

The Ontological Import of Adding Proper Classes

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Abstract

In this article, we analyse the ontological import of adding classes to set theories. We assume that this increment is well represented by going from ZF system to NBG. We thus consider the standard techniques of reducing one system to the other. Novak proved that from a model of ZF we can build a model of NBG (and vice versa), while Shoenfield have shown that from a proof in NBG of a set-sentence we can generate a proof in ZF of the same formula. We argue that the first makes use of a too strong metatheory. Although meaningful, this symmetrical reduction does not equate the ontological content of the theories. The strong metatheory levels the two theories. Moreover, we will modernize Shoenfields proof, emphasizing its relation to Herbrands theorem and that it can only be seen as a partial type of reduction. In contrast with symmetrical reductions, we believe that asymmetrical relations are powerful tools for comparing ontological content. In virtue of this, we prove that there is no interpretation of NBG in ZF, while NBG trivially interprets ZF. This challenges the standard view that the two systems have the same ontological conten

1 How can we compare the ontological content of different theories?

Within the context of a formal theory, an assertion is ontologically committing if it expresses a closure property of the intended models. For example, the power set and the union axioms express closure properties in set theories.

If X and Y are elements in our set theory model, then the axioms guarantee that their union and their power sets are also in the model. This property represents how an axiom generates ontological import. A detailed analysis of this conception leads to an understanding of existential aspects in formal theories (see [4]). But it is of no help if we want to compare different theories regarding ontology. We propose a different approach for this kind of comparison: If T_1 and T_2 are formal theories and T_2 can reduce T_1 but T_1 cannot reduce T_2 using a natural method of ontological reduction given in a metatheory, then the ontological content of T_2 is greater than that of T_1 .

Now, what is a natural method of ontological reduction? We expect that if T_1 reduces T_2 , then the consistency of T_1 can be proved from the consistency of T_2 in the corresponding metatheory. This and other related questions are dealt with in [1], which is the source of the main results in this paper. Nevertheless, we will not be concerned with this question in its full generality here, as we will concentrate on methods of interpretations between first-order theories in a finitary metatheory. More precisely, using the method of interpretations and the above conception of ontological comparison, we prove that the ontological content of NBG is greater than the ontological content of ZF. This result goes against the received view according to which those theories are equivalent.

We will analyze the received view and point out its insufficiencies. It is based on the folklore conservativity result: NBG is conservative with respect to ZF. A detailed finitary proof of this result will be provided, as there is basically no modern proof for this in the literature. We will argue that the ontological reduction operating here is not conclusive, for the corresponding reduction method is partial. Hence, it is not clear if it gives a right transposition of the ontology. Also, it is always possible to strengthen the metatheory and weaken the reduction method in order to trivialize the comparison.

Therefore, for the equivalence claim, it is not enough to have one mutual comparison according to some reduction method in some metatheory. In the opposite direction, it is very significant to have an asymmetric comparison by a standard method of reduction, and this result will be proved subsequently, together with other related results. The absence of interpretation from NBG to ZF thus understood is a strong evidence for the thesis that the ontological content of the former surpasses that of the latter.

2 Novak's model-theoretic reduction of NBG to ZF

From every model of ZF, a model of NBG can be obtained by the addition of definable proper classes. This construction has the important feature that the resulting model has the same *sets* as the original model. Therefore, the following reduction of NBG to ZF is obtained: Assume that α is a ZF-sentence and that $NBG \vdash \alpha$. Let \mathcal{M} be a model of ZF, and let \mathcal{N} be the model of NBG obtained from \mathcal{M} by the addition of the definable proper classes. Since $NBG \vdash \alpha$, it follows that $\mathcal{N} \models \alpha$. However, if $\mathcal{N} \models \alpha$, then $\mathcal{M} \models \alpha$, for α is a ZF-sentence – it is about sets only – and \mathcal{M} and \mathcal{N} have the same sets. Now, the completeness theorem gives that $ZF \vdash \alpha$, for \mathcal{M} is an arbitrary model of ZF.

The above reduction gives, in particular, a model-theoretic proof that if ZF is consistent, so is NBG. However, a bolder philosophical conclusion from this is that the ontological content of NBG is already present in ZF, as it can be easily fulfilled by the definable classes lurking in models of ZF. We claim that the bolder conclusion is unwarranted. The problem is that the metatheory in which the reduction takes place is too strong and the difference may be obliterated by its excessive strength. To make the point clear, assume that we were interested in comparing theories T_1 and T_2 in a metatheory which happens to be strong enough to prove the consistency of both theories. The equivalence between the consistency of T_1 and that of T_2 is, therefore, valid in such a metatheory, but no ontological comparison can be drawn.

From a proof of equiconsistency in some metatheory, one cannot conclude ontological equivalence. However, there is another reduction of NBG to ZF providing a finitary equiconsistency result due to Shoenfield. Although the above argument does not apply to Shoenfield's reduction, which takes place in a finitary metatheory, we still claim that the bolder conclusion that the ontological content of NBG is already present in ZF is unwarranted. The problem is within the reduction method itself: It does not provide a reduction of the quantification over class variables, which gives NBG its extra ontological content, but rather just shows that the quantificational reasoning with class variables in NBG is dispensable and can be avoided in proofs of ZF-sentences. Therefore, there is no real reduction taking place here, there is no transposition of ontology, and the ontological equivalence does not follow.

Shoenfield's proof will be given in the next section and meticulously analyzed to support our claim.

3 Shoefield's finitary reduction

The finitary proof of equiconsistency between NBG and ZF was provided by Shoefield in the article "A relative consistency proof" ([9]). This article is written in the language of *Principia Mathematica* by Whitehead and Russell, and makes extensive use of the techniques developed in *Grundlagen der Mathematik* by Hilbert and Bernays. For this reason, we have developed this section by an excavation, a reverse engineering, in which the tools used were unraveled by the clues left in the article. In addition, changing the axiomatic system (see [7]) poses several additional difficulties and, in many instances, the proof changes significantly.

