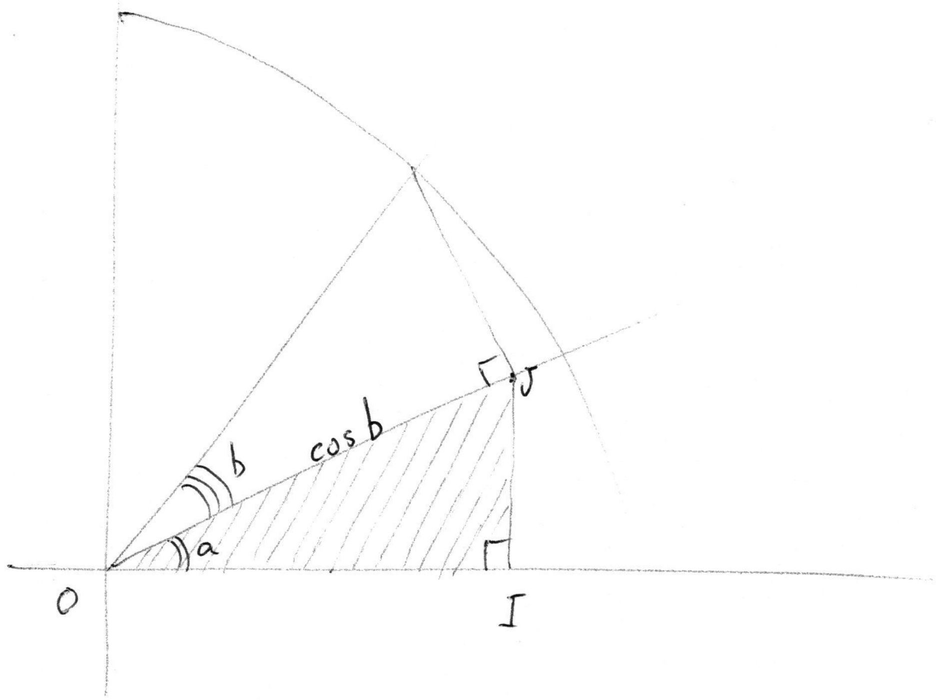
$$\overline{KP} = \sin b \cdot \cos a$$



$$\cos b = \frac{\text{adj.}}{\text{hip}} = \frac{\overline{OJ}}{1}$$

So ...

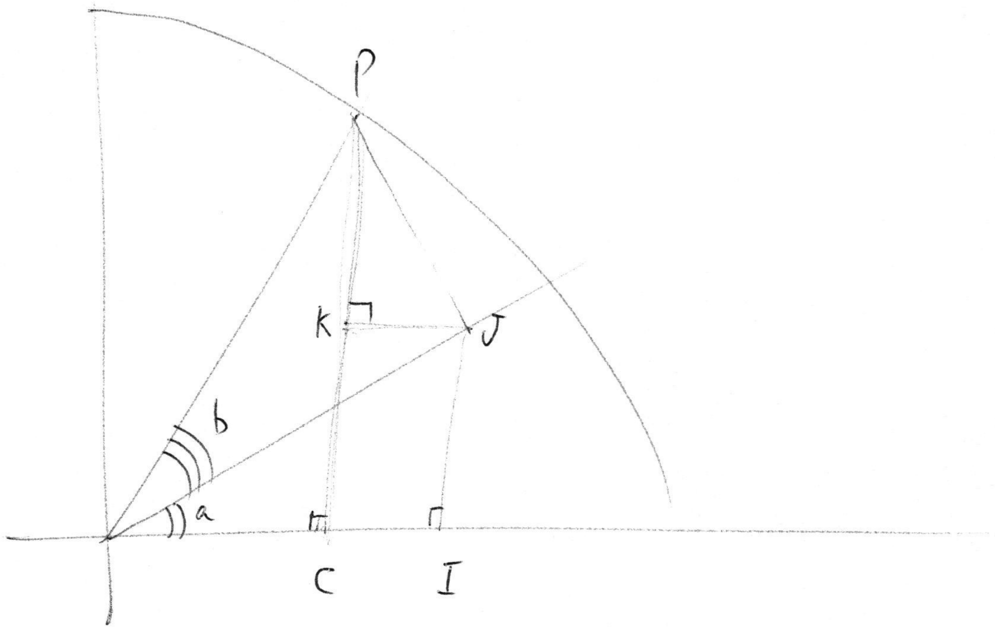
$$\overline{OJ} = \cos b$$



$$\sin a = \frac{\text{opp}}{\text{hip}} = \frac{\overline{IJ}}{\cos b}$$

So ...

$$\overline{IJ} = \sin a \cdot \cos b$$



from page 117, we know that :

$$\sin(a+b) = \overline{IJ} + \overline{KP}$$

$$\begin{cases} \overline{IJ} = \sin a \cos b \\ \overline{KP} = \sin b \cos a \end{cases}$$ (see previous pages)

Conclusion :

$$\boxed{\sin(a+b) = \sin a \cos b + \sin b \cos a}$$

This formula (with its proof) is very important, and I really encourage you to study it carefully.