We will expose here the technique used for the equiconsistency proof. Before doing that, however, we need to remember the finitary proof technique of Herbrand's theorem. For this, we will go through the necessary definitions, then we will enunciate Hilbert-Ackermann's theorem and, finally, the necessary part of the strategy for establishing Herbrand's theorem.

Definition 1 *A formula α is open if all variables occurring in the formula are free.*

Definition 2 *A theory T is open if all its axioms are open formulas.*

Definition 3 *α is a quasi-tautology if, and only if, α is tautological consequence of instances of identity and equality axioms.*

Theorem 4 (Hilbert-Ackermann) *A open theory T is inconsistent if, and only if, there is a quasi-tautology α of the form $\neg\beta_1 \vee \neg\beta_2 \vee \dots \vee \neg\beta_k$, such that β_i is an instance of some axiom in T for each $i \leq k$.*

The finitary proof of this theorem can be found in [5, p. 48 - 52]. It is important to bear in mind this theorem since it is equivalent to the existential case of Herbrand's theorem.

Definition 5 *Let Q be a quantifier \forall or \exists , then a prenex formula is of the form:*

$$Qx_1Qx_2\dots Qx_n\theta,$$

being θ a open formula.

We call θ the matrix of $Qx_1Qx_2\dots Qx_n\theta$.

We can write the prenex form, without loss in generality, with explicit quantifiers \forall e \exists , instead of using the Q . The general formula is of the form:

$$\exists \bar{z}_1 \forall \bar{y}_1 \dots \exists \bar{z}_k \forall \bar{y}_k \theta[\bar{x}, \bar{z}_1, \bar{y}_1, \dots, \bar{z}_k, \bar{y}_k],$$

being \bar{x} the sequence of free variables in the matrix θ . Using this notation will simplify the proof, since each quantifier \forall and \exists is treated differently.

Definition 6 A formula α is existential when α is a prenex formula of the form $\exists x_1 \exists x_2 \dots \exists x_n \theta$, being θ an open formula.

From prenex formulas, we have the following theorem:

Theorem 7 For any formula α , there is a α' , such that:

1. α' is a prenex formula;
2. $\vdash \alpha \leftrightarrow \alpha'$;
3. α' is obtained by a primitive recursive procedure;

We call α' the prenex form of α .

Next we will expose Herbrand's normal form. It can be understood as the representation of any formula by an existential formula through a procedure of elimination of universal quantifiers. Such elimination is due to the introduction of function symbols in language.

Definition 8 (Herbrand's normal form) Let α be any formula, we build α_H through the following procedure:

1. α_0 is the prenex form of α ;
2. If α_i is an existential formula, then α_H is α_i ;

3. If α_i is of the form $\exists x_1 \exists x_2 \dots \exists x_n \forall y \gamma$, we introduce a function symbol f , such that:

$$\alpha_{i+1} \quad \exists x_1 \exists x_2 \dots \exists x_n \gamma_y(f(x_1, x_2, \dots, x_n)).$$

If α_i is of the form $\forall y \gamma$, we introduce a constant symbol c , such that:

$$\alpha_{i+1} \quad \gamma_y(c);$$

We can represent Herbrand's normal form of a prenex formula

$$\exists \bar{z}_1 \forall \bar{y}_1 \dots \exists \bar{z}_k \forall \bar{y}_k \theta[\bar{x}, \bar{z}_1, \bar{y}_1, \dots, \bar{z}_k, \bar{y}_k],$$

for

$$\exists \bar{z}_1 \dots \exists \bar{z}_k \theta[\bar{x}, \bar{z}_1, \overline{f_1(\bar{z}_1)}, \dots, \bar{z}_k, \overline{f_k(\bar{z}_1, \dots, \bar{z}_k)}].$$

Theorem 9 (Herbrand) *Let T be a theory without non logical axioms in the language \mathcal{L} . Then, for any prenex formula α in \mathcal{L} , it holds that:*

$T \vdash \alpha$ in the language $\mathcal{L} \iff$ there is a quasi-tautology $\beta_1 \vee \beta_2 \dots \vee \beta_k$, for which, for each i , β_i is an instance of the matrix α_H .

Proof. (Detailed strategy) The procedure for proving Herbrand's theorem follows the steps:

1. If α is an existential formula, the theorem is a corollary of Hilbert-Ackermann's theorem:

We suppose that α is of the form $\exists x_1 \exists x_2 \dots \exists x_n \beta$, with β as an open formula. In this case, $\neg \alpha$ is logically equivalent to $\forall x_1 \forall x_2 \dots \forall x_n \neg \beta$. Thus, $T \vdash \neg \alpha \leftrightarrow \neg \beta$.

Because of that, $T \vdash \alpha \iff$ the theory $\{\neg \beta\}$ is inconsistent. By the Hilbert-Ackermann's theorem, for $\{\neg \beta\}$ is a open theory,

$\{\neg \beta\}$ is inconsistent \iff there is a quasi-tautology $\neg(\neg \beta_1) \vee \neg(\neg \beta_2) \vee \dots \vee \neg(\neg \beta_k)$, with β_i an instance of β for all i .

But this is equivalent to $\beta_1 \vee \beta_2 \vee \dots \vee \beta_k$, finalizing the proof.

2. We take, for the general case, a prenex formula α in the language \mathcal{L}

$$\exists \bar{z}_1 \forall \bar{y}_1 \dots \exists \bar{z}_k \forall \bar{y}_k \theta[\bar{x}, \bar{z}_1, \bar{y}_1, \dots, \bar{z}_k, \bar{y}_k].$$

3. Let $\mathcal{L}_{\mathcal{H}}$ be the language \mathcal{L} extended with functions used to built α_H and T_H the theory without logical axioms in the language \mathcal{L}_H , we should prove that:

$$T \vdash \alpha \iff T_H \vdash \alpha_H.$$

4. Since α_H is an existential formula, we obtain a quasi-tautology for the extended language $\mathcal{L}_{\mathcal{H}}$. Hence,

$$T \vdash \alpha \iff T_H \vdash \alpha_H \iff \text{there is a quasi-tautology with instances of } \alpha_H$$

For this reason, the proof of item 3 finish the proof.

5. To prove item 3, we only show the strategy for the converse implication, that is, $T \vdash \alpha \Leftarrow T' \vdash \alpha_H$, since the direct proof is relatively simple.
6. Let T_c be the Henkin extension of T , defined in [5, p. 46], and T_{c+eq} be the addition of equivalence axioms¹ of Herbrand [5, p. 52] in T_c , we show that:

(a) T_c is a conservative extension of T .

(b) T_{c+eq} is a conservative extension of T_c

These two fact will be important to the following steps.

7. We suppose that there is a quasi-tautology $\beta_1 \vee \beta_2 \dots \vee \beta_q$ in $\mathcal{L}_{\mathcal{H}}$, with β_i being a instance of the matrix α_H . We now do a procedure of replacement of functions introduced for α_H by constants in \mathcal{L}_c . More specifically, if α_H is of the form

$$\theta[\bar{x}, \bar{z}_1, \overline{f_1(\bar{z}_1)}, \dots, \bar{z}_k, \overline{f_k(\bar{z}_1, \dots, \bar{z}_k)}]$$

and β_i if of the form

$$\theta[\bar{t}, \bar{u}_1, \overline{f_1(\bar{u}_1)}, \dots, \bar{u}_k, \overline{f_k(\bar{u}_1, \dots, \bar{u}_k)}],$$

we define the sequence of special constants $\overline{d(\bar{u}_1)}, \overline{d(\bar{u}_1, \bar{u}_2)}, \dots, \overline{d(\bar{u}_1, \dots, \bar{u}_k)}$ for eliminating the functions in the following manner:

¹If c_1 and c_2 are special constant for the formulas $\exists x \alpha_1(x)$ and $\exists x \alpha_2(x)$, then the equivalence axiom for these constants is $\forall x (\alpha_1(x) \leftrightarrow \alpha_2(x)) \rightarrow c_1 = c_2$.

Notation 10 .

- (a) $\bar{z}^{i \rightarrow}$ is a sequence of terms of the sequence \bar{z} from the index i onward;
- (b) $\bar{z}^{i \rightarrow j}$ is a sequence of terms of \bar{z} from the index i up to the index j ;
- (c) $(\bar{z})_i$ is the i 'th term in the sequence \bar{z} ;

The constant $(\overline{d(\bar{u}_1)})_i$ is a special constant for the formula

$$\exists(\bar{y}_1)_i(\neg \forall \bar{y}_1^{(i+1) \rightarrow} \exists \bar{z}_2 \forall \bar{y}_2 \dots \exists \bar{z}_k \forall \bar{y}_k \theta[\bar{t}, \bar{u}_1, \overline{d(\bar{u}_1)}^{1 \rightarrow (i-1)}, \overline{f_1(z_1)}^{i \rightarrow}, *]),$$

begin $*$ a abbreviation that indicates the remaining sequence $\bar{u}_2, f_2(\bar{u}_1), \dots, \bar{u}_k, f_k(\bar{u}_1, \dots, \bar{u}_k)$.

And, generally, $(\overline{d(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_i)})_j$ is a special constant for the formula

$$\exists(\bar{y}_i)_j(\neg \forall \bar{y}_i^{(j+1) \rightarrow} \exists \bar{z}_{i+1} \forall \bar{y}_{i+1} \dots \exists \bar{z}_k \forall \bar{y}_k \theta[\bar{t}, *, \overline{d(\bar{u}_1, \dots, \bar{u}_i)}^{1 \rightarrow (j-1)}, \overline{f_1(z_i)}^{j \rightarrow}, *]).$$

By successively applying the substitution axiom and *modus ponens*, we have that, se β'_i is the formula

$$\theta[\bar{t}, \bar{u}_1, \overline{d_1(\bar{u}_1)}, \dots, \bar{u}_k, \overline{d_k(\bar{u}_1, \dots, \bar{u}_k)}],$$

then $\vdash_{T_{c+eq}} \beta'_i \rightarrow \alpha$.

8. Note that variables can occur in \bar{u} . However, to ensure the use of the equivalence axiom properties in T_{c+eq} , we need all variables to be replaced by special constants. This will be necessary for the completion of the proof.

So we make a second transformation in the quasi-tautology. Here equivalence axioms play an important role: they ensure that any two equivalent formulas refer to a single constant ². With the addition of only

²Realizing this characteristic was instrumental in establishing the relationship between the techniques used in [9] and the techniques presented in the book *Grundlagen der Mathematik* [10]. In this paper, prior to his textbook of mathematical logic [5], Shoenfield makes use of epsilon Hilbert's calculus as a way of guaranteeing the uniqueness of the special constants. Modern techniques make the conservative introduction of equivalence axioms.

the special axioms, we do not have this guarantee, since we could have two distinct elements satisfying the same formula of the form $\exists x\alpha$.

We introduce distinct special constants for each variable in \bar{u} , obtaining \bar{u}' . Subsequently, we prove, using the special equivalence axioms, that

$$\begin{aligned} \vdash_{T_{c+eq}} \bar{u}_1 = \bar{u}_1' &\rightarrow \overline{d(\bar{u}_1)} = \overline{d(\bar{u}_1')} \\ \vdash_{T_{c+eq}} \bar{u}_1 = \bar{u}_1' \wedge \bar{u}_2 = \bar{u}_2' &\rightarrow \overline{d(\bar{u}_1, \bar{u}_2)} = \overline{d(\bar{u}_1', \bar{u}_2')} \\ &\vdots \\ \vdash_{T_{c+eq}} \bar{u}_1 = \bar{u}_1' \wedge \bar{u}_k = \bar{u}_k' &\rightarrow \overline{d(\bar{u}_1, \dots, \bar{u}_k)} = \overline{d(\bar{u}_1', \dots, \bar{u}_k')} \end{aligned}$$

9. We use a similar procedure as in [5, p. 55] to obtain a formula $\beta_1'' \vee \beta_2'' \dots \vee \beta_q''$, being β_i'' the replacement of the functions and variables introduced by the special constants shown above. Recall that the second transformation ensure us that $\vdash_{T_{c+eq}} \beta_i'' \rightarrow \alpha$ by the same procedure as in item 7.