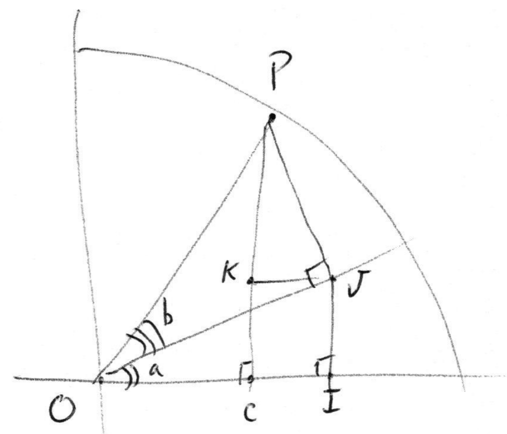
Moreover, the process of mastering its proof would be an excellent exercise, since it gives us a good idea of how the sequence of arguments takes place in such context.

For the $\cos(a+b)$

The proof is very similar,

and we can use the same picture

we did in the case of the sine.

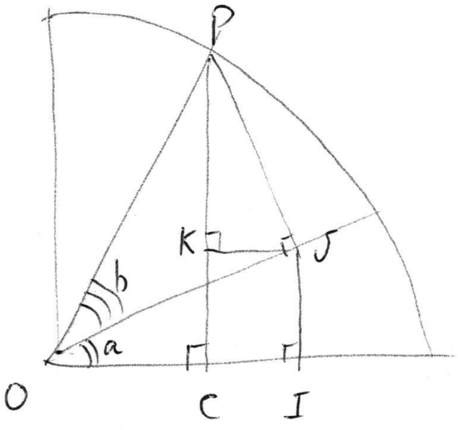


The difference is that now, (in the cosine case),

all we need to observe is that $\overline{OC}$ would be

the representative of $\cos(a+b)$.

So, the idea is to do a similar proof, but, now, (in this case), we would subtract the segments $\overline{OI} - \overline{CI}$ in order to get $\overline{OC}$:



$$\cos(a+b) = \overline{OC}$$

$$\cos(a+b) = \overline{OI} - \overline{CI}$$

$$\cos(a+b) = \overline{OI} - \overline{KJ}$$

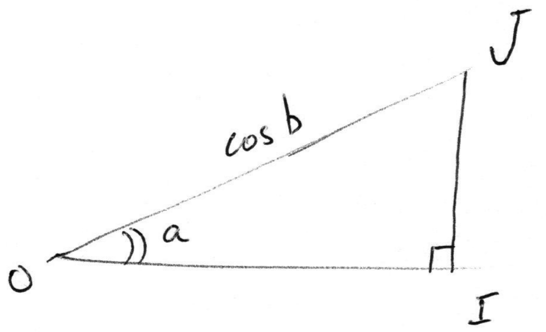
From our previous analysis

we already know that $\overline{OJ} = \cos b$

so, $\overline{OI}$ would be equal to $\cos b \cos a$

$$\left(\overline{OI} = \cos b \cos a \right)$$

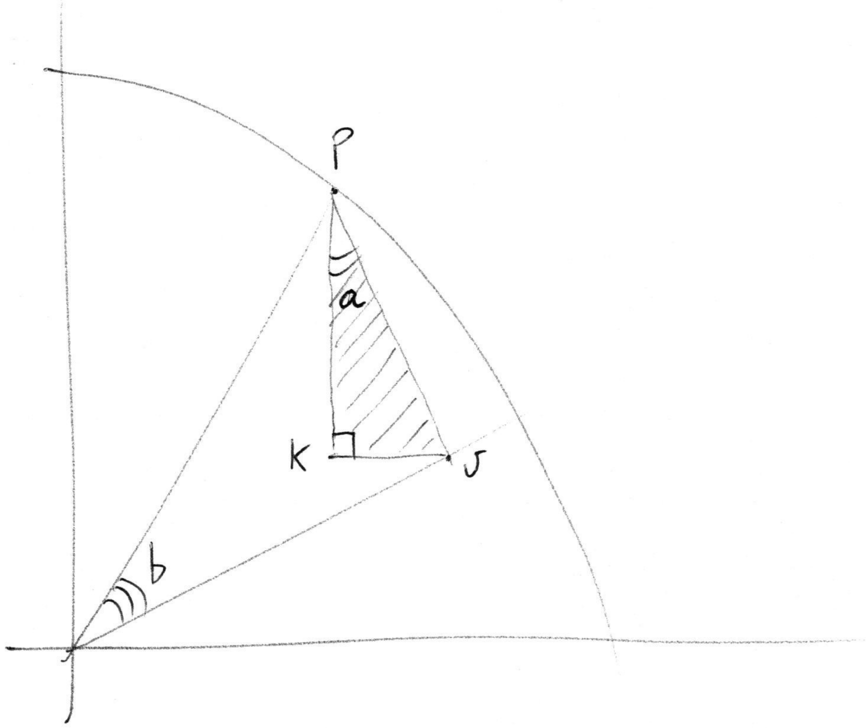
because



$$\cos a = \frac{adj}{hip} = \frac{\overline{OI}}{\cos b}$$

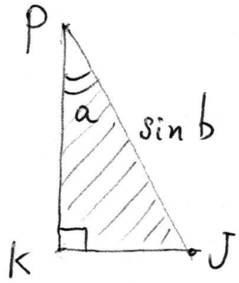
So ...

$$\overline{OI} = \cos a \cos b$$



Again ...

From our previous analysis we already knew that $\overline{JP} = \sin b$



So...

$$\sin a = \frac{\text{opp}}{\text{hip}} = \frac{\overline{KJ}}{\sin b}$$

and so...

$$\overline{KJ} = \sin a \sin b$$

Recalling that our goal is to arrive at

$$\cos(a+b) = \overline{OI} - \overline{KJ}$$, (see page 127),

we have:

$$\cos(a+b) = \overline{OI} - \overline{KJ}$$

$$\begin{cases} \overline{OI} = \cos a \cos b \\ \overline{KJ} = \sin a \sin b \end{cases}$$

(see previous pages)

So ...

$$\boxed{\cos(a+b) = \cos a \cos b - \sin a \sin b}$$

Besides being very useful,
These last Two formulas

$$\boxed{\begin{aligned} \sin(a+b) &= \sin a \cos b + \sin b \cos a \\ \cos(a+b) &= \cos a \cos b - \sin a \sin b \end{aligned}}$$

can be used as a starting point for the development of other formulas, such as, for instance;

$$\sin(2a) = \sin(a+a) = \sin a \cos a + \sin a \cos a$$

$$= 2 \sin a \cos a$$

In other words: $\sin(2a) = 2 \sin a \cos a$

—— // ——

(Another example):

$$\cos(2a) = \cos a \cos a - \sin a \sin a$$

$$= \cos^2 a - \sin^2 a$$

(That is:) $\cos(2a) = \cos^2 a - \sin^2 a$

—— // ——

— // —

Now, we can go beyond the context of the above formulas (for the sum of two angles), and take a look at another interesting "phenomenon" known as the "law of sines".

I decided to end these notes, contemplating it with you, because:

first: its proof is not difficult, at all.

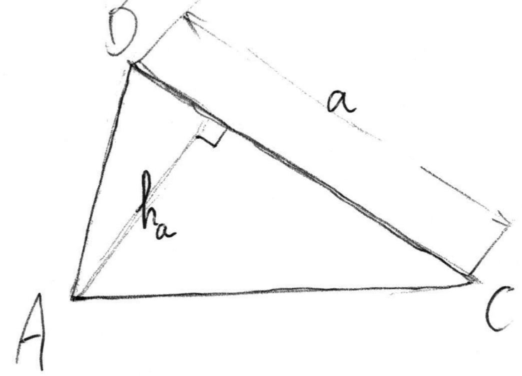
second: The theorem presents itself with a certain touch of "beauty" in the sense that it says:

Given a triangle ABC with opposite sides a, b, c, then

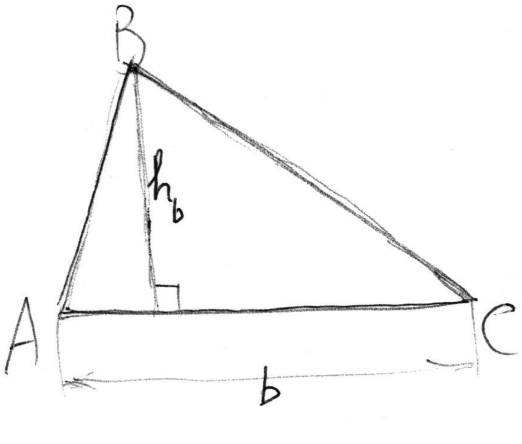
$$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$$