We suppose C_1, C_2, \dots, C_m to be the proof sequence for the quasi-tautology that uses only identity and equality in T . Thus we prove that the transformation that have led $\beta_1 \vee \beta_2 \dots \vee \beta_k$ to become $\beta_1'' \vee \beta_2'' \dots \vee \beta_k''$ preserves tautologically the sequence. Therefore, $\beta_1'' \vee \beta_2'' \dots \vee \beta_k''$ is tautological consequence of $C_1'', C_2'', \dots, C_m''$.

It remains to prove that each C_i'' is a theorem of T_{c+eq} , when C_i is an axiom of T . This will result in the proof of $\beta_1'' \vee \beta_2'' \dots \vee \beta_k''$ in T_{c+eq} . Each C_i is an axiom of identity, equality, an instance of identity or an instance of equality. We note that, when we get C_i'' , we have transformed C_i into axioms of identity or equality, unless C_i is an instance of identity. This last case, though, is easily proved in T_{c+eq} from the propositions shown in the end of last item.

10. As $T_{c+eq} \vdash \beta_1'' \vee \beta_2'' \dots \vee \beta_k''$ and $T_{c+eq} \vdash \beta_i'' \rightarrow \alpha$ for each i , then $T_{c+eq} \vdash \alpha$.
11. Since T_{c+eq} is a conservative extension of the logic in \mathcal{L} , we lastly obtain that $T \vdash \alpha$.

□

3.1 Finitary proof of equiconsistency

In order to understand the finitary proof, it is important to consider procedures 7 and 8 of the previous section. In them, syntactic transformations are performed to provide the formulas β_i'' , eliminating the functions introduced to obtain the Herbrand's normal formula.

Each introduction of a function to eliminate a universal quantification is performed independently of the other introductions. In this sense, we can restrict the elimination procedure to certain universal quantifications. In fact, we can eliminate one, some or all functions introduced to obtain the Herbrand's normal form.

For the proof of the equiconsistency theorem, we will use the direct part of Herbrand's theorem to obtain the quasi-tautology. Subsequently, we will make the procedure of the converse proof of Herbrand's theorem restricted to the variables limited to sets.

Before performing this procedure, we will eliminate all universal quantifications that are not restricted to sets in the NBG axioms. We will therefore prove that unrestricted universal quantifications are easily eliminated from the axiomatization presented.

Proposition 11 (Extensionality)

$$\vdash (\forall z \in V(z \in x \leftrightarrow z \in y) \rightarrow x = y) \leftrightarrow (\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)).$$

Proof. By substitution, we have

$$\{\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)\} \vdash \forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y.$$

And, by generalization,

$$\{\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y\} \vdash \forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y).$$

We should now prove that

$$\vdash (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y) \leftrightarrow (\forall z \in V(z \in x \leftrightarrow z \in y) \rightarrow x = y),$$

We know that $(\alpha \rightarrow \beta \wedge \neg\alpha \rightarrow \beta) \rightarrow \theta$ is tautologically equivalent to $\beta \rightarrow \theta$. Thus, $\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y$ is tautologically equivalent to $(\forall z \in V(z \in x \leftrightarrow z \in y) \wedge \forall z \notin V(z \in x \leftrightarrow z \in y)) \rightarrow x = y$. Nevertheless, since we know that, by definition, $z \in V \leftrightarrow \exists w(z \in w)$, then $z \notin V$ is equivalent to $\forall w(z \notin w)$. From this, we have that $\forall z \notin V(z \in x \leftrightarrow z \in y)$ is a tautology. We thus obtain:

$$\begin{aligned} & \forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y \iff \\ & (\forall z \in V(z \in x \leftrightarrow z \in y) \wedge \forall z \notin V(z \in x \leftrightarrow z \in y)) \rightarrow x = y \iff \\ & \forall z \in V(z \in x \leftrightarrow z \in y) \rightarrow x = y. \end{aligned}$$

□

Note that we can use a similar procedure to limit other quantifications to sets, whenever we have a universal quantifier $\forall z$ for a formula in which $z \in x$ occurs. Although this technique does not work in all cases, for the axiomatization of NBG used in this article the procedure is effective. We will not expose here the corresponding proof for each axiom, for they are all very similar. In this sense, we have the following theorem:

Theorem 12 *Let α be an axiom of NBG, then there is a formula α' such that α' is the elimination or restriction (to V) of all universal quantifications occurring in α and $\vdash \alpha' \leftrightarrow \alpha$.*

We modify NBG yet one more time. The unrestricted existential quantifiers can be eliminated from instances of the scheme axiom for classes. We add, for each instance $\forall \bar{v} \in V \exists z \forall x (x \in z \leftrightarrow (x \in V \wedge \alpha(x, \bar{v})))$, the constant c_α and replace the axiom for $\forall \bar{v} \in V \forall x \in V (x \in c_\alpha \leftrightarrow (x \in V \wedge \alpha(x, \bar{v})))$. Subsequently, we replace all other axiom of NBG by the version obtained by the successive application of theorem 12. We call the resulting theory U .

We should prove the following theorem:

Theorem 13 *For every formula γ , if $NBG \vdash \gamma$, then $U \vdash \gamma$.*

Proof. In order to proof this theorem, we should just treat the case in which γ is the scheme axiom for classes. From this and from theorem 12, we obtain easily that U proves all other axioms of NBG.

We take a formula α with n free variables and in which all quantifications are bounded to V . We then prove that

$$U \vdash \forall \bar{v} \in V \exists z \forall x (x \in z \leftrightarrow (x \in V \wedge \alpha(x, \bar{v}))) \quad (1)$$

(We call this formula θ).

By replacement and generalization, we can eliminate the universal quantifications from the scheme axiom for classes. Thus we have

$$U \vdash \theta \iff U \vdash (\bar{v} \in V) \rightarrow \exists z \forall x (x \in z \leftrightarrow (x \in V \wedge \alpha(x, \bar{v}))). \quad (2)$$

Since x and z do not occur in the left side of the implication, then

$$U \vdash \theta \iff U \vdash \exists z \forall x ((\bar{v} \in V) \rightarrow (x \in z \leftrightarrow (x \in V \wedge \alpha(x, \bar{v}))). \quad (3)$$

Let θ' be the formula $\forall \bar{v} \in V \forall x \in V (x \in c_\alpha \leftrightarrow (x \in V \wedge \alpha(x, \bar{v})))$.

We now eliminate the quantifiers of θ' , obtaining

$$U \vdash \forall x((\bar{v} \in V) \rightarrow (x \in c_\alpha \leftrightarrow (x \in V \wedge \alpha(x, \bar{v}))))).$$

Thus, by replacement and *modus ponens*,

$$U \vdash \exists z \forall x((\bar{v} \in V) \rightarrow (x \in z \leftrightarrow (x \in V \wedge \alpha(x, \bar{v}))))).$$

Therefore, $U \vdash \theta$.

□

We continue by proving the lemma:

Lemma 14 *Let α be a sentence without variables for classes. If α is a theorem of U , then there is a proof of α in U that is free of unrestricted quantifications.*

Proof.

By hypothesis, we have a proof of α in U . Let $\gamma[\bar{x}]$ be the conjunction of axioms used in the given proof, in which \bar{x} is the sequence of variables for classes that occur in the axioms. In this case, be the reduction theorem [5, p. 42]:

$$T \vdash \forall \bar{x} \gamma[\bar{x}] \rightarrow \alpha,$$

or, equivalently,

$$T \vdash \exists \bar{x} (\gamma[\bar{x}] \rightarrow \alpha), \text{ for } \bar{x} \text{ does not occur in } \alpha,$$

in which T is the theory without non logical axioms in the language of U .

Let $\theta[\bar{x}]$ be a prenex form of $\gamma[\bar{x}] \rightarrow \alpha$. The formula $\theta[\bar{x}]$ is of the form

$$\exists \bar{z}_1 \forall \bar{y}_1 \dots \exists \bar{z}_k \forall \bar{y}_k \beta[\bar{x}, \bar{z}_1, \bar{y}_1, \dots, \bar{z}_k, \bar{y}_k].$$

From Herbrand's theorem provability equivalence,

$$T \vdash \exists \bar{x} \theta[\bar{x}]$$

if, and only if,

$$\vdash_{T_H} \exists \bar{x} \exists \bar{z}_1 \dots \exists \bar{z}_k \beta[\bar{x}, \bar{z}_1, \overline{f_1(\bar{x}, \bar{z}_1)}, \dots, \bar{z}_k, \overline{f_k(\bar{x}, \bar{z}_1, \dots, \bar{z}_k)}],$$

in which T_H is the theory obtained by the addition of function symbols in T , according to Herbrand's normal form $\exists \bar{x} \theta[\bar{x}]$.

Let β' be an open formula $\beta[\bar{x}, \bar{z}_1, \overline{f_1(\bar{x}, \bar{z}_1)}, \dots, \bar{z}_k, \overline{f_k(\bar{x}, \bar{z}_1, \dots, \bar{z}_k)}]$.

By Herbrand's theorem, there is a quasi-tautology $\beta'_1 \vee \dots \vee \beta'_m$, in which each β'_i is an instance of β' in the language of T_H .

Let's build the appropriate proof of α in U_{c+eq} , obtained from U by the addition of special constants, special axioms and special axioms of equivalence.

From start, we replace each free variable in $\beta'_1 \vee \dots \vee \beta'_m$ for special constants. The result is a disjunction of m sentences that are quasi-tautological in U_{c+eq} . This quasi-tautology is the starting point of the proof of α in U_{c+eq} .

Subsequently, we replace the occurrences of $f_1(\bar{a}, \bar{b}_1), \dots, f_k(\bar{a}, \bar{b}_1, \dots, \bar{b}_k)$ for sequences of appropriate special constants. For this, we follow the items 7 and 8 in the proof of Herbrand's theorem.

The result is the disjunction $\beta'_{1c} \vee \dots \vee \beta'_{mc}$, tautological consequence of instances of identity, equality, special axioms and special equivalence axioms, *having quantifications only for the variables $\bar{z}_1, \bar{y}_1, \dots, \bar{z}_k, \bar{y}_k$* .

Let θ_{ic} be the formula

$$\exists \bar{z}_1 \forall \bar{y}_1 \dots \exists \bar{z}_k \forall \bar{y}_k \beta[\bar{t}_i, \bar{z}_1, \bar{y}_1, \dots, \bar{z}_k, \bar{y}_k],$$

obtained from β_{ic} by the restating the quantifications $\exists \bar{z}_1 \forall \bar{y}_1 \dots \exists \bar{z}_k \forall \bar{y}_k$. The closed terms in \bar{t}_i are in U_{c+eq} .

Each disjunction θ_{ic} implies the corresponding β'_{ic} in U_{c+eq} , as we have seen in the first paragraph of item 9 in Herbrand's theorem proof. To show this, we use only simple properties of implication, of the quantifiers $\exists \bar{z}_1 \dots \exists \bar{z}_k$ and the special axioms used in the replacement of the occurrences $f_1(\bar{a}, \bar{b}_1), \dots, f_k(\bar{a}, \bar{b}_1, \dots, \bar{b}_k)$ described above. From what we have exposed, it follows that

$$U_{c+eq} \vdash \theta_{1c} \vee \dots \vee \theta_{mc},$$

without using quantifications over classes.

However, since $\theta[\bar{x}]$ is a prenex form of $\gamma[\bar{x}] \rightarrow \alpha$, we have

$$U_{c+eq} \vdash \theta[\bar{x}] \leftrightarrow (\gamma[\bar{x}] \rightarrow \alpha),$$

using only variations of quantifications that occur in $\gamma[\bar{x}] \rightarrow \alpha$. Therefore, no quantification for class variables. On the other hand, by the rule of substitution,

$$U_{c+eq} \vdash \theta[\overline{t_i}] \leftrightarrow (\gamma[\overline{t_i}] \rightarrow \alpha).$$

As $\theta[\overline{t_i}] \theta_{ic}$, we conclude, using tautological consequence, that

$$U_{c+eq} \vdash \gamma[\overline{t_1}] \wedge \dots \wedge \gamma[\overline{t_m}] \rightarrow \alpha,$$

without using quantification for class variables.