The proof is based on the idea of expressing the area of Triangle ABC, $\left(\frac{\text{base} \times \text{height}}{2}\right)$

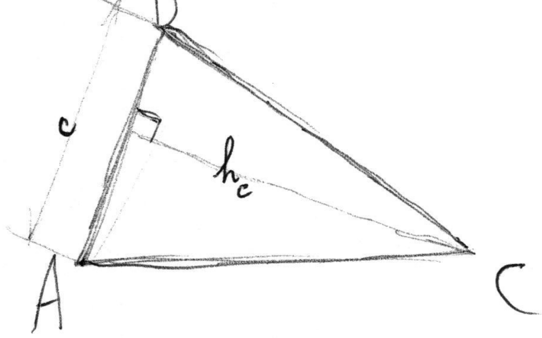
Through three different viewpoints:



$$\text{Area} = \frac{a \cdot h_a}{2}$$



$$\text{Area} = \frac{b \cdot h_b}{2}$$



$$\text{Area} = \frac{c \cdot h_c}{2}$$

Since all three "areas" are (of course) the same, then:

$$\frac{ah_a}{2} = \frac{bh_b}{2} = \frac{ch_c}{2}$$

(That is:) $$ah_a = bh_b = ch_c$$

—//—

Well ... since the heights are not present in our final formula $\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$

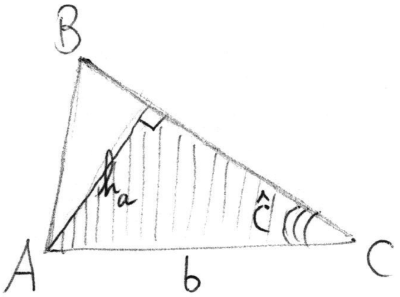
we should now, try to make an effort to "eliminate" h_a, h_b and h_c from the expression $ah_a = bh_b = ch_c$.

In order to do that,

let's use (once more),

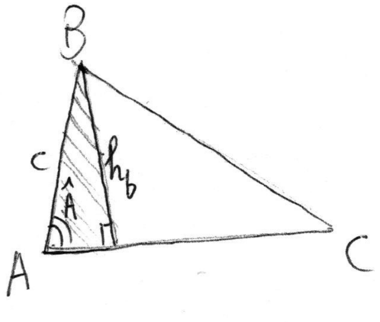
the basic definitions of Trigonometry,

in the following pictures:



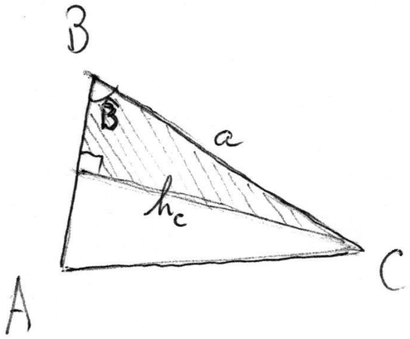
$$\sin \hat{C} = \frac{\text{opp}}{\text{hip}} = \frac{h_a}{b}$$

so... $\boxed{h_a = b \sin \hat{C}}$



$$\sin \hat{A} = \frac{\text{opp}}{\text{hip}} = \frac{h_b}{c}$$

so... $\boxed{h_b = c \sin \hat{A}}$



$$\sin \hat{B} = \frac{\text{opp}}{\text{hip}} = \frac{h_c}{a}$$

so... $\boxed{h_c = a \sin \hat{B}}$

Now, we can write down our (already familiar) expression:

$$a\,h_a = b\,h_b = c\,h_c$$

and replace h_a, h_b and h_c by their respective "mini-formulas" just developed in the previous page.

$$ab \sin \hat{C} = bc \sin \hat{A} = ca \sin \hat{B}$$

Dividing everybody by abc, we get:

$$\frac{\cancel{a}\cancel{b} \sin \hat{C}}{\cancel{a}\cancel{b}\, c} = \frac{\cancel{b}\cancel{c} \sin \hat{A}}{a\,\cancel{b}\cancel{c}} = \frac{\cancel{c}\cancel{a} \sin \hat{B}}{\cancel{a}\, b\, \cancel{c}}$$

(That is:) $$\frac{\sin \hat{C}}{c} = \frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b}$$

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