On the other hand, $\gamma[\overline{x}]$ is the conjunction of axioms in U_{c+eq} . Each $\gamma[\overline{t_1}]$ can be proved in U_{c+eq} using only tautological consequences and instances of the substitution rule. Therefore,

$$U_{c+eq} \vdash \alpha,$$

without using quantification for class variables.

Finally, we observe that any proof of α of U in U_{c+eq} can be transformed in a proof in U of the same α , and that this transformation introduces only quantifications directly related to the special axioms used [5, p. 52]. Since we haven't used special axioms for class variables, the result follows. \square

From this result, we prove by finitary means the equiconsistency result:

Theorem 15 *Let α be a sentence with all its quantifiers bounded to sets and such that $NBG \vdash \alpha$. Then, $ZF \vdash \alpha$.*

Proof. Let α be a sentence with all its quantifiers bounded to sets and such that $NBG \vdash \alpha$. Thus, by theorem 14, there is a proof in U in which no unbounded quantifications occur. We call this proof sequence *Seq*.

We make transformations in this proof sequence that preserve tautological consequences, do not affect α and such that the transformed axioms are theorems of ZF.

Let x_1, x_2, \dots, x_k be the free variables that occur in *Seq*. We add the following initial segment to the proof sequence $x_1 \in V, x_2 \in V, \dots, x_k \in V$, obtaining *Seq*^{*1}.

Next, we apply the transformation $*^{ZF}$ in the formulas in *Seq*^{*1}.

Every occurrence

1. $c_\theta = c_\alpha$, are replaced by $\forall y \in V (y \in c_\theta \leftrightarrow y \in c_\alpha)$
2. $c_\theta \in x$, are replaced by $\exists y \in V (y = c_\theta \wedge y \in x)$
3. $c_\theta \in c_\alpha$, are replaced by $\exists y \in V (y = c_\theta \wedge y \in c_\alpha)$

4. $x = c_\alpha$, are replaced by $\forall y \in V (y \in x \leftrightarrow y \in c_\alpha)$
5. $x \in c_\theta$, are replaced by $\theta(x)$

The successive application of this transformation to Seq^{*1} eliminate all occurrences of c_θ , forming the sequence Seq^{*2} .

Recall that, for each logical axiom $Axiom_j$ in Seq^{*1} , we should verify whether $Axiom_j^{*ZF}$ is also a logical axiom or a consequence of ZF together with formulas $x_1 \in V, x_2 \in V, \dots, x_k \in V$. In the second case, we replace each axiom $Axiom_j^{*ZF}$ in Seq^{*2} by the proof sequence of $Axiom_j^{*ZF}$. Hence, we obtain Seq^{*3} .

We know the following proposition about functors [5, p. 30]:

Proposition 16 *A functor $*$ of formulas to formulas satisfies for every formula α and β :*

1. $(\neg\alpha)^* \neg\alpha^*$
2. $(\alpha \vee \beta)^* \alpha^* \vee \beta^*$

Thus, if δ is tautological consequence of $\gamma_1, \gamma_2, \dots, \gamma_n$, then δ^ is tautological consequence of $\gamma_1^*, \gamma_2^*, \dots, \gamma_n^*$.*

Therefore, all transformations described above do not affect the proof tautologically. However, some instances of the logical axioms may not be logical axioms after the transformation. We now investigate what happen with logical axioms in which constants of U occur.

1. Substitution axiom

There is no instance of the substitution $\theta_x(c) \rightarrow \exists x\theta$ in Seq^{*1} since unrestricted quantifications do not occur in the initial proof in U .

2. Identity axiom

Note that $(c_\alpha = c_\alpha)^{*ZF}$ is $\forall y \in V (\alpha(y) \leftrightarrow \alpha(y))$. And this last one is a tautology.

3. Equality axiom

- (a) If the axiom is of the form $x_1 = c_\alpha \wedge x_2 = y_2 \rightarrow x_1 \in x_2 \leftrightarrow c_\alpha \in y_2$, then, after the transformation:

$$(\forall z \in V(z \in x_1 \leftrightarrow \alpha(z)) \wedge x_2 = y_2) \\ \rightarrow (x_1 \in x_2 \leftrightarrow \exists w \in V(\forall z \in V(z \in w \leftrightarrow \alpha(z)) \wedge w \in y_2))$$

We show that this formula is theorem of ZF and of the formulas $x_1 \in V, x_2 \in V, \dots, x_k \in V$.

If we have that $\forall z \in V(z \in x_1 \leftrightarrow \alpha(z)) \wedge x_2 = y_2$ and we suppose that $x_1 \in x_2$, then, by corollary 2 of the identity theorem in [5, p. 36], we obtain

$$\forall z \in V(z \in x_1 \leftrightarrow \alpha(z)) \wedge x_1 \in y_2.$$

Then, by the substitution axiom in e $x_1 \in V$,

$$\exists x_1 \in V(\forall z \in V(z \in x_1 \leftrightarrow \alpha(z)) \wedge x_1 \in y_2).$$

Using the variant theorem [5, p. 35],

$$\exists w \in V(\forall z \in V(z \in w \leftrightarrow \alpha(z)) \wedge w \in y_2).$$

On the other hand, if we suppose that $\exists w \in V(\forall z \in V(z \in w \leftrightarrow \alpha(z)) \wedge w \in y_2)$, then, since $\forall z \in V(z \in x_1 \leftrightarrow \alpha(z))$ and by extensionality in ZF, we prove that $\exists w \in V(w = x_1 \wedge w \in y_2)$. Hence, we obtain that $x_1 \in y_2$. As, from hypothesis, $x_2 = y_2$, we conclude that $x_1 \in x_2$, finalizing the proof.

- (b) $x_1 = y_1 \wedge c_\alpha = y_2 \rightarrow x_1 \in c_\alpha \leftrightarrow y_1 \in y_2$. The strategy for item *b* is similar to item *a*.
- (c) $x_1 = y_1 \wedge c_\alpha = c_\beta \rightarrow x_1 \in c_\alpha \leftrightarrow y_1 \in c_\beta$. For this case, we obtain from the transformation:
 $x_1 = y_1 \wedge \forall z \in V(\alpha(z) \leftrightarrow \beta(z)) \rightarrow \alpha(x_1) \leftrightarrow \beta(y_1)$.
 As x_1 and y_1 are free variables in the formula, the transformed formula is tautological consequence of $\forall x_1 \in V \forall x_2 \in V$ and instances of identity axioms.
- (d) $c_\beta = c_\alpha \wedge c_\gamma = y_2 \rightarrow c_\beta \in c_\gamma \leftrightarrow c_\alpha \in y_2$.
- (e) $c_\beta = c_\alpha \wedge x_2 = y_2 \rightarrow c_\beta \in x_2 \leftrightarrow c_\alpha \in y_2$.
- (f) $c_\beta = c_\alpha \wedge c_\gamma = c_\psi \rightarrow c_\beta \in c_\gamma \leftrightarrow c_\alpha \in c_\psi$.

We prove items d , e and f by a simple combination of the strategies used in a and c .

Let's now evaluate what occur to the axioms in U after the transformation $*_{ZF}$:

The instances of extensionality, scheme for classes, replacement for classes and foundations are the only ones we affect by applying $*_{ZF}$ (only in those axioms constants of U occur).

1. **Extensionality.** We evaluate two cases:

$$\forall y \in V (y \in x \leftrightarrow y \in c_\theta) \rightarrow x = c_\theta.$$

$$\forall y \in V (y \in c_\alpha \leftrightarrow y \in c_\theta) \rightarrow c_\alpha = c_\theta.$$

After $*_{ZF}$, we have respectively

$$\forall y \in V (y \in x \leftrightarrow \theta(x)) \rightarrow \forall y \in V (y \in x \leftrightarrow \theta(x))$$

$$\forall y \in V (\alpha(x) \leftrightarrow \theta(x)) \rightarrow \forall y \in V (\alpha(x) \leftrightarrow \theta(x)).$$

Both are tautologies.

2. **Scheme for classes.** Let v_1, v_2, \dots, v_n be free variables occurring in θ

$$\forall v_1 v_2 \dots v_n \in V \forall y \in V (y \in c_\theta \leftrightarrow \theta(y))$$

After $*_{ZF}$, we have

$$\forall v_1 v_2 \dots v_n \in V \forall y \in V (\theta(y) \leftrightarrow \theta(y))$$

And this is a tautology.

3. **Replacement for classes.**

$$\forall x \in V (func(c_\theta) \rightarrow \exists y \in V \forall w (w \in y \leftrightarrow \exists v \in x ((v, w) \in c_\theta)))$$

becomes

$$\forall x \in V (\forall v_1 v_2 v_3 \in V (\theta(v_1, v_2) \wedge \theta(v_1, v_3) \rightarrow v_2 = v_3) \rightarrow \exists y \in V \forall w_1 (w_1 \in y \leftrightarrow \exists w_2 \in x (\theta(w_2, w_1))))$$

Which is precisely the replacement axiom for ZF.

4. **Foundation.**

$$(c_\theta \neq \emptyset \rightarrow \exists y \in V (y \in c_\theta \rightarrow c_\theta \cap y = \emptyset))$$

becomes

$\exists x \in V\theta(x) \rightarrow \exists y \in V(\theta(y) \rightarrow \forall w(w \notin y \vee \neg\theta(w)))$;

We suppose $\exists x \in V\theta(x)$. Then $\vdash \exists x(x \in V_a \wedge \theta(x))$, for some ordinal a . Let $A = \{x \mid x \in V_a \wedge \theta(x)\}$, it follows that $ZF \vdash A \neq \emptyset$. By the axiom of foundation, $\exists y(y \in A \rightarrow A \cap y = \emptyset)$.

This is equivalent to

$$\begin{aligned} \exists y(\theta(y) \wedge y \in V_a \rightarrow \forall w \neg(w \in y \wedge \theta(w) \wedge w \in V_a)) \\ \text{(as } w \in y \rightarrow w \in V_a) \\ \exists y(\theta(y) \wedge y \in V_a \rightarrow \forall w \neg(w \in y \wedge \theta(w))) \end{aligned}$$

That is, the formula is a theorem of ZF.

When axioms of U occur without constants, they can be understood as axioms of ZF, since we have added the formulas $x_i \in V$.

1. Extensionality $\forall y \in V(y \in x \leftrightarrow y \in z) \rightarrow x = z$, since we have $x \in V$ and $z \in V$, represents extensionality in ZF.
2. The same is true for the axiom of replacement and foundation.
3. The axiom scheme for classes do not occur without constants in U .

Therefore, in the sequence Seq^{*3} , we have:

1. formulas of the form $x \in V$,
2. logical axioms,
3. axioms of ZF,
4. and all others are consequence of logical inferences from previous formulas in the sequence.

This is a proof in ZF. Since no transformation affect α , we have proved that $ZF \vdash \alpha$. □

Corollary 17 *If there is α such that $NBG \vdash \alpha \wedge \neg\alpha$, then there is a procedure that generate β such that $ZF \vdash \beta \wedge \neg\beta$.*

Proof. If $NBG \vdash \alpha \wedge \neg\alpha$, then NBG proves any formula. In particular, it proves a formula $\beta \rightarrow \beta$ in which all quantifications are bounded to sets. Thus, by the theorem, $ZF \vdash \beta \wedge \neg\beta$. □

4 There is no interpretation of NBG in ZF

Before we prove there is no interpretation of NBG in ZF, we show some definitions as propositions:

Definition 18 *Let V be a model of ZF and M a class V -definable, we say that the model $\mathcal{M} = (M, \epsilon^V)$ is a V -natural model.*

Definition 19 *Let \mathcal{M} be a model in the language \mathcal{L}_{ZF} (the only non logical symbol is membership) and an interpretation $I = \langle U, \phi \rangle$ of \mathcal{L}_{ZF} in \mathcal{L}_{ZF} (we write ϵ^I for $\phi(\epsilon)$), then we define the model $\mathcal{M}^I = (A, \epsilon^{I^{\mathcal{M}}})$ in \mathcal{L} as*

$$A = \{x \mid \mathcal{M} \models U(x)\} \text{ and } \epsilon^{I^{\mathcal{M}}} = \{\langle x, y \rangle \mid \mathcal{M} \models U(x) \wedge U(y) \rightarrow x \epsilon^I y\}$$

From this definition, we can easily prove by induction that:

Proposition 20 *Let \mathcal{M} be a model in \mathcal{L}_{ZF} and $I = \langle U, \epsilon^I \rangle$ be an interpretation of \mathcal{L}_{ZF} in \mathcal{L}_{ZF} , then, for all sentences α*

$$\mathcal{M} \models \alpha^I \iff \mathcal{M}^I \models \alpha$$

The following proposition is a strengthening of a result in [?]. They show that if the existence of a transitive model of ZF is consistent with ZF, then there cannot be a set-interpretation of ZF in ZF. Here, we replace that consistency condition for “ZF does not prove the inconsistent sentence for ZF itself”.

Proposition 21 *Let $I = \langle U, \epsilon^I \rangle$ be an interpretation of ZF in ZF, if $ZF \vdash \{x \mid U(x)\}$ is a set, then $ZF \vdash \neg \text{Con}(ZF)$.*

Proof. Take I as in the proposition and suppose $ZF \vdash \{x \mid U(x)\}$ is a set.

Let $\ulcorner \alpha \urcorner$ be the Gödel number of the formula α represented in ZF. Since \mathcal{M}^I is a set for every $\mathcal{M} \models ZF$, we define recursively the set T for each $\mathcal{M} \models ZF$:

Notation 22 $\bar{a}([k] = b)$ is the replacement of the k 'th element of the sequence \bar{a} for b .

$\langle \ulcorner \alpha \urcorner, \bar{a} \rangle \in T$ if, and only if, $\bar{a} \in M^I$ and

- (1) if α is atomic of the form $x_i \in x_j, \langle a_i, a_j \rangle \in (\in^{M^I})$
- (2) if α is of the form $\beta \wedge \gamma : \langle \ulcorner \beta \urcorner, \bar{a} \rangle \in T$ and $\langle \ulcorner \gamma \urcorner, \bar{a} \rangle \in T$
- (3) if α is of the form $\neg\beta : \langle \ulcorner \beta \urcorner, \bar{a} \rangle \notin T$
- (4) if α is of the form $\neg\exists x_k \beta : \langle \ulcorner \beta \urcorner, \bar{a}([k] = b) \rangle \in T$ for some $b \in M^I$

By finite induction over the formula complexity, we prove that

$$\mathcal{M} \models \langle \ulcorner \varphi \urcorner, \bar{a} \rangle \in T \text{ if, and only if, } \mathcal{M}^I \models \varphi(\bar{a})$$

Take $Pr_{ZF}(x, y)$ to be the provability predicate for ZF defined in ZF and representing the statement “ x is the number of the proof y ”. Then, we say that $Th(ZF) = \{y \mid \exists x Pr_{ZF}(x, y)\}$. Hence, since $\mathcal{M}^I \models ZF$, $\mathcal{M} \models Th(ZF) \subseteq T$. As $\mathcal{M}^I \models \emptyset \in \emptyset$, we have $\ulcorner \emptyset \in \emptyset \urcorner \notin T$. From this, we obtain $\mathcal{M} \models \ulcorner \emptyset \in \emptyset \urcorner \notin Th(ZF)$. Once \mathcal{M} is arbitrary, by the completeness theorem, if ZF has a model, then $ZF \vdash \ulcorner \emptyset \in \emptyset \urcorner \notin Th(ZF)$. This is an absurd by Gödel’s incompleteness theorem. Thus, $ZF \vdash \neg Con(ZF)$. □

Definition 23 *Let V be a model of ZF and \mathcal{M} a V -natural model. We say that \mathcal{M} reflect a formula $\varphi(\bar{x})$ if, and only if, for every $\bar{a} \in M$*

$$V \models \varphi(\bar{a}) \iff \mathcal{M} \models \varphi(\bar{a})$$

Theorem 24 *[Reflection theorem] [6, p. 168] Let V be a model of ZF and $\phi_1, \phi_2, \dots, \phi_n$ any sequence of formulas, then, there is an ordinal α such that $\mathcal{M} = (V_\alpha, \in)$ reflect ϕ_i for i between 1 and n .*

We show that the desired result is a consequence of the reflection theorem and the fact that NBG is finitely axiomatizable:

Theorem 25 *There is no interpretation of NBG in ZF.*

Proof.

Suppose there is an interpretation I of NBG in ZF.

Since the number of axioms in NBG can be said to be finite, there is a formula α that is equivalent to the conjunction of all NBG’s axioms:

$$\alpha \text{ is } Axiom_1 \wedge Axiom_2 \wedge \dots \wedge Axiom_n.$$

Thus, $NBG \vdash \alpha$ and, from the interpretation theorem for first-order logic, $ZF \vdash \alpha^I$.

Suppose $V \models ZF$, then $V \models \alpha^I$. By the reflection theorem, there is an ordinal a such that $V_a \models \alpha^I$. It follows that $V_a^I \models \alpha$.

Since V_a is a set, we obtain that the domain in V_a^I is also a set.

We define the model V^* in \mathcal{L}_{ZF} :

1. The domain D in V^* is such that $D = \{x \mid V_a^I \models \exists y(x \in y)\}$.
2. The predicate $\epsilon^{V^*} = \{\langle x, y \rangle \mid V_a^I \models x \in y\}$.

Since NBG proves the restriction to sets of all ZF axioms, we have that $V^* \models ZF$. Since the domain V_a^I is a set, it follows that D is also a set. It means that $ZF \vdash Con(ZF)$, absurd by the incompleteness theorem. □

